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On Quasi-Hermitian Varieties in Even Characteristic and Related Orthogonal Arrays

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ABSTRACT

In this article, we study the BM quasi-Hermitian varieties, laying in the three-dimensional Desarguesian projective space of even order. After a brief investigation of their combinatorial properties, we first show that all of these varieties are projectively equivalent, exhibiting a behavior which is strikingly different from what happens in odd characteristic. This completes the classification project started there. Here we prove more; indeed, by using previous results, we explicitly determine the structure of the full collineation group stabilizing these varieties. Finally, as a byproduct of our investigation, we also construct a family of simple orthogonal arrays $O(q^5, q^4, q, 2)$, with entries in $\mathbb{F}_q$, where q is an even prime power. Orthogonal arrays (OA's) are principally used to minimize the number of experiments needed to investigate how variables in testing interact with each other.

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1 | Introduction

Units in a finite projective plane of order q^2 are sets of $q^3 + 1$ points which have the same intersection numbers as Hermitian curves with respect to lines, that is, they meet every line in either 1 or $q + 1$ points. Quasi-Hermitian varieties are a natural generalization of unitals to higher dimensions; namely they are set of points in the finite projective space $\mathrm{PG}(n, q^2)$ which have the same size and the same intersection numbers as Hermitian varieties with respect to hyperplanes.

Actually, a point set S of $\mathrm{PG}(n, q^2)$, $n > 2$, having the same intersection numbers with respect to hyperplanes as a non-singular Hermitian variety has also the same number of points; for $n = 2$, the size of S can be either $q^3 + 1$, that is the size of a Hermitian curve (also called a classical unital), or $q^2 + q + 1$, which is the number of points of a Baer subplane of $\mathrm{PG}(2, q^2)$; see [3, 4].

It is a classical problem in finite geometry to characterize point-sets in term of their incidence properties with respect to subspaces. For instance, the notion of arc in a plane is born by abstracting the incidence properties of a conic in a Desarguesian plane $\mathrm{PG}(2, q)$. A celebrated theorem by Segre states that for q odd all $(q + 1)$-arcs are complete and turn out to be conics. As mentioned above, the case for Hermitian curves is different, as for $q > 2$ there exist also nonclassical unitals in planes of order q^2 (i.e., they are not sets of points of a Hermitian curve).

Indeed, important families of unitals were found by Buekenhout [5] in every two-dimensional (projective) translation plane; Metz [6] showed how to use Buekenhout's method to construct a nonclassical unital in the Desarguesian plane $\mathrm{PG}(2, q^2)$ for any prime power $q > 2$. The unitals of this family are called Buekenhout-Metz (BM) unitals. For a careful description of these unitals see [7, 8] whereas for a thorough survey of the research literature on embedded unitals see, for example, [9].

As in the case of unitals, several constructions are also known for quasi-Hermitian varieties in higher projective dimensions; see, for example, [1, 10–12]. In particular, in [1] a large family of quasi-Hermitian varieties of $\mathrm{PG}(n, q^2)$, depending on two parameters in the finite field $\mathbb{F}_{q^2}$ of order q^2, has been introduced. In dimension $n = 2$ these varieties are BM–unitals and thus they will be called BM quasi-Hermitian varieties.

In [2], the two authors studied the equivalence classes, up to projectivities, of BM quasi-Hermitian varieties for $n = 3$ and q odd and they enumerated these classes, using a technique similar to the one employed to determine the equivalence classes number of the BM–unitals in the plane.

In the present paper, we consider BM quasi-Hermitian varieties in $\mathrm{PG}(3, q^2)$ with q even, case which was left open in [2], completing the classification project started there and, more explicitly, determining the structure of the full collineation group that stabilizes these varieties.

Precisely, in Section 2 we explicitly recall the construction of BM quasi-Hermitian varieties from [1]. In Section 3 we determine some geometric properties of the BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$ of $\mathrm{PG}(3, q^2)$ for q even; in particular we observe that it is possible to choose a projective reference in such a way that through each affine point of $\mathcal{M}_{a,b}$ there is exactly one line contained in the variety, and these lines are all parallel to a given plane. It is shown in Section 4 that in even characteristic all of the varieties $\mathcal{M}_{a,b}$ are projectively equivalent. This is in marked contrast with the behavior for q odd. Combining this result together with some geometric features of $\mathcal{M}_{a,b}$ and specific properties of suitable subgroups of $P\Gamma L_4(q^2)$, the stabilizer in $P\Gamma L_4(q^2)$ of the quasi-Hermitian variety $\mathcal{M}_{a,b}$ is determined in Section 5. Further, its structure and its action on the points of $\mathcal{M}_{a,b}$ is discussed.

Our long-term aim is to try to find a characterization of the BM quasi-Hermitian varieties among all possible quasi-Hermitian varieties in spaces of the same dimension and order.

Finally, in Section 6 simple orthogonal arrays $OA(q^5, q^4, q, 2)$ of index q^3, q even, are constructed from the BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$.

Orthogonal arrays (OA's) are principally used to minimize the number of experiments needed to investigate how variables in testing interact with each other and, consequently, determine the required parameters. For instance, OA's are used to calibrate the flight parameters of drones, to optimize their performance; see, for example, [13].

2 | Background on Quasi-Hermitian Varieties

Quasi-Hermitian varieties were introduced in [10] as a generalization of non-singular Hermitian varieties through the following definition.

A point-set H in $\mathrm{PG}(n, q^2)$ is a *quasi-Hermitian variety* if has the same size and the same intersection numbers with hyperplanes as a non-singular Hermitian variety $H(n, q^2)$ of $\mathrm{PG}(n, q^2)$.

In particular, a quasi-Hermitian variety is a set of size $(q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$ of $\mathrm{PG}(n, q^2)$ meeting the hyperplanes in either

$$(q^n + (-1)^{n-1})(q^{n-1} - (-1)^{n-1})/(q^2 - 1)$$

or

$$1 + q^2(q^{n-1} + (-1)^n)(q^{n-2} - (-1)^n)/(q^2 - 1)$$

points; see [14].

Remark 2.1. As pointed out in the Introduction, it is possible to drop the requirement on the size of H if $n \geq 3$.

There are a few known families of quasi-Hermitian varieties; some of them turn out to be sort of higher-dimension analogues to the known families of unitals or can be obtained from them by pivoting; see [4]. Here we point out that there are also families for $n \geq 3$ which are quite different from those; see [15].

The quasi-Hermitian varieties we are considering in the present paper are BM quasi-Hermitian varieties as for $n = 2$ they turn out to be Buekenhout-Metz unitals; see, for example, [9]. They are defined as follows.

Let $\mathcal{B}_{a,b}$ be the surface of $\mathrm{PG}(3, q^2)$ of projective equation

$$\begin{aligned}\mathcal{B}_{a,b} : Z^q J^q - Z J^{2q-1} + a^q(X^{2q} + Y^{2q}) - a(X^2 + Y^2)J^{2q-2} \\ = (b^q - b)(X^{q+1} + Y^{q+1})J^{q-1},\end{aligned} \quad (1)$$

with $a \in \mathbb{F}_{q^2}^*$ and $b \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Denote by Σ_∞ the hyperplane at infinity with equation $J = 0$ of $\mathrm{PG}(3, q^2)$, put

$$\mathcal{F} := \{(0, X, Y, Z) \in \mathrm{PG}(3, q^2) | X^{q+1} + Y^{q+1} = 0\} \quad (2)$$

and

$$\mathcal{B}_\infty := (\mathcal{B}_{a,b} \cap \sum_\infty).$$

In [1] it was proved that the following point set of $\mathrm{PG}(3, q^2)$

$$\mathcal{M}_{a,b} := (\mathcal{B}_{a,b} \setminus \mathcal{B}_\infty) \cup \mathcal{F} \quad (3)$$

is a quasi-Hermitian variety for $q \geq 4$ even or for q odd and $4a^{q+1} + (b^q - b)^2 \neq 0$. We call it a *BM quasi-Hermitian variety*.

Clearly, the affine points of $\mathcal{M}_{a,b}$ satisfy the affine equation:

$$\begin{aligned}\mathcal{B}_{a,b} : Z^q - Z + a^q(X^{2q} + Y^{2q}) - a(X^2 + Y^2) \\ = (b^q - b)(X^{q+1} + Y^{q+1}).\end{aligned} \quad (4)$$

If $q=2$ then $b+b^q=1$; hence $\mathcal{M}_{a,b}$ is an Hermitian variety with equation

$$Z^qJ+ZJ^q+a^q(X+Y)J^q+a(X^q+Y^q)J + (X^{q+1}+Y^{q+1})=0. \tag{5}$$

Therefore, in the sequel we assume that $q>2$.

Finally we recall that any two BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$ and $\mathcal{M}_{\alpha,\beta}$ of $\mathrm{PG}(3,q^2)$ are *projectively equivalent* if there exists a collineation $\psi\in P\Gamma L_4(q^2)$ such that $\psi(\mathcal{M}_{a,b})=\mathcal{M}_{\alpha,\beta}$.

3 | Preliminaries on BM Quasi-Hermitian Varieties in $\mathrm{PG}(3,q^2)$, q Even

Let $q>2$ be an even prime power, $a\in\mathbb{F}_{q^2}^*$ and $b\in\mathbb{F}_{q^2}\backslash\mathbb{F}_q$. In the present section we first determine the number of lines through a point of $\mathcal{B}_{a,b}$ with Equation (1) which are contained in $\mathcal{B}_{a,b}$, next we deduce some related combinatorial properties of the BM quasi-Hermitian variety $\mathcal{M}_{a,b}$ (see Equation (3)) and some information about the stabilizer of $\mathcal{M}_{a,b}$ in the projective linear group $\mathrm{PGL}_4(q^2)$.

Theorem 3.1. *Let $\mathcal{B}_{a,b}$ be the surface of Equation (1) in $\mathrm{PG}(3,q^2)$, q an even prime power and put $\mathcal{B}_\infty=\mathcal{B}_{a,b}\cap[J=0]$ and $P_\infty=(0,0,0,1)$. Then,*

i. *for any affine point Q of $\mathcal{B}_{a,b}$ there is exactly one line of $\mathrm{PG}(3,q^2)$ passing through Q and contained in $\mathcal{B}_{a,b}$;*
ii. *for any point R in $\mathcal{B}_\infty\backslash P_\infty$ there are $q+1$ lines contained in $\mathcal{B}_{a,b}$ passing through R (and exactly one of these lines is contained in $\mathcal{B}_\infty$);*
iii. *there is exactly one line, among the ones contained in $\mathcal{B}_{a,b}$, that passes through P_∞ and it consists of the points of $\mathcal{B}_\infty$.*

Proof. Observe that for q even

$$\mathcal{B}_\infty:\begin{cases}J=0\\(X+Y)^{2q}=0.\end{cases}$$

This is the line in $\Sigma_\infty:=[J=0]$ of equations $X+Y=0=J$. To stress this fact we shall call it ℓ_∞. We refer to the points in ℓ_∞ as $M_\infty=(0,1,1,0)$, $P_\infty=(0,0,0,1)$ and $L_\infty^m=(0,m,m,1)$ with $m\in\mathbb{F}_{q^2}^*$.

Any line $\ell\in\mathcal{B}_{a.b}$ which is not contained in Σ_∞, must contain one of the points in $\ell_\infty=\mathcal{B}_\infty$.

Take $P\in\mathcal{B}_{a,b}\cap\mathrm{AG}(3,q^2)$, it is known [1] that the collineation group of $\mathcal{B}_{a,b}$ acts transitively on the points of $\mathcal{B}_{a,b}\cap\mathrm{AG}(3,q^2)$, thus we can assume without loss of generality $O=(0,0,0)$ and that ℓ has the following affine parametric equations:

$$\begin{cases}x=m_1t\\y=m_2t\\z=m_3t\end{cases}$$

for $t\in\mathbb{F}_{q^2}$ and $(m_1,m_2,m_3)\in\{(0,0,1),(m,m,1),(1,1,0)\}$. One can easily notice that:

- $(m_1,m_2,m_3)\neq(0,0,1)$, for otherwise the line would not be contained in $\mathcal{B}_{a,b}$;
- $(m_1,m_2,m_3)\neq(m,m,1)$ because $\mathrm{char}(\mathbb{K})=2$ and again ℓ would not be contained in $\mathcal{B}_{a,b}$.

So, we conclude that the only possible line contained in $\mathcal{B}_{a,b}$ and passing through P_∞ has affine representation

$$\begin{cases}x=t\\y=t\\z=0,\end{cases}\quad t\in\mathbb{F}_{q^2}.$$

Inspection of Equation (1) shows that this line is actually contained in $\mathcal{B}_{a,b}$. Using now the transitivity of the collineation group on the affine points of $\mathcal{B}_{a,b}$ we obtain that for any point in $\mathcal{B}_{a,b}\cap\mathrm{AG}(3,q^2)$ passes one and only one line contained in $\mathcal{B}_{a,b}$.

Now we turn our attention at the points of $\ell_\infty=\mathcal{B}_\infty$, in particular we count the lines in $\mathcal{B}_{a,b}$ that contain $L_\infty^m=(0,m,m,1)$ and are not ℓ_∞. The general line r with this property has affine parametric equations

$$r:\begin{cases}x=\bar{x}+mt\\y=\bar{y}+mt\\z=\bar{z}+t,\end{cases}$$

where $(\bar{x},\bar{y},\bar{z})\in\mathcal{B}_{a,b}\cap\mathrm{AG}(3,q^2)$, which means that:

$$\bar{z}^q+\bar{z}+a^q(\bar{x}^{2q}+\bar{y}^{2q})+a(\bar{x}^2+\bar{y}^2) = (b^q+b)(\bar{x}^{q+1}+\bar{y}^{q+1}). \tag{6}$$

We now write the condition for the whole line r to be contained in $\mathcal{B}_{a,b}$:

$$\begin{aligned}&\bar{z}^q+t^q+\bar{z}+t+a^q(\bar{x}^{2q}+\bar{y}^{2q}+\underbrace{(mt)^{2q}+(mt)^{2q}}_{=0})\\&\quad+a(\bar{x}+\bar{y})^2+\underbrace{(mt)^2+(mt)^2}_{=0})\\&\quad=(b^q-b)[(\bar{x}^q+m^qt^q)(\bar{x}+mt)+(\bar{y}^q+m^qt^q)(\bar{y}+mt)].\end{aligned}$$

Simplifying Equation (6) we obtain

$$t^q[(m^q(b^q+b)(\bar{x}+\bar{y}))+1]+t[m(b^q+b)(\bar{x}+\bar{y})^q+1] =0.$$

To have the latter equation satisfied for any $t\in\mathbb{F}_{q^2}$, we must have

$$(\bar{x}+\bar{y})^q=\frac{1}{m(b^q+b)}\text{equivalently, }(\bar{x}+\bar{y}) =\frac{1}{m^q(b^q+b)}. \tag{7}$$

Given any m, there are q^2 possible pairs $(\bar{x}, \bar{y})$ that satisfy $(\bar{x} + \bar{y})^q = \frac{1}{m(b^q + b)}$. For any such pair $(\bar{x}, \bar{y})$, there are q possible values of $\bar{z}$ that satisfy $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}_{a,b}$. We deduce that the number of lines passing through L_∞^m contained in $\mathcal{B}_{a,b}$ is $\frac{q^2 q}{q^2} + 1 = q + 1$.

We can repeat the same argument for $M_\infty = (0, 1, 1, 0)$ and count the lines in $\mathcal{B}_{a,b}$ through M_∞. Consider the general affine line r such that $M_\infty \in r$ and $r \neq \ell_\infty$:

$$r : \begin{cases} x = \bar{x} + t \\ y = \bar{y} + t \\ z = \bar{z}, \end{cases}$$

with $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}_{a,b} \cap \mathrm{AG}(3, q^2)$. Reasoning as for L_∞^m we obtain

$$t^q[((b^q + b)(\bar{x} + \bar{y}))] + t[(b^q + b)(\bar{x} + \bar{y})^q] = 0.$$

This equality is satisfied for every $t \in \mathbb{F}_{q^2}$ if and only if $\bar{x} = \bar{y}$. Notice that for every $\bar{x} = \bar{y}$ there are q possible $\bar{z}$ such that $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}_{a,b} \cap \mathrm{AG}(3, q^2)$. So, we obtain q possible lines passing through M_∞ with $r \neq \ell_\infty$.

The general line passing through P_∞ and not entirely contained in Σ_∞ has affine equation

$$r : \begin{cases} x = \bar{x} \\ y = \bar{y} \\ z = \bar{z} + t. \end{cases}$$

We require that $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}_{a,b} \cap \mathrm{AG}(3, q^2)$ and $r \subset \mathcal{B}_{a,b}$. This implies $t^q + t = 0$ for any $t \in \mathbb{F}_{q^2}$, which is not true. We conclude that the only line contained in $\mathcal{B}_{a,b}$ passing through P_∞ is ℓ_∞. □

Remark 3.2. Observe that for every point L_∞^m or M_∞ the q affine lines in $\mathcal{B}_{a,b}$ containing it are coplanar. In particular, the general affine line of $\mathcal{B}_{a,b}$ through M_∞ is contained in the plane of equation $x + y = 0$, while the general affine line of $\mathcal{B}_{a,b}$ through L_∞^m is contained in the affine plane $x + y = \frac{1}{m^q(b^q + b)}$. Furthermore $\ell_\infty \subset \mathcal{F}$, where $\mathcal{F}$ is the Hermitian cone defined in Equation (2).

Theorem 3.3. *Let $\mathcal{M}_{a,b}$ be the BM quasi-Hermitian variety of* $\mathrm{PG}(3, q^2)$, *q even, described by Equation* (3). *Then through each affine point of $\mathcal{M}_{a,b}$ there passes one line of $\mathcal{M}_{a,b}$, whereas through a point at infinity of $\mathcal{M}_{a,b} \cap \ell_\infty$ there pass $q + 1$ lines of a pencil contained in $\mathcal{M}_{a,b}$;*

Proof. We observe that the affine points of $\mathcal{M}_{a,b}$ are the same as those of $\mathcal{B}_{a,b}$, whereas the set $\mathcal{F}$ of points at infinity of $\mathcal{M}_{a,b}$ consists of the points $P = (0, x, y, z)$ such that $x^{q+1} + y^{q+1} = 0$ and it contains the points at infinity of $\mathcal{B}_\infty = \ell_\infty$. Hence, from Theorem 3.1 and Remark 3.2 we get the result. □

Now, denote by G the stabilizer of $\mathcal{M}_{a,b}$ in the projective linear group $\mathrm{PGL}_4(q^2)$.

Lemma 3.4. *The group G stabilizes the affine points of $\mathcal{M}_{a,b}$, fixes the point P_∞ and preserves both the line ℓ_∞ and the hyperplane Σ_∞.*

Proof. By Theorem 3.3, the points of ℓ_∞ are the only points of $\mathcal{M}_{a,b}$ through which more than one line of $\mathcal{M}_{a,b}$ passes. So, any element of G must map a point of ℓ_∞ onto a point of ℓ_∞. We also know by [1, Corollary 4.3] that G acts transitively on the affine points of $\mathcal{M}_{a,b}$. In particular, since for any affine point Q of $\mathcal{M}_{a,b}$ there is exactly one line ℓ_Q meeting ℓ_∞ in a point different from P_∞, we get that G is also transitive on the points of $\ell_\infty \setminus \{P_\infty\}$ and fixes P_∞ itself. Finally, as P_∞ is fixed by G, any collineation in G must send lines through P_∞ to lines through P_∞. However, all lines though P_∞ in $\mathcal{M}_{a,b}$ are contained in Σ_∞ (and they actually span this hyperplane). So G stabilizes Σ_∞ too. □

Remark 3.5. By Lemma 3.4, the group G is an affine group of collineations, as it fixes the hyperplane at infinity. As such, we can represent the elements of G by 4×4 matrices with elements in $\mathbb{F}_{q^2}$ of the form

$$M = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & d & e & h \\ 0 & f & g & i \\ 0 & 0 & 0 & c \end{pmatrix},$$

where $c(dg + ef) \neq 0$ and $d + f = e + g$.

The first column of M is $(1, 0, 0, 0)^t$ because $\phi(\Sigma_\infty) = \Sigma_\infty$; the last row of M is $(0, 0, 0, c)$ because $\phi(P_\infty) = P_\infty$. Furthermore $d + f = e + g$ since ϕ preserves the line ℓ_∞.

4 | Projective Equivalence of $\mathcal{M}_{a,b}$'s

In this section we are going to prove that the BM quasi-Hermitian varieties in $\mathrm{PG}(3, q^2)$, $q > 2$ even are equivalent. Let ϕ in $P\Gamma L_4(q^2)$. We represent ϕ by a non–singular matrix M together with a field automorphism σ. By convention, to apply ϕ to some point we first apply σ to each entry of the row vector representing the point and then multiply on the right by the matrix M. The maps $\mathrm{Tr} : \mathbb{F}_{q^2} \to \mathbb{F}_q, x \mapsto x + x^q$ and $N : \mathbb{F}_{q^2} \to \mathbb{F}_q, x \mapsto x^{q+1}$ are the $\mathbb{F}_q$-trace and the $\mathbb{F}_q$-norm, respectively.

Lemma 4.1. *$\mathcal{M}_{a,b}$ and $\mathcal{M}_{a',b'}$ are equivalent as quasi-Hermitian varieties if and only if there is a collineation $\phi : \mathcal{M}_{a,b} \mapsto \mathcal{M}_{a',b'}$ with associated field automorphism σ and represented by a matrix*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & e & 0 \\ 0 & \lambda_1 e & \lambda_2 d & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

where $c \in \mathbb{F}_q^*$, $d, e \in \mathbb{F}_{q^2}$, $\lambda_1, \lambda_2 \in \mathbb{F}_{q^2}$ such that $\lambda_1^{q+1} = \lambda_2^{q+1} = 1$ and one of the following holds

I. *$e = 0, d \neq 0$ and $\lambda_2 = 1$;*

II. $e \neq 0, d = 0$ *and* $\lambda_1 = 1$;

III. $e \neq 0 \neq d, \lambda_1 = \lambda_2 = 1$, *and* $d/e \in \mathbb{F}_q \setminus \{1\}$;

IV. $e \neq 0 \neq d, \lambda_2 \neq 1 \neq \lambda_1, \lambda_1 \neq \lambda_2$ *and* $d = \frac{(1+\lambda_1)}{(1+\lambda_2)} e$.

Proof. It is enough to prove the necessary condition. So assume that there is $\phi \in P\Gamma L_4(q^2)$ such that $\phi(\mathcal{M}_{a,b}) = \mathcal{M}_{a',b'}$. Let $G_1 = \mathrm{Aut}(\mathcal{M}_{a,b})$ and $G_2 = \mathrm{Aut}\left(\mathcal{M}_{a',b'}\right)$, then $G_1^{\phi} = G_2$. Since $(P_\infty, \ell_\infty, \Sigma_\infty)$ are the unique subspaces of $PG_3(q^2)$, preserved by G_i, $i = 1, 2$, having nonempty intersection with $\mathcal{M}_{a,b}$ and $\mathcal{M}_{a',b'}$, then ϕ preserves $(P_\infty, \ell_\infty, \Sigma_\infty)$. Further, ϕ preserves the Hermitian cone $\mathcal{C} = \mathcal{M}_{a,b} \cap \Sigma_\infty = \mathcal{M}_{a',b'} \cap \Sigma_\infty$, hence ϕ preserves each of the subsets $P_\infty, \ell_\infty, \mathcal{F}, \Sigma_\infty$.

Since both G_1 and G_2 act transitively on $\mathcal{M}_{a,b} \setminus \Sigma_\infty$ and on $\mathcal{M}_{a',b'} \setminus \Sigma_\infty$ in [1, Corollary 4.3], there is $g_i \in G_i$ such that $g_2 \phi g_1$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{a',b'}$ fixing O. Thus, w.l.o.g. ϕ fixes $O = (1, 0, 0, 0)$, and hence from Remark 3.5 it is represented by the non-singular matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & e & h \\ 0 & f & g & i \\ 0 & 0 & 0 & c \end{pmatrix}. \tag{8}$$

where $d + f = e + g$. For any $\alpha \in \mathbb{F}_q$, $(1, 0, 0, \alpha) \in \mathcal{M}_{a,b}$. So we deduce from Equation (8) that necessarily $\phi(1, 0, 0, \alpha) = (1, 0, 0, c\alpha) \in \mathcal{M}_{a',b'}$ and hence $c \in \mathbb{F}_q^*$.

We consider now the plane of equation $Y = 0$. Its intersection with $\mathcal{M}_{a,b}$ is given by a set of points $(1, x, 0, z)$ such that

$$ax^2 + bx^{q+1} + z \in \mathbb{F}_q,$$

which implies that

$$a^\sigma x^{2\sigma} + b^\sigma x^{\sigma(q+1)} + z^\sigma \in \mathbb{F}_q. \tag{9}$$

Suppose $\phi(1, x, 0, z) \in \mathcal{M}_{a',b'}$; then

$$a' \frac{(d^2 + e^2)}{c} x^{2\sigma} + b' \frac{(d^{q+1} + e^{q+1})}{c} x^{\sigma(q+1)} + \frac{h}{c} x^\sigma + z^\sigma \in \mathbb{F}_q;$$

this and Equation (9) together give:

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2))x^{2\sigma} + (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1})) \\ &x^{\sigma(q+1)} + \frac{h}{c} x^\sigma \in \mathbb{F}_q. \end{aligned} \tag{10}$$

As $q > 2$, we can choose a primitive element δ of $\mathbb{F}_{q^2}$ such that $\delta^q = 1 + \delta, \delta^2 + \delta + \lambda = 0, \delta^{q+1} = \lambda \neq 0$ and the absolute trace of λ equals 1.

We substitute in Equation (10) the following values of x^σ:

i. $x^\sigma = 1$; thus

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2)) + (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1})) \\ &+ \frac{h}{c} \in \mathbb{F}_q; \end{aligned}$$

ii. $x^\sigma = \delta$; thus

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2))\delta^2 \\ &+ (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1}))\delta^{q+1} + \frac{h}{c}\delta \in \mathbb{F}_q; \end{aligned}$$

iii. $x^\sigma = \delta^q$ thus

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2))\delta^{2q} \\ &+ (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1}))\delta^{q+1} + \frac{h}{c}\delta^q \in \mathbb{F}_q; \end{aligned}$$

iv. $x^\sigma = \lambda$ thus

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2))\lambda^2 \\ &+ (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1}))\lambda^2 + \frac{h}{c}\lambda \in \mathbb{F}_q; \end{aligned}$$

v. $x^\sigma = \lambda\delta$ thus

$$\begin{aligned} &(a^\sigma + c^{-1}a'(d^2 + e^2))(\delta\lambda^2) \\ &+ (b^\sigma + c^{-1}b'(d^{q+1} + e^{q+1}))\lambda^2\delta^{q+1} + \frac{h}{c}\lambda\delta \in \mathbb{F}_q. \end{aligned}$$

Let $A = a^\sigma + a' \frac{(d^2+e^2)}{c}$; $B = (b^\sigma + b' \frac{(d^{q+1}+e^{q+1})}{c})$; $C = \frac{h}{c}$.

From (i) we obtain that $A = B + C + s$ where $s \in \mathbb{F}_q$, substituting it in (iv) we obtain

$$\begin{aligned} &\cancel{B\lambda^2} + C\lambda^2 + s\lambda^2 + \cancel{B\lambda^2} + C\lambda \in \mathbb{F}_q \Rightarrow C\lambda(\lambda+1) \in \mathbb{F}_q \Rightarrow \\ &C(\lambda+1) \in \mathbb{F}_q \Rightarrow C \in \mathbb{F}_q. \end{aligned}$$

Summing up (ii) and (iii) we obtain

$$A \underbrace{(\delta^2 + \delta^{2q})}_{=1} + C \underbrace{(\delta + \delta^q)}_{=1} \in \mathbb{F}_q \Rightarrow A + C \in \mathbb{F}_q \Rightarrow A, C \in \mathbb{F}_q.$$

Furthermore from (ii) and $\delta^2 = \lambda + \delta$

$$A(\delta + \lambda) + B\lambda + C\delta \in \mathbb{F}_q \Rightarrow A\delta + C\delta \in \mathbb{F}_q \underset{\delta \notin \mathbb{F}_q}{\Rightarrow} A = C.$$

Finally, from (v)

$$A\lambda\delta + c\delta \in \mathbb{F}_q \Rightarrow A\lambda + C = 0, A = C = 0.$$

It follows that

$$a^\sigma + a' \frac{(d^2 + e^2)}{c} = 0; \tag{11}$$

$$\left(b^\sigma + b'\frac{(d^{q+1}+e^{q+1})}{c}\right) \in \mathbb{F}_q; \tag{12}$$

and

$$h = 0.$$

With a very similar argument with respect to the plane of equation $X = 0$ we can conclude

$$a^\sigma + a'\frac{(f^2+g^2)}{c} = 0; \left(b^\sigma + b'\frac{(f^{q+1}+g^{q+1})}{c}\right) \in \mathbb{F}_q;\ i = 0.$$

So,

$$d^2 + e^2 = f^2 + g^2 \neq 0 \text{ and } f^{q+1} + g^{q+1} = d^{q+1} + e^{q+1} \neq 0. \tag{13}$$

We now know that $(1, x, y, z) \in \mathcal{M}_{a,b}$ if and only if $\phi(1, x, y, z) \in \mathcal{M}_{a',b'}$ and also $(1, x, y, z) \in \mathcal{M}_{a,b}$ if and only if

$$a(x^2 + y^2) + b(x^{q+1} + y^{q+1}) + z \in \mathbb{F}_q. \tag{14}$$

On the other hand, $\phi(1, x, y, z) = (1, dx^\sigma + fy^\sigma, ex^\sigma + gy^\sigma, cz^\sigma)$, so $\phi(1, x, y, z) \in \mathcal{M}_{a',b'}$ if and only if

$$c^{-1}(a'(dx^\sigma + fy^\sigma)^2 + (ex^\sigma + gy^\sigma)^2)$$

$$+ c^{-1}b'((dx^\sigma + fy^\sigma)^{q+1} + (ex^\sigma + gy^\sigma)^{q+1}) + z^\sigma \in \mathbb{F}_q.$$

This together with $(1, x, y, z) \in \mathcal{M}_{a,b}$ leads to

$$a^\sigma(x^{2\sigma} + y^{2\sigma}) + a'\left(\frac{(dx^\sigma + fy^\sigma)^2}{c} + \frac{(ex^\sigma + gy^\sigma)^2}{c}\right)$$

$$b^\sigma(x^{\sigma(q+1)} + y^{\sigma(q+1)}) + b'\frac{(dx^\sigma + fy^\sigma)^{q+1}}{c} + \frac{(ex^\sigma + gy^\sigma)^{q+1})}{c} \in \mathbb{F}_q.$$

Using Equations (11), (12), and (13) we obtain

$$b'[(d^q f + e^q g)x^{\sigma q}y^\sigma + (df^q + eg^q)x^\sigma y^{\sigma q}] \in \mathbb{F}_q. \tag{15}$$

Let $\omega \in \mathbb{F}_{q^2}$ be a solution of $\xi^{q+1} = 1$. Since ϕ has to leave invariant the Hermitian cone $\mathcal{M}_{a,b} \cap \Sigma_\infty$, we have $\phi(0, x, \omega x, z) \in \mathcal{M}_{a,b} \cap \Sigma_\infty$. Again using Equation (13) we have:

$$(d^q f + e^q g)\omega^\sigma + (df^q + eg^q)\omega^{\sigma q} = 0$$

for any of the $q + 1$ values ω such that $\omega^{q+1} = 1$. This means that we found $q + 1$ solutions to an equation of degree q, so it must be

$$d^q f + e^q g = 0 \tag{16}$$

If $e = 0$, since $d \neq 0$, we get $f = 0$ and from $e + d = f + g$ we obtain $d = g$ that is (I). Suppose $e \neq 0$. If $d = 0$ then also $g = 0$ and $e = f$, that is (II). If $d \neq 0$, as also $f \neq 0$ from Equation (16) we have $(\frac{d}{e})^q = \frac{g}{f}$.

From Equation (13) we have:

$$\begin{aligned}(d + e)^{q+1} &= (f + g)^{q+1}\\ d^{q+1} + d^q e + e^q d + e^{q+1} &= f^{q+1} + f^q g + g^q f + g^{q+1}\\ d^q e + e^q d &= f^q g + g^q f = \lambda \in \mathbb{F}_q.\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{d^q}{e^q} &= \frac{\lambda}{e^{q+1}} + \frac{d}{e}\\ \frac{g^q}{f^q} &= \frac{\lambda}{f^{q+1}} + \frac{g}{f}\end{aligned}$$

and hence $\frac{d}{e} + \frac{g}{f} = \frac{\lambda}{e^{q+1}} = \frac{\lambda}{f^{q+1}}$, which implies $\lambda = 0$ or $e^{q+1} = f^{q+1}$ and $d^{q+1} = g^{q+1}$.

In the case in which $\lambda = 0$ then $df + ge = 0$. Put $d/e = g/f = \alpha$. Then $\alpha \neq 1$ and from $d + e = g + f$ we get $e(\alpha + 1) = (\alpha + 1)f$. Hence $e = f$ and $d = g$, that is case (III) holds.

If $\lambda \neq 0$ then $f = \lambda_1 e$ and $g = \lambda_2 d$ such that $\mathrm{N}(\lambda_1) = \mathrm{N}(\lambda_2) = 1$. If $\lambda_1 = \lambda_2 = 1$, then we get again case (III). If $\lambda_1 \neq 1$, then also $\lambda_2 \neq 1$ and we get $(1 + \lambda_2)d = (1 + \lambda_1)e$, furthermore if $\lambda_1 = \lambda_2$ then $d = e$ and M would be singular so $\lambda_1 \neq \lambda_2$, that is case (IV). This concludes the proof. □

From the previous Lemma, taking into account conditions (11) and (12), we get the following.

Lemma 4.2. *Let $(a, b), (a', b') \in \mathbb{F}^*_{q^2} \times (\mathbb{F}_{q^2} \backslash \mathbb{F}_q)$ with $(a', b') \neq (a, b)$. There is $\phi \in P\Gamma L_4(q^2)$ such that $\mathcal{M}^\phi_{a,b} = \mathcal{M}_{a',b'}$ if and only if*

$$\begin{cases} a' = ca^\sigma/(d^2 + e^2)\\ b' = cb^\sigma/(d^{q+1} + e^{q+1}) + u\end{cases}$$

for some $c \in \mathbb{F}^*_q u \in \mathbb{F}_q$, and d, e satisfying the conditions of Lemma 4.1.

Assume that $\mathcal{M}_{a,b}$ and $\mathcal{M}_{a',b'}$ are projectively equivalent. In this case we write $(a, b) \sim (a', b')$ where $\sim$ is in particular an equivalence relation on the ordered pairs $(a, b) \in \mathbb{F}^2_{q^2}$ such that $a \neq 0$ and $b \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$.

Lemma 4.3. *Let $\mathcal{M}_{a,b}$ be a BM quasi-Hermitian variety of* $\mathrm{PG}(3, q^2)$, *q even and ε be a primitive element of $\mathbb{F}_{q^2}$. Then, there exists $\alpha \in \mathbb{F}^*_{q^2}$ such that $\mathcal{M}_{a,b}$ is equivalent to $\mathcal{M}_{\alpha,\varepsilon}$.*

Proof. Write $b = b_0 + \varepsilon b_1$, with $b_0, b_1 \in \mathbb{F}_q$ and $b_1 \neq 0$. Then, there exists $d \in \mathbb{F}^*_{q^2}$, such that $b_1/d^{q+1} = 1$. Therefore one can apply Lemma 4.2 with $c = 1, e = 0, u = b_0/d^{q+1}$ and obtain $(a, b) \sim (a/d^2, b/d^{q+1} + b_0/d^{q+1}) = (a/d^2, \varepsilon)$. □

Theorem 4.4. *All BM quasi-Hermitian varieties of* $\mathrm{PG}(3, q^2)$, *q even, are equivalent.*

Proof. In light of Lemma 4.3, to determine the equivalence classes of BM quasi-Hermitian varieties it is enough to determine when two varieties $\mathcal{M}_{a,\varepsilon}$ and $\mathcal{M}_{a',\varepsilon}$ are linearly equivalent. In particular, we consider the case $\sigma = id$. Then, $\mathcal{M}_{a,\varepsilon}$ and $\mathcal{M}_{a',\varepsilon}$ are equivalent if and only if

$$a' = ca/(d^2 + e^2);$$

$$\varepsilon(1 + c/(d^{q+1} + e^{q+1})) = u.$$

As $1 + c/(d^{q+1} + e^{q+1}) \in \mathbb{F}_q$ and $u \in \mathbb{F}_q$, we must have $c/(d^{q+1} + e^{q+1}) = 1$ for the second equation to be possible. Replacing in the first equation we get

$$a' = a\frac{d^{q+1} + e^{q+1}}{d^2 + e^2}.$$

We claim that this yields just one equivalence class; this is the same as to say that the function

$$(d, e) \mapsto \frac{d^{q+1} + e^{q+1}}{d^2 + e^2}$$

is surjective on $\mathbb{F}^*_{q^2}$.

We know that $d(1 + \lambda_2) = e(1 + \lambda_1)$ with $\mathrm{N}(\lambda_1) = \mathrm{N}(\lambda_2) = 1$. Assume $\lambda_2 \neq 1$ and put $\beta = \frac{1+\lambda_1}{1+\lambda_2}$ Hence we have to prove that for each $m^2 \in \mathbb{F}_{q^2}$ (recall that in characteristic 2 the map $x \mapsto x^2$ is bijective) there are $e, \lambda_1, \lambda_2 \in \mathbb{F}_{q^2}$ such that

$$e^{q-1} = m^2\frac{(1 + \beta)^2}{(1 + \beta^{q+1})}.$$

This is possible if and only if

$$m^{2(q+1)}\frac{(1 + \beta)^{2(q+1)}}{(1 + \beta^{q+1})^2} = 1$$

that is

$$m^{q+1}\frac{(1 + \beta)^{(q+1)}}{(1 + \beta^{q+1})} = 1,$$

whence

$$m^{q+1}\left(\lambda_1^q + \lambda_1\right) = \lambda_2^q + \lambda_2 + \lambda_1^q + \lambda_1.$$

For a chosen λ_1 we have to find λ_2 such that

$$\lambda_2^q + \lambda_2 = (1 + m^{q+1})\left(\lambda_1^q + \lambda_1\right),$$

that is

$$\lambda_2^2 + (1 + m^{q+1})\left(\lambda_1^q + \lambda_1\right)\lambda_2 + 1 = 0.$$

Since the absolute trace of $\frac{1}{(1+m^{q+1})\left(\lambda_1^q+\lambda_1\right)^2}$ is zero we can find λ_2 with the desired properties. □

5 | The Stabilizer of $\mathcal{M}_{a,b}$

In this section we shall provide a full description of the stabilizer in $P\Gamma L_4(q^2)$, $q > 2$ even, of the quasi-Hermitian variety $\mathcal{M}_{a,b}$. Throughout the section we shall adopt the notation and the conventions of Conway et al. [16]. In particular, C_m is the cyclic group with m elements, while E_m is the elementary Abelian group of order m. If A and B are two groups, we denote by $A \times B$ the direct product of A and B, $A.B$ the upward extension of A by B (i.e., the group G with $A \trianglelefteq G$ such that $G/A \cong B$) and $A : B$ the semidirect product between A and B (where A is normal in $A : B$ and B acts by conjugation as an automorphism group of A).

Let $\phi_s, \psi_\gamma, \mu_\delta$ where $s \in \mathbb{F}_q, \delta \in \mathbb{F}^*_q, \gamma = (\gamma_1, \gamma_2) \in \mathbb{F}^2_{q^2}$, be the elations associated with the following non-singular matrices:

$$\phi_s : \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \psi_\gamma(a, b)$$

$$: \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & a\left(\gamma_1^2 + \gamma_2^2\right) + b\left(\gamma_1^{q+1} + \gamma_2^{q+1}\right) \\ 0 & 1 & 0 & (b + b^q)\gamma_1^q \\ 0 & 0 & 1 & (b + b^q)\gamma_2^q \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\mu_\delta : \mathrm{diag}(1, \delta, \delta, \delta^2).$$

It follows from [1, Corollary 4.3] that

$$H := \left\langle \phi_s, \psi_\gamma(a, b), \mu_\delta : s \in \mathbb{F}_q, \gamma \in \mathbb{F}^2_{q^2}, \delta \in \mathbb{F}^*_q \right\rangle$$

is a subgroup of G preserving the Hermitian cone

$$\mathcal{M}_{a,b} \cap \Sigma_\infty = \left\{(0, 1, \omega^{i(q-1)}, k) : i = 1, ..., q + 1, k \in \mathbb{F}^*_{q^2}\right\} \cup P_\infty$$

where ω is a primitive element of $\mathbb{F}_{q^2}$. The subgroup

$$S = \left\langle \phi_s, \psi_\gamma(a,b) : s \in \mathbb{F}_q, \gamma \in \mathbb{F}_{q^2}^2 \right\rangle$$

is a normal Sylow 2-subgroup of H and

$$K = \langle \phi_s : s \in \mathbb{F}_q \rangle$$

is the kernel of the action of H on Σ_∞. The group

$$D := \left\{ \mu_\delta : \delta \in \mathbb{F}_q^* \right\}$$

is cyclic of order $q-1$. Also, S acts regularly on the q^5 points of $\mathcal{M}_{a,b} \backslash \Sigma_\infty$.

It can be immediately deduced from [1, Section 4] that the induced group $\overline{H}$ on Σ_∞ is a Frobenius group $\overline{H} = \overline{S} : \overline{D}$ of order $q^4(q-1)$ where

1. $\overline{S}$ is an elementary Abelian 2-group of order q^4. It is the kernel of $\overline{H}$ and consists of the elations of Σ_∞ with center P_∞;
2. $\overline{D}$ is a group of (P_∞, m_∞)-homologies of Σ_∞, where m_∞ is the line $J = Z = 0$.

Thus $\overline{H} \trianglelefteq \overline{G} \leq \overline{S} : (C_{q+1} \times \mathrm{GL}_2(q))$ since the second one is the stabilizer in $\mathrm{PGL}_3(q^2)$ of the Hermitian cone being q even.

Let $U = \{\tau_e : e \in \mathbb{F}_q\}$, where τ_e is the elation of $\mathrm{PGL}_4(q^2)$ represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e+1 & e & 0 \\ 0 & e & e+1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then U induces on Σ_∞ a group of (V_∞, ℓ_∞)-homologies, where $V_\infty = (0,1,1,0)$. Further $(0, 1, \omega^{i(q-1)}, k)^{\tau_e} = (0, 1 + (1+\omega^{i(q-1)})e, \omega^{i(q-1)} + (1+\omega^{i(q-1)})e, k)$ with

$$\begin{aligned} \frac{\omega^{i(q-1)} + (1+\omega^{i(q-1)})e}{1 + (1+\omega^{i(q-1)})e} &= \omega^{i(q-1)} \frac{1 + (1+\omega^{-i(q-1)})e}{1 + (1+\omega^{i(q-1)})e} \\ &= \omega^{i(q-1)} \frac{1 + (1+\omega^{qi(q-1)})e}{1 + (1+\omega^{i(q-1)})e} \\ &= \omega^{i(q-1)} [1 + (1+\omega^{i(q+1)})e]^{q-1} \end{aligned}$$

since $e \in \mathbb{F}_q$. Thus τ_e, and hence U, preserves $\mathcal{M}_{a,b} \cap \Sigma_\infty$ fixing ℓ_∞. Further, U preserves $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. Indeed, if $(1, x_0, y_0, z_0) \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$,

$$\begin{aligned} &z_0^q + z_0 + a^q\left(x_0^{2q} + y_0^{2q}\right) + a\left(x_0^2 + y_0^2\right) \\ &\quad = (b^q + b)\left(x_0^{q+1} + y_0^{q+1}\right) + (b^q + b)\mathrm{Tr}(e)(x_0 + y_0)^{q+1} \\ &\quad = (b^q + b)\left(x_0^{q+1} + y_0^{q+1}\right) \end{aligned}$$

since $e \in \mathbb{F}_q$, and so $(1, x_0, y_0, z_0)^{\tau_e} \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$. Therefore $U = \{\tau_e : e \in \mathbb{F}_q\}$ is an elementary Abelian 2-group preserving $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. Then U preserves both $\mathcal{M}_{a,b}$ and $\mathcal{B}_{a,b}$ since U preserves $\mathcal{M}_{a,b} \cap \Sigma_\infty$ fixing ℓ_∞.

Lemma 5.1. *K is the kernel of the action of G on Σ_∞.*

Proof. The point-wise stabilizer N in $\mathrm{PGL}_4(q^2)$ of Σ_∞ consists of the elations represented by matrices of the form

$$\begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{a} & 0 & 0 \\ 0 & 0 & \mathfrak{a} & 0 \\ 0 & 0 & 0 & \mathfrak{a} \end{pmatrix}$$

with $\mathfrak{a} \neq 0$. Clearly $K \leq N \cap G$. Let $\alpha \in N$ and for each $\theta \in \mathbb{F}_{q^2}$ and $\lambda \in \mathbb{F}_q$ consider the point $P_{\theta,\lambda} = (1, \theta, \theta, \lambda)$ in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. Then $P_{\theta,\lambda}^\alpha = (1, \mathfrak{b} + \mathfrak{a}\theta, \mathfrak{c} + \mathfrak{a}\theta, \mathfrak{a}\lambda + \mathfrak{d})$ which lies in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$ if and only if

$$\begin{aligned} &(\mathfrak{a}\lambda + \mathfrak{d})^q + (\mathfrak{a}\lambda + \mathfrak{d}) + a^q(\mathfrak{b}^{2q} + \mathfrak{c}^{2q}) + a(\mathfrak{b}^2 + \mathfrak{c}^2) \\ &= (b^q + b)(\mathfrak{b}^{q+1} + \mathfrak{c}^{q+1}) + (b^q + b)(\mathfrak{b} + \mathfrak{c})\mathfrak{a}^q\theta^q \\ &\quad + (b^q + b)(\mathfrak{b} + \mathfrak{c})^q\mathfrak{a}\theta \end{aligned}$$

is satisfied for each $\theta \in \mathbb{F}_{q^2}$ and $\lambda \in \mathbb{F}_q$. Thus $\mathfrak{b} = \mathfrak{c}$ and $\mathfrak{a}, \mathfrak{d} \in \mathbb{F}_q$ since $\mathfrak{a} \neq 0$.

Now, let $Q = (1, x_0, y_0, z_0)$ in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$, then

$$z_0^q + z_0 + a^q\left(x_0^{2q} + y_0^{2q}\right) + a\left(x_0^2 + y_0^2\right) = (b^q + b)\left(x_0^{q+1} + y_0^{q+1}\right).$$

Now, $Q^\alpha = (1, \mathfrak{b} + \mathfrak{a}x_0, \mathfrak{b} + \mathfrak{a}y_0, \mathfrak{d} + \mathfrak{a}z_0)$ lies in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$ if and only if

$$\begin{aligned} &\mathfrak{a}\mathrm{Tr}(z_0) + \mathfrak{a}^2a^q\left(x_0^{2q} + y_0^{2q}\right) + \mathfrak{a}^2a\left(x_0^2 + y_0^2\right) \\ &= \mathfrak{a}^2(b^q + b)\left(x_0^{q+1} + y_0^{q+1}\right) + (b^q + b)\mathfrak{a}\mathrm{Tr}((x_0 + y_0)\mathfrak{b}^q) \end{aligned}$$

and hence

$$(\mathfrak{a} + 1)\mathrm{Tr}(z_0) = (b^q + b)\mathrm{Tr}((x_0 + y_0)\mathfrak{b}^q) \tag{17}$$

since $Q \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$. We may repeat the previous argument by choosing two distinct points $Q_i = (1, x_i, x_i + \mu, z_i)$ with $\mathrm{Tr}(x_1) \neq \mathrm{Tr}(x_2), \mu \in \mathbb{F}_q^*$ and z_i such that $\mathrm{Tr}(z_i) = \mu^2\mathrm{Tr}(a) + \mathrm{Tr}(b)(\mathrm{Tr}(x_i)\mu + \mu^2)$. Therefore, $\mathrm{Tr}(z_1) \neq \mathrm{Tr}(z_2)$ since $b \notin \mathbb{F}_q$, and $Q_1, Q_2 \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$. Since $Q_i^\alpha \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$, we may argue as above with Q_i in the role of Q and hence Equation (17) becomes

$$(\mathfrak{a} + 1)\mathrm{Tr}(z_i) = (b^q + b)\mu\mathrm{Tr}(\mathfrak{b}^q),$$

which leads to $\mathfrak{a} = 1$ and $\mathfrak{b} \in \mathbb{F}_q$. Now, let $R = (1, x_0, y_0, z_0)$ with $x_0 + y_0 \notin \mathbb{F}_q$. Then $R^\alpha \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$ implies $\mathfrak{b}\mathrm{Tr}(x_0 + y_0) = 0$.

Therefore $\mathfrak{b} = 0$ as $x_0 + y_0 \notin \mathbb{F}_q$. Thus $G \cap N = K$, which is the assertion. □

Proposition 5.2. *$S : U$ is a Sylow p-subgroup of G.*

Proof. Clearly, $U \cap S = 1$. It is easy to check that

$$\tau_e \phi_s = \phi_s \tau_e$$
$$\tau_e^{-1} \psi_\gamma(a,b) \tau_e = \psi_{\gamma'}(a,b), \text{ where } \gamma' = (\gamma_1 + (\gamma_1 + \gamma_2) e, \gamma_2 + (\gamma_1 + \gamma_2) e)$$

and hence $S : U$ lies in a Sylow p-subgroup W of G. Note that $W = SW_O$, where $O = (1,0,0,0)$, since S acts transitively on $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. Let $\alpha \in W_O$, α is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathfrak{f}+1 & \mathfrak{f} & \mathfrak{g}+\mathfrak{e} \\ 0 & \mathfrak{f} & \mathfrak{f}+1 & \mathfrak{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for suitable $\mathfrak{e}, \mathfrak{f}, \mathfrak{g} \in \mathbb{F}_{q^2}$ as a consequence of Lemma 3.4. Let $(1, x_0, y_0, z_0) \in \mathcal{M}_{a,b} \backslash \Sigma_\infty$, then

$$(1, x_0, y_0, z_0)^\alpha = (1, x_0 + \mathfrak{f}(x_0 + y_0), y_0 + \mathfrak{f}(x_0 + y_0), z_0 + \mathfrak{e}(x_0 + y_0) + \mathfrak{g} x_0)$$

which actually lies in $\mathcal{M}_{a,b}$ if and only if

$$\mathrm{Tr}(\mathfrak{e}(x_0 + y_0) + \mathfrak{g} x_0) = (b^q + b)\Big[\mathrm{Tr}\Big(x_0^q \mathfrak{f}(x_0 + y_0)\Big) + \mathrm{Tr}\Big(y_0^q \mathfrak{f}(x_0 + y_0)\Big)\Big]$$

and hence

$$\mathrm{Tr}(\mathfrak{e}(x_0 + y_0) + \mathfrak{g} x_0) = (b^q + b)(x_0 + y_0)^{q+1} \mathrm{Tr}(\mathfrak{f})$$

Since $(1, \omega^i, \omega^i, 1) \in \mathcal{M}_{a,b}$ for each $i = 0, \ldots, q^2 - 2$ given ω a primitive element of $\mathbb{F}_{q^2}$, it follows that $(1, \omega^i, \omega^i, 1)^\alpha \in \mathcal{M}_{a,b}$ if and only if $\mathrm{Tr}(\mathfrak{g}\omega^i) = 0$, hence $\mathfrak{g} = 0$.

Let $\lambda \in \mathbb{F}_{q^2}$ and $z_\lambda \in \mathbb{F}_{q^2}$ such that $\mathrm{Tr}(z_\lambda) = \mathrm{Tr}(a\lambda^2) + (b^q + b)(\omega^{i(q+1)} + (\lambda - \omega^i)^{q+1})$, then $(1, \omega^i, \lambda - \omega^i, z_\lambda) \in \mathcal{M}_{a,b}$ and so $(1, \omega^i, \lambda - \omega^i, z_\lambda)^\alpha \in \mathcal{M}_{a,b}$ if and only if $\mathrm{Tr}(\lambda \mathfrak{e}) = (b^q + b)\lambda^{q+1} \mathrm{Tr}(\mathfrak{f})$. Now, choosing distinct λ's in $\mathbb{F}_q^*$ we obtain $\mathrm{Tr}(\mathfrak{e}) = \mathrm{Tr}(\mathfrak{f}) = 0$ and so $\mathfrak{e}, \mathfrak{f} \in \mathbb{F}_q$. Therefore, $\mathfrak{e}\mathrm{Tr}(x_0 + y_0) = 0$. Finally, choosing λ in $\mathbb{F}_{q^2} \backslash \mathbb{F}_q$, we obtain $\mathfrak{e} = 0$. Thus $\alpha \in U$ and so $W = S : U$. □

Proposition 5.3. *Let $W = S : U$, then $W \lhd G$.*

Proof. Assume that W is not normal in G. Then there is $g \in G$ such that $W^g \neq W$. Nevertheless, $S \leq W^g$ since $K \leq S$, $K \lhd G$ by Lemma 5.1, and since $S/K \cong E_{q^4}$ induces the (full) elation group of center P_∞ on Σ_∞. Thus the group induced by $\langle W^g, W \rangle$ on the Hermitian cone $\mathcal{M}_{a,b} \cap \Sigma_\infty$ contains two distinct Sylow p-subgroups and hence contains $SL_2(q)$ since $\overline{G} \leq \overline{S} : (C_{q+1} \times \mathrm{GL}_2(q))$ and $q > 2$, where $\overline{G}$ is the the group induced on $\mathcal{M}_{a,b} \cap \Sigma_\infty$ by G. Let R be subgroup of $\langle W^g, W \rangle$ inducing a cyclic subgroup of $SL_2(q)$ of order $q + 1$. Note that R can be chosen in a way that $R \cap S = 1$ since S is a p-group. Then R fixes a point $\mathcal{M}_{a,b} \backslash \Sigma_\infty$ since $|\mathcal{M}_{a,b} \backslash \Sigma_\infty| = q^5$ and permutes regularly the $q + 1$ lines of the Hermitian cone. Since $S \lhd G$ and S acts transitively on $\mathcal{M}_{a,b} \backslash \Sigma_\infty$, possibly substituting R with a suitable conjugate in $S : R$, we may assume that that R fixes $O = (1,0,0,0)$. Now, also U fixes O thus $\langle U, R \rangle$ fixes O and hence $\langle U, R \rangle \cap S = 1$. Thus $\langle U, R \rangle$ acts faithfully on the Hermitian cone and contains subgroups of order q and $q + 1$, and the last one acts transitively on the lines of the Hermitian cone. Thus $\langle U, R \rangle$ contains a copy of $SL(2,q)$. Then $\langle U, R \rangle$ contains a conjugate of R, say $\langle \zeta \rangle$ with ζ represented by the matrix $Diag(1, \omega^{q-1}, \omega^{1-q}, 1)$ with ω a primitive element of $\mathbb{F}_{q^2}^*$.

Let $P_{\theta,\lambda} = (1, \theta, \theta, \lambda)$ with $\theta \in \mathbb{F}_{q^2}$ such that $\mathrm{Tr}(a\theta^2) \neq 0$ and $\lambda \in \mathbb{F}_q$. Then $P_{\theta,\lambda}$ lies in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$ and hence $P_{\theta,\lambda}^\zeta = (1, \omega^{q-1}\theta, \omega^{1-q}\theta, \lambda)$ must lie in $\mathcal{M}_{a,b} \backslash \Sigma_\infty$.

Thus

$$\mathrm{Tr}(a\theta^2(\omega^{2(q-1)} + \omega^{2(1-q)})) = 0$$
$$\mathrm{Tr}\left(a\theta^2 \frac{\omega^{4q} + \omega^4}{\omega^{2(q+1)}}\right) = \mathrm{Tr}(a\theta^2) \frac{\mathrm{Tr}(\omega^4)}{\mathrm{N}(\omega^2)} = 0$$

and so $\mathrm{Tr}(a\theta^2) = 0$ because $\mathrm{Tr}(\omega^4) \neq 0$. Thus W is normal in G. □

Theorem 5.4. *$G = \langle \phi_s, \psi_\gamma(a,b), \tau_e, \mu_\delta : \gamma \in \mathbb{F}_{q^2}^2, s, e, \delta \in \mathbb{F}_q, \delta \neq 0 \rangle$. It has order $q^6(q-1)$.*

Proof. First, we observe that $E_{q^4} : (C_{q-1} \times E_q) = \langle W, D \rangle / K \leq G/K$ and hence $E_{q^4} : (C_{q-1} \times E_q) \trianglelefteq G_{\Sigma_\infty}/K \trianglelefteq E_{q^4} : (C_{q-1} . (E_q : C_{q-1}))$.

Assume that there is an element of odd order ϱ in G such that $\bar{\varrho} \notin E_{q^4} : (C_{q-1} \times E_q)$. Then $\bar{\varrho}$ preserves ℓ_∞ and fixes P_∞, and two further points since $o(\bar{\varrho}) \mid q - 1$, namely one on $\ell_\infty \backslash \{P_\infty\}$ and the other on $(\mathcal{M}_{a,b} \cap \Sigma_\infty) \backslash \ell_\infty$. Recall that S/K is the group of (P_∞, P_∞)-elations of Σ_∞, then it acts transitively on $\ell_\infty \backslash \{P_\infty\}$, and hence we may assume that $\bar{\varrho}$ fixes the point $\{V_\infty\} = \ell_\infty \cap m_\infty$, where m_∞ is the line $J = Z = 0$. Moreover, the stabilizer in S/K of V_∞ acts regularly on the set q^2 lines of Σ_∞ which are incident with V_∞ and are distinct from ℓ_∞, thus we may also assume that $\bar{\varrho}$ preserves m_∞. Therefore, $\langle D/K, \bar{\varrho} \rangle$ is a subgroup of the stabilizer of a triangle in Σ_∞ with P_∞ and V_∞ as two of its vertices and ℓ_∞ and m_∞ as two of its sides. Actually, $\langle D/K, \bar{\varrho} \rangle \leq C_{q-1} \times C_{q-1}$, where the group $C_{q-1} \times C_{q-1}$ is generated by a cyclic subgroup of homologies in a triangular configuration. Since D/K is a cyclic group of order $q - 1$ consisting of (P_∞, m_∞)-homologies of Σ_∞, by suitably multiplying $\bar{\varrho}$ with an element of D/K we may assume that $\bar{\varrho}$ is a (Q, ℓ_∞)-homology of Σ_∞, where $Q = (0, 1, w, 0)$ with w a fixed element of $\mathbb{F}_{q^2} \backslash \{1\}$ such that $\mathrm{N}(w) = 1$. Thus, ϱ fixes ℓ_∞ point-wise, fixes Q and preserves the points on the Hermitian cone $\mathcal{M}_{a,b} \cap \Sigma_\infty$ with apex P_∞ and $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. In particular, ϱ preserves $\mathcal{M}_{a,b} \cap m_\infty = \{(0, 1, x, 0) : \mathrm{N}(x) = 1\}$.

Now, we are going to determine the matrix representation of ϱ. By Remark 3.5 we have

$$(0 \quad 1 \quad 1 \quad 0)\begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{f} & \mathfrak{g} & \mathfrak{h} \\ 0 & \mathfrak{j} & \mathfrak{k} & \mathfrak{l} \\ 0 & 0 & 0 & \mathfrak{p} \end{pmatrix} = (0 \quad \mathfrak{f}+\mathfrak{j} \quad \mathfrak{g}+\mathfrak{k} \quad \mathfrak{h}+\mathfrak{l}),$$

$$\begin{aligned}(0 \quad 1 \quad w \quad 0)&\begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{f} & \mathfrak{g} & \mathfrak{h} \\ 0 & \mathfrak{f}+\mathfrak{g}+\mathfrak{k} & \mathfrak{k} & \mathfrak{h} \\ 0 & 0 & 0 & \mathfrak{p} \end{pmatrix} \\ &= (0 \quad \mathfrak{f}(w+1)+\mathfrak{g}w+\mathfrak{k}w \quad \mathfrak{g}+\mathfrak{k}w \quad \mathfrak{h}(w+1)).\end{aligned}$$

Since the collineation ρ exists and $w \neq 1$ we have $\mathfrak{f}(w+1)+\mathfrak{g}w+\mathfrak{k}w \neq 0$, $\mathfrak{g}+\mathfrak{k}w \neq 0$ and $\mathfrak{h}=0$. Furthermore,

$$\begin{aligned}(\mathfrak{f}+\mathfrak{f}w+\mathfrak{g}w+\mathfrak{k}w)w+\mathfrak{g}+\mathfrak{k}w = \mathfrak{f}(w^2+w)+\mathfrak{g}(w^2+1) \\ +\mathfrak{k}(w^2+w)=0\end{aligned}$$

$$\mathfrak{k}=\mathfrak{f}+\mathfrak{g}(w^q+1).$$

In particular $\mathfrak{f} \neq \mathfrak{g}, \mathfrak{g}w^q$ because the matrix associated to ϱ is non-singular. Consider the point of $\mathcal{M}_{a,b}$ with coordinates $(1,1,1,\theta)$, where $\theta \in \mathbb{F}_q$. Then

$$\begin{aligned}(1 \quad 1 \quad 1 \quad \theta)&\begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{f} & \mathfrak{g} & 0 \\ 0 & \mathfrak{g}w^q & \mathfrak{f}+\mathfrak{g}(w^q+1) & 0 \\ 0 & 0 & 0 & \mathfrak{p} \end{pmatrix} \\ &= (1 \quad \mathfrak{b}+\mathfrak{f}+\mathfrak{g}w^q \quad \mathfrak{c}+\mathfrak{f}+\mathfrak{g}w^q \quad \mathfrak{d}+\mathfrak{p}\theta),\end{aligned}$$

$$\begin{aligned}&\mathrm{Tr}(\mathfrak{d})+\mathrm{Tr}(\mathfrak{p})\theta+\mathrm{Tr}(a(\mathfrak{b}+\mathfrak{c})^2 \\ &\quad = \mathrm{Tr}(b)[\mathfrak{b}^{q+1}+\mathfrak{c}^{q+1}+\mathrm{Tr}(\mathfrak{b}+\mathfrak{c})(\mathfrak{f}^q+\mathfrak{g}^q w)].\end{aligned} \tag{18}$$

which must be fulfilled for each $\theta \in \mathbb{F}_q$ and hence $\mathrm{Tr}(\mathfrak{p})=0$ and

$$\begin{aligned}\mathrm{Tr}(\mathfrak{d})+\mathrm{Tr}(a\mathfrak{b}+\mathfrak{c})^2 = \mathrm{Tr}(b)[\mathfrak{b}^{q+1}+\mathfrak{c}^{q+1} \\ +\mathrm{Tr}((\mathfrak{b}+\mathfrak{c})(\mathfrak{f}^q+\mathfrak{g}^q w))].\end{aligned}$$

Thus, $\mathfrak{p} \in \mathbb{F}_q^*$.

Now ϱ lies G, so $\mu_\delta\varrho$ does. Now, possibly after choosing $\delta = \mathfrak{p}^{-1/2}$ since $\mathfrak{p} \in \mathbb{F}_q$, we may consider

$$\varrho' = \mu_{\mathfrak{p}^{-1/2}}\varrho = \begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{f} & \mathfrak{g} & 0 \\ 0 & \mathfrak{g}w^q & \mathfrak{f}+\mathfrak{g}(w^q+1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which no longer induces a (Q, ℓ_∞)-homology of Σ_∞, but still has odd order and fixes the triangle vertices P_∞, V_∞ and Q, and preserves the points on the Hermitian cone $\mathcal{M}_{a,b} \cap \Sigma_\infty$ with apex P_∞ and $\mathcal{M}_{a,b} \backslash \Sigma_\infty$. In particular, ϱ' preserves $\mathcal{M}_{a,b} \cap m_\infty = \{(0,1,y,0) : \mathrm{N}(y)=1\}$, thus

$$\begin{aligned}(0 \quad 1 \quad y \quad 0)&\begin{pmatrix} 1 & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{f} & \mathfrak{g} & 0 \\ 0 & \mathfrak{g}w^q & \mathfrak{f}+\mathfrak{g}(w^q+1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= (0 \quad \mathfrak{f}+\mathfrak{g}w^q y \quad \mathfrak{g}+\mathfrak{f}y+\mathfrak{g}y+\mathfrak{g}w^q y \quad 0),\end{aligned}$$

with $\mathfrak{f} \neq \mathfrak{g}w^q y$ for each $y \in \mathbb{F}_{q^2}$ with $N(y)=1$, and hence

$$\mathrm{N}\left(\frac{(\mathfrak{f}+\mathfrak{g}+\mathfrak{g}w^q)y+\mathfrak{g}}{\mathfrak{g}w^q y+\mathfrak{f}}\right)=1.$$

Thus, $\varsigma : x \mapsto \frac{(\mathfrak{f}+\mathfrak{g}+\mathfrak{g}w^q)x+\mathfrak{g}}{\mathfrak{g}w^q x+\mathfrak{f}}$ is an element of $\mathrm{PGL}_2(q^2)$ fixing $1, w$ and preserving the Baer subline $\{y \in \mathbb{F}_{q^2} : \mathrm{N}(y)=1\}$ of $PG_1(q^2)$. Hence, ς lies in the cyclic subgroup of order $q-1$ of $\mathrm{PGL}_2(q^2)$ fixing $1, w$ and preserving $\{y \in \mathbb{F}_{q^2} : \mathrm{N}(y)=1\}$.

For each $c \in \mathbb{F}_q^*$ consider $\alpha_c \in \mathrm{PGL}_2(q^2)$ defined by

$$\alpha_c : x \mapsto \frac{x(c+cw+w^q+1)+c+cw+w+1}{x(c+cw^q+w^q+1)+c+cw^q+w+1}.$$

Then $1^{\alpha_c}=1$, $w^{\alpha_c}=w$ and

$$(w^q)^{\alpha_c} = w(w(c+cw^q+w+1)+c+cw^q+w^q+1)^{q-1}.$$

Indeed,

$$\begin{aligned}&\frac{w^q(c+cw+w^q+1)+c+cw+w+1}{w^q(c+cw^q+w^q+1)+c+cw^q+w+1} \\ &\quad = \frac{(w(c+cw^q+w+1)+c+cw^q+w^q+1)^q}{w^q(w(c+cw^q+w+1)+c+cw^q+w^q+1)} \\ &\quad = w(w(c+cw^q+w+1)+c+cw^q+w^q+1)^{q-1}.\end{aligned}$$

Thus $\left\{\alpha_c : c \in \mathbb{F}_q^*\right\}$ is the cyclic subgroup of order $q-1$ of $\mathrm{PGL}_2(q^2)$ fixing $1, w$ and preserving the Baer subline $\{y \in \mathbb{F}_{q^2} : \mathrm{N}(y)=1\}$ of $PG_1(q^2)$, and hence

$$\mathfrak{f} = \frac{c+w+cw^q+1}{d}, \quad \mathfrak{g} = \frac{c+cw+w+1}{d}$$

for some suitable $c \in \mathbb{F}_q^*$ and $d \in \mathbb{F}_{q^2}^*$. It is easy to check that, such $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\mathrm{N}(\mathfrak{f}) \neq \mathrm{N}(\mathfrak{g})$ and hence $\mathfrak{f}$ and $\mathfrak{g}$ fulfill $\mathfrak{f} \neq \mathfrak{g}w^q y$ for each $y \in \mathbb{F}_{q^2}$ with $N(y)=1$. Further,

$$\begin{aligned}\mathfrak{f}+\mathfrak{g}w^q+\mathfrak{g} &= \frac{c+w+cw^q+1}{d}+\left(\frac{c+cw+w+1}{d}\right)w^q \\ &\quad +\frac{c+cw+w+1}{d} = \frac{c+cw+w^q+1}{d}\end{aligned}$$

and by multiplying each term of the matrix representing ϱ' by d, we may assume that ϱ' is represented by

$$\begin{pmatrix} d & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & c+w+cw^q+1 & c+cw+w+1 & 0 \\ 0 & c+cw^q+w^q+1 & c+cw+w^q+1 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Note that $\tau_{\frac{c+1}{w+w^q}}\varrho'$ is an element of G represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c+1}{w+w^q}+1 & \frac{c+1}{w+w^q} & 0 \\ 0 & \frac{c+1}{w+w^q} & \frac{c+1}{w+w^q}+1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & c+w+cw^q+1 & c+cw+w+1 & 0 \\ 0 & c+cw^q+w^q+1 & c+cw+w^q+1 & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} d & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & w+cw^q & w+cw & 0 \\ 0 & cw^q+w^q & cw+w^q & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Set $\mathfrak{r} = w + cw^q$, then $\tau_{\frac{c+1}{w+w^q}}\varrho'$ is of the form

$$\begin{pmatrix} d & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{r} & \mathfrak{r}+\lambda & 0 \\ 0 & \mathfrak{r}^q+\lambda & \mathfrak{r}^q & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

where $\lambda = c\mathrm{Tr}(w) \neq 0$ and $\mathrm{Tr}(\mathfrak{r}) = \mathrm{Tr}(w)(c+1) = \lambda + \mathrm{Tr}(w)$.

The point $(1, x_0, x_0, c_0)$ with $c_0 \in \mathbb{F}_q$ lies in $\mathcal{M}_{a,b}\backslash\Sigma_\infty$, and hence

$$\begin{aligned}(1\quad x_0\quad x_0\quad c_0)\begin{pmatrix} d & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \\ 0 & \mathfrak{r} & \mathfrak{r}+\lambda & 0 \\ 0 & \mathfrak{r}^q+\lambda & \mathfrak{r}^q & 0 \\ 0 & 0 & 0 & d \end{pmatrix} &= (d\quad \mathfrak{b}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0\quad \mathfrak{c}+x_0\lambda+\mathfrak{r}^q x_0 \\ &\quad +\mathfrak{r}x_0\quad c_0 d+\mathfrak{d}),\end{aligned}$$

which is equivalent to

$$\left(1\quad \frac{\mathfrak{b}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0}{d}\quad \frac{\mathfrak{c}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0}{d}\quad c_0+\frac{\mathfrak{d}}{d}\right),$$

where

$$\left(\frac{\mathfrak{b}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0}{d}\right)+\left(\frac{\mathfrak{c}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0}{d}\right)=\frac{1}{d}(\mathfrak{b}+\mathfrak{c}),$$

$$\begin{aligned}\frac{1}{d^{q+1}}&\left[(\mathfrak{b}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0)^{q+1}+(\mathfrak{c}+x_0\lambda+\mathfrak{r}^q x_0+\mathfrak{r}x_0)^{q+1}\right]\\ &=\frac{1}{d^{q+1}}\left[\mathfrak{b}^{q+1}+\mathfrak{c}^{q+1}+(\lambda+\mathrm{Tr}(\mathfrak{r}))\mathrm{Tr}\left[(\mathfrak{b}+\mathfrak{c})x_0^q\right]\right]\\ &=\frac{1}{d^{q+1}}\left[\mathfrak{b}^{q+1}+\mathfrak{c}^{q+1}+\mathrm{Tr}(w)\mathrm{Tr}\left[(\mathfrak{b}+\mathfrak{c})x_0^q\right]\right].\end{aligned}$$

Now, since the image of the point $(1, x_0, x_0, c_0)$ under $\tau_{\frac{c+1}{w+w^q}}\varrho'$ must lie in $\mathcal{M}_{a,b}\backslash\Sigma_\infty$, it follows that

$$\begin{aligned}&\mathrm{Tr}(c_0)+\mathrm{Tr}(\mathfrak{d}/d)+\mathrm{Tr}(a(\mathfrak{b}/d+\mathfrak{c}/d)^2)\\ &=\mathrm{Tr}(b)[(\mathfrak{b}/d)^{q+1}+(\mathfrak{c}/d)^{q+1}]+\mathrm{Tr}(b)\mathrm{Tr}(w)\mathrm{Tr}\\ &\quad\left[(\mathfrak{b}+\mathfrak{c})x_0^q\right]/d^{q+1},\\ &\mathrm{Tr}(\mathfrak{d}/d)+\mathrm{Tr}(a(\mathfrak{b}/d+\mathfrak{c}/d)^2)\\ &=\mathrm{Tr}(b)[(\mathfrak{b}/d)^{q+1}+(\mathfrak{c}/d)^{q+1}]+\mathrm{Tr}(b)\mathrm{Tr}(w)\mathrm{Tr}\\ &\quad\left[(\mathfrak{b}+\mathfrak{c})x_0^q\right]/d^{q+1}.\end{aligned}$$

The previous equation must be fulfilled for each value of x_0 in $\mathbb{F}_{q^2}$, then $\mathrm{Tr}(b)\mathrm{Tr}(w)(\mathfrak{b}+\mathfrak{c}) = 0$ and

$$\mathrm{Tr}(\mathfrak{d}/d)+\mathrm{Tr}(a(\mathfrak{b}/d+\mathfrak{c}/d)^2)=\mathrm{Tr}(b)[(\mathfrak{b}/d)^{q+1}+(\mathfrak{c}/d)^{q+1}]. \tag{19}$$

Therefore $\mathfrak{b} = \mathfrak{c}$ since $b, w \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$, and hence $\mathfrak{d}/d \in \mathbb{F}_q$. Thus

$$\varrho'' = \phi_{\mathfrak{d}/d}\tau_{\frac{c+1}{w+w^q}}\varrho' = \begin{pmatrix} d & \mathfrak{b} & \mathfrak{b} & 0 \\ 0 & \mathfrak{r} & \mathfrak{r}+\lambda & 0 \\ 0 & \mathfrak{r}^q+\lambda & \mathfrak{r}^q & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

lies in G, and hence it preserves $\mathcal{M}_{a,b}\backslash\Sigma_\infty$.

Since the trace is surjective, any point with coordinates $(1, x, 0, z)$, where x is any element of $\mathbb{F}_{q^2}$, and z is a suitable element of $\mathbb{F}_{q^2}$ depending on the choice of x, lies in $\mathcal{M}_{a,b}\backslash\Sigma_\infty$, and hence

$$z^q + z + a^q x^{2q} + ax^2 = (b^q + b)x^{q+1}, \tag{20}$$

Then $(1, x, 0, z)^{\varrho''}$, which is given by

$$(1\quad x\quad 0\quad z)\begin{pmatrix} d & \mathfrak{b} & \mathfrak{b} & 0 \\ 0 & \mathfrak{r} & \mathfrak{r}+\lambda & 0 \\ 0 & \mathfrak{r}^q+\lambda & \mathfrak{r}^q & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = (d\quad \mathfrak{b}+\mathfrak{r}x\quad \mathfrak{b}+x\lambda+\mathfrak{r}x\quad zd),$$

lies in $\mathcal{M}_{a,b}\backslash\Sigma_\infty$, and hence

$$\begin{aligned}d^{q+1}(z^q+z)+a^q\lambda^2 d^{1-q}x^{2q}+a\lambda^2 d^{q-1}x^2\\ =(b^q+b)[\lambda^2 x^{q+1}+\lambda\mathrm{Tr}[(\mathfrak{b}+\mathfrak{r}x)x^q]],\end{aligned} \tag{21}$$

Now, combining Equations (20) with (21) one obtains

$$\begin{aligned}&a^q(\lambda^2 d^{1-q}+d^{q+1})x^{2q}+a(\lambda^2 d^{q-1}+d^{q+1})x^2\\ &=(b^q+b)[(\lambda^2+d^{q+1}+\lambda\mathrm{Tr}(\mathfrak{r}))x^{q+1}+\lambda\mathrm{Tr}(\mathfrak{b}x^q))].\end{aligned} \tag{22}$$

Consequently, equality in Equation (22) must be fulfilled for each $x \in \mathbb{F}_{q^2}$. Thus $\lambda^2 d^{1-q} + d^{q+1} = \lambda^2 d^{q-1} + d^{q+1} = \mathfrak{b} = \mathrm{Tr}(\mathfrak{r}) = 0$, and hence $c = 1$, $\mathfrak{r} = \mathrm{Tr}(w) = \lambda = d$ since $\lambda = c\mathrm{Tr}(w) \neq 0$, $\mathfrak{r} = w + cw^q$ and $\mathrm{Tr}(\mathfrak{r}) = \mathrm{Tr}(w)(c+1) = \lambda + \mathrm{Tr}(w)$, where $w \in \mathbb{F}_{q^2} \backslash \{1\}$ is such that $\mathrm{N}(w) = 1$. So $\varrho'' = 1$, and hence $\varrho = \left(\phi_{\mathfrak{d}/d} \tau_{\frac{c+1}{w+w^q}} \mu_{\mathfrak{p}^{-1/2}}\right)^{-1} \in \langle W, D\rangle$, which is a contradiction. □

From now on, we denote the stabilizer in $PGL_4(q^2)$ and in $P\Gamma L_4(q^2)$ of $\mathcal{M}_{a,b}$ by $G(a,b)$ and $\Gamma(a,b)$ respectively.

Theorem 5.5. *Let σ be an element $P\Gamma L_4(q^2)$ induced by a generator of $\mathrm{Aut}(\mathbb{F}_{q^2})$, and let β be an element $P\Gamma L_4(q^2)$ of the form as in Lemma 4.1 mapping $\mathcal{M}_{1,\epsilon}$ with $Tr(\epsilon) = 1$ onto $\mathcal{M}_{a,b}$, which exists by Theorem 4.4. Then*

$$\Gamma(a,b) = \left\langle \phi_s, \psi_\gamma(a,b), \tau_e, \mu_\delta, \sigma^\beta : \gamma \in \mathbb{F}_{q^2}^2, s, e, \delta \in \mathbb{F}_q, \delta \neq 0 \right\rangle,$$

and its order is $q^6(q-1)\log_2 q$.

Proof. We may assume that $\sigma : (j, x, y, z) \mapsto (j^2, x^2, y^2, z^2)$. Clearly, σ fixes $\Sigma_\infty, m_\infty P_\infty$, where $m_\infty : J = Z = 0$. Also σ permutes the points $(0, 1, \omega^j, 0)$, where $j = 0, \ldots, q-1$ fixing $(0, 1, 1, 0)$. Thus, $\langle\sigma\rangle$ preserves the Hermitian cone $\mathcal{M}_{1,\epsilon} \cap \Sigma_\infty$ fixing ℓ_∞.

Now, let $(1, x, y, z) \in \mathcal{M}_{1,\epsilon} \backslash \Sigma_\infty$ then

$$\mathrm{Tr}(z) + \mathrm{Tr}(x^2 + y^2) + N(x) + N(y) = 0,$$

thus

$$(\mathrm{Tr}(z) + \mathrm{Tr}(x^2 + y^2) + N(x) + N(y))^{2^i} = 0,$$

and hence

$$(\mathrm{Tr}(z^{2^i}) + \mathrm{Tr}((x^{2^i})^2 + (y^{2^i})^2) + N(x^{2^i}) + N(y^{2^i}) = 0.$$

Therefore, σ preserves $\mathcal{M}_{1,\epsilon}$ and hence $\Lambda(1,\varepsilon) \leq \Gamma(1,\varepsilon)$, where

$$\Lambda(1,\varepsilon) = G(1,\varepsilon)\langle\sigma\rangle = \left\langle \phi_s, \psi_\gamma(1,\epsilon), \tau_e, \mu_\delta, \sigma : \gamma \in \mathbb{F}_{q^2}^2, s, e, \delta \in \mathbb{F}_q, \delta \neq 0 \right\rangle.$$

Let $\xi \in \Gamma(1,\varepsilon)$, then $\xi \in P\Gamma L_4(q^2)$ and hence $\xi = \sigma^j \alpha$ for some $j = 0, \ldots, \log_2 q - 1$ and $\alpha \in PGL_4(q^2)$. Then $\sigma^{-j}\xi \in \Gamma(1,\varepsilon) \cap PGL_4(q^2) = G(1,\varepsilon)$ by Theorem 5.4 since σ preserves $\mathcal{M}_{1,\epsilon}$. Thus, $\xi \in G(1,\varepsilon)\langle\sigma\rangle = \Lambda(1,\varepsilon)$, and hence $\Lambda(1,\varepsilon) = \Gamma(1,\varepsilon)$, whose order clearly is $q^6(q-1)\log_2 q$. Then $\Gamma(a,b)$ has order $q^6(q-1)\log_2 q$ since $\Gamma(a,b) = \Gamma(1,\varepsilon)^\beta$. Therefore,

$$\Gamma(a,b) = \left\langle \phi_s, \psi_\gamma(a,b), \tau_e, \mu_\delta, \sigma^\beta : \gamma \in \mathbb{F}_{q^2}^2, s, e, \delta \in \mathbb{F}_q, \delta \neq 0 \right\rangle$$

by Theorem 5.4 since $\sigma^\beta \in \Gamma(a,b)$, $o(\sigma^\beta) = \log_2 q$ and $\langle\sigma^\beta\rangle \cap G(a,b) = 1$, which is the assertion. □

6 | Some Orthogonal Arrays

Let S be a set with $v := |S|$ elements. An $N \times k$ array with entries in S is an *orthogonal array* $OA(N, k, v, t)$ with v levels, strength t and index $\lambda := N/v^t$ if every $N \times t$ subarray of A contains each t-uple of elements of S exactly λ times; see [17]. Well–known examples of orthogonal arrays are latin squares and Hadamard matrices.

There is a very rich literature about orthogonal arrays, as they play an important role in statistics (where they are used in devising experimental designs), cryptography (e.g., in constructing threshold schemes) as well as in computer science (where they are used both for quality control and for optimizing the placement and routing of elements on PCBs). More recent applications have been found in the calibration of the flight parameters of drones to optimize their performance in the detection of some prescribed features; see [13].

A general geometric procedure for constructing an orthogonal array is as follows: let $f_1, \ldots, f_k$ be homogeneous forms in $n+1$ unknowns defining some algebraic varieties $V(f_1), \ldots, V(f_k)$, let also $\mathcal{W} \subseteq \mathbb{F}_q^{n+1}$ be a set of representatives of distinct points of $\Sigma = \mathrm{PG}(n,q)$ with $|\mathcal{W}| = N$. The array

$$A(f_1, \ldots, f_k; \mathcal{W}) = \{(f_1(x) \quad f_2(x) \quad \ldots \quad f_k(x)) : x \in \mathcal{W}\},$$

with an arbitrary order of rows, is orthogonal if the size of the intersection $V(f_i) \cap V(f_j) \cap \mathcal{W}$ for distinct varieties $V(f_i)$ and $V(f_j)$, is independent of the choice of i, j. This procedure was applied to linear functions by [18], to quadratic functions by [19, 20] and to Hermitian forms by [21].

In general, it is possible to generate functions f_i starting from homogeneous polynomials in $n+1$ variables and considering the action of a suitable subgroup of the projective group $\mathrm{PGL}_{n+1}(q)$. Recall that, the image $V(f)^g$ of $V(f)$ under the action of an element $g \in \mathrm{PGL}_{n+1}(q)$ is a variety $V(f^g)$ of Σ, associated with the polynomial f^g. In [20], the authors used a subgroup of $\mathrm{PGL}_4(q)$, to obtain suitable quadratic functions in four variables; then, the domain $\mathcal{W}$ of these functions was appropriately restricted to a set of q^3 representatives, thus producing an orthogonal array of type $OA(q^3, q^2, q, 2)$.

Here, we construct a simple $OA(q^5, q^4, q, 2) = \mathcal{A}_0$, with entries in $\mathbb{F}_q$, $q > 2$ an even prime power, using the above procedure with forms related to the BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$. To do this we look into the action of a large subgroup of $\mathrm{PGL}_4(q^2)$ on a set of BM quasi-Hermitian varieties in $\mathrm{PG}(3, q^2)$.

As seen before, $\mathcal{M}_{a,b}$ has the same affine points as the variety $\mathcal{B}_{a,b}$ associated to the form

$$F = Z^q J^q + Z J^{2q-1} + a^q (X^{2q} + Y^{2q}) - a(X^2 + Y^2) J^{2q-2} + (b + b^q)(X^{q+1} + Y^{q+1}) J^{q-1}.$$

We shall now construct an array by choosing suitable varieties of the form $\mathcal{M}_{a,b}$ lying in the orbit of a suitable set of affine collineations.

Take G as the subgroup of $\mathrm{PGL}_4(q^2)$ consisting of all elations represented by

$$c(j', x', y', z') = (j, x, y, z)M$$

where $c \in \mathbb{F}^*_{q^2}$, and

$$M = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 1 & 0 & \gamma_4 \\ 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

with $\gamma_i \in \mathbb{F}_{q^2}$. Then, the group G has order q^{10}, it stabilizes the hyperplane Σ_∞, fixes the point $P_\infty(0, 0, 0, 1)$ and acts transitively on $\mathrm{AG}(3, q^2)$ (i.e. it acts as an affine group of collineations).

Let now Ψ be the subgroup of G consisting of all elations whose matrices are of the form

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_2 & a\left(\gamma_1^2 + \gamma_2^2\right) + b\left(\gamma_1^{q+1} + \gamma_2^{q+1}\right) + s \\ 0 & 1 & 0 & (b^q + b)\gamma_1^q \\ 0 & 0 & 1 & (b^q + b)\gamma_2^q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

with $\gamma_1, \gamma_2 \in \mathbb{F}_{q^2}, s \in \mathbb{F}_q$. The group Ψ contains q^5 elations, preserves $\mathcal{M}_{a,b}$ and acts on the affine points of $\mathcal{M}_{a,b}$, that is to say the affine points of $\mathcal{B}_{a,b}$, as a sharply transitive permutation group. Let also $C = \{a_1 = 0, \ldots, a_q\}$ be a set of representatives for the elements of $\mathbb{F}_{q^2}/\mathbb{F}_q$ (regarding them both as their additive groups). Denote now by $\mathcal{R}$ the subset of G whose elations are induced by

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where $\gamma_1, \gamma_2 \in \mathbb{F}_{q^2}$, and γ_3 is the unique solution in C of the equation

$$\begin{aligned} &\gamma_3^q + \gamma_3 + a^q\left(\gamma_1^{2q} + \gamma_2^{2q}\right) + a\left(\gamma_1^2 + \gamma_2^2\right) \\ &\quad + (b + b^q)\left(\gamma_1^{q+1} + \gamma_2^{q+1}\right) = 0. \end{aligned} \tag{26}$$

The set $\mathcal{R}$ has cardinality q^4 and it can be seen that its elements belong to a transversal of the group Ψ; in particular $\mathcal{R}$ can be used to construct a set $\{F^g | g \in \mathcal{R}\}$ of forms whose related $\mathcal{B}_{a,b}$'s (and thus $\mathcal{M}_{a,b}$'s) are pairwise distinct.

Theorem 6.1. *For any prime power q, the matrix $\mathcal{A}_0 = A(F^g, g \in \mathcal{R}, \mathcal{W}_0)$, where*

$$\mathcal{W}_0 = \{(1, x, y, z) : x, y \in \mathbb{F}_{q^2}, z \in C\}$$

is a simple $OA(q^5, q^4, q, 2)$ of index $\lambda = q^3$.

Proof. We start by showing that the number of solutions in $\mathcal{W}_0$ to the system

$$\begin{cases} F(J, X, Y, Z) = t \\ F^g(J, X, Y, Z) = t' \end{cases} \tag{27}$$

is q^3 for any $t, t' \in \mathbb{F}_q, g \in \mathcal{R}\backslash\{id\}$. By definition of $\mathcal{W}_0$, this system is equivalent to

$$\begin{cases} Z^q + Z + a^q(X^{2q} + Y^{2q}) + a(X^2 + Y^2) \\ \qquad + (b + b^q)(X^{q+1} + Y^{q+1}) = t \\ Z^q + Z + a^q(X^{2q} + Y^{2q}) + a(X^2 + Y^2) + (b + b^q) \\ \qquad \left(X^{q+1} + Y^{q+1} + \gamma_1^q X + \gamma_2^q Y + \gamma_1 X^q + \gamma_2 Y^q\right) = t' \end{cases} \tag{28}$$

Subtracting the first equation from the second we get

$$\mathrm{Tr}\left(\gamma_1^q X + \gamma_2^q Y\right) = \frac{t + t'}{(b + b^q)}, \tag{29}$$

Since g is not the identity, $(\gamma_1, \gamma_2) \neq (0, 0)$; hence, Equation (29) is equivalent to the union of q linear equations in X, Y over $\mathbb{F}_{q^2}$. Thus, there are q^3 pairs (x, y) satisfying (29). For each such a pair, Equation (28) has q solutions in Z, corresponding to a coset of $\mathbb{F}_q$ in $\mathbb{F}_{q^2}$ and only one of these q solutions is in C. Therefore, System (27) has q^3 solutions in $\mathcal{W}_0$.

Next, we show that $\mathcal{A}_0$ does not contain any repeated row. Let us index its rows by the corresponding elements in $\mathcal{W}_0$. Observe that the row (x, y, z) is the same as (x_1, y_1, z_1) in $\mathcal{A}_0$ if, and only if,

$$F^g(1, x, y, z) = F^g(1, x_1, y_1, z_1),$$

for any $g \in \mathcal{R}$. We thus obtain a system of q^4 equations in the 6 indeterminates x_1, y_1, z_1, x, y, z. Each equation is of the form

$$\begin{aligned} &(z + z_1)^q + (z + z_1) + a^q((x + x_1)^{2q} + (y + y_1)^{2q}) \\ &\quad + a((x + x_1)^2 + (y + y_1)^2) + (b + b^q)((x + x_1)^{q+1} \\ &\quad + (y + y_1)^{q+1}) = (b^q + b)\left(\gamma_2^q(y + y_1) + \gamma_2(y + y_1)^q\right. \\ &\quad \left.+ \gamma_1^q(x + x_1) + \gamma_1(x + x_1)^q\right) \end{aligned} \tag{30}$$

where the elements γ_i vary in $\mathbb{F}_{q^2}$ in all possible ways. In particular, for $\gamma_i = 0$ we have that the left hand side of the equations of (30) equals zero. Thus,

$$\begin{aligned} &(b^q + b)\left(\gamma_2^q(y + y_1) + \gamma_2(y + y_1)^q + \gamma_1^q(x + x_1)\right. \\ &\quad \left.+ \gamma_1(x + x_1)^q\right) = 0 \end{aligned} \tag{31}$$

Choosing $\gamma_1 = 1$ and $\gamma_2 = 0$ it follows from Equation (31) that x and x_1 must be in the same coset of $\mathbb{F}_q$. If we choose $\gamma_1 = 0$ and $\gamma_2 = 1$ in Equation (31) we get that y and y_1 are as well in the same coset of $\mathbb{F}_q$; so $x + x_1, y + y_1 \in \mathbb{F}_q$. So, Equation (30) becomes

$$\mathrm{Tr}(z + z_1) + \mathrm{Tr}(a + b)((x + x_1) + (y + y_1))^2 = 0 \quad (32)$$

and Equation (31) becomes

$$\mathrm{Tr}(b)(\mathrm{Tr}(\gamma_2)(y + y_1) + \mathrm{Tr}(\gamma_1)(x + x_1)) = 0. \quad (33)$$

By assumption $\mathrm{Tr}(b) \neq 0$ and by the arbitrariness of γ_1 and γ_2 Equation (33) gives $x = x_1$ as well as $y = y_1$.

Then, Equation (32) implies $\mathrm{Tr}(z + z_1) = 0$ that is, z and z_1 are in the same coset of $\mathbb{F}_q$. Thus, there are no two distinct vectors in $\mathcal{W}_0$ whose difference is of the required form; hence, $\mathcal{A}_0$ does not contain repeated rows and the theorem follows. □

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Data Availability Statement

The authors have nothing to report.

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