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Nash Equilibria in Traffic Networks with Multiple Populations and Origins–Destinations

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Abstract
Different populations of vehicles travel along a network. Each population has its origin, destination and travel costs — which may well be unbounded. Under the only requirement of the continuity of the travel costs, we prove the existence of a Nash equilibrium for all populations. Conditions for its uniqueness are also provided. A few cases are treated in detail to show specific situations of interest.

Keywords Nash Equilibria on Traffic Networks · Multi-Populations Braess Paradox · Multi-Populations Models on Networks

Mathematics Subject Classification 90B20 · 91A80 · 91B74

1 Introduction

A road network is used by different vehicles, grouped into *populations* according to their origin, their destination and their travel costs. For each population, the origin is connected to the destination by different *routes*, each consisting of a sequence of adjacent *roads*. Traveling along a road bears a *cost*, in general different for each population, which may be related to gas consumption, pollutant production or travel time. In the latter case, it is reasonable to admit that it can become infinite, for instance when congestions occur. A road can be shared by drivers from various populations, with travel costs varying according to the level of congestion. Each traveler chooses a

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route among those available to its population and, correspondingly, pays a *route cost* which is the sum of the costs to its population of the roads constituting that route.

It is then natural to consider several *concurrent* and *interdependent games*, each played by the vehicles in a single population. According to the *selfish paradigm* [25], [26], [27], each individual chooses a route such that no different choice could be more convenient. In other words, all populations distribute among the different routes so that each game is at a *Nash equilibrium* and the whole network is at a *global* Nash equilibrium.

Below, under rather general assumptions, we prove the existence of such global Nash equilibria. At these configurations, for each population, all vehicles in that population pay the same cost. Moreover, at the equilibria we exhibit, no individual, whatever the population it belongs to, finds convenient to change its route choice. The costs of different populations may well be unrelated, in the sense that a population can be interested in minimizing travel time, another in gas consumption, and so on.

In the case of a single population, frameworks essentially equivalent to the one presented below are classical, see for instance [10], [13], [18], [26], [27]. In general, existence proofs of Nash equilibria rely on the considered game being a *potential game*, see [2], [10], a property that here fails due to the presence of multiple populations. A different approach to multiple populations competing on the same network is presented in [15], where a measure based framework is employed. Nash equilibria in multi-agent systems are considered, for instance, in [17].

As the number of populations and the complexity increase, it is realistic to expect that multiple (global) Nash equilibria may occur. Thus, conditions ensuring uniqueness in general situations become more intricate and require stricter assumptions. First, we present an equality satisfied by all (global) Nash equilibria. As long as the number of populations is small and the networks are simple, this condition may ensure uniqueness, as examples show. In the Appendix, we also present a general uniqueness theorem.

We stress that in the present, multi-population, setting, a definition of a *globally optimal*[1] strategy appears as quite arbitrary, due to the presence of multiple cost functions. It is then open to further investigations to extend to the present case the many relevant results, see e.g. [26], [27], about the comparison between globally optimal strategies and Nash equilibria.

The present multi-population setting allows the appearance of the well known Braess paradox [6] in a variety of new situations. Since we comprehend also the case of populations using the same network but consisting of different vehicles, say trucks and cars, we show that Braess paradox may appear for both populations, see § 3.3.1. Otherwise, the insertion of a new road for only one population may well increase travel cost at Nash equilibrium for another population, not directly affected by the new road, see § 3.3.2.

To reduce formal complexity, we consider the case of 2 populations, i.e., of 2 origins and 2 destinations. The 2 populations are referred to as the *hat* and *check* population. The extension to finitely many populations requires merely notational modifications.

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[1] Also referred to as *social optimum*, conforming to Wardrop's *Second principle*, see [29, p. 345].

The next Section 2 presents the formal setting and the general existence result. All analytic proofs are left to Section 4. Specific cases are presented in Section 3. The final Appendix is devoted to a general uniqueness result.

2 Formal Framework and Existence Result

We consider N one way roads $r_1, \ldots, r_N$ having at least one end point (a junction) in common with other roads. At each junction there is at least one entering and one exiting road. Two roads are adjacent if the end point of one of the two roads is the initial point of the other one. A route γ is an m–tuple of adjacent (pairwise distinct) roads, say $\gamma \equiv (r_{h_1}, \ldots, r_{h_m})$, such that the end point of r_{h_ℓ} is the initial point of $r_{h_{\ell+1}}$, for $\ell \in \{1, \ldots, m-1\}$. By *network* $\mathcal{N}$ we mean the set of routes.

We assume there are 2 origin–destination pairs, say $(\widehat{\mathcal{O}}, \widehat{\mathcal{D}})$ and $(\widecheck{\mathcal{O}}, \widecheck{\mathcal{D}})$. Correspondingly, we call $\widehat{\mathcal{N}}$ the sub–network of $\mathcal{N}$ consisting of routes $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_{\widehat{n}}$ connecting $\widehat{\mathcal{O}}$ to $\widehat{\mathcal{D}}$ and, similarly we call $\widecheck{\mathcal{N}}$ the sub-network of $\mathcal{N}$ consisting of routes $\widecheck{\gamma}_1, \ldots, \widecheck{\gamma}_{\widecheck{n}}$ connecting $\widecheck{\mathcal{O}}$ to $\widecheck{\mathcal{D}}$. Clearly, $\widehat{n} \leq N$ and $\widecheck{n} \leq N$.

In the terminology of graph theory, both networks $\widehat{\mathcal{N}}$ and $\widecheck{\mathcal{N}}$ are (connected) directed acyclic graphs (DAG) with a single sink and a single source.

As it is usual, see e.g. [8], construct the $N \times \widehat{n}$ matrix $\widehat{\Gamma}$ and the $N \times \widecheck{n}$ matrix $\widecheck{\Gamma}$ setting

$$\widehat{\Gamma}_{hi} := \begin{cases} 1 & \text{road } r_h \text{ belongs to route } \widehat{\gamma}_i, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for } h \in \{1, \ldots, N\} \text{ and } i \in \{1, \ldots, \widehat{n}\}\,;$$

$$\widecheck{\Gamma}_{hi} := \begin{cases} 1 & \text{road } r_h \text{ belongs to route } \widecheck{\gamma}_i, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for } h \in \{1, \ldots, N\} \text{ and } i \in \{1, \ldots, \widecheck{n}\}\,.$$

A single road may well belong to more than one route and to one or both the subnetworks $\widehat{\mathcal{N}}$ and $\widecheck{\mathcal{N}}$. A structural assumption of use below on the subnetworks $\widehat{\mathcal{N}}$ and $\widecheck{\mathcal{N}}$ is the following condition:

(Γ) Each route in $\widehat{\mathcal{N}}$ contains a road that is not contained in any other route in $\widehat{\mathcal{N}}$. Similarly, each route in $\widecheck{\mathcal{N}}$ contains a road that is not contained in any other route in $\widecheck{\mathcal{N}}$.

Clearly, the above condition implies that both matrices $\widehat{\Gamma}$ and $\widecheck{\Gamma}$ contain a copy of the identity $\mathrm{Id}_{\widehat{n}}$ or $\mathrm{Id}_{\widecheck{n}}$. Hence $\widehat{\Gamma}$ and $\widecheck{\Gamma}$ are both full rank: $\mathrm{rnk}\widehat{\Gamma} = \widehat{n}$ and $\mathrm{rnk}\widecheck{\Gamma} = \widecheck{n}$.

Up to a renormalization, see [4, Section 1], we assume that the total amount of vehicles driving along each of the networks $\widehat{\mathcal{N}}$ and $\widecheck{\mathcal{N}}$ is 1. According to their choices, $\widehat{\vartheta}_i$ travelers of the first population choose the route $\widehat{\gamma}_i$ for $i \in \{1, \ldots, \widehat{n}\}$ and $\widecheck{\vartheta}_i$ travelers of the second population choose $\widecheck{\gamma}_i$ for $i \in \{1, \ldots, \widecheck{n}\}$. Clearly, $\widehat{\vartheta} \equiv (\widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_{\widehat{n}})$, respectively $\widecheck{\vartheta} \equiv (\widecheck{\vartheta}_1, \ldots, \widecheck{\vartheta}_{\widecheck{n}})$, is a point in the $\widehat{n} - 1$ dimensional simplex $S^{\widehat{n}}$, respectively in the $\widecheck{n} - 1$ dimensional simplex $S^{\widecheck{n}}$. We identify $\widehat{\vartheta}$ and $\widecheck{\vartheta}$ with column vectors. Thus, the total number of vehicles along the road r_h is $\widehat{\Gamma}_h\, \widehat{\vartheta} + \widecheck{\Gamma}_h\, \widecheck{\vartheta} = \sum_{i=1}^{\widehat{n}} \widehat{\Gamma}_{hi}\, \widehat{\vartheta}_i + \sum_{i=1}^{\widecheck{n}} \widecheck{\Gamma}_{hi}\, \widecheck{\vartheta}_i$.

In general, each road r_h is equipped with its *costs* $\widehat{\tau}_h = \widehat{\tau}_h(\widehat{\eta}, \widecheck{\eta})$ for the hat population and $\widecheck{\tau}_h = \widecheck{\tau}_h(\widehat{\eta}, \widecheck{\eta})$ for the check population, where $\widehat{\eta}$, respectively $\widecheck{\eta}$, is the amount of hat, respectively check, population traveling along r_h. When dealing with, say, trucks and cars they may be the costs (travel times) for each of the two populations, for instance. In the case of populations consisting of homogeneous vehicles differing only in their origins–destinations, it is reasonable for example to set $\widehat{\tau}_h = \widecheck{\tau}_h$ for all $h \in \{1, \dots, N\}$. Whenever the vehicles of the two populations are homogeneous and the total numbers of vehicles of each population are the same, it is likely that $\widehat{\tau}_h$ and $\widecheck{\tau}_h$ are functions of the sum $\widehat{\eta} + \widecheck{\eta}$. However, the normalization of different total numbers of travelers to 1 suggests to consider also general functions $\widehat{\tau}_h = \widehat{\tau}_h(\widehat{\eta}, \widecheck{\eta})$ and $\widecheck{\tau}_h = \widecheck{\tau}_h(\widehat{\eta}, \widecheck{\eta})$.

It is of interest to allow also for the case of fully congested roads. Therefore, we allow these costs to attain the value $+\infty$. To this aim, we introduce the following classes of functions.

We call a map $\tau\colon [0,1]^2 \to \mathbb{R}_+ \cup \{+\infty\}$ *weakly increasing in both variables* if

$$\begin{array}{lll} \forall\, \widehat{\eta}, \widecheck{\eta}', \widecheck{\eta}'' \in [0,1] & \widecheck{\eta}' \le \widecheck{\eta}'' \implies \tau(\widehat{\eta}, \widecheck{\eta}') \le \tau(\widehat{\eta}, \widecheck{\eta}'')\,; \\ \forall\, \widehat{\eta}', \widehat{\eta}'', \widecheck{\eta} \in [0,1] & \widehat{\eta}' \le \widehat{\eta}'' \implies \tau(\widehat{\eta}', \widecheck{\eta}) \le \tau(\widehat{\eta}'', \widecheck{\eta})\,. \end{array}$$

Then, we introduce the class of continuous functions weakly increasing in both variables attaining values in $\mathbb{R}_+ \cup \{+\infty\}$:

$$\mathcal{C} := \left\{ \tau \in \mathbf{C}^0([0,1]^2; \mathbb{R}_+ \cup \{+\infty\}) \colon \tau \text{ is weakly increasing in both variables} \right\}.$$

Below, most results require one of the following assumptions on all cost functions:

(C1) For all $h \in \{1, \dots, N\}$, both travel times $\widehat{\tau}_h, \widecheck{\tau}_h$ are in $\mathcal{C}$ and admit a continuous derivative at every point η where they attain finite values.
(C2) For all $h \in \{1, \dots, N\}$, both travel times $\widehat{\tau}_h, \widecheck{\tau}_h$ are in $\mathcal{C}$ and are convex.

It is then useful to introduce the maps,

$$\begin{array}{rcl} \widehat{\tau}\,\colon [0,1]^N \times [0,1]^N & \to & \mathbb{R}_+^N \\ \widehat{\eta}, \widecheck{\eta} & \mapsto & \widehat{\tau}(\widehat{\eta}, \widecheck{\eta}) \text{ where } \big(\widehat{\tau}(\widehat{\eta}, \widecheck{\eta})\big)_h = \widehat{\tau}_h(\widehat{\eta}_h, \widecheck{\eta}_h) \quad \text{for} \quad h \in \{1, \dots, N\}\,; \\ \widecheck{\tau}\,\colon [0,1]^N \times [0,1]^N & \to & \mathbb{R}_+^N \\ \widehat{\eta}, \widecheck{\eta} & \mapsto & \widecheck{\tau}(\widehat{\eta}, \widecheck{\eta}) \text{ where } \big(\widecheck{\tau}(\widehat{\eta}, \widecheck{\eta})\big)_h = \widecheck{\tau}_h(\widehat{\eta}_h, \widecheck{\eta}_h) \quad \text{for} \quad h \in \{1, \dots, N\}\,. \end{array} \tag{2.1}$$

To shorten the notation, we often set $\tau \equiv (\widehat{\tau}, \widecheck{\tau})$, $\eta \equiv (\widehat{\eta}, \widecheck{\eta})$ and $\vartheta \equiv (\widehat{\vartheta}, \widecheck{\vartheta})$.

Denote by $\widehat{T}_i(\vartheta)$ and $\widecheck{T}_i(\vartheta)$ the route travel times along the routes $\widehat{\gamma}_i$ and $\widecheck{\gamma}_i$, while $\widehat{T}(\vartheta)$ and $\widecheck{T}(\vartheta)$ are the vectors of all travel times of each population, so that

$$\begin{array}{l} \widehat{T}(\vartheta) := \widehat{\Gamma}^{\intercal}\, \widehat{\tau}(\widehat{\Gamma}\,\widehat{\vartheta}, \widecheck{\Gamma}\,\widecheck{\vartheta}) \ \text{ and } \ \widehat{T}_i(\vartheta) = \sum_{h=1}^N \widehat{\Gamma}_{hi}\, \widehat{\tau}_h(\widehat{\Gamma}_h\,\widehat{\vartheta}, \widecheck{\Gamma}_h\,\widecheck{\vartheta})\,,\ i \in \{1, \dots, \widehat{n}\}\,; \\ \widecheck{T}(\vartheta) := \widecheck{\Gamma}^{\intercal}\, \widecheck{\tau}(\widehat{\Gamma}\,\widehat{\vartheta}, \widecheck{\Gamma}\,\widecheck{\vartheta}) \ \text{ and } \ \widecheck{T}_i(\vartheta) = \sum_{h=1}^N \widecheck{\Gamma}_{hi}\, \widecheck{\tau}_h(\widehat{\Gamma}_h\,\widehat{\vartheta}, \widecheck{\Gamma}_h\,\widecheck{\vartheta})\,,\ i \in \{1, \dots, \widecheck{n}\}\,. \end{array} \tag{2.2}$$

From a global point of view, it is natural to evaluate the quality of a network through the mean route travel times[2] resulting from the partitions $\widehat{\vartheta}$ and $\widecheck{\vartheta}$:

$$\begin{array}{lll} \widehat{T}_M(\vartheta) = \widehat{\vartheta}^\intercal\ \widehat{T}(\vartheta) & \text{or, equivalently,} & \widehat{T}_M(\vartheta) := \sum_{i=1}^{\widehat{n}} \widehat{\vartheta}_i\ \widehat{T}_i(\vartheta)\,; \\ \widecheck{T}_M(\vartheta) = \widecheck{\vartheta}^\intercal\ \widecheck{T}(\vartheta) & \text{or, equivalently,} & \widecheck{T}_M(\vartheta) := \sum_{i=1}^{\widecheck{n}} \widecheck{\vartheta}_i\ \widecheck{T}_i(\vartheta)\,. \end{array} \tag{2.3}$$

Recall the following basic definition inspired from [8, Definition 3.1].

Definition 2.1 Given a state $\vartheta \in S^{\widehat{n}} \times S^{\widecheck{n}}$, we call *hat relevant*, respectively *check relevant* those route travel times $\widehat{T}_i(\vartheta)$, respectively $\widecheck{T}_i(\vartheta)$, such that $\widehat{\vartheta}_i \neq 0$, respectively $\widecheck{\vartheta}_i \neq 0$. A state $\vartheta^* \in S^{\widehat{n}} \times S^{\widecheck{n}}$ is an *equilibrium state* if all relevant route travel times of each population coincide, i.e.,

$$\begin{array}{lll} \text{for all } i, j \in \{1, \ldots, \widehat{n}\} & \text{if } \widehat{\vartheta}_i^* \neq 0 \text{ and } \widehat{\vartheta}_j^* \neq 0, & \text{then } \widehat{T}_i(\vartheta^*) = \widehat{T}_j(\vartheta^*)\,; \\ \text{for all } i, j \in \{1, \ldots, \widecheck{n}\} & \text{if } \widecheck{\vartheta}_i^* \neq 0 \text{ and } \widecheck{\vartheta}_j^* \neq 0, & \text{then } \widecheck{T}_i(\vartheta^*) = \widecheck{T}_j(\vartheta^*)\,. \end{array}$$

The common value of the relevant hat/check route travel times is the *hat/check equilibrium time.*

In other words, at equilibrium all drivers of the same population need the same time to go from that population's origin to that population's destination. Clearly, if $\widehat{\vartheta}^*$ is an extreme point of $S^{\widehat{n}}$ and $\widecheck{\vartheta}^*$ is an extreme point of $S^{\widecheck{n}}$, then ϑ^* is an equilibrium. Moreover, clearly, at equilibrium the common value of the relevant travel times is the mean route travel time. However, at equilibrium, the mean route travel time needs not be optimal.

The idea of Nash equilibrium [19], see also [7, Definition 1.3], corresponds to a situation where no player finds convenient to change strategy. If an equilibrium ϑ^* lies along the boundary of $S^{\widehat{n}} \times S^{\widecheck{n}}$, so that $\widehat{\vartheta}_j^* = 0$ (or $\widecheck{\vartheta}_i^* = 0$) simple continuity considerations ensure that if $\widehat{T}_j(\vartheta^*) < \widehat{T}_M(\vartheta^*)$ (or $\widecheck{T}_j(\vartheta^*) < \widecheck{T}_M(\vartheta^*)$) then passing to the j-th route is convenient for some hat (or check) drivers.

Definition 2.2 An equilibrium state ϑ^* is a *Nash equilibrium* if for $j \in \{1, \ldots, n\}$

$$\begin{array}{l} \forall\, i \in \{1, \ldots, \widehat{n}\}\ \widehat{\vartheta}_i^* = 0 \implies \widehat{T}_i(\vartheta^*) \geq \widehat{T}_M(\vartheta^*)\,; \\ \forall\, i \in \{1, \ldots, \widecheck{n}\}\ \widecheck{\vartheta}_i^* = 0 \implies \widecheck{T}_i(\vartheta^*) \geq \widecheck{T}_M(\vartheta^*)\,. \end{array} \tag{2.4}$$

In the search for equilibria, comparing travel times corresponding to different distributions of drivers plays a key role. Assume that ε drivers pass from the i-th route $\widehat{\gamma}_i$ to the j-th route $\widehat{\gamma}_j$, so that $\widehat{\vartheta}$ becomes $\widehat{\vartheta} - \varepsilon\, e_i + \varepsilon\, e_j$[3]. Then, the present framework is compatible with the obvious observation that the travel time $\widehat{T}_i(\vartheta)$ along the i-th route $\widehat{\gamma}_i$ does not increase.

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[2] Also referred to as *average latency* of the system or *social cost* of the network.

[3] As usual, $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$, and we use the same notation in $\mathbb{R}^{\widehat{n}}$ and $\mathbb{R}^{\widecheck{n}}$.

Lemma 2.1 *Let (C1) hold. For all* $i, j \in \{1, \ldots, \widehat{n}\}$ *and for all* $\vartheta \in S^{\check{n}} \times S^{\widehat{n}}$,

$$\begin{aligned} \widehat{\vartheta}_i > 0 \text{ and } \widehat{T}_i(\vartheta) < +\infty &\implies \forall\, \varepsilon \text{ sufficiently small } \quad \widehat{T}_i(\vartheta) \geq \widehat{T}_i(\widehat{\vartheta} - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta})\,; & (2.5)\\ \widehat{\vartheta}_j < 1 \text{ and } \widehat{T}_j(\vartheta) < +\infty &\implies \forall\, \varepsilon \text{ sufficiently small } \quad \widehat{T}_j(\vartheta) \leq \widehat{T}_j(\widehat{\vartheta} + \varepsilon\, e_j - \varepsilon\, e_i, \check{\vartheta}) & (2.6)\\ & \qquad \text{and} \quad \widehat{T}_j(\widehat{\vartheta} + \varepsilon\, e_j - \varepsilon\, e_i, \check{\vartheta}) < +\infty\,. & (2.7) \end{aligned}$$

An entirely analogous statement holds for the check population.

A further general monotonicity property is provided by Lemma 4.3.

In the literature, see e.g. [26, Definition 2.2.1], Nash equilibria are also related to a small percentage of players changing strategy. Differently from (2.5)–(2.6), this condition requires a comparison between travel times along different routes, as defined below. The following definition is inspired, for instance, by [8, Definition 3.3] or [27, Definition 2.1].

Definition 2.3 An equilibrium state $\vartheta^* \in S^{\widehat{n}} \times S^{\check{n}}$ is an ε-*-Nash Equilibrium* if for all sufficiently small ε and

$$\begin{aligned} &\text{if } i, j \in \{1, \ldots, \widehat{n}\} \text{ and } \widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j \in S^{\widehat{n}} && \text{then} && \widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) \geq \widehat{T}_i(\vartheta^*)\,;\\ &\text{if } i, j \in \{1, \ldots, \check{n}\} \text{ and } \check{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j \in S^{\check{n}} && \text{then} && \check{T}_j(\widehat{\vartheta}^*, \check{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j) \geq \check{T}_i(\vartheta^*)\,. \end{aligned}$$

In other words, for ε drivers there is no gain in changing from route $\widehat{\gamma}_i$ to route $\widehat{\gamma}_j$. Note that in the literature, the term "*ε–Nash Equilibrium*" often refers to approximate Nash equilibria, where no player would lower its cost by more than ε when changing strategy, see [21, § 2.6.6].

The following theorem ensures the equivalence of Definition 2.2 and Definition 2.3 in all cases of interest.

Theorem 2.1 *If travel times are continuous, ε–Nash equilibria are also Nash equilibria. Moreover, either of the conditions (C1) or (C2) ensures that any Nash equilibrium is also an ε–Nash equilibrium.*

The extension to $\widehat{\tau}_h \in \mathbf{C}^1$ and $\check{\tau}_h$ convex for all h, or *vice versa*, is immediate.

Intuitive connections between globally optimal configurations and Nash equilibria are confirmed and formalized by the next Proposition.

Proposition 2.1 *Let (C1) hold.*

(NE1) If $\vartheta^* \in S^{\widehat{n}} \times S^{\check{n}}$ is an equilibrium such that

$$\widehat{T}_M(\vartheta^*) = \min_{\widehat{\vartheta} \in S^{\widehat{n}}} \widehat{T}_M(\widehat{\vartheta}, \check{\vartheta}^*) \quad \text{and} \quad \check{T}_M(\vartheta^*) = \min_{\check{\vartheta} \in S^{\check{n}}} \check{T}_M(\widehat{\vartheta}^*, \check{\vartheta})\,, \tag{2.8}$$

then ϑ^* is a Nash equilibrium.

(NE2) If both $\widehat{\vartheta}^*$ and $\check{\vartheta}^*$ are extreme points of $S^{\widehat{n}}$ and $S^{\check{n}}$ satisfying (2.8), then ϑ^* is a Nash equilibrium.

The following result, inspired by [19, Theorem 1], is based on Brouwer Fixed Point Theorem, see e.g. [20, Theorem 1.6.2]. Continuity of the road travel times suffices to ensure the existence of Nash equilibria.

Theorem 2.2 *Let all road travel times* $(\widehat{\tau}_1, \check{\tau}_1), \ldots, (\widehat{\tau}_N, \check{\tau}_N)$ *be in* $\mathbf{C}^0([0,1]^2; (\mathbb{R}_+ \cup \{+\infty\})^2)$. *Then, there exists a Nash equilibrium* $\vartheta^* \in S^{\widehat{n}} \times S^{\check{n}}$, *in the sense of Definition 2.2.*

We now proceed to present a condition satisfied by multiple concurrent Nash equilibria. As shown in § 3, it is a useful tool to prove the uniqueness of Nash equilibria.

Proposition 2.2 *Let (C1) hold. If* ϑ' *and* ϑ'' *are Nash equilibria, then*

$$(\widehat{\vartheta}'' - \widehat{\vartheta}')^\intercal \left(\widehat{T}(\vartheta'') - \widehat{T}(\vartheta')\right) = 0 \quad \textit{and} \quad (\check{\vartheta}'' - \check{\vartheta}')^\intercal \left(\check{T}(\vartheta'') - \check{T}(\vartheta')\right) = 0\,. \tag{2.9}$$

Note that whenever $\widehat{n} = \check{n} = 2$, the above condition (2.9), complemented with the constraints $\widehat{\vartheta}'_1 + \widehat{\vartheta}'_2 = \widehat{\vartheta}''_1 + \widehat{\vartheta}''_2$ and $\check{\vartheta}'_1 + \check{\vartheta}'_2 = \check{\vartheta}''_1 + \check{\vartheta}''_2$ leads to a (possibly non-linear) system of 4 equations in 4 variables and, hence, may ensure the uniqueness of the Nash equilibrium.

3 Specific Cases

Here we highlight the meaning of the analytic structure introduced in Section 2.

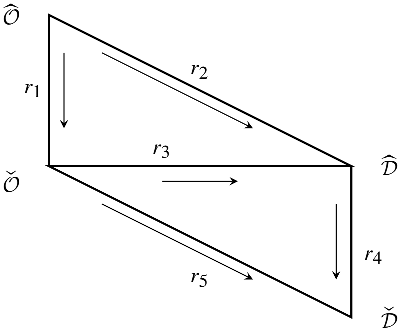
As a first example we present a pathological situation where, in the case of a single population, the equilibria in Definition 2.2 and Definition 2.3 are different. Let $N = 2$, $n = 2$, $\Gamma = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ and set $\tau_1(\eta) = 1 + 3\eta$, $\tau_2(\eta) = 3 - \eta$. Then, $(1,0)$ and $(0,1)$ are equilibria but not Nash equilibria; $(1/2, 1/2)$ is a Nash equilibrium but not an ε–Nash equilibrium. In particular, there is no ε–Nash equilibrium.

Below, to simplify the presentation, we define only those travel times that are necessary, i.e., if the road r_h does not belong to $\widehat{\mathcal{N}}$, then $\widehat{\tau}_h$ is not defined.

3.1 Nash Equilibria – a Simple Case

A well known feature of traffic on networks [14], [23], [30] are the *non-local* consequences of changes in a road's travel time. We see below that the insurgence of a delay on a road may have effects on routes geographically distant and, apparently, completely independent.

Assume that two different populations of vehicles move in the following network, where $N = 5$, $\widehat{n} = 2$, $\check{n} = 2$:



$$\begin{aligned} \widehat{\gamma}_1 &= (r_1, r_3) \\ \widehat{\gamma}_2 &= r_2 \\ \check{\gamma}_1 &= (r_3, r_4) \\ \check{\gamma}_2 &= r_5 \end{aligned} \qquad \begin{aligned} \widehat{\tau}_1(\eta) &= 1 + \widehat{\eta} \\ \widehat{\tau}_2(\eta) &= 3 + \widehat{\eta} \\ \widehat{\tau}_3(\eta) &= 1 + \widehat{\eta} + \check{\eta} \end{aligned} \qquad \begin{aligned} \check{\tau}_4(\eta) &= 1 + \check{\eta} \\ \check{\tau}_5(\eta) &= 3 + \check{\eta} \\ \check{\tau}_3(\eta) &= 1 + \widehat{\eta} + \check{\eta} \end{aligned}$$

$$\widehat{\Gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \check{\Gamma} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The first population moves from the origin $\widehat{\mathcal{O}}$ to the destination $\widehat{\mathcal{D}}$; $\widehat{\vartheta}_1$ travelers go along the central route $\widehat{\gamma}_1 = (r_1, r_3)$, while $\widehat{\vartheta}_2$ travelers move along the northern highway $\widehat{\gamma}_2 = r_2$. The second population moves from the origin $\check{\mathcal{O}}$ to the destination $\check{\mathcal{D}}$ and $\check{\vartheta}_1$ travelers move from the origin $\check{\mathcal{O}}$ to the destination $\check{\mathcal{D}}$ along the central route $\check{\gamma}_1 = (r_3, r_4)$, while $\check{\vartheta}_2$ travelers move along the southern highway $\check{\gamma}_2 = r_5$. Road r_3 is used by both populations. We thus have

$$\begin{aligned} \widehat{T}_1(\vartheta) &= 2 + 2\widehat{\vartheta}_1 + \check{\vartheta}_1\,; \qquad & \check{T}_1(\vartheta) &= 2 + \widehat{\vartheta}_1 + 2\check{\vartheta}_1\,; \\ \widehat{T}_2(\vartheta) &= 3 + \widehat{\vartheta}_2\,; & \check{T}_2(\vartheta) &= 3 + \check{\vartheta}_2\,. \end{aligned}$$

The point $\vartheta^* \equiv ((1/2, 1/2), (1/2, 1/2))$ is a Nash equilibrium with corresponding travel times $\widehat{T}_i(\vartheta^*) = \check{T}_j(\vartheta^*) = 7/2$ for $i, j \in \{1, 2\}$. A straightforward application of Proposition 2.2 ensures that ϑ^* is the unique Nash equilibrium.

Assume now that travel time increases on the northern highway $\widehat{\gamma}_2 = r_2$ by a fixed delay Δ due, for instance, to construction works or to an accident, so that we have

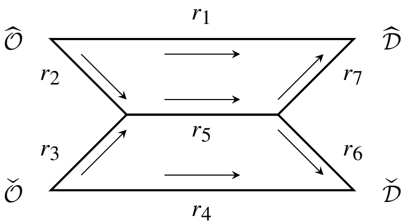
$$\begin{aligned} \widehat{\tau}_1(\eta) &= 1 + \widehat{\eta} \\ \widehat{\tau}_2(\eta) &= 3 + \widehat{\eta} + \Delta \\ \widehat{\tau}_3(\eta) &= 1 + \widehat{\eta} + \check{\eta} \end{aligned} \qquad \begin{aligned} \check{\tau}_4(\eta) &= 1 + \check{\eta} \\ \check{\tau}_5(\eta) &= 3 + \check{\eta} \\ \check{\tau}_3(\eta) &= 1 + \widehat{\eta} + \check{\eta} \end{aligned} \qquad \begin{aligned} \widehat{T}_1(\vartheta) &= 2 + 2\widehat{\vartheta}_1 + \check{\vartheta}_1 \\ \widehat{T}_2(\vartheta) &= 3 + \widehat{\vartheta}_2 + \Delta \end{aligned} \qquad \begin{aligned} \check{T}_1(\vartheta) &= 2 + \widehat{\vartheta}_1 + 2\check{\vartheta}_1 \\ \check{T}_2(\vartheta) &= 3 + \check{\vartheta}_2\,. \end{aligned}$$

Straightforward computations show that $\vartheta_\Delta \equiv (1/2 + 3\Delta/8, 1/2 - 3\Delta/8), (1/2 - \Delta/8, 1/2 + \Delta/8))$ is a Nash equilibrium. The same computations as above, on the basis of Proposition 2.2, ensure that Θ_Δ is the unique Nash equilibrium. The travel times at ϑ_Δ are $\widehat{T}_1(\vartheta_\Delta) = \widehat{T}_2(\vartheta_\Delta) = 7/2 + 5\Delta/8$ and $\check{T}_1(\vartheta_\Delta) = \check{T}_2(\vartheta_\Delta) = 7/2 + \Delta/8$: they are worse than before for all routes, also for those not directly influenced by the delay.

3.2 Nash Equilibria with Unbounded Travel Times

The non-local effects presented in § 3.1 are here more relevant due to the congestion caused on a road by a delay appearing on an apparently independent road.

Two populations of vehicles move in the following network, where $N = 7, \widehat{n} = 2$, $\check{n} = 2$:



$$\widehat{\tau}_1(\eta) = 2 + \widehat{\eta} + \Delta \qquad \check{\tau}_3(\eta) = 1$$
$$\widehat{\tau}_2(\eta) = 1 \qquad \check{\tau}_4(\eta) = 2 + \check{\eta}$$
$$\widehat{\tau}_5(\eta) = \frac{\widehat{\eta}+\check{\eta}}{1-(\widehat{\eta}+\check{\eta})} \qquad \check{\tau}_5(\eta) = \frac{\widehat{\eta}+\check{\eta}}{1-(\widehat{\eta}+\check{\eta})}$$
$$\widehat{\tau}_7(\eta) = 1 \qquad \check{\tau}_6(\eta) = 1$$

$$\widehat{\gamma}_1 = (r_2, r_5, r_7)$$
$$\widehat{\gamma}_2 = r_1$$
$$\check{\gamma}_1 = (r_3, r_5, r_6)$$
$$\check{\gamma}_2 = r_4$$

$$\widehat{T}_1(\vartheta) = 2 + \frac{\widehat{\vartheta}_1+\check{\vartheta}_1}{1-(\widehat{\vartheta}_1+\check{\vartheta}_1)}$$
$$\widehat{T}_2(\vartheta) = 2 + \widehat{\vartheta}_2 + \Delta$$
$$\check{T}_1(\vartheta) = 2 + \frac{\widehat{\vartheta}_1+\check{\vartheta}_1}{1-(\widehat{\vartheta}_1+\check{\vartheta}_1)}$$
$$\check{T}_2(\vartheta) = 2 + \check{\vartheta}_2$$

$$\widehat{\Gamma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \check{\Gamma} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

In this network, one population moves from the origin $\widehat{\mathcal{O}}$ to the destination $\widehat{\mathcal{D}}$; $\widehat{\vartheta}_1$ travelers move along the central route $\widehat{\gamma}_1 = (r_2, r_5, r_7)$, while $\widehat{\vartheta}_2$ travelers move along the northern highway $\widehat{\gamma}_2 = r_1$. In the other population (which moves from the origin $\check{\mathcal{O}}$ to the destination $\check{\mathcal{D}}$), $\check{\vartheta}_1$ travelers move along the central route $\check{\gamma}_1 = (r_3, r_5, r_6)$ and $\check{\vartheta}_2$ travelers move along the southern highway $\check{\gamma}_2 = r_4$. Road r_5 is used by both populations.

For $\Delta \in [0, 3/2[$, a Nash equilibrium and the corresponding travel times are:

$$\begin{aligned}
\widehat{\vartheta}_\Delta &= \left(\frac{5 + 5\Delta - \sqrt{\Delta^2 + 26\Delta + 17}}{4}, \frac{-1 - 5\Delta + \sqrt{\Delta^2 + 26\Delta + 17}}{4}\right), \\
\check{\vartheta}_\Delta &= \left(\frac{5 + \Delta - \sqrt{\Delta^2 + 26\Delta + 17}}{4}, \frac{-1 - \Delta + \sqrt{\Delta^2 + 26\Delta + 17}}{4}\right), \\
\widehat{T}_1(\vartheta_\Delta) = \widehat{T}_2(\vartheta_\Delta) &= \frac{7 - \Delta + \sqrt{\Delta^2 + 26\Delta + 17}}{4}, \\
\check{T}_1(\vartheta_\Delta) = \check{T}_2(\vartheta_\Delta) &= \frac{7 - \Delta + \sqrt{\Delta^2 + 26\Delta + 17}}{4}.
\end{aligned} \tag{3.1}$$

Proposition 2.2 ensures that $\vartheta_\Delta \equiv (\widehat{\vartheta}_\Delta, \check{\vartheta}_\Delta)$ is the unique Nash equilibrium. For $\Delta = 0$, ϑ_0 is in the interior of $S^2 \times S^2$ and satisfies $(\widehat{\vartheta}_0)_1 + (\check{\vartheta}_0)_1 < 1$, so that all travel times are finite. As Δ increases, traveling along r_1 gets less and less convenient, so that $(\widehat{\vartheta}_\Delta)_2$ decreases, $(\widehat{\vartheta}_\Delta)_1$ increases and the equilibrium travel time also increases.

In turn, for the check population, traveling along $\check{\gamma}_1$ is less and less convenient, so that $\check{\vartheta}_1$ decreases. When $\Delta = 1/2$, the Nash equilibrium for the check population is at $(\check{\vartheta}_{1/2})_1 = 0$, $(\check{\vartheta}_{1/2})_2 = 1$, while for the hat population it is in the interior of S^2, due to the unboundedness of the travel time along $\widehat{\gamma}_1$.

This example shows that a delay along a route for the hat population may radically change the optimal choice for the check population.

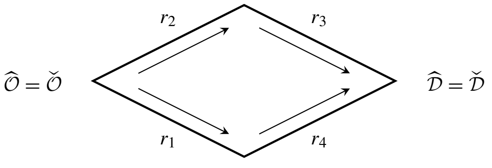
3.3 Braess Paradox for 2 Populations

In the present multi-population setting, phenomena like the celebrated Braess paradox [6] are possible and may have effects on all the populations on the network, even in the case only one population is affected by the introduction of a new road. The literature on Braess paradox is extremely rich. We refer here for instance to the textbook [12, § 8.2] for a general introduction; a stochastic study is in [5]; queue theory is employed in [16]; a variational inequality model is used in [18] while a non-stationary approach is considered in [9]. A different multi-population approach is in [24].

3.3.1 Trucks and Cars on the Same Network

We consider here two different populations traveling along the same network. The hat population consists of, say, trucks while the check population consists of cars. The former are slow and not influenced by the latter, which are faster and slowed down by the presence of the former. Thus, both populations travel along the same roads, but with different travel times.

Consider the classical network (3.2) leading to Braess paradox [6], where $N = 4$, $\widehat{n} = 2$, $\check{n} = 2$ and all roads are one way from left to right.

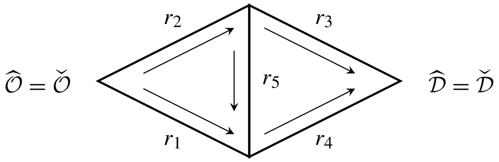


$$\begin{array}{ll} \widehat{\gamma}_1 = (r_2, r_3) & \check{\gamma}_1 = (r_2, r_3) \\ \widehat{\gamma}_2 = (r_1, r_4) & \check{\gamma}_2 = (r_1, r_4) \end{array} \qquad \begin{array}{ll} \widehat{\tau}_1(\eta) = 45 & \check{\tau}_1(\eta) = 30 \\ \widehat{\tau}_2(\eta) = 40\,\widehat{\eta} & \check{\tau}_2(\eta) = 20\,\check{\eta} + 8\,\widehat{\eta} \\ \widehat{\tau}_3(\eta) = 45 & \check{\tau}_3(\eta) = 30 \\ \widehat{\tau}_4(\eta) = 40\,\widehat{\eta} & \check{\tau}_4(\eta) = 20\,\check{\eta} + 8\,\widehat{\eta} \\ \widehat{T}_1(\vartheta) = 45 + 40\,\widehat{\vartheta}_1 & \check{T}_1(\vartheta) = 30 + 20\,\check{\vartheta}_1 + 8\,\widehat{\vartheta}_1 \\ \widehat{T}_2(\vartheta) = 45 + 40\,\widehat{\vartheta}_2 & \check{T}_2(\vartheta) = 30 + 20\,\check{\vartheta}_2 + 8\,\widehat{\vartheta}_2. \end{array} \tag{3.2}$$

Straightforward computations show that a Nash equilibrium and the corresponding route travel times are

$$\vartheta^* \equiv \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right) \qquad \begin{array}{l} \widehat{T}_1(\vartheta^*) = \widehat{T}_2(\vartheta^*) = 65\,; \\ \check{T}_1(\vartheta^*) = \check{T}_2(\vartheta^*) = 44\,. \end{array}$$

The uniqueness of this Nash equilibrium follows from Proposition 2.2. Then, we add a new road, r_5, as in (3.3), characterized by a negligible travel time. Thus we have



$$\begin{array}{ll}
\widehat{\tau}_1(\eta) = 45 & \check{\tau}_1(\eta) = 30 \\
\widehat{\tau}_2(\eta) = 40\,\widehat{\eta} & \check{\tau}_2(\eta) = 20\,\check{\eta} + 8\,\widehat{\eta} \\
\widehat{\tau}_3(\eta) = 45 & \check{\tau}_3(\eta) = 30 \\
\widehat{\tau}_4(\eta) = 40\,\widehat{\eta} & \check{\tau}_4(\eta) = 20\,\check{\eta} + 8\,\widehat{\eta} \\
\widehat{\tau}_5(\eta) = 0 & \check{\tau}_5(\eta) = 0
\end{array}$$

$$\begin{array}{ll}
\widehat{\gamma}_1 = (r_2, r_3) & \widehat{T}_1(\vartheta) = 45 + 40\,(\widehat{\vartheta}_1 + \widehat{\vartheta}_3) \\
\widehat{\gamma}_2 = (r_1, r_4) & \widehat{T}_2(\vartheta) = 45 + 40\,(\widehat{\vartheta}_2 + \widehat{\vartheta}_3) \\
\widehat{\gamma}_3 = (r_2, r_5, r_4) & \widehat{T}_3(\vartheta) = 40(\widehat{\vartheta}_1 + \widehat{\vartheta}_3) + 40(\widehat{\vartheta}_2 + \widehat{\vartheta}_3) \\
\check{\gamma}_1 = (r_2, r_3) & \check{T}_1(\vartheta) = 30 + 20\,(\check{\vartheta}_1 + \check{\vartheta}_3) + 8(\widehat{\vartheta}_1 + \widehat{\vartheta}_3) \\
\check{\gamma}_2 = (r_1, r_4) & \check{T}_2(\vartheta) = 30 + 20\,(\check{\vartheta}_2 + \check{\vartheta}_3) + 8(\widehat{\vartheta}_2 + \check{\vartheta}_3) \\
\check{\gamma}_2 = (r_2, r_5, r_4) & \check{T}_3(\vartheta) = 20(\check{\vartheta}_1 + \check{\vartheta}_3) + 8(\widehat{\vartheta}_1 + \widehat{\vartheta}_3) + 20(\check{\vartheta}_2 + \check{\vartheta}_3) + 8(\widehat{\vartheta}_2 + \widehat{\vartheta}_3)\,.
\end{array} \tag{3.3}$$

A Nash equilibrium and the corresponding route travel times are

$$\vartheta^* \equiv ((0, 0, 1), (0, 0, 1)) \qquad \begin{array}{lll}
\widehat{T}_1(\vartheta^*) = 95 & \widehat{T}_2(\vartheta^*) = 95 & \widehat{T}_3(\vartheta^*) = 80 \\
\check{T}_1(\vartheta^*) = 58 & \check{T}_2(\vartheta^*) = 58 & \check{T}_3(\vartheta^*) = 56\,.
\end{array}$$

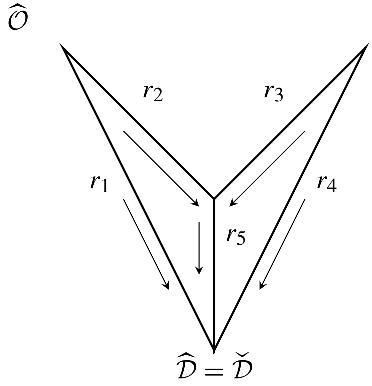
The uniqueness of this Nash equilibrium directly follows form Proposition 2.2.

The insertion of the new road r_5 for both populations causes an increase in the travel time at the unique global Nash equilibrium for all network users.

3.3.2 Braess Paradox Induced by a Different Population

In this example two different populations, the hat and the check population, move from different origins to the same destination. Our aim is to show that the addition of a road, aimed at helping the hat population, has a negative effect on both the hat and the check population.

In more details, consider the situation (3.4), where $N = 5$, $\widehat{n} = 2$, $\check{n} = 2$ and all roads are one way downwards. In the hat population, $\widehat{\vartheta}_1$ travelers move from the origin $\widehat{\mathcal{O}}$ to the destination $\widehat{\mathcal{D}}$ along the highway $\widehat{\gamma}_1 = r_1$, while $\widehat{\vartheta}_2$ move along $\widehat{\gamma}_2 = (r_2, r_5)$. In the check population, $\check{\vartheta}_1$ travelers move from the origin $\check{\mathcal{O}}$ to the destination $\check{\mathcal{D}}$ along $\check{\gamma}_1 = (r_3, r_5)$, while $\check{\vartheta}_2$ move along the highway $\check{\gamma}_2 = r_4$. Road r_5 is used by both populations. Thus



$$\begin{array}{ll} \widehat{\gamma}_1 = r_1 & \widehat{\tau}_1(\eta) = 4 \\ \widehat{\gamma}_2 = (r_2, r_5) & \widehat{\tau}_2(\eta) = 1 + \widehat{\eta} \\ & \widehat{\tau}_5(\eta) = 1 + \widehat{\eta} + \widecheck{\eta} \\ & \widecheck{\tau}_3(\eta) = \widecheck{\eta} \\ \widecheck{\gamma}_1 = (r_3, r_5) & \widecheck{\tau}_4(\eta) = 5\,\widecheck{\eta} \\ \widecheck{\gamma}_2 = r_4 & \widecheck{\tau}_5(\eta) = 1 + \widehat{\eta} + \widecheck{\eta}\,. \end{array} \tag{3.4}$$

The route travel times are

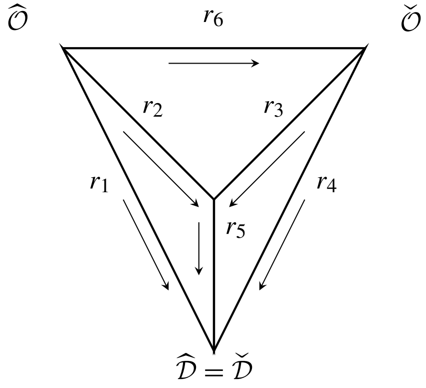
$$\begin{array}{ll} \widehat{T}_1(\vartheta) = 4 & \widecheck{T}_1(\vartheta) = 1 + \widehat{\vartheta}_2 + 2\widecheck{\vartheta}_1 \\ \widehat{T}_2(\vartheta) = 2 + 2\widehat{\vartheta}_2 + \widecheck{\vartheta}_1 & \widecheck{T}_2(\vartheta) = 5\,\widecheck{\vartheta}_2\,, \end{array}$$

so that a Nash equilibrium and its corresponding travel times are

$$\begin{array}{lll} \widehat{\vartheta}^* = (0, 1) & \widehat{T}_1(\vartheta^*) = 4 & \widehat{T}_2(\vartheta^*) = 19/6 \\ \widecheck{\vartheta}^* = (5/6, 1/6) & \widecheck{T}_1(\vartheta^*) = 5/6 & \widecheck{T}_2(\vartheta^*) = 5/6\,. \end{array} \tag{3.5}$$

Proposition 2.2 ensures the uniqueness of this equilibrium.

Introduce now a new road r_6 from $\widehat{\mathcal{O}}$ to $\widecheck{\mathcal{O}}$, the new route $\widehat{\gamma}_3$ and the travel time $\widehat{\tau}_6$, so that now $N = 6$, $\widehat{n} = 3$ and $\widecheck{n} = 2$. Modify accordingly the travel time $\widehat{\tau}_6$ as follows:



$$\begin{array}{ll} \widehat{\gamma}_1 = r_1 & \widehat{\tau}_1 = 4 \\ \widehat{\gamma}_2 = (r_2, r_5) & \widehat{\tau}_2 = 1 + \widehat{\eta} \\ \widehat{\gamma}_3 = (r_6, r_4) & \widehat{\tau}_4 = 5\,\widehat{\eta} + 5\,\widecheck{\eta} \\ & \widehat{\tau}_5 = 1 + \widehat{\eta} + \widecheck{\eta} \\ & \widehat{\tau}_6 = 1 \\ \widecheck{\gamma}_1 = (r_3, r_5) & \widecheck{\tau}_3 = \widecheck{\eta} \\ \widecheck{\gamma}_2 = r_4 & \widecheck{\tau}_4 = 5\,\widehat{\eta} + 5\,\widecheck{\eta} \\ & \widecheck{\tau}_5 = 1 + \widehat{\eta} + \widecheck{\eta}\,, \end{array}$$

so that the route travel times are

$$\begin{array}{lll}\widehat{T}_1(\vartheta)=4 & \widehat{T}_2(\vartheta)=2+2\widehat{\vartheta}_2+\check{\vartheta}_1 & \widehat{T}_3(\vartheta)=1+5\,\widehat{\vartheta}_3+5\,\check{\vartheta}_2 \\ \check{T}_1(\vartheta)=1+\widehat{\vartheta}_2+2\check{\vartheta}_1 & \check{T}_2(\vartheta)=5\,\widehat{\vartheta}_3+5\,\check{\vartheta}_2\,. & \end{array} \tag{3.6}$$

A Nash equilibrium and the corresponding travel times are

$$\begin{array}{llll}\widehat{\vartheta}^*=(1/15,2/3,4/15) & \widehat{T}_1(\vartheta^*)=4 & \widehat{T}_2(\vartheta^*)=4 & \widehat{T}_3=4 \\ \check{\vartheta}^*=(2/3,\,1/3) & \check{T}_1(\vartheta^*)=3 & \check{T}_2(\vartheta^*)=3\,. & \end{array} \tag{3.7}$$

Again, simple computations based on Proposition 2.2 ensure the uniqueness of this equilibrium.

The introduction of the new road for the hat population actually worsens all travel times at the new Nash equilibrium $\vartheta^* = ((1/15, 2/3, 4/15),\ (2/3, 1/3))$. Indeed, these travel times pass from $\widehat{T}^* = 19/6$ and $\check{T}^* = 5/6$, see (3.5), to $\widehat{T}^* = 4$ and $\check{T}^* = 3$, see (3.7).

4 Proofs and Further Remarks

The following notation is used throughout. $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_- =]-\infty, 0]$. $(e_1, \ldots, e_n)$ is the canonical basis in $\mathbb{R}^n$. $\mathbb{1}_n$ is the column vector in $\mathbb{R}^n$ whose components are all 1: $\mathbb{1}_n = \sum_{i=1}^n e_i$. The identity matrix of order n is Id_n. Occasionally, we write $\mathbb{1}_n$ for the null vector in $\mathbb{R}^n$. The simplex S^n is defined as $S^n := \left\{\vartheta \in \mathbb{R}^n_+ \colon \sum_{i=1}^n \vartheta_i = 1\right\}$. The cardinality of the set I is $\sharp I$.

The next result is a simple computation repeatedly used in the sequel.

Lemma 4.1 *Let (C1) hold. Then, with the notation (2.2), for all $\widehat{\vartheta}, \widehat{\vartheta}', \widehat{\vartheta}'' \in S^{\widehat{n}}$ and $\check{\vartheta}, \check{\vartheta}', \check{\vartheta}'' \in S^{\check{n}}$ the following equalities hold, whenever the terms in the left hand side are finite:*

$$\begin{array}{l}\widehat{T}(\widehat{\vartheta}'',\check{\vartheta})-\widehat{T}(\widehat{\vartheta}',\check{\vartheta})=\widehat{\Gamma}^{\intercal}\,\widehat{Q}\,\widehat{\Gamma}\,(\widehat{\vartheta}''-\widehat{\vartheta}')\ \textit{where}\ \widehat{Q}=\int_0^1 D_{\widehat{\eta}}\widehat{\tau}\left(\widehat{\Gamma}(s\,\widehat{\vartheta}''+(1-s)\widehat{\vartheta}'),\check{\Gamma}\,\check{\vartheta}\right)\mathrm{d}s\,; \\ \widehat{T}(\widehat{\vartheta},\check{\vartheta}'')-\widehat{T}(\widehat{\vartheta},\check{\vartheta}')=\widehat{\Gamma}^{\intercal}\,\widehat{P}\,\check{\Gamma}\,(\check{\vartheta}''-\check{\vartheta}')\ \textit{where}\ \widehat{P}=\int_0^1 D_{\check{\eta}}\widehat{\tau}\left(\widehat{\Gamma\vartheta},\check{\Gamma}(s\,\check{\vartheta}''+(1-s)\check{\vartheta}')\right)\mathrm{d}s\,; \\ \check{T}(\widehat{\vartheta},\check{\vartheta}'')-\check{T}(\widehat{\vartheta},\check{\vartheta}')=\check{\Gamma}^{\intercal}\,\check{Q}\,\check{\Gamma}\,(\check{\vartheta}''-\check{\vartheta}')\ \textit{where}\ \check{Q}=\int_0^1 D_{\check{\eta}}\check{\tau}\left(\check{\Gamma}\,\widehat{\vartheta},\check{\Gamma}(s\,\check{\vartheta}''+(1-s)\check{\vartheta}')\right)\mathrm{d}s\,; \\ \check{T}(\widehat{\vartheta}'',\check{\vartheta})-\check{T}(\widehat{\vartheta}',\check{\vartheta})=\check{\Gamma}^{\intercal}\,\check{P}\,\widehat{\Gamma}\,(\widehat{\vartheta}''-\widehat{\vartheta}')\ \textit{where}\ \check{P}=\int_0^1 D_{\widehat{\eta}}\check{\tau}\left(\widehat{\Gamma}(s\,\widehat{\vartheta}''+(1-s)\widehat{\vartheta}'),\check{\Gamma}\,\check{\vartheta}\right)\mathrm{d}s\,.\end{array} \tag{4.1}$$

Moreover, $\widehat{Q}$, $\widehat{P}$, $\check{Q}$ and $\check{P}$ are diagonal $N \times N$ matrices with non-negative entries.

Proof Consider the first 2 lines in (4.1) Standard calculus applied to (2.2) leads to

$$\begin{aligned}\widehat{T}(\widehat{\vartheta}'',\check{\vartheta})-\widehat{T}(\widehat{\vartheta}',\check{\vartheta}) &= \widehat{\Gamma}^{\intercal}\left(\widehat{\tau}(\widehat{\Gamma\vartheta}'',\check{\Gamma}\,\check{\vartheta})-\widehat{\tau}(\widehat{\Gamma\vartheta}',\check{\Gamma}\,\check{\vartheta})\right) \\ &= \widehat{\Gamma}^{\intercal}\int_0^1 D_{\widehat{\eta}}\widehat{\tau}\left(\widehat{\Gamma}(s\,\widehat{\vartheta}''+(1-s)\widehat{\vartheta}',\check{\Gamma}\,\check{\vartheta})\right)\mathrm{d}s\ \ \widehat{\Gamma}\,(\widehat{\vartheta}''-\widehat{\vartheta}')\,.\end{aligned}$$

$$\begin{aligned}\widehat{T}(\widehat{\vartheta}, \check{\vartheta}'') - \widehat{T}(\widehat{\vartheta}, \check{\vartheta}') &= \widehat{\Gamma}^\intercal \left(\widehat{\tau}(\widehat{\Gamma}\,\widehat{\vartheta}, \check{\Gamma}\,\check{\vartheta}'') - \widehat{\tau}(\widehat{\Gamma}\,\widehat{\vartheta}, \check{\Gamma}\,\check{\vartheta}')\right)\\ &= \widehat{\Gamma}^\intercal \int_0^1 D_{\check{\eta}}\widehat{\tau}\left(\widehat{\Gamma}\,\widehat{\vartheta}, \check{\Gamma}(s\check{\vartheta}'' + (1-s)\check{\vartheta}')\right) \mathrm{d}s\ \ \check{\Gamma}(\check{\vartheta}'' - \check{\vartheta}')\,.\end{aligned}$$

Moreover, (2.1) ensures that the two $N \times N$ matrices $\widehat{Q}$ and $\widehat{P}$ are diagonal and, by (C1),

$$\begin{aligned}\widehat{Q}_{hh} &= \int_0^1 \partial_{\widehat{\eta}_h}\widehat{\tau}_h\left(\widehat{\Gamma}(s\,\widehat{\vartheta}'' + (1-s)\widehat{\vartheta}', \widehat{\Gamma}\,\check{\vartheta})\right) \mathrm{d}s \geq 0\\ \widehat{P}_{hh} &= \int_0^1 \partial_{\check{\eta}_h}\widehat{\tau}_h\left(\widehat{\Gamma}\,\widehat{\vartheta}, \check{\Gamma}(s\,\check{\vartheta}'' + (1-s)\check{\vartheta}')\right) \mathrm{d}s \geq 0\end{aligned}$$

proving the first 2 lines in (4.1). The last 2 lines of (4.1) are proved analogously. □

The following observation is of use below.

Remark 4.1 The matrices $\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma}$, $\widehat{\Gamma}\,\widehat{P}\,\check{\Gamma}$, $\check{\Gamma}^\intercal\,\check{Q}\,\check{\Gamma}$ and $\check{\Gamma}\,\check{P}\,\widehat{\Gamma}$ in (4.1) are invariant with respect to any renumbering of roads or routes.

Indeed, renumbering the roads amounts to a permutation of the rows of $\widehat{\Gamma}$ and of the rows and the columns of $\widehat{Q}$ and $\widehat{P}$ i.e., we have $\widehat{\Gamma} \to \Pi\,\widehat{\Gamma}$ and $\widehat{Q} \to \Pi\,\widehat{Q}\,\Pi^{-1}$, for a suitable $N \times N$ permutation matrix Π. Then, since Π is orthogonal, $\Gamma^\intercal\,\widehat{Q}\,\Gamma \to (\Pi\,\Gamma)^\intercal(\Pi\,\widehat{Q}\,\Pi^{-1})(\Pi\,\Gamma) = \Gamma^\intercal\,\widehat{Q}\,\Gamma$, and similarly for the other matrices.

Lemma 4.2 *Let* Γ *be an* $N \times n$ *matrix whose entries are* 0 *or* 1. *Let* M *be an* $N \times N$ *diagonal matrix with non-negative entries. Then, for all* $i, j \in \{1, \ldots, n\}$

$$\left(\Gamma^\intercal\,M\,\Gamma\right)_{jj} \geq \left(\Gamma^\intercal\,M\,\Gamma\right)_{ji}\ .$$

Proof Let $i, j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, N\}$. Then, $(\Gamma^\intercal\,M\,\Gamma)_{ij} = \sum_{k=1}^N \Gamma_{ki}\,(M\,\Gamma)_{kj} = \sum_{k=1}^N \Gamma_{ki}\,M_{kk}\,\Gamma_{kj}$. Recalling that $(\Gamma_{kj})^2 = \Gamma_{kj}$, we have

$$\left(\Gamma^\intercal\,M\,\Gamma\right)_{jj} = \sum_{k=1}^N \Gamma_{kj}\,M_{kk} \qquad \text{and} \qquad \left(\Gamma^\intercal\,M\,\Gamma\right)_{ji} = \sum_{k=1}^N \Gamma_{ki}\,\Gamma_{kj}\,M_{kk}$$

showing that the latter sum contains either part or all of the terms in the former sum. Since all summands are non-negative, the proof follows. □

Proof of Lemma 2.1 Straightforward computations yield:

$$\begin{aligned}\widehat{T}_i(\vartheta) - \widehat{T}_i(\widehat{\vartheta} - \varepsilon\,e_i + \varepsilon\,e_j, \check{\vartheta}) &= e_i^\intercal\,\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma}\,(\varepsilon\,e_i - \varepsilon\,e_j) && \text{[By Lemma 4.1]}\\ &= \varepsilon\left(e_i^\intercal\,\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma}\,e_i - e_i^\intercal\,\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma}\,e_j\right)\\ &= \varepsilon\left((\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma})_{ii} - (\widehat{\Gamma}^\intercal\,\widehat{Q}\,\widehat{\Gamma})_{ij}\right)\\ &\geq 0 && \text{[By Lemma 4.2]}\end{aligned}$$

proving (2.5). Inequality (2.6) is proved similarly and (2.7) follows by continuity. □

The next Lemma provides a further general monotonicity property.

Lemma 4.3 *Let (C1) hold. Fix $j \in \{1, \ldots, \widehat{n}\}$, $i \in \{1, \ldots, \check{n}\}$ and any $\vartheta \in S^{\widehat{n}} \times S^{\check{n}}$. Then, both the maps*

$$\begin{array}{rcl} \widehat{\Upsilon}_j : [0,1] & \to & \mathbb{R}_+ \\ \sigma & \mapsto & \widehat{T}_j\left(\sigma\, e_j + (1-\sigma)\widehat{\vartheta}, \check{\vartheta}\right) \end{array} \qquad \begin{array}{rcl} \check{\Upsilon}_i : [0,1] & \to & \mathbb{R}_+ \\ \sigma & \mapsto & \check{T}_i\left(\widehat{\vartheta}, \sigma\, e_i + (1-\sigma)\check{\vartheta}\right) \end{array}$$

are weakly increasing (i.e., non-decreasing).

Proof Fix $\sigma', \sigma'' \in [0,1]$ with $\sigma' < \sigma''$. Write $\widehat{\vartheta}' = \sigma'\, e_j + (1-\sigma')\widehat{\vartheta}$ and $\widehat{\vartheta}'' = \sigma''\, e_j + (1-\sigma'')\widehat{\vartheta}$. Use (2.2) to compute:

$$\begin{aligned} \widehat{\Upsilon}_j(\sigma'') - \widehat{\Upsilon}_j(\sigma') &= \widehat{T}_j(\widehat{\vartheta}'', \check{\vartheta}) - \widehat{T}_j(\widehat{\vartheta}', \check{\vartheta}) && [\text{ Definition of } \widehat{\Upsilon}_j] \\ &= e_j^{\mathsf{T}} \left(\widehat{T}(\widehat{\vartheta}'', \check{\vartheta}) - \widehat{T}(\widehat{\vartheta}', \check{\vartheta})\right) \\ &= {e_j}^{\mathsf{T}}\, \widehat{\Gamma}^{\mathsf{T}}\, \widehat{Q}\, \widehat{\Gamma}\, (\widehat{\vartheta}'' - \widehat{\vartheta}') && [\text{Lemma } 4.1] \\ &= {e_j}^{\mathsf{T}}\, \widehat{\Gamma}^{\mathsf{T}}\, \widehat{Q}\, \widehat{\Gamma}\, (\sigma'' - \sigma')(e_j - \widehat{\vartheta}) && [\text{ Definition of } \widehat{\vartheta}', \widehat{\vartheta}''] \\ &= (\sigma'' - \sigma')\, {e_j}^{\mathsf{T}}\, \widehat{\Gamma}^{\mathsf{T}}\, \widehat{Q}\, \widehat{\Gamma}\, (e_j - \widehat{\vartheta}) \\ &= (\sigma'' - \sigma') \sum_{i=1}^{n} \widehat{\vartheta}_i \left((\widehat{\Gamma}^{\mathsf{T}}\, \widehat{Q}\, \widehat{\Gamma})_{jj} - (\widehat{\Gamma}^{\mathsf{T}}\, \widehat{Q}\, \widehat{\Gamma})_{ji}\right) \\ &\geq 0\,. && [\text{by Lemma } 4.2] \end{aligned}$$

The case of $\check{\Upsilon}_i$ is entirely similar and the proof is completed. □

Proof of Theorem 2.1 The implication from ε–Nash equilibrium to Nash equilibrium under the continuity of the travel times follows by a limiting procedure.

Let ϑ^* be a Nash equilibrium. Define $\bar{\varepsilon} = \frac{1}{2} \min\left\{\widehat{\vartheta}_i^* : \widehat{\vartheta}_i^* > 0\right\}$. Following Definition 2.1, fix $i, j \in \{1, \ldots, n\}$ so that $\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j \in S^n$ for $\varepsilon \in [0, \bar{\varepsilon}]$. We thus have $\widehat{\vartheta}_i^* > 0$ so that $\widehat{T}_i(\vartheta^*) = \widehat{T}_j(\vartheta^*)$ if $\widehat{\vartheta}_j^* > 0$, or $\widehat{T}_i(\vartheta^*) \leq \widehat{T}_j(\vartheta^*)$ by Definition 2.2 if $\widehat{\vartheta}_j^* = 0$. In both cases, using also (2.2),

$$\begin{aligned} \widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}_i(\vartheta^*) &\geq \widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}_j(\widehat{\vartheta}^*, \check{\vartheta}^*) \\ &= {e_j}^{\mathsf{T}} \left(\widehat{T}(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}(\vartheta^*)\right). \end{aligned} \tag{4.2}$$

Under Assumption (C1), define $\widehat{Q} = \int_0^1 D_{\widehat{\eta}}\widehat{\tau}\left(\widehat{\Gamma}(s\, \widehat{\vartheta}^* + (1-s)\widehat{\vartheta}^*), \check{\Gamma}\, \check{\vartheta}^*\right) \mathrm{d}s$ and use Lemma 4.1:

$$\begin{aligned} \widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}_i(\widehat{\vartheta}^*, \check{\vartheta}^*) &\geq e_j{}^{\intercal} \left(\widehat{T}(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}(\vartheta^*)\right) \\ &= e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal}\, \widehat{Q}\, \widehat{\Gamma}(-\varepsilon\, e_i + \varepsilon\, e_j)\,. \end{aligned} \tag{4.3}$$

On the other hand, under Assumption (C2), we have

$$\begin{aligned} &\forall h \in \{1, \ldots, N\}\ \forall \eta_o \in [0,1]\ \exists\, \ell^h_{\eta_o} \in \mathbb{R}_+ \colon \forall \eta \in [0,1]\ \widehat{\tau}_h(\eta, \check{\vartheta}^*) \\ &\quad \geq \widehat{\tau}_h(\eta_o, \check{\vartheta}^*) + \ell^h_{\eta_o}(\eta - \eta_o)\,. \end{aligned} \tag{4.4}$$

In the terminology of convex analysis, $\ell^h_{\eta_o}$ is a *subgradient* of $\eta \mapsto \widehat{\tau}_h(\eta, \check{\vartheta}^*)$ at η_o. Introduce the $N \times N$ diagonal matrix L with entries $L_{hh} = \ell^h_{\widehat{\vartheta}^*_j}$, for $h = 1, \ldots, N$, as defined in (4.4), so that $L_{hh} \geq 0$. All entries in e_j, $\Gamma^{\intercal}$ and Γ are non-negative, actually either 0 or 1. Continue from (4.2) using (2.2):

$$\begin{aligned} &\widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}_i(\widehat{\vartheta}^*, \check{\vartheta}^*) \\ &\geq e_j{}^{\intercal} \left(\widehat{T}(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}(\vartheta^*)\right) \\ &\geq e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal} \left(\widehat{\tau}(\widehat{\Gamma}\widehat{\vartheta}^*, \check{\Gamma}\check{\vartheta}^*) + L\, \Gamma\left(-\varepsilon\, e_i + \varepsilon\, e_j\right) - \widehat{\tau}(\widehat{\Gamma}\widehat{\vartheta}^*, \check{\Gamma}\check{\vartheta}^*)\right) \\ &= e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\left(-\varepsilon\, e_i + \varepsilon\, e_j\right)\,. \end{aligned} \tag{4.5}$$

Both, (4.3) and (4.5) yield, by Lemma 4.2:

$$\begin{aligned} \widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) - \widehat{T}_i(\widehat{\vartheta}^*, \check{\vartheta}^*) &\geq e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\left(-\varepsilon\, e_i + \varepsilon\, e_j\right) \\ &= \varepsilon\left(e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\, e_j - e_j{}^{\intercal}\, \widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\, e_i\right) \\ &= \varepsilon\left(\left(\widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\right)_{jj} - \left(\widehat{\Gamma}^{\intercal}\, L\, \widehat{\Gamma}\right)_{ji}\right) \\ &\geq 0\,. \end{aligned}$$

An entirely similar argument applies to $\check{T}$. Thus, ϑ^* is an ε–Nash equilibrium and the proof is completed. □

Proof of Proposition 2.1 Consider (NE1). Let $i, j \in \{1, \ldots, \widehat{n}\}$ be such that $\widehat{\vartheta}^*_i > 0$ and $\widehat{\vartheta}^*_j > 0$. Then, define $\widehat{\vartheta}^{\varepsilon} = \widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j$ and compute:

$$\begin{aligned} 0 &\geq \widehat{T}_M(\vartheta^*) - \widehat{T}_M(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*) && [\text{By } (2.8)] \\ &= \sum_{\ell=1}^{n}\left(\widehat{\vartheta}^*_\ell\, \widehat{T}_\ell(\vartheta^*) - \widehat{\vartheta}^{\varepsilon}_\ell\, \widehat{T}_\ell(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*)\right) && [\text{By } (2.3)] \\ &= \widehat{\vartheta}^*_i\, \widehat{T}_i(\vartheta^*) - \widehat{\vartheta}^{\varepsilon}_i\, \widehat{T}_i(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*) && [\widehat{T}_i(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*) \leq \widehat{T}_i(\vartheta^*) \text{ by } (2.5)] \\ &\quad + \widehat{\vartheta}^*_j\, \widehat{T}_j(\vartheta^*) - \widehat{\vartheta}^{\varepsilon}_j\, \widehat{T}_j(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*) && [\widehat{T}_j(\vartheta^*) = \widehat{T}_i(\vartheta^*),\ \vartheta^* \text{ equilibrium}] \\ &\quad + \sum_{\ell \neq i,j} \widehat{\vartheta}^*_\ell \left(\widehat{T}_\ell(\vartheta^*) - \widehat{T}_\ell(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*)\right) && [\widehat{T}_\ell(\widehat{\vartheta}^*, \check{\vartheta}^*) = \widehat{T}_\ell(\widehat{\vartheta}^{\varepsilon}, \check{\vartheta}^*) \text{ by Def. } 2.1] \end{aligned}$$

$$
\begin{aligned}
&\geq (\widehat{\vartheta}^*_i - \widehat{\vartheta}^\varepsilon_i + \widehat{\vartheta}^*_j)\, \widehat{T}_i(\vartheta^*) - \widehat{\vartheta}^\varepsilon_j\, \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*) && [\widehat{\vartheta}^*_i - \widehat{\vartheta}^\varepsilon_i + \widehat{\vartheta}^*_j = \widehat{\vartheta}^\varepsilon_j] \\
&= \widehat{\vartheta}^\varepsilon_j \left(\widehat{T}_i(\vartheta^*) - \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*)\right)
\end{aligned}
$$

proving that $\widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) \geq \widehat{T}_i(\vartheta^*)$, so that ϑ^* is an ε–Nash equilibrium by Definition 2.3.

If on the other hand $\widehat{\vartheta}^*_j = 0$, then the same computations as above lead to

$$
\begin{aligned}
0 &\geq \widehat{\vartheta}^*_i\, \widehat{T}_i(\vartheta^*) - \widehat{\vartheta}^\varepsilon_i\, \widehat{T}_i(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*) - \widehat{\vartheta}^\varepsilon_j\, \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*) && [\widehat{\vartheta}^\varepsilon_j = \varepsilon] \\
&= (\widehat{\vartheta}^\varepsilon_i + \varepsilon)\, \widehat{T}_i(\widehat{\vartheta}^*, \check{\vartheta}^*) - \widehat{\vartheta}^\varepsilon_i\, \widehat{T}_i(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*) - \varepsilon\, \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*) \\
&= \widehat{\vartheta}^\varepsilon_i \left(\widehat{T}_i(\vartheta^*) - \widehat{T}_i(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*)\right) + \varepsilon \left(\widehat{T}_i(\vartheta^*) - \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*)\right) && [\text{by Lemma 2.1}] \\
&\geq \varepsilon \left(\widehat{T}_i(\vartheta^*) - \widehat{T}_j(\widehat{\vartheta}^\varepsilon, \check{\vartheta}^*)\right)
\end{aligned}
$$

which also implies $\widehat{T}_j(\widehat{\vartheta}^* - \varepsilon\, e_i + \varepsilon\, e_j, \check{\vartheta}^*) \geq \widehat{T}_i(\vartheta^*)$. The proof of (NE1) is completed, once the same computations are repeated for $\check{T}$.

Item (NE2) follows, since extreme points are equilibrium points.

Proof of Theorem 2.2 Introduce the functions

$$
\begin{array}{rccc}
\bar{\varphi}\, : & \mathbb{R}_+ \cup \{+\infty\} & \to & [0,1] \\
& \xi & \mapsto & \begin{cases} \xi/(1-\xi) & \xi \in \mathbb{R}_+ \\ 1 & \xi = +\infty \end{cases} \\
\varphi\, : & (\mathbb{R}_+ \cup \{+\infty\})^n & \to & [0,1]^n \\
& (\xi_1, \ldots, \xi_n) & \mapsto & (\bar{\varphi}(\xi_1), \ldots, \bar{\varphi}(\xi_n))\ .
\end{array}
\tag{4.6}
$$

Call C^n_+ the set of those $\vartheta \in \mathbb{R}^n$ such that $\vartheta_i > 0$ for at least one index $i \in \{1, \ldots, n\}$. Define for all $\vartheta \in C^n_+$ the normalization $\mathcal{N}(\vartheta) \in S^n$ by $\mathcal{N}(\vartheta)_i = \dfrac{\max\{0, \vartheta_i\}}{\sum_{j=1}^n \max\{0, \vartheta_j\}}$, for $i \in \{1, \ldots, n\}$.

With a slight abuse of notation, we use the same letters φ or $\bar{\varphi}$ independently of whether they are defined on $\mathbb{R}^{\widehat{n}} \cup \{+\infty\}$ or $\mathbb{R}^{\check{n}} \cup \{+\infty\}$. The same simplification applies to $\mathcal{N}$.

Recalling (2.2)–(2.3) and Definition 2.1, define the map $\mathcal{F}$: $S^{\widehat{n}} \times S^{\check{n}} \to S^{\widehat{n}} \times S^{\check{n}}$ through its components $\widehat{\mathcal{F}}$ and $\check{\mathcal{F}}$:

$$
\begin{array}{rcc}
\widehat{\mathcal{F}} : S^{\widehat{n}} \times S^{\check{n}} \to & & S^{\widehat{n}} \\
\vartheta & \mapsto & \mathcal{N}\Big(\widehat{\vartheta} - \lambda \left((\varphi \circ \widehat{T})(\vartheta) - \widehat{\vartheta}^{\mathsf{T}} (\varphi \circ \widehat{T})(\vartheta)\, \mathbb{1}_{\widehat{n}}\right)\Big) \\
\check{\mathcal{F}} : S^{\widehat{n}} \times S^{\check{n}} \to & & S^{\check{n}} \\
\vartheta & \mapsto & \mathcal{N}\left(\check{\vartheta} - \lambda \left((\varphi \circ \check{T})(\vartheta) - \check{\vartheta}^{\mathsf{T}} (\varphi \circ \check{T})(\vartheta)\, \mathbb{1}_{\widehat{n}}\right)\right)
\end{array}
\tag{4.7}
$$

where

$$\lambda = \frac{1}{2}\min\left\{\frac{1}{\widehat{n}}, \frac{1}{\check{n}}\right\}. \tag{4.8}$$

Claim 1: $\mathcal{F} \in \mathbf{C}^0(S^{\widehat{n}} \times S^{\check{n}}; S^{\widehat{n}} \times S^{\check{n}})$. Given the definition of $\mathcal{F}$ and $\mathcal{N}$, it is sufficient to verify that the denominator in the expression of each component of $\mathcal{N}$ does not vanish. Proceed by contradiction:

$$\begin{aligned}
&\sum_{j=1}^{\widehat{n}} \max\left\{0, \widehat{\vartheta}_j - \lambda\left((\bar{\varphi}\circ\widehat{T}_j)(\vartheta) - \vartheta^\top(\varphi\circ\widehat{T})(\vartheta)\right)\right\} = 0\\
&\quad\iff \forall\, j = 1,\ldots,\widehat{n} \quad \widehat{\vartheta}_j - \lambda\left((\bar{\varphi}\circ\widehat{T}_j)(\vartheta) - \vartheta^\top(\varphi\circ\widehat{T})(\vartheta)\right) \le 0\\
&\quad\implies \sum_{j=1}^{\widehat{n}}\left[\widehat{\vartheta}_j - \lambda\left((\bar{\varphi}\circ\widehat{T}_j)(\vartheta) - \vartheta^\top(\varphi\circ\widehat{T})(\vartheta)\right)\right] \le 0\\
&\quad\iff \lambda\sum_{j=1}^{\widehat{n}}\left((\bar{\varphi}\circ\widehat{T}_j)(\vartheta) - \vartheta^\top(\varphi\circ\widehat{T})(\vartheta)\right) \ge 1\\
&\quad\implies \lambda\sum_{j=1}^{\widehat{n}}(\bar{\varphi}\circ\widehat{T}_j)(\vartheta) \ge 1
\end{aligned}$$

which contradicts the choice (4.8) of λ, since $\sum_{j=1}^{\widehat{n}}(\bar{\varphi}\circ\widehat{T}_j)(\vartheta) \le \widehat{n}$. An entirely similar argument applies to the second component $\check{\mathcal{F}}$ of $\mathcal{F}$, proving Claim 1. ✓

By Brouwer Fixed Point Theorem [20, Theorem 1.6.2], $\mathcal{F}$ admits a fixed point, say ϑ^*.

Claim 2: $\widehat{\vartheta}^*_i = 0 \implies (\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) \ge (\widehat{\vartheta}^*)^\top(\varphi\circ\widehat{T})(\vartheta^*)$ and $\check{\vartheta}^*_i = 0 \implies (\bar{\varphi}\circ\check{T}_i)(\vartheta^*) \ge (\check{\vartheta}^*)^\top(\varphi\circ\check{T})(\vartheta^*)$.

Direct computations yield:

$$\begin{aligned}
\widehat{\vartheta}^* = \widehat{\mathcal{F}}(\vartheta^*) \quad\text{and}\quad \widehat{\vartheta}^*_i = 0 &\implies \max\left\{0, 0 - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - \widehat{\vartheta}^\top(\varphi\circ\widehat{T})(\vartheta^*)\right)\right\} = 0\\
&\iff (\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) \ge (\vartheta^*)^\top\,\widehat{T}(\vartheta^*)\,.
\end{aligned}$$

The second implication is identical. Claim 2 is proved. ✓

Claim 3: $\widehat{\vartheta}^*_i > 0 \implies \widehat{\vartheta}^*_i > \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^\top(\varphi\circ\widehat{T})(\vartheta^*)\right)$ and $\check{\vartheta}^*_i > 0 \implies \check{\vartheta}^*_i > \lambda\left((\bar{\varphi}\circ\check{T}_i)(\vartheta^*) - (\check{\vartheta}^*)^\top(\varphi\circ\check{T})(\vartheta^*)\right)$.

Similar computations give:

$$\begin{aligned}
\widehat{\vartheta}^* = \widehat{\mathcal{F}}(\vartheta^*) \quad\text{and}\quad \widehat{\vartheta}^*_i > 0 &\implies \max\left\{0, \widehat{\vartheta}^*_i - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^\top(\varphi\circ\widehat{T})(\vartheta^*)\right)\right\} > 0\\
&\iff \widehat{\vartheta}^*_i > \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^\top(\varphi\circ\widehat{T})(\vartheta^*)\right),
\end{aligned}$$

proving Claim 3, since the other implication is identical. ✓

Claim 4: $\widehat{\vartheta}_i^* > 0$ and $\widehat{\vartheta}_j^* > 0 \implies \widehat{T}_i(\vartheta^*) = \widehat{T}_j(\vartheta^*)$.

By the above claims:

$$\begin{aligned}
\widehat{\vartheta}_i^* &= \frac{\max\left\{0, \widehat{\vartheta}_i^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)\right\}}{\sum_{k=1}^{n}\max\left\{0, \widehat{\vartheta}_k^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_k)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)\right\}} && [\vartheta^* \text{ is a fixed point}]\\
&= \frac{\widehat{\vartheta}_i^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)}{\sum_{k\colon \widehat{\vartheta}_k^*>0}\max\left\{0, \widehat{\vartheta}_k^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_k)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)\right\}} && [\widehat{\vartheta}_i^* > 0 \text{ and Claim 3}]\\
&= \frac{\widehat{\vartheta}_i^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)}{\sum_{k\colon \widehat{\vartheta}_k^*>0}\left(\widehat{\vartheta}_k^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_k)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)\right)} && [\widehat{\vartheta}_k^* > 0 \text{ and Claim 3}]\\
&= \frac{\widehat{\vartheta}_i^* - \lambda\left((\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)}{1 - \lambda\sum_{k\colon \widehat{\vartheta}_k^*>0}\left((\bar{\varphi}\circ\widehat{T}_k)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right)} && \Big[\sum_{k\colon \widehat{\vartheta}_k^*>0}\widehat{\vartheta}_k^* = 1\Big]
\end{aligned}$$

so that

$$(\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*) = \widehat{\vartheta}_i^* \sum_{k\colon \widehat{\vartheta}_k^*>0}\left((\bar{\varphi}\circ\widehat{T}_k)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\right). \tag{4.9}$$

Repeating the same procedure with $\widehat{\vartheta}_j^*$, we have that for all j such that $\widehat{\vartheta}_j^* > 0$, the differences $(\bar{\varphi}\circ T_i)(\vartheta^*) - (\widehat{\vartheta}^*)^{\mathsf{T}}(\varphi\circ T)(\vartheta^*)$ all have the same sign. Since $(\widehat{\vartheta}^*)^{\mathsf{T}}(\varphi\circ T)(\vartheta^*)$ is a convex combination of these $(\bar{\varphi}\circ T_j)(\vartheta^*)$, we have that for all j such that $\widehat{\vartheta}_j^* > 0$, $(\bar{\varphi}\circ T_j)(\vartheta^*) = (\widehat{\vartheta}^*)^{\mathsf{T}}(\varphi\circ T)(\vartheta^*)$. Claim 4 now follows by (4.6), the other implication being analogous. ✓

Hence, by Claim 4., ϑ^* is an equilibrium point in the sense of Definition 2.1. To prove property (2.4), by Claim 2 if $\widehat{\vartheta}_i^* = 0$, we have:

$$\begin{aligned}
(\bar{\varphi}\circ\widehat{T}_i)(\vartheta^*) &\geq (\widehat{\vartheta}^*)^{\mathsf{T}}\,(\varphi\circ\widehat{T})(\vartheta^*)\\
&= \sum_{j\colon \widehat{\vartheta}_j^*>0}\widehat{\vartheta}_j^*\,(\bar{\varphi}\circ\widehat{T}_j)(\vartheta^*)\\
&= (\bar{\varphi}\circ\widehat{T}_j)(\vartheta^*) \quad \forall\, j\colon \widehat{\vartheta}_j^* > 0 && [\text{By Claim 4}]
\end{aligned}$$

and the monotonicity of $\bar{\varphi}$ ensures that (2.4) holds. □

Observe that by the above proof any Nash equilibrium is a fixed point for $\mathcal{F}$ in (4.7).

Proof of Proposition 2.2 If ϑ^* is any Nash equilibrium, define the sets of indexes

$$\begin{aligned} \widehat{I}(\vartheta^*) &:= \left\{ i \in \{1, \dots, \widehat{n}\} \colon \widehat{T}_i(\vartheta^*) = \widehat{T}_M(\vartheta^*) \right\} \\ \check{I}(\vartheta^*) &:= \left\{ i \in \{1, \dots, \check{n}\} \colon \check{T}_i(\vartheta^*) = \check{T}_M(\vartheta^*) \right\} \end{aligned} \tag{4.10}$$

and note that, by Definition 2.2,

$$\begin{aligned} i \in \{1, \dots, \widehat{n}\} \setminus \widehat{I}(\vartheta^*) &\implies \widehat{\vartheta}_i^* = 0 \quad \text{and} \quad \widehat{T}_i(\vartheta^*) > \widehat{T}_M(\vartheta^*)\,; \\ i \in \{1, \dots, \check{n}\} \setminus \check{I}(\vartheta^*) &\implies \check{\vartheta}_i^* = 0 \quad \text{and} \quad \check{T}_i(\vartheta^*) > \check{T}_M(\vartheta^*). \end{aligned}$$

Recall that ϑ' and ϑ'' are Nash equilibria. With reference to (4.10), introduce the sets

$$\begin{array}{lll} \widehat{\mathcal{I}} := \widehat{I}(\vartheta') \cap \widehat{I}(\vartheta'')\,, & \widehat{\mathcal{J}} := \widehat{I}(\vartheta') \setminus \widehat{I}(\vartheta'')\,, & \widehat{\mathcal{K}} := \widehat{I}(\vartheta'') \setminus \widehat{I}(\vartheta') \\ \mathcal{I} := \check{I}(\vartheta') \cap \check{I}(\vartheta'')\,, & \check{\mathcal{J}} := \check{I}(\vartheta') \setminus \check{I}(\vartheta'')\,, & \check{\mathcal{K}} := \check{I}(\vartheta'') \setminus \check{I}(\vartheta') \end{array}$$

so that

$$\widehat{\mathcal{I}} \cup \widehat{\mathcal{J}} \cup \widehat{\mathcal{K}} = \{1, \dots, n\}\,, \tag{4.11}$$

$$i \in \widehat{\mathcal{I}} \implies \qquad \widehat{T}_i(\vartheta') = \widehat{T}_M(\vartheta') \qquad \widehat{T}_i(\vartheta'') = \widehat{T}_M(\vartheta'')\,, \tag{4.12}$$

$$j \in \widehat{\mathcal{J}} \implies \qquad \widehat{T}_j(\vartheta') = \widehat{T}_M(\vartheta') \qquad \widehat{T}_j(\vartheta'') > \widehat{T}_M(\vartheta'') \qquad \vartheta_j'' = 0\,, \tag{4.13}$$

$$k \in \widehat{\mathcal{K}} \implies \qquad \widehat{T}_k(\vartheta') > \widehat{T}_M(\vartheta') \qquad \widehat{T}_k(\vartheta'') = \widehat{T}_M(\vartheta'') \qquad \vartheta_k' = 0\,. \tag{4.14}$$

We then have

$$\begin{aligned} & (\widehat{\vartheta}'' - \widehat{\vartheta}')^\intercal \left(\widehat{T}(\vartheta'') - \widehat{T}(\vartheta')\right) \\ = & \sum_{i \in \widehat{\mathcal{I}}} (\widehat{\vartheta}_i'' - \widehat{\vartheta}_i') \left(\widehat{T}_i(\vartheta'') - \widehat{T}_i(\vartheta')\right) && \text{[By (4.11)]} \\ & + \sum_{j \in \widehat{\mathcal{J}}} (\widehat{\vartheta}_j'' - \widehat{\vartheta}_j') \left(\widehat{T}_j(\vartheta'') - \widehat{T}_j(\vartheta')\right) + \sum_{k \in \widehat{\mathcal{K}}} (\widehat{\vartheta}_k'' - \widehat{\vartheta}_k') \left(\widehat{T}_k(\vartheta'') - \widehat{T}_k(\vartheta')\right) \\ = & \sum_{i \in \widehat{\mathcal{I}}} (\widehat{\vartheta}_i'' - \widehat{\vartheta}_i') \left(\widehat{T}_M(\vartheta'') - \widehat{T}_M(\vartheta')\right) && \text{[By (4.12)]} \\ & + \sum_{j \in \widehat{\mathcal{J}}} -\widehat{\vartheta}_j' \left(\widehat{T}_j(\vartheta'') - \widehat{T}_M(\vartheta')\right) + \sum_{k \in \widehat{\mathcal{K}}} \widehat{\vartheta}_k'' \left(\widehat{T}_M(\vartheta'') - \widehat{T}_k(\vartheta')\right) && \text{[By (4.13) and (4.14)]} \\ \leq & \sum_{i \in \widehat{\mathcal{I}}} (\widehat{\vartheta}_i'' - \widehat{\vartheta}_i') \left(\widehat{T}_M(\vartheta'') - \widehat{T}_M(\vartheta')\right) \\ & + \sum_{j \in \widehat{\mathcal{J}}} -\widehat{\vartheta}_j' \left(\widehat{T}_M(\vartheta'') - \widehat{T}_M(\vartheta')\right) + \sum_{k \in \widehat{\mathcal{K}}} \widehat{\vartheta}_k'' \left(\widehat{T}_M(\vartheta'') - \widehat{T}_M(\vartheta')\right) && \text{[By (4.13) and (4.14)]} \\ = & \left(\sum_{i \in \widehat{\mathcal{I}} \cup \widehat{\mathcal{K}}} \widehat{\vartheta}_i'' - \sum_{i \in \widehat{\mathcal{I}} \cup \widehat{\mathcal{J}}} \widehat{\vartheta}_i' \right) \left(\widehat{T}_M(\vartheta'') - \widehat{T}_M(\vartheta')\right) \\ = & \, 0\,. \end{aligned}$$

An entirely similar procedure yields that also $(\check{\vartheta}'' - \check{\vartheta}')^\intercal \left(\check{T}(\check{\vartheta}'') - \check{T}(\check{\vartheta}')\right) = 0.$ □

5 Conclusions

The present framework is amenable to a variety to extensions. For instance, the presence of a two way road in the network fits in the present setting by considering the two directions as two independent roads with reverse orientation. The present construction applies to the so obtained undirected graphs. However, in this model the travel times in each direction result to be independent from the traffic in the opposite direction. Hence, a possible extension may consist in road travel times depending also on the traffic density along other roads.

Inherent to the dynamics of vehicular traffic is the presence of stochastic disturbances. Therefore, an extension of the present setting to that of stochastic games [28] is definitely worth pursueing, possibly on the basis of the recent works [1], [3].

As it is well known, the complexity of the actual computation of Nash equilibria grows incredibly fast as the numbers of routes in the network and of populations grow, see [21, Chapter 2] or [22, Chapter 14]. Notably, Nash equilibria can be found in entirely different ways, ranging from convex optimization methods to those based on Brouwer fixed point theorem. Nevertheless, the complexity of all of these methods are essentially equivalent, see [11].

A Appendix: A Uniqueness Result

We now ensure the uniqueness of Nash equilibria. A key role is played by Condition (Γ) together with a sort of non-degeneracy assumption, which we require together with assumption (C1).

Lemma A.1 *Let $\widehat{Q}$, $\widehat{P}$, $\check{Q}$ and $\check{P}$ be $N \times N$ diagonal matrices with non-negative entries. Then,*

$$\begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \widehat{Q} \end{bmatrix} \textit{ is positive semidefinite} \iff \forall\, h \in \{1, \dots, N\} \begin{array}{ll} \textit{either:} & \widehat{Q}_{hh}{=}0,\ \check{Q}_{hh}{=}0,\ \widehat{P}_{hh}{=}0,\ \check{P}_{hh}{=}0; \\ \textit{or:} & \widehat{Q}_{hh}{>}0,\ \check{Q}_{hh}{=}0,\ \widehat{P}_{hh}{=}0,\ \check{P}_{hh}{=}0; \\ \textit{or:} & \widehat{Q}_{hh}{=}0,\ \check{Q}_{hh}{>}0,\ \widehat{P}_{hh}{=}0,\ \check{P}_{hh}{=}0; \\ \textit{or:} & \widehat{Q}_{hh}{>}0,\ \check{Q}_{hh}{>}0,\ 4\,\widehat{Q}_{hh}\,\check{Q}_{hh}{\geq}(\widehat{P}_{hh} + \check{P}_{hh})^2. \end{array}$$

Note that in the case of more than 2 populations the above statement remains substantially unaltered, while a quite different proof is necessary.

Proof of Lemma A.1 Write a $v \in \mathbb{R}^{2N}$ as a pair $\left[\begin{smallmatrix}\widehat{v}\\ \check{v}\end{smallmatrix}\right]$ with $\widehat{v}, \check{v} \in \mathbb{R}^N$. Then,

$$\begin{bmatrix} \widehat{v}^\intercal & \check{v}^\intercal \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \widehat{Q} \end{bmatrix} \begin{bmatrix} \widehat{v} \\ \check{v} \end{bmatrix} = \sum_{h=1}^{N} \left(\widehat{Q}_{hh}\, (\widehat{v}_h)^2 + (\widehat{P}_{hh} + \check{P}_{hh}) \widehat{v}_h\, \check{v}_h + \check{Q}_{hh}\, (\check{v}_h)^2 \right). \tag{A.1}$$

Hence, the form $\begin{bmatrix} \widehat{Q} & \widehat{P} \\ \widecheck{P} & \widehat{Q} \end{bmatrix}$ is positive semidefinite if and only if each summand in the right hand side above is a non-negative second order polynomial, i.e., if and only if any one of the conditions in the hypothesis holds. □

Theorem A.1 *Let (Γ), (C1) hold. Define for $h \in \{1, \ldots, N\}$, $\widehat{\vartheta}, \widehat{\vartheta}', \widehat{\vartheta}'' \in S^{\widehat{n}}$, $\widecheck{\vartheta}, \widecheck{\vartheta}', \widecheck{\vartheta}'' \in S^{\widecheck{n}}$,*

$$\begin{aligned}
\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}) &= \int_0^1 \partial_{\widehat{\eta}} \widehat{\tau}_h \left(\widehat{\Gamma}\left(s\,\widehat{\vartheta}'' + (1-s)\widehat{\vartheta}'\right), \widecheck{\Gamma}\,\widecheck{\vartheta}\right) \mathrm{d}s \\
\widehat{P}_h(\widehat{\vartheta}, \widecheck{\vartheta}', \widecheck{\vartheta}'') &= \int_0^1 \partial_{\widecheck{\eta}} \widehat{\tau}_h \left(\widehat{\Gamma}\,\widehat{\vartheta}, \widecheck{\Gamma}\left((1-s)\widecheck{\vartheta}'' + s\widecheck{\vartheta}'\right)\right) \mathrm{d}s \\
\widecheck{P}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}) &= \int_0^1 \partial_{\widehat{\eta}} \widecheck{\tau}_h \left(\widehat{\Gamma}\left(s\,\widehat{\vartheta}'' + (1-s)\,\widehat{\vartheta}'\right), \widecheck{\Gamma}\,\widecheck{\vartheta}\right) \mathrm{d}s \\
\widecheck{Q}_h(\widehat{\vartheta}, \widecheck{\vartheta}', \widecheck{\vartheta}'') &= \int_0^1 \partial_{\widecheck{\eta}} \widecheck{\tau}_h \left(\widehat{\Gamma}\,\widehat{\vartheta}, \widecheck{\Gamma}\left(s\,\widecheck{\vartheta}'' + (1-s)\,\widecheck{\vartheta}'\right)\right) \mathrm{d}s\ .
\end{aligned} \tag{A.2}$$

Assume moreover that

(H) For all $\widehat{\vartheta}', \widehat{\vartheta}'' \in S^{\widehat{n}}$, $\widecheck{\vartheta}', \widecheck{\vartheta}'' \in S^{\widecheck{n}}$, and for all $h \in \{1, \ldots, N\}$, with the exception of at most one $\bar{h}$

(H0) $\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'') > 0$, $\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') > 0$ *and*
$4\,\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'')\,\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') > \left(\widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') + \widecheck{P}_h(\widehat{\vartheta}'', \widehat{\vartheta}', \widecheck{\vartheta}'')\right)^2$,

while at $\bar{h}$ one of the following conditions holds:

(H1) $\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'') = 0$, $\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = 0$, $\widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = 0$ *and* $\widecheck{P}_h(\widehat{\vartheta}'', \widehat{\vartheta}', \widecheck{\vartheta}'') = 0$;
(H2) $\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'') > 0$, $\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = 0$, $\widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = 0$ *and* $\widecheck{P}_h(\widehat{\vartheta}'', \widehat{\vartheta}', \widecheck{\vartheta}'') = 0$;
(H3) $\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'') = 0$, $\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') > 0$, $\widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = 0$ *and* $\widecheck{P}_h(\widehat{\vartheta}'', \widehat{\vartheta}', \widecheck{\vartheta}'') = 0$;
(H4) $\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'') > 0$, $\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') > 0$ *and*
$4\,\widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'')\,\widecheck{Q}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') = \left(\widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'') + \widecheck{P}_h(\widehat{\vartheta}'', \widehat{\vartheta}', \widecheck{\vartheta}'')\right)^2$.

Then, there exists at most one Nash equilibrium.

Proof of Theorem A.1 Assume ϑ' and ϑ'' are Nash equilibria and using Lemma 4.1 write

$$\begin{aligned}
\widehat{T}(\vartheta'') - \widehat{T}(\vartheta') &= \widehat{T}(\widehat{\vartheta}'', \widecheck{\vartheta}'') - \widehat{T}(\widehat{\vartheta}', \widecheck{\vartheta}') \\
&= \widehat{T}(\widehat{\vartheta}'', \widecheck{\vartheta}'') - \widehat{T}(\widehat{\vartheta}', \widecheck{\vartheta}'') + \widehat{T}(\widehat{\vartheta}', \widecheck{\vartheta}'') - \widehat{T}(\widehat{\vartheta}', \widecheck{\vartheta}') \\
&= \widehat{\Gamma}^{\intercal}\,\widehat{Q}\,\widehat{\Gamma}(\widehat{\vartheta}'' - \widehat{\vartheta}') + \widehat{\Gamma}^{\intercal}\,\widehat{P}\,\widecheck{\Gamma}(\widecheck{\vartheta}'' - \widecheck{\vartheta}') \\
&= \left[\widehat{\Gamma}^{\intercal}\,\widehat{Q}\,\widehat{\Gamma} \quad \widehat{\Gamma}^{\intercal}\,\widehat{P}\,\widecheck{\Gamma}\right] (\vartheta'' - \vartheta'),
\end{aligned}$$

where, with the notation in (A.2), the diagonal $N \times N$ matrices $\widehat{Q}$ and $\widehat{P}$ are given by $\widehat{Q}_{hh} = \widehat{Q}_h(\widehat{\vartheta}', \widehat{\vartheta}'', \widecheck{\vartheta}'')$ and $\widehat{P}_{hh} = \widehat{P}_h(\widehat{\vartheta}', \widecheck{\vartheta}', \widecheck{\vartheta}'')$, for $h \in \{1, \ldots, N\}$. Similarly,

$\check{T}(\vartheta'') - \check{T}(\vartheta') = \begin{bmatrix} \check{\Gamma}^{\intercal}\, \check{P}\, \widehat{\Gamma} & \check{\Gamma}^{\intercal}\, \check{Q}\, \check{\Gamma} \end{bmatrix} (\vartheta'' - \vartheta')$, where, with the notation in (A.2), the diagonal $N \times N$ matrices $\check{Q}$ and $\check{P}$ are given by $\check{Q}_{hh} = \check{Q}_h(\widehat{\vartheta}', \check{\vartheta}', \check{\vartheta}'')$ and $\check{P}_{hh} = \check{P}_h(\widehat{\vartheta}'', \check{\vartheta}', \check{\vartheta}'')$, for $h \in \{1, \ldots, N\}$.

Hence,

$$\begin{aligned} \begin{bmatrix} \widehat{T}(\vartheta'') - \widehat{T}(\vartheta') \\ \check{T}(\vartheta'') - \check{T}(\vartheta') \end{bmatrix} &= \begin{bmatrix} \widehat{\Gamma}^{\intercal}\, \widehat{Q}\, \widehat{\Gamma} & \widehat{\Gamma}^{\intercal}\, \widehat{P}\, \check{\Gamma} \\ \check{\Gamma}^{\intercal}\, \check{P}\, \widehat{\Gamma} & \check{\Gamma}^{\intercal}\, \check{Q}\, \check{\Gamma} \end{bmatrix} (\vartheta'' - \vartheta') \\ &= \begin{bmatrix} \widehat{\Gamma}^{\intercal} & 0 \\ 0 & \check{\Gamma}^{\intercal} \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} (\vartheta'' - \vartheta')\,. \end{aligned} \tag{A.3}$$

Claim 1: The vectors $v = \left[\begin{smallmatrix} \widehat{v} \\ \check{v} \end{smallmatrix}\right]$ isotropic with respect to $v \mapsto v^{\intercal} \left[\begin{smallmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{smallmatrix}\right] v$ are such that at most one component of $\widehat{v}$ is non-zero and at most one component of $\check{v}$ is non-zero. The form $v \mapsto v^{\intercal} \left[\begin{smallmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{smallmatrix}\right] v$ is positive semidefinite on $\mathbb{R}^{2N}$ by Lemma A.1, which can be applied by (H). Call $\mathcal{I}$ the subset of $\mathbb{R}^{2N}$ consisting of isotropic vectors for the quadratic, though not necessarily symmetric, form $v \mapsto v^{\intercal} \left[\begin{smallmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{smallmatrix}\right] v$. Call $v = \left[\begin{smallmatrix} \widehat{v} \\ \check{v} \end{smallmatrix}\right]$ a vector in $\mathcal{I}$. Fix $h \in \{1, \ldots, N\}$. If (H0) holds, then $\widehat{v}_h = 0$ and $\check{v}_h = 0$. Hence, if there is no $\bar{h}$ where equality in (H) holds, then necessarily $\mathcal{I} = \{0\}$ and the Claim follows.

If there is a $\bar{h}$, where (H0) does not hold, then 4 cases are in order, and by means of the representation (A.1), we have:

Case (H1): $\mathcal{I} = \operatorname{span}\{e_{\bar{h}}, e_{N+\bar{h}}\}$.
Case (H2): $\mathcal{I} = \operatorname{span}\{e_{N+\bar{h}}\}$.
Case (H3): $\mathcal{I} = \operatorname{span}\{e_{\bar{h}}\}$.
Case (H4): $\mathcal{I} = \operatorname{span}\left\{\sqrt{\check{Q}_{\bar{h}}}\; e_{\bar{h}} - \sqrt{\widehat{Q}_{\bar{h}}}\; e_{N+\bar{h}}\right\}$.

Therefore, any vector $v = \left[\begin{smallmatrix} \widehat{v} \\ \check{v} \end{smallmatrix}\right]$ in $\mathcal{I}$ is such that at most one component of $\widehat{v}$ is non-zero and at most one component of $\check{v}$ is non-zero. ✓

Claim 2: If $\vartheta' \equiv (\widehat{\vartheta}', \check{\vartheta}')$, $\vartheta'' \equiv (\widehat{\vartheta}'', \widehat{\vartheta}'') \in S^N$ and $\vartheta' \neq \vartheta''$, then at least one of the 2 vectors $\widehat{\Gamma}(\widehat{\vartheta}'' - \widehat{\vartheta}')$ or $\check{\Gamma}(\check{\vartheta}'' - \check{\vartheta}')$ has at least 2 non-zero components.

We are assuming $\vartheta' \neq \vartheta''$. Then, $\widehat{\vartheta}' \neq \widehat{\vartheta}''$ or $\check{\vartheta}' \neq \check{\vartheta}''$. Since $\sum_{i=1}^{\widehat{n}} \widehat{\vartheta}'_i = 1 = \sum_{j=1}^{\check{n}} \widehat{\vartheta}''_j$, at least one of the 2 vectors $\widehat{\vartheta}'' - \widehat{\vartheta}'$ or $\check{\vartheta}'' - \check{\vartheta}'$ has at least 2 non-zero components.

By Condition (Γ) and Remark 4.1, $\widehat{\Gamma}$ admits the decomposition $\widehat{\Gamma} = \left[\begin{array}{c} \mathrm{Id}_{\widehat{n}} \\ \hdashline \widehat{\gamma} \end{array}\right]$, where the matrix $\widehat{\gamma}$ has order $(N - \widehat{n}) \times \widehat{n}$ and its entries are either 0 or 1. Similarly, $\check{\Gamma}$ admits the decomposition $\check{\Gamma} = \left[\begin{array}{c} \mathrm{Id}_{\check{n}} \\ \hdashline \check{\gamma} \end{array}\right]$, where the matrix $\check{\gamma}$ has order $(N - \check{n}) \times \check{n}$ and its entries are either 0 or 1.

Thus,

$$\widehat{\Gamma}(\widehat{\vartheta}'' - \widehat{\vartheta}') = \left[\begin{array}{c} \widehat{\vartheta}'' - \widehat{\vartheta}' \\ \hdashline \widehat{\gamma}(\widehat{\vartheta}'' - \widehat{\vartheta}') \end{array}\right] \quad \text{and similarly} \quad \check{\Gamma}(\check{\vartheta}'' - \check{\vartheta}') = \left[\begin{array}{c} \check{\vartheta}'' - \check{\vartheta}' \\ \hdashline \check{\gamma}(\check{\vartheta}'' - \check{\vartheta}') \end{array}\right].$$

This proves that at least one of the 2 vectors $\widehat{\Gamma}(\widehat{\vartheta}'' - \widehat{\vartheta}')$ or $\check{\Gamma}(\check{\vartheta}'' - \check{\vartheta}')$ has at least 2 non-zero components. ✓

Claim 3: If ϑ', ϑ'' are distinct Nash equilibria, then

$$\begin{aligned} &(\widehat{\vartheta}'' - \widehat{\vartheta}')^\intercal \, \left(\widehat{T}(\vartheta'') - \widehat{T}(\vartheta')\right) \geq 0\,, \qquad (\check{\vartheta}'' - \check{\vartheta}')^\intercal \, \left(\check{T}(\vartheta'') - \check{T}(\vartheta')\right) \geq 0 \\ \text{and} \quad &\max\left\{(\widehat{\vartheta}'' - \widehat{\vartheta}')^\intercal \, \left(\widehat{T}(\vartheta'') - \widehat{T}(\vartheta')\right)\,,\ (\check{\vartheta}'' - \check{\vartheta}')^\intercal \, \left(\check{T}(\vartheta'') - \check{T}(\vartheta')\right)\right\} > 0\,. \end{aligned} \tag{A.4}$$

The form $v \mapsto v^\intercal \begin{bmatrix} \widehat{\Gamma}^\intercal & 0 \\ 0 & \check{\Gamma}^\intercal \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} v$ is positive semidefinite on $\mathbb{R}^{\widehat{n}+\check{n}}$. Indeed, if there exists a $v \in \mathbb{R}^{\widehat{n}+\check{n}}$ such that $v^\intercal \begin{bmatrix} \widehat{\Gamma}^\intercal & 0 \\ 0 & \check{\Gamma}^\intercal \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} v < 0$, then $\left(\begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} v\right)^\intercal \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \left(\begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} v\right) < 0$, which contradicts the fact that $v \mapsto v^\intercal \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} v$ is positive semidefinite on $\mathbb{R}^{2N}$.

Call $\mathcal{J}$ the subset of $\mathbb{R}^{\widehat{n}+\check{n}}$ consisting of the vectors v isotropic relative to the form $v \mapsto v^\intercal \begin{bmatrix} \widehat{\Gamma}^\intercal & 0 \\ 0 & \check{\Gamma}^\intercal \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} v$. Clearly, $v \in \mathcal{J}$ with $v = \begin{bmatrix} \widehat{v} \\ \check{v} \end{bmatrix}$, if and only if $\begin{bmatrix} \widehat{\Gamma}\,\widehat{v} \\ \check{\Gamma}\,\check{v} \end{bmatrix} \in \mathcal{I}$.

This shows that $\begin{bmatrix} \widehat{\Gamma}\,\widehat{v} \\ \check{\Gamma}\,\check{v} \end{bmatrix}$ can not be in $\mathcal{I}$. Therefore

$$(\vartheta'' - \vartheta')^\intercal \, \begin{bmatrix} \widehat{\Gamma}^\intercal & 0 \\ 0 & \check{\Gamma}^\intercal \end{bmatrix} \begin{bmatrix} \widehat{Q} & \widehat{P} \\ \check{P} & \check{Q} \end{bmatrix} \begin{bmatrix} \widehat{\Gamma} & 0 \\ 0 & \check{\Gamma} \end{bmatrix} (\vartheta'' - \vartheta') > 0$$

and by (A.3) we have (A.4).

The contradiction between Claim 3 and Proposition 2.2 shows that $\widehat{\vartheta}' = \widehat{\vartheta}''$. □

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