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ON THE VISCOPLASTIC APPROXIMATION OF A RATE-INDEPENDENT COUPLED ELASTOPLASTIC DAMAGE MODEL*

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Abstract. In this paper we study a rate-independent system for the propagation of damage and plasticity. To construct solutions we resort to approximation in terms of viscous evolutions, where viscosity affects *both* damage and plasticity with the same rate. The main difficulty arises from the fact that the available estimates do not provide sufficient regularity on the limiting evolutions to guarantee that forces and velocities are in a duality pairing, and hence we cannot use a chain rule for the driving energy. Nonetheless, via careful techniques we can characterize the limiting rate-independent evolution by means of an energy-dissipation balance, which encodes the onset of viscous effects in the behavior of the system at jumps.

Key words. rate-independent systems, variational models, vanishing viscosity, balanced viscosity solutions, damage, elastoplasticity

MSC codes. 35A15, 35Q74, 74C05

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1. Introduction.

In order to predict and prevent degradation and failure of materials, it is crucial to capture the interplay between different phenomena, such as damage and plasticity. In several applications, for instance to load-bearing structures, the propagation of such phenomena is very slow if compared to the scale of internal oscillations of the body under examination. Thus, the system is considered as being in equilibrium at every instant. From a mathematical point of view, this amounts to the concept of *quasi-static,* or *rate-independent, evolution.* In turn, rate-independent evolutions are idealized descriptions of processes where some phenomena are neglected, for instance the effects of viscosity. The analysis of quasi-static evolutions and their approximation by viscous evolutions has been the object of extensive mathematical literature in recent years. In fact, in order to understand how damage and plasticity grow, it is paramount to analyze their interaction already at a viscous level.

In this paper we study a rate-independent system for the propagation of damage and plasticity, by means of approximation in terms of viscous evolutions, where viscosity affects both damage and plasticity. Specifically, in the setting of linear

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elasticity the body is determined by its reference configuration $\Omega \subset \mathbb{R}^n$, $n=2,3$, and the *displacement* $u:(0,T)\times\Omega\to\mathbb{R}^n$, where $(0,T)$ is the time interval during which the process is observed. The degradation of the material is described by the *damage* variable $z:(0,T)\times\Omega\to[0,1]$ and by the *plastic strain* $p:(0,T)\times\Omega\to\mathbb{M}^{n\times n}_{\mathrm{D}}$, where $\mathbb{M}^{n\times n}_{\mathrm{D}}$ is the subspace of symmetric matrices $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ with null trace. Together with the *elastic strain* $e:(0,T)\times\Omega\to\mathbb{M}^{n\times n}_{\mathrm{sym}}$, the plastic strain p complies with the kinematic admissibility condition for the *strain* $\mathrm{E}(u)=\frac{\nabla u+\nabla u^T}{2}$, namely,

$$\mathrm{E}(u)=e+p \quad \text{in } (0,T)\times\Omega. \tag{1.1}$$

The rate-independent system under consideration consists of

- the momentum balance

$$-\mathrm{div}\,\sigma=f \quad \text{in } (0,T)\times\Omega, \tag{1.2a}$$

where σ is the *stress* tensor

$$\sigma=\mathbf{C}(z)e \quad \text{in } (0,T)\times\Omega, \tag{1.2b}$$

with $\mathbf{C}:[0,1]\to\mathbb{R}^{n\times n\times n\times n}$ the elastic stress tensor, and $f:(0,T)\times\Omega\to\mathbb{R}^n$ a given time-dependent external force;

- the flow rule for the damage variable z

$$\partial\mathrm{R}(\dot z)+A_{\mathrm{m}}(z)+W'(z)\ni-\frac{1}{2}\mathbf{C}'(z)e:e \quad \text{in } (0,T)\times\Omega, \tag{1.2c}$$

where $\partial\mathrm{R}:\mathbb{R}\rightrightarrows\mathbb{R}$ denotes the convex analysis subdifferential of the function

$$\mathrm{R}:\mathbb{R}\to[0,+\infty] \ \text{ defined by } \ \mathrm{R}(\eta):=\begin{cases}\kappa|\eta| & \text{if } \eta\le 0,\\ +\infty & \text{otherwise,}\end{cases} \tag{1.2d}$$

with $\kappa>0$ the toughness of the material. In (1.2c), the *nonlocal* m-Laplacian operator $A_{\mathrm{m}}:H^{\mathrm{m}}(\Omega)\to H^{\mathrm{m}}(\Omega)^*$ features an exponent $\mathrm{m}>\frac{n}{2}$. Finally, $W:[0,1]\to\mathbb{R}$ is a suitable nonlinear, possibly nonsmooth, function;

- the flow rule for the plastic tensor

$$\partial_{\dot p}\mathrm{H}(z,\dot p)\ni\sigma_{\mathrm{D}} \quad \text{in } \Omega\times(0,T), \tag{1.2e}$$

where σ_{D} is the deviatoric part of the stress tensor σ, i.e., its orthogonal projection on $\mathbb{M}^{n\times n}_{\mathrm{D}}$, and $\mathrm{H}(z,\cdot)$ is the density of the plastic dissipation potential.

System (1.2a)–(1.2e) is complemented by the boundary conditions

$$u=w \text{ on } (0,T)\times\Gamma_{\mathrm{Dir}}, \qquad \sigma\mathrm{n}=g \text{ on } (0,T)\times\Gamma_{\mathrm{Neu}}, \qquad \partial_{\mathrm{n}}z=0 \text{ on } (0,T)\times\partial\Omega, \tag{1.2f}$$

where Γ_{Dir} is the Dirichlet part of the boundary $\partial\Omega$ and $w:\Omega\to\mathbb{R}^n$ a time-dependent Dirichlet loading, while Γ_{Neu} is the Neumann part of $\partial\Omega$ (disjoint from Γ_{Dir}), n its exterior unit normal, and $g:\Omega\to\mathbb{R}^n$ an assigned traction.

The elastoplastic damage model (1.2), which was first proposed and studied in [AMV14, AMV15], includes rate-independent flow rules for damage and plasticity, both given in terms of threshold conditions: propagation starts when the damage variable or the deviatoric part of the stress reaches the boundary of the stability set. Note that R is the density of the dissipation potential for damage; hence, the flow rule (1.2c) encompasses the unidirectionality in the evolution of damage through the constraint $\dot z\le 0$ in $\Omega\times(0,T)$. In turn, $\mathrm{H}(z,\dot p)$ is the density of the plastic dissipation potential. The coupling of the system is apparent both from the dependence of the elasticity tensor $\mathbf{C}$ on the damage variable and from the z-dependence of H. In particular, along the footsteps of [AMV14, AMV15] we encompass softening in the model, which consists in the reduction of the yield stress as plastic deformation proceeds (see (2.8b) below, as well as [AMV14, equation (30)] and [DMDMM08]).

The main impact of the analysis of such a coupled model arises in the study of cohesive fracture. Indeed, in parts of the material where plastic strain has been cumulated, one may observe nucleation of cohesive cracks and thus the emergence of fatigue phenomena (see [AMV14, section 5], as well as [Cri16, CL16] and references therein.

Rate-independent damage processes with plasticity have been extensively studied in recent years. Among phase-field type models (i.e, featuring the damage parameter and the plastic strain as internal variables), we mention, e.g., [BMR12, BRRT16, RV17] for damage coupled with plasticity with hardening, [Cri17] for damage and strain-gradient plasticity, [RV16] also accounting for damage healing, [MSZ19] for finite-strain plasticity with damage, and [DRS19] for perfect plasticity and damage in a dynamical setting.

The vanishing-viscosity approach and our results. The existence of *quasi-static evolutions* (or *energetic solutions*) to the Cauchy problem for system (1.2) was proved in [Cri16]. In this paper we will instead address the following viscous approximation of (1.2) for $\varepsilon > 0$ given:

$$-\operatorname{div}\sigma = f \quad \text{with } \sigma = \mathbf{C}(z)e \qquad \text{in } \Omega\times(0,T), \tag{1.3a}$$

$$\partial\mathrm{R}(\dot z) + \varepsilon\dot z + A_{\mathrm{m}}(z) + W'(z) \ni -\frac{1}{2}\mathbf{C}'(z)e : e \qquad \text{in } \Omega\times(0,T), \tag{1.3b}$$

$$\partial_{\dot p}\mathrm{H}(z,\dot p) + \varepsilon\dot p \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega\times(0,T), \tag{1.3c}$$

supplemented by the boundary conditions (1.2f). System (1.3) thus pertains to the class of *rate-dependent* damage models, which have also been widely studied. Existence results for rate-dependent damage processes are indeed available both in the case when plasticity effects are neglected (starting from the pioneering papers [BS04, BSS05]) and for models encompassing plasticity and even temperature [RT15, Ros17]. A hallmark of these rate-dependent systems is the gradient regularization of the damage parameter, ensuring sufficient spatial regularity for z: typically, p-Laplacian operators with $p > n$ feature in the damage flow rule. In systems (1.2) and (1.3), along the footsteps of [KRZ13] we have instead resorted to a nonlocal, but *linear,* operator for analytical reasons.

The existence of a solution $(u_\varepsilon, z_\varepsilon, p_\varepsilon)$ to the Cauchy problem for (1.3) can be proven, for instance, by time discretization; cf. Theorem 2.4 ahead. In the approximated system, both flow rules for damage and plasticity feature a viscous regularization; specifically, they contain the time derivatives of the damage variable and of the plastic strain. Such regularization can be tuned through the parameter $\varepsilon > 0$. In fact, we study the limit as $\varepsilon \downarrow 0$, expecting to find (a version of) the rate-independent system (1.2) in the limit. This reflects the fact that, again, both rate-independent damage and rate-independent plasticity are idealized processes where first-order terms are neglected. We emphasize that, in (1.3), the viscosities in p and in z *both* vanish with the same rate.

The strategy to derive (1.2) from (1.3) follows the *vanishing-viscosity approach* explored in a wide literature; see [MR15] and references therein. Exploiting suitable a priori estimates on the approximate solutions, uniform with respect to ε, one then passes to the limit as $\varepsilon \downarrow 0$, finding a so-called *balanced viscosity (*BV*) solution*

[MRS12, MRS16a] to (1.2). This technique is by now standard: it is based on reparameterization and was pioneered in [EM06]. Indeed, the reparameterized trajectories, defined in a new interval $[0,S]$, are uniformly Lipschitz in the new time scale, and so it is their limiting trajectory; for the system under consideration, a BV solution is in fact a quadruple $(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$, Lipschitz as a function of the (artificial) time $s\in[0,S]$, where the rescaling function $\mathsf{t}:[0,S]\to[0,T]$ records the original process time. Now it is to be expected that solutions to system (1.2) jump as functions of the (true) time $t\in[0,T]$: the parameterized solutions $(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$ keep track of this, in that jumps in the original time correspond to the regime in which $\mathsf{t}'=0$; namely, the function t is frozen. The notion of BV solution then provides additional information on the behavior of the rate-independent system in a jump regime. In fact, it encodes the possibility that, between two stable states, there occurs either a slow transition, corresponding to quasi-static propagation in the original time scale, or a fast transition, where the system displays viscous behavior. All of this is encompassed in the (single) energy-dissipation balance

$$
\begin{aligned}
&\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s))+\int_0^s \mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\
&\quad=\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(0),\mathsf{q}(0))+\int_0^s \partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r \qquad \text{for all } s\in[0,S];
\end{aligned} \tag{1.4}
$$

cf. Definition 3.6, where $\mathcal{E}_{\mathrm{PP}}$ is the (overall) driving energy functional for system (1.2) and q is a place-holder for the rescaled triple $(\mathsf{u},\mathsf{z},\mathsf{p})$. In fact, in (1.4) the energy released between the initial time and a given final time s (in the artificial time scale) is balanced by the work of the external forces, and by a term involving the "vanishing-viscosity contact potential" $\mathcal{M}_0$ (see (3.20) for its explicit definition), which keeps track of the occurrence of slow/fast transitions in the jump regime. The main result of this paper, **Theorem 3.8**, states the convergence of (reparameterized) viscous trajectories to a BV solution $(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$ satisfying (1.4).

When applying the vanishing-viscosity approach to our setup, one major analytical difficulty arises already when dealing with the a priori estimates, uniform w.r.t. the parameter ε, needed for the vanishing-viscosity limit. Indeed, we may deduce the first set of basic estimates for the viscous solutions from an energy-dissipation balance that is tightly connected with the gradient structure of (1.3) (cf. Proposition 2.5). Relying on such estimates, we prove convergence as $\varepsilon\downarrow 0$ of the reparameterized viscous trajectories (along a suitable subsequence), to a quadruple $(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$ that fulfills the inequality $\le$ in (1.4), where the energy at the current (artificial) time s is estimated from above by the initial energy. However, the available estimates do *not* provide sufficient regularity on the limiting parameterized curves to guarantee that forces and velocities are in a duality pairing. Hence, we are *not* able to prove the validity of a chain rule for the energy $\mathcal{E}_{\mathrm{PP}}$ evaluated along the curve $(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$; in turn, chain rules are the tool usually employed to obtain the energy inequality $\ge$ in (1.4) (hereafter, we shall refer to it as the *lower* energy inequality, as therein the energy evaluated at the current time is estimated from below by the initial energy).

In order to prove the energy-dissipation balance, in this paper we have to carry out finer arguments, based on the *sole* validity of those estimates, uniform w.r.t. the viscosity parameter, that are deducible from the viscous energy-dissipation balance. This is in the spirit of the analysis carried out in [MRS16a], whose results are, however, not directly applicable here because in the present functional setup we do not have a suitable chain rule at our disposal; cf. Remark 3.7. Rather, the techniques advanced here in fact revisit the approach developed in [DDS11]; see also [BFM12].

More precisely, we are going to obtain the desired lower energy inequality by exploiting the fact that the all terms featuring in (1.4) are the limit of suitable Riemann sums, defined on carefully chosen subdivisions of the time interval that distinguish between the quasi-static regime and the jump regime, and by using that an approximate form of the lower energy inequality holds along the intervals associated with the partitions. A remarkable difference between this work and the previous contributions is that now the chain rule is not available even in the jump regime, which forces us to also perform a discrete approximation there; cf. Remark 5.2.

We finally mention that the vanishing-viscosity approximation of system (1.2) has already been addressed by means of different viscously regularizing systems, whose structure allowed for enhanced estimates on the reparameterized viscous trajectories, leading to better regularity for the limiting curve and, ultimately, allowing for the validity of the chain rule. First of all, in [CL16], (1.2) was approximated by an elasto-plastic damage system in which the viscous regularization only involved the damage variable, while the evolution of the plastic variable was kept rate-independent; in particular, following this approach, the possible emergence of viscous behavior for the plastic strain at jumps is *not* included in the mathematical description of the evolution.

An alternative approximation scheme was studied in [CR21] in a setting with multiple rates (in the spirit of the approach in [MRS16b, MR23]). In fact, in the viscous system addressed in [CR21], the momentum equation was also viscously regularized, and the displacement and the plastic strain were set to converge to elastic equilibrium and rate-independent evolution, respectively, at a faster rate than the damage variable. Furthermore, the vanishing-viscosity regularization was combined with a vanishing-hardening approximation of the flow rule for technical reasons related to the validity of the aforementioned *enhanced* a priori estimates. The notion of BV solution in [CR21] thus enjoys better regularity properties than those obtained in Theorem 3.8 ahead. In particular, relying on the validity of the chain rule for the driving energy functional, in [CLR23] we have provided a characterization of the BV solutions from [CR21] in terms of a system of subdifferential inclusions that illustrates in a precise way the possible onset of viscous behavior for the system at jumps. Such a differential characterization, tightly related to the validity of a chain rule, seems to be out of reach for the present notion of BV solution. However, exploiting that very same differential characterization, in [CLR23] we have proved that the BV solutions of [CR21] “essentially” coincide (after an initial transition layer) with those from [CL16], where viscous regularization in p was *not* present.

For this reason, in this paper we consider the alternative scheme (1.3), which is a natural regularization of the rate-independent system (1.2). Indeed, system (1.3) reflects the fact that plasticity and damage are tightly connected [AMV14, AMV15]: loosely speaking, setting $\varepsilon = 0$ formally leads to removing viscous perturbation both in plasticity and damage (which is not the case with the multirate regularization chosen in [CLR23]). From this viewpoint, a single-rate regularization like the one adopted in this paper seems more adequate. On the other hand, due to poor regularity and lack of chain rule, it remains an open problem to further characterize the BV solutions fulfilling (1.4), e.g., by means of a differential characterization as in [CLR23]; see also Remark 3.7 ahead. Thus, we leave open the question of whether the solutions found in the present paper are different from those of [CLR23, CL16], in particular if in this case the presence of the viscous regularization in p could be effectively detected in the limit. Anyhow, we maintain that the techniques developed in this paper, as a revisitation of [DDS11, BFM12], are interesting on their own and could be useful in other contexts where the lack of a chain rule in any regime is a hindrance from the analysis of BV solutions.

Plan of the paper. In section 2 we settle most of the notation and preliminary results for our analysis. In section 3, after introducing the reparameterization technique for the vanishing-viscosity limit (section 3.1) and the energetics for the perfectly plastic system (section 3.2), we give the notion of *(parameterized) balanced viscosity* solution to the system for perfect plasticity and damage and state the convergence of (reparameterized) viscous trajectories to a BV solution in Theorem 3.8. Its proof is carried out throughout three sections. Indeed, section 4 contains the compactness arguments and the proof of the upper energy-dissipation inequality. Its converse is proved in sections 5–6. Finally, the appendix collects some auxiliary results used in proofs scattered throughout the paper.

2. Preliminaries. First of all, let us fix some notation that will be used throughout the paper.

Notation 2.1 (general notation and preliminaries). Given a Banach space $\mathbf{X}$, we will denote by $\langle\cdot,\cdot\rangle_{\mathbf{X}}$ both the duality pairing between $\mathbf{X}^*$ and $\mathbf{X}$ and that between $(\mathbf{X}^n)^*$ and $\mathbf{X}^n$; we will just write $\langle\cdot,\cdot\rangle$ for the inner Euclidean product in $\mathbb{R}^n$. Analogously, we will indicate by $\|\cdot\|_{\mathbf{X}}$ the norm in $\mathbf{X}$ and, most often, use the same symbol for the norm in $\mathbf{X}^n$, while we will just write $|\cdot|$ for the Euclidean norm in $\mathbb{R}^n$. We will denote by $B_r(0)$ the open ball of radius r, centered at 0, in $\mathbb{R}^n$. For the Lebesgue measure of a set $A\subset\mathbb{R}^n$ we will use both notations $|A|$ and $\mathcal{L}^n(A)$.

We will denote by $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ the space of the symmetric $(n\times n)$-matrices and by $\mathbb{M}^{n\times n}_{\mathrm{D}}$ the subspace of the deviatoric matrices with null trace. Any $\eta\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be written as $\eta=\eta_{\mathrm{D}}+\frac{\mathrm{tr}(\eta)}{n}I$, with η_{D} the orthogonal projection of η into $\mathbb{M}^{n\times n}_{\mathrm{D}}$; η_{D} will be referred to as the deviatoric part of η. We write $\mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ for the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_{\mathrm{D}}$.

We will often use the short-hand notation $\|\cdot\|_{L^p}$, $1\le p<+\infty$, for the L^p-norm on the space $L^p(O;\mathbb{R}^m)$, with O a measurable subset of $\mathbb{R}^n$, and analogously we will write $\|\cdot\|_{H^1}$. Furthermore, we will denote by $\mathcal{M}_{\mathrm{b}}(O;\mathbb{R}^m)$ the space of bounded Radon measures on O with values in $\mathbb{R}^m$.

Let $v:\Omega\times(0,T)\to\mathbb{R}$ be differentiable w.r.t. time a.e. on $\Omega\times(0,T)$. We will denote by $\dot v:\Omega\times(0,T)\to\mathbb{R}$ its (almost everywhere defined) partial time derivative. However, as soon as we consider v as a (Bochner) function from $(0,T)$ with values in a suitable Lebesgue/Sobolev space $\mathbf{X}$ (possessing the Radon–Nikodým property), and v is in the space $\mathrm{AC}([0,T];\mathbf{X})$, we will denote by $v':(0,T)\to\mathbf{X}$ its (almost everywhere defined) time derivative.

Finally, we will use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities.

Functions of bounded deformation. The space $\mathrm{BD}(\Omega)$ of *functions of bounded deformation* is defined by

$$\mathrm{BD}(\Omega):=\{u\in L^1(\Omega;\mathbb{R}^n):\ \mathrm{E}(u)\in\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\},\tag{2.1}$$

where $\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ is the space of bounded Radon measures on Ω with values in $\mathbb{M}^{n\times n}_{\mathrm{sym}}$, with norm $\|\lambda\|_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}:=|\lambda|(\Omega)$, and $|\lambda|$ the variation of the measure. By the Riesz representation theorem, $\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ can be identified with the dual of the space $\mathrm{C}_0(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. The space $\mathrm{BD}(\Omega)$ is endowed with the graph norm

$$\|u\|_{\mathrm{BD}(\Omega)}:=\|u\|_{L^1(\Omega;\mathbb{R}^n)}+\|\mathrm{E}(u)\|_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})},$$

which makes it a Banach space. In fact, $\mathrm{BD}(\Omega)$ is the dual of a normed space; cf. [TS80].

In addition to the strong convergence induced by $\|\cdot\|_{\mathrm{BD}(\Omega)}$, the duality from [TS80] defines a notion of weak* convergence on $\mathrm{BD}(\Omega)$: a sequence $(u_k)_k$ converges weakly* to u in $\mathrm{BD}(\Omega)$ if $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^n)$ and $\mathrm{E}(u_k) \overset{*}{\rightharpoonup} \mathrm{E}(u)$ in $\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$; every bounded sequence in $\mathrm{BD}(\Omega)$ has a weakly* converging subsequence. The space $\mathrm{BD}(\Omega)$ is contained in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$; every bounded sequence in $\mathrm{BD}(\Omega)$ has a subsequence converging weakly in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$ and strongly in $L^p(\Omega;\mathbb{R}^n)$ for every $1\le p<\frac{n}{n-1}$.

Finally, we recall that for every $u\in\mathrm{BD}(\Omega)$ the trace $u|_{\partial\Omega}$ is well defined as an element in $L^1(\partial\Omega;\mathbb{R}^n)$ and that (cf. [Tem83, Proposition 2.4, Remark 2.5]) a Poincaré-type inequality holds:

(2.2)
$$\exists\, C>0 \quad \forall\, u\in\mathrm{BD}(\Omega): \quad \|u\|_{L^1(\Omega;\mathbb{R}^n)} \le C\left(\|u\|_{L^1(\Gamma_{\mathrm{Dir}};\mathbb{R}^n)} + \|\mathrm{E}(u)\|_{\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}\right).$$

2.1. Assumptions. We now detail the standing assumptions for our analysis.

Hypothesis 1 (the reference configuration). We suppose that $\Omega\subset\mathbb{R}^n$, $n\in\{2,3\}$, is a bounded Lipschitz domain satisfying the so-called *Kohn–Temam condition,* namely $\partial\Omega=\Gamma_{\mathrm{Dir}}\cup\Gamma_{\mathrm{Neu}}\cup\Sigma$ with

– Γ_{Dir}, Γ_{Neu}, Σ pairwise disjoint;
– Γ_{Dir} and Γ_{Neu} relatively open in $\partial\Omega$, and $\Sigma=\partial\Gamma_{\mathrm{Dir}}=\partial\Gamma_{\mathrm{Neu}}$ their relative boundary in $\partial\Omega$;
– Σ of class C^2 and $\mathcal{H}^{n-1}(\Sigma)=0$, and $\partial\Omega$ Lipschitz and of class C^2 in a neighborhood of Σ.

We will work with the space

$$H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) := \left\{u\in H^1(\Omega;\mathbb{R}^n) : u=0 \text{ on } \Gamma_{\mathrm{Dir}}\right\}.$$

A divergence operator. Any $\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ such that $\operatorname{div}\sigma\in L^2(\Omega;\mathbb{R}^n)$ induces the distribution $[\sigma\mathrm{n}]$ defined by

$$\langle[\sigma\mathrm{n}],\psi\rangle_{\partial\Omega} := \langle\operatorname{div}\sigma,\psi\rangle_{L^2} + \langle\sigma,\mathrm{E}(\psi)\rangle_{L^2} \qquad \text{for every } \psi\in H^1(\Omega;\mathbb{R}^n). \tag{2.3}$$

By [KT83, Theorem 1.2] and [DMDM06, (2.24)] we have that $[\sigma\mathrm{n}]\in H^{-1/2}(\partial\Omega;\mathbb{R}^n)$; moreover, if $\sigma\in\mathrm{C}^0(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{sym}})$, then the distribution $[\sigma\mathrm{n}]$ fulfills $[\sigma\mathrm{n}]=\sigma\mathrm{n}$, where the right-hand side is the standard pointwise product of the matrix σ and the normal vector n in $\partial\Omega$.

For the treatment of the perfectly plastic system for damage, it will be crucial to work with the space

$$\Sigma(\Omega) := \{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) : \operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n),\ \sigma_{\mathrm{D}}\in L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\}. \tag{2.4}$$

Furthermore, our choice of external loadings will ensure that the stress fields σ that we consider have, at equilibrium, the additional property that $[\sigma\mathrm{n}]\in L^\infty(\Omega;\mathbb{R}^n)$ and $\sigma\in\Sigma(\Omega)$. Therefore, any of such fields induces a functional $-\operatorname{Div}\sigma\in\mathrm{BD}(\Omega)^*$ via

(2.5)
$$\langle-\operatorname{Div}\sigma,v\rangle_{\mathrm{BD}(\Omega)} := \langle-\operatorname{div}\sigma,v\rangle_{L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + \langle[\sigma\mathrm{n}],v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)} \qquad \text{for all } v\in\mathrm{BD}(\Omega).$$

With slight abuse of notation, we shall denote by $-\mathrm{Div}\,\sigma$ also the restriction of the above functional to $H^1(\Omega;\mathbb{R}^n)$.

Hypothesis 2 (the elasticity tensor). We assume that $\mathbf{C}:[0,+\infty)\to\mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}})$ fulfills the following conditions:

$$\mathbf{C}\in\mathrm{C}^{1,1}([0,+\infty);\mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}}))\,, \tag{2.6a}$$

$$\exists\,\gamma_1,\,\gamma_2>0\quad\forall\,z\in[0,+\infty)\ \forall\,\xi\in\mathbb{M}^{n\times n}_{\mathrm{sym}}:\quad \gamma_1|\xi|^2\le\mathbf{C}(z)\xi:\xi\le\gamma_2|\xi|^2\,, \tag{2.6b}$$

$$\forall\,z_o\in(0,1)\exists\,\alpha_{\mathbf{C}}>0\quad\forall\,z\in[z_o,1]\quad\forall\,\xi\in\mathop{\mathbb{M}}^{n\times n}_{\mathrm{sym}}:\mathbf{C}'(z)\xi:\xi\ge\alpha_{\mathbf{C}}|\xi|^2. \tag{2.6c}$$

Furthermore, we require that

(2.6d)

$$\exists\,\mathrm{V}\in\mathrm{C}^{1,1}([0,+\infty);[0,+\infty))\,,\quad \mathrm{V}\text{ bounded from below},\quad \inf_{z\in[0,+\infty)}\mathrm{V}(z)\ge c_{\mathrm{V}}>0\,,$$

$$\exists\,\mathbb{C}\in\mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}})\text{ positive definite, symmetric, }\textit{isotropic},\text{ such that}$$
$$\forall\,z\in[0,+\infty):\ \mathbf{C}(z)=\mathrm{V}(z)\,\mathbb{C}\,,$$

where *isotropic* implies that $\mathbb{C}A=2\mu A+\lambda\mathrm{tr}(A)I$ for all $A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$, with $\lambda,\,\mu>0$ the Lamé constants.

Remark 2.2 (on condition (2.6d)). In view of the required structure $\mathbf{C}(z)=\mathrm{V}(z)\,\mathbb{C}$, conditions (2.6a)–(2.6c) could be reformulated in terms of the function V. Nonetheless, we have opted for stating (2.6a)–(2.6c) in general, regardless of (2.6d), because the latter structure condition will be exploited *solely* in the proof of Proposition A.1 ahead; cf. also Remark A.2. In turn, Proposition A.1 will be used to prove the lower energy-dissipation inequality, i.e., $\ge$ in (1.4).

Hypothesis 3 (stored energy for damage). The stored energy for damage encompasses a gradient regularizing contribution, featuring the bilinear form $a_{\mathrm{m}}:H^{\mathrm{m}}(\Omega)\times H^{\mathrm{m}}(\Omega)\to\mathbb{R}$,

$$a_{\mathrm{m}}(z_1,z_2):=\int_\Omega\int_\Omega\frac{\big(\nabla z_1(x)-\nabla z_1(y)\big)\cdot\big(\nabla z_2(x)-\nabla z_2(y)\big)}{|x-y|^{n+2(\mathrm{m}-1)}}\,\mathrm{d}x\mathrm{d}y\quad\text{with }\mathrm{m}\in\Big(\frac{n}{2},2\Big)\,,$$

and an additional integral term, with density W satisfying

$$W\in\mathrm{C}^2((0,+\infty);\mathbb{R}^+)\,\cap\,\mathrm{C}^0([0,+\infty);\mathbb{R}^+\cup\{+\infty\})\,, \tag{2.7a}$$

$$s^{2n}W(s)\to+\infty\ \text{ as }\ s\to0^+\,, \tag{2.7b}$$

where $W\in\mathrm{C}([0,+\infty);\mathbb{R}^+\cup\{+\infty\})$ means that $W(0)=\infty$ and $W(z)\to+\infty$ if $z\to0$ as prescribed by (2.7b).

We will denote by $A_{\mathrm{m}}:H^{\mathrm{m}}(\Omega)\to H^{\mathrm{m}}(\Omega)^*$ the operator associated with the bilinear form a_{m}, namely

$$\langle A_{\mathrm{m}}(z),w\rangle_{H^{\mathrm{m}}(\Omega)}:=a_{\mathrm{m}}(z,w)\quad\text{for every }z,\,w\in H^{\mathrm{m}}(\Omega).$$

We recall that $H^{\mathrm{m}}(\Omega)$ is a Hilbert space with the inner product

$$\langle z_1,z_2\rangle_{H^{\mathrm{m}}(\Omega)}:=\int_\Omega z_1z_2\,\mathrm{d}x+a_{\mathrm{m}}(z_1,z_2).$$

Since we assume that $m>\frac{n}{2}$, we have the compact embedding $H^{\mathrm{m}}(\Omega)\Subset\mathrm{C}(\overline{\Omega})$.

Hypothesis 4 (plastic dissipation). The plastic dissipation density $H:[0,+\infty)\times\mathbb{M}^{n\times n}_{\mathrm{D}}\to[0,+\infty)$ is continuous and enjoys the following properties:

(2.8a)
$$\pi \mapsto H(z,\pi) \text{ is convex and 1-positively homogeneous for all } z\in[0,1]\,,$$
(2.8b)
$$0 \le H(z_2,\pi) - H(z_1,\pi) \qquad \text{for all } 0\le z_1\le z_2 \text{ and all } \pi\in\mathbb{M}^{n\times n}_{\mathrm{D}} \text{ with } |\pi|=1\,,$$
(2.8c)
$$\exists\, C_K>0\ \forall\, z_1,\, z_2\in[0,+\infty)\ \forall\,\pi\in\mathbb{M}^{n\times n}_{\mathrm{D}} \quad |H(z_2,\pi)-H(z_1,\pi)|\le C_K|\pi||z_2-z_1|\,,$$
(2.8d)
$$\exists\,\bar r,\, \bar R>0\ \ \forall\,(z,\pi)\in[0,+\infty)\times\mathbb{M}^{n\times n}_{\mathrm{D}}: \quad \bar r|\pi|\le H(z,\pi)\le\bar R|\pi|\,.$$

Remark 2.3 (constraint sets). We point out that a damage-dependent plastic dissipation density fulfilling (2.8) can be obtained as support function $H(z,\pi) := \sup_{\sigma\in K(z)}\sigma{:}\pi$ associated with a family $(K(z))_{z\in[0,+\infty)}$ of closed and convex constraint sets in $\mathbb{M}^{n\times n}_{\mathrm{D}}$ fulfilling

$$\begin{aligned} &\exists\, C_K>0 \quad \forall\, z_1,\, z_2\in[0,+\infty): && d_{\mathscr{H}}(K(z_1),K(z_2))\le C_K|z_1-z_2|,\\ &\exists\, 0<\bar r<\overline{R} \quad \forall\, 0\le z_1\le z_2: && B_{\bar r}(0)\subset K(z_1)\subset K(z_2)\subset B_{\overline{R}}(0),\end{aligned}$$

with $d_{\mathscr{H}}$ the Hausdorff distance.

In fact, any dissipation density as in (2.8) arises from a family of constraint sets with the above properties: it suffices to set $K(z):=\partial_\pi H(z,0)$, with $\partial_\pi H:\mathbb{M}^{n\times n}_{\mathrm{D}}\rightrightarrows\mathbb{M}^{n\times n}_{\mathrm{D}}$ the subdifferential of H w.r.t. its second variable.

Hypothesis 5 (forces and data). We consider initial data

$$\begin{aligned} &u_0\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n),\\ &z_0\in H^{\mathrm{m}}(\Omega) \text{ with } W(z_0)\in L^1(\Omega) \text{ and } z_0\le 1 \text{ in } \overline{\Omega},\\ &p_0\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}).\end{aligned} \tag{2.9}$$

We require that the volume force f and the assigned traction g fulfill

$$f\in H^1(0,T;L^n(\Omega;\mathbb{R}^n)),\quad g\in H^1(0,T;L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)). \tag{2.10a}$$

Furthermore, as is customary for perfect plasticity, we shall impose a *uniform safe load condition,* namely that there exists

$$\rho\in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})) \quad \text{with} \quad \rho_{\mathrm{D}}\in H^1(0,T;L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})) \tag{2.10b}$$

and there exists $\alpha>0$ such that for every $t\in[0,T]$ (recall (2.3))

$$-\mathrm{div}\,\varrho(t)=f(t) \text{ a.e. on } \Omega, \qquad [\varrho(t)\mathrm{n}]=g(t) \text{ on } \Gamma_{\mathrm{Neu}}, \tag{2.10c}$$

$$\rho_{\mathrm{D}}(t,x)+\xi\in K \qquad \text{for a.a. } x\in\Omega \text{ and for every } \xi\in\mathbb{M}^{n\times n}_{\mathrm{sym}} \text{ s.t. } |\xi|\le\alpha. \tag{2.10d}$$

As for the time-dependent Dirichlet boundary condition w, we assume that

$$w\in H^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n)). \tag{2.10e}$$

By the properties of W, the requirement $W(z_0)\in L^1(\Omega)$ already encodes the condition $z_0\ge 0$ in $\overline{\Omega}$. The body and surface forces f and g define the total load function

$$F:[0,T]\to\mathrm{BD}(\Omega)^*, \qquad \langle F(t),v\rangle_{\mathrm{BD}(\Omega)} := \langle f(t),v\rangle_{L^{n/(n-1)}(\Omega;\mathbb{R}^n)} + \langle g(t),v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)}$$

for all $v \in \mathrm{BD}(\Omega)$. Observe that combining (2.10a) with (2.10b)–(2.10d) yields

$$F \in H^1(0,T;\mathrm{BD}(\Omega)^*) \quad \text{and} \quad -\operatorname{Div}\varrho(t) = F(t) \text{ for all } t \in [0,T]. \tag{2.11}$$

In order to shorten the discussion, in sections 4, 5, and 6 we carry out the analysis in the case $f = 0$, $g = 0$, so that the only external loading will be given by w from (2.10e). However, all of our results also hold with non-null volume and surface forces satisfying assumptions (2.10). An extended version of the proofs with volume and surface forces is available in the preprint version of this paper [CLRpre].

2.2. Energetics for the viscous system. The damage dissipation density $\mathrm{R}: \mathbb{R} \to [0,+\infty]$ from (1.2d) induces the dissipation potential

$$\mathcal{R}: L^1(\Omega) \to [0,+\infty], \qquad \mathcal{R}(\eta) := \int_\Omega \mathrm{R}(\eta(x))\mathrm{d}x. \tag{2.12}$$

We will work with the subdifferential of the restriction of $\mathcal{R}$ to $H^{\mathrm{m}}(\Omega)$, namely with the operator $\partial_{H^{\mathrm{m}}}\mathcal{R}: H^{\mathrm{m}}(\Omega) \rightrightarrows H^{\mathrm{m}}(\Omega)^*$ defined by

$$\zeta \in \partial_{H^{\mathrm{m}}}\mathcal{R}(\eta) \quad \text{if and only if} \quad \mathcal{R}(\omega) - \mathcal{R}(\eta) \geq \langle \zeta, \omega - \eta \rangle_{H^{\mathrm{m}}(\Omega)} \quad \text{for all } \omega \in H^{\mathrm{m}}(\Omega).$$

In what follows, we will simply denote the above subdifferential by $\partial\mathcal{R}$. The viscous system will also feature the viscously perturbed dissipation potential

$$\mathcal{R}_\varepsilon: L^1(\Omega) \to [0,+\infty], \qquad \mathcal{R}_\varepsilon(\eta) := \mathcal{R}(\eta) + \frac{\varepsilon}{2}\|\eta\|^2_{L^2(\Omega)}, \tag{2.13}$$

and $\partial\mathcal{R}_\varepsilon : H^{\mathrm{m}}(\Omega) \rightrightarrows H^{\mathrm{m}}(\Omega)^*$ will denote its subdifferential in the duality pairing $\langle\cdot,\cdot\rangle_{H^{\mathrm{m}}(\Omega)}$.

The *plastic dissipation potential* $\mathcal{H}: \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to \mathbb{R}$ is defined by

$$\mathcal{H}(z,\pi) := \int_\Omega H(z(x),\pi(x))\mathrm{d}x. \tag{2.14}$$

Its convex analysis subdifferential w.r.t. the second variable is the operator

$$\begin{aligned} &\partial_\pi\mathcal{H}: \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \text{ defined by} \\ &\omega \in \partial_\pi\mathcal{H}(z,\pi) \quad \text{if and only if} \quad \omega(x) \in \partial_\pi H(z(x),\pi(x)) \quad \text{for a.a. } x \in \Omega. \end{aligned} \tag{2.15}$$

Recall that the dissipation potential density H is associated with a family $(K(z))_{z\in[0,+\infty)}$ of convex subsets of Ω; cf. Remark 2.3. Then, for a given $z \in \mathrm{C}^0(\overline{\Omega})$, we set

(2.16)
$$\begin{aligned} \mathcal{K}_z(\Omega) &= \{\omega \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) : \omega(x) \in K(z(x)) \text{ for a.a. } x \in \Omega\} \\ &= \{\omega \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) : \omega(x) \in \partial_\pi H(z(x),0) \text{ for a.a. } x \in \Omega\} = \partial_\pi\mathcal{H}(z,0), \end{aligned}$$

and observe that $\mathcal{H}(z,\cdot)$ is the support function of the set $\mathcal{K}_z(\Omega)$, namely

$$\mathcal{H}(z,\pi) = \sup_{\omega\in\mathcal{K}_z(\Omega)} \int_\Omega \omega(x):\pi(x)\mathrm{d}x \qquad \text{for all } \pi \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}). \tag{2.17}$$

We will also work with the viscously perturbed potential

(2.18)
$$\mathcal{H}_\varepsilon: \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to [0,+\infty), \quad \mathcal{H}_\varepsilon(z,\pi) := \mathcal{H}(z,\pi) + \frac{\varepsilon}{2}\|\pi\|^2_{L^2(\Omega)}.$$

We introduce the stored elastic energy $\mathcal{Q}: L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathrm{C}^0(\overline{\Omega}) \to \mathbb{R}$:

$$\mathcal{Q}(e,z) := \frac{1}{2}\int_\Omega \mathbf{C}(z)e{:}e\,\mathrm{d}x\,. \tag{2.19}$$

The energy functional driving the evolution of the *viscous* system is

(2.20)

$$\begin{aligned} &\mathcal{E}: [0,T]\times H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\times H^{\mathrm{m}}(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \to (-\infty,+\infty] \text{ defined by}\\ &\mathcal{E}(t,u,z,p) := \mathcal{Q}(e(t),z) + \Phi(z) - \langle F(t), u+w(t)\rangle_{H^1(\Omega;\mathbb{R}^n)}\\ &\quad\text{with} \qquad e(t) := \mathrm{E}(u+w(t)) - p\,, \qquad \Phi(z) := \frac{1}{2}a_{\mathrm{m}}(z,z) + \int_\Omega W(z)\,\mathrm{d}x\,. \end{aligned}$$

In (2.20) we have incorporated the boundary loading w in the elastic energy and in the term with the external force F. This reflects the fact that we will indeed impose the Dirichlet condition for the displacements on Γ_{Dir} (cf. (1.2f), by working with a state variable in $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, so that $u+w$ in fact satisfies the desired boundary condition. The new variable $u+w$ will thus feature in the driving energy functional and in the statement of our existence theorem for the viscous system; cf. Theorem 2.4 ahead.

2.3. Existence and a priori estimates for the viscous problem. The state space for the viscous system is

$$\mathbf{Q} := H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\times H^{\mathrm{m}}(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\,.$$

It was proved in [CR21, Lemma 3.3] that for every $t\in[0,T]$ the functional $q := (u,z,p) \mapsto \mathcal{E}(t,q)$ is Fréchet differentiable on its domain $[0,T]\times\mathscr{D}$, with

$$\mathscr{D} = \left\{ (u,z,p)\in\mathbf{Q} : \min_{x\in\overline{\Omega}} z(x) > 0 \text{ in } \overline{\Omega} \right\}.$$

It also follows from Hypothesis 5 that for all $t\in[0,T]$ the function $q\mapsto\mathcal{E}(t,q)$ is Fréchet-differentiable on $\mathscr{D}$, with

(2.21a)

$$\mathrm{D}_q\mathcal{E}(t,q) = \left(-\mathrm{Div}\,\sigma(t) - F(t), A_{\mathrm{m}}z + W'(z) + \frac{1}{2}\mathbf{C}'(z)e(t){:}e(t), -\sigma_{\mathrm{D}}(t)\right) \in \mathbf{Q}^*$$

for all $(t,q)\in[0,T]\times\mathbf{Q}$, where $\sigma_{\mathrm{D}}(t)$ is the deviatoric part of the stress tensor $\sigma(t) = \mathbf{C}(z)e(t)$. Furthermore, for all $q\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\times H^{\mathrm{m}}(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ the function $t\mapsto\mathcal{E}(t,q)$ belongs to $H^1(0,T)$, with

$$\partial_t\mathcal{E}(t,q) = \int_\Omega \sigma(t) : \mathrm{E}(w'(t))\mathrm{d}x - \langle F'(t), u+w(t)\rangle_{H^1(\Omega)} - \langle F(t), w'(t)\rangle_{H^1(\Omega)} \tag{2.21b}$$

for a.a. $t\in(0,T)$. Relying on this, it is easy to check that system (1.3) reformulates as the generalized gradient system

$$\partial_{q'}\Psi_\varepsilon(q(t),q'(t)) + \mathrm{D}_q\mathcal{E}(t,q(t)) \ni 0 \qquad \text{in } \mathbf{Q}^* \quad \text{for a.a. } t\in(0,T), \tag{2.22}$$

involving the overall dissipation potential (degenerate w.r.t. the variable u)

$$\Psi_\varepsilon : \mathbf{Q}\times\mathbf{Q} \to [0,+\infty], \qquad \Psi_\varepsilon(q,q') := \mathcal{R}_\varepsilon(z') + \mathcal{H}_\varepsilon(z,p')\,.$$

A by now standard argument (cf., e.g., [AGS08, MRS13]) based on the validity of the chain rule for the driving energy $\mathcal{E}$ shows that a curve $q = (u, z, p) \in H^1(0, T; \mathbf{Q})$ is a solution to the generalized gradient system (2.22) if and only if it satisfies for every $[s, t] \subset [0, T]$ the energy-dissipation balance

$$\begin{aligned} &\mathcal{E}(t, q(t)) + \int_s^t (\Psi_\varepsilon(q(r), q'(r)) + \Psi^*_\varepsilon(q(r), -\mathrm{D}_q\mathcal{E}(r, q(r))))\,\mathrm{d}r \\ &\quad = \mathcal{E}(0, q(0)) + \int_0^t \partial_t\mathcal{E}(r, q(r))\mathrm{d}r. \end{aligned} \tag{2.23}$$

Indeed, (2.23) features the Fenchel–Moreau conjugate $\Psi^*_\varepsilon : \mathbf{Q}^* \to [0, +\infty)$ of Ψ_ε, which is defined, for $\xi = (\eta, \zeta, \omega) \in \mathbf{Q}^* = H^1_{\mathrm{Dir}}(\Omega; \mathbb{R}^n)^* \times H^{\mathrm{m}}(\Omega)^* \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, by

$$\begin{aligned} &\Psi^*_\varepsilon(q, \xi) = \mathcal{R}^*_\varepsilon(\zeta) + \mathcal{H}^*_\varepsilon(z, \omega) \qquad \text{with} \\ &\begin{cases} \mathcal{R}^*_\varepsilon(\zeta) = \dfrac{1}{2\varepsilon} \inf\limits_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_{L^2}(\zeta - \gamma)^2\,, & \zeta \in H^{\mathrm{m}}(\Omega)^*, \\ \mathcal{H}^*_\varepsilon(z, \omega) = \dfrac{1}{2\varepsilon} \min\limits_{\rho\in\partial_\pi\mathcal{H}(z,0)} \|\omega - \rho\|^2_{L^2(\Omega)}\,, & \omega \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}), \end{cases} \end{aligned} \tag{2.24}$$

where

$$\mathfrak{f}_{L^2} : H^{\mathrm{m}}(\Omega)^* \to [0, +\infty] \text{ is given by } \mathfrak{f}_{L^2}(\eta) = \begin{cases} \|\eta\|_{L^2(\Omega)} & \text{if } \eta \in L^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Now it can be easily checked that if $\mathcal{R}^*_\varepsilon(\zeta) < +\infty$, then the inf in its definition is indeed attained and thus $\mathcal{R}^*_\varepsilon(\zeta) = \frac{1}{2\varepsilon} \min_{\gamma\in\partial\mathcal{R}(0)} \|\zeta - \gamma\|^2_{L^2(\Omega)}$. In what follows, we will use the short-hand notation

$$\begin{aligned} &\mathsf{d_z}(t, q) := \begin{cases} \min_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_{L^2}(-\mathrm{D}_z\mathcal{E}(t, q) - \gamma) & \text{if } \inf_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_{L^2}(-\mathrm{D}_z\mathcal{E}(t, q) - \gamma) < +\infty, \\ +\infty & \text{otherwise,} \end{cases} \\ &\mathsf{d_p}(t, q) := \min_{\rho\in\partial_\pi\mathcal{H}(z,0)} \| - \mathrm{D}_p\mathcal{E}(t, q) - \rho\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.25}$$

Clearly, the notation (2.25) hints at the fact that both objects are in fact the distances of $-\mathrm{D}_z\mathcal{E}$ and $-\mathrm{D}_p\mathcal{E}$ (for $-\mathrm{D}_z\mathcal{E}$, one in fact has to consider the *extended* distance) from the respective stable sets $\partial\mathcal{R}(0) \subset H^{\mathrm{m}}(\Omega)^*$ and $\partial_\pi\mathcal{H}(z, 0) = \mathcal{K}_z(\Omega)$ (the latter is in fact a subset of $L^\infty(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$).

Slightly adapting the arguments from [CR21, Theorem 5.1], we prove that the viscous system (1.3) admits a solution additionally satisfying the energy-dissipation balance (2.28) below.

THEOREM 2.4. *Assume Hypotheses* 1–5. *Then there exists a triple* $q = (u, z, p)$, *with*

$$\begin{aligned} &u \in H^1_{\mathrm{Dir}}(0, T; H^1(\Omega; \mathbb{R}^n))\,, \\ &z \in H^1(0, T; H^{\mathrm{m}}(\Omega))\,, \qquad 0 \le z(t, x) \le 1 \quad \textit{for every } (t, x) \in [0, T] \times \overline{\Omega}\,, \\ &p \in H^1(0, T; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}))\,, \end{aligned} \tag{2.26}$$

such that $(u + w, z, p)$ *satisfies for almost all* $t \in (0, T)$ *system* (1.3) *in the sense that*

$$- \mathrm{Div}\big(\mathbf{C}(z(t))e(t)\big) = F(t) \qquad \textit{in } H^1_{\mathrm{Dir}}(\Omega; \mathbb{R}^n)^*, \tag{2.27a}$$

with $e(t) = \mathrm{E}(u(t)+w(t)) - p(t)$,

$$\partial\mathcal{R}_\varepsilon(z'(t)) + A_{\mathrm{m}}(z(t)) + W'(z(t)) \ni -\frac{1}{2}\mathbf{C}'(z)e(t):e(t) \qquad in\ H^{\mathrm{m}}(\Omega)^*, \tag{2.27b}$$

$$\partial_p H(z(t),p'(t)) + \varepsilon p'(t) \ni \sigma_{\mathrm{D}}(t) \qquad a.e.\ in\ \Omega, \tag{2.27c}$$

joint with the initial conditions

$$u(0) = u_0 \ in\ H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\,, \qquad z(0) = z_0 \ in\ H^{\mathrm{m}}(\Omega)\,, \qquad p(0) = p_0 \ in\ L^2(\Omega)\,. \tag{2.27d}$$

In fact, (2.27a) *holds at all* $t \in [0,T]$. *Furthermore, the curve* $q = (u,z,p)$ *satisfies the energy-dissipation balance for every* $t \in [0,T]$*:*

$$\begin{aligned}&\mathcal{E}(t,q(t)) + \int_0^t (\mathcal{R}(z'(r)) + \mathcal{H}(z(r),p'(r)))\,\mathrm{d}r + \int_0^t \Big(\frac{\varepsilon}{2}\|z'(r)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|p'(r)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}r \\ &\quad + \int_0^t \Big(\frac{1}{2\varepsilon}\mathsf{d}^2_{\mathsf{z}}(r,q(r)) + \frac{1}{2\varepsilon}\mathsf{d}^2_{\mathsf{p}}(r,q(r))\Big)\,\mathrm{d}r = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(r,q(r))\mathrm{d}r\,.\end{aligned} \tag{2.28}$$

Sketch of the proof. As mentioned earlier, the proof of Theorem 2.4 basically relies on the same arguments devised for [CR21, Theorem 5.1]. Indeed, the main difference between system (1.3) and the viscous regularization of (1.2) addressed in [CR21] resides in the fact that, here, no viscosity regularization is considered for the momentum balance. It is then worthwhile to comment the temporal regularity $u \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$. Formally, it can be justified by testing (2.27a) by u', (2.27b) by z', and (2.27c) by p'. Rigorously, the existence of solutions to (2.27) can be shown by time discretization as in [CR21]; the abovementioned estimate can then be performed on the discrete system. ☐

Now let $(q_\varepsilon)_\varepsilon = (u_\varepsilon, z_\varepsilon, p_\varepsilon)_\varepsilon \subset H^1(0,T;\mathbf{Q})$ be a family of solutions to the Cauchy problem (2.27). The following result collects the bounds that they enjoy uniformly w.r.t. the parameter ε. Such estimates are a direct consequence of the energy-dissipation balance (cf., e.g., the arguments in [CR21, Proposition 4.3]). In particular, the strict positivity property (2.33) below derives from the energy estimate (2.29) via the growth condition (2.7b) on the potential W; cf. [CL16, Lemma 3.3].

PROPOSITION 2.5. *There exists a constant* $\widetilde{S} > 0$ *such that for every* $\varepsilon > 0$

$$\sup_{t\in[0,T]} |\mathcal{E}(t,q_\varepsilon(t))| \le \widetilde{S}\,, \tag{2.29}$$

$$\int_0^T \|z'_\varepsilon(t)\|_{L^1}\,\mathrm{d}t = \frac{1}{\kappa}\int_0^T \mathcal{R}(z'(t))\mathrm{d}t \le \widetilde{S}\,, \tag{2.30}$$

$$\int_0^T \|p'_\varepsilon(t)\|_{L^1}\,\mathrm{d}t \le \frac{1}{r}\int_0^T \mathcal{H}(z_\varepsilon(t),p'_\varepsilon(t))\,\mathrm{d}t \le \widetilde{S}\,, \tag{2.31}$$

$$\int_0^T \sqrt{\|z'_\varepsilon(t)\|^2_{L^2} + \|p'_\varepsilon(t)\|^2_{L^2}}\,\sqrt{\mathsf{d}^2_{\mathsf{z}}(t,q_\varepsilon(t)) + \mathsf{d}^2_{\mathsf{p}}(t,q_\varepsilon(t))}\mathrm{d}t \le \widetilde{S}\,. \tag{2.32}$$

Moreover,

$$\exists\, m_0 > 0 \quad \forall\,\varepsilon > 0 \quad \forall\, t\in[0,T]: \quad \min_{x\in\overline{\Omega}} z_\varepsilon(t,x) \ge m_0\,. \tag{2.33}$$

3. Vanishing-viscosity analysis and main result.

3.1. Reparameterization. Let $(q_\varepsilon)_\varepsilon$ be a family of solutions to the Cauchy problem (2.27). Relying on Proposition 2.5, we are going to reparameterize them by the *energy-dissipation arclength* $\widetilde{s}_\varepsilon : [0,T] \to [0,\widetilde{S}_\varepsilon]$ (with $\widetilde{S}_\varepsilon := \widetilde{s}_\varepsilon(T)$) defined by

$$\begin{aligned}\widetilde{s}_\varepsilon(t) := \int_0^t \Big(&1 + \|z_\varepsilon'(\tau)\|_{L^1} + \|p_\varepsilon'(\tau)\|_{L^1} \\ &+ \sqrt{\|z_\varepsilon'(\tau)\|^2_{L^2} + \|p_\varepsilon'(\tau)\|^2_{L^2}}\,\sqrt{\mathsf{d}^2_{\mathsf{z}}(,\tau)q_\varepsilon(\tau) + \mathsf{d}^2_{\mathsf{p}}(,\tau)q_\varepsilon(\tau)}\Big)\,\mathrm{d}\tau\,.\end{aligned} \tag{3.1a}$$

"Energy-dissipation" refers to the interplay between the dissipation term $\sqrt{\|z'\|^2_{L^2} + \|p'\|^2_{L^2}}$ and the term $\sqrt{\mathsf{d}^2_{\mathsf{z}} + \mathsf{d}^2_{\mathsf{p}}}$, which contains the forces $-\mathrm{D}_z\mathcal{E}$ and $-\mathrm{D}_p\mathcal{E}$. From the estimates of Proposition 2.5 it follows that $\sup_\varepsilon \widetilde{S}_\varepsilon \le C$. We set

$$\mathsf{t}_\varepsilon := (\widetilde{s}_\varepsilon)^{-1}, \qquad \mathsf{q}_\varepsilon := q_\varepsilon \circ \mathsf{t}_\varepsilon = (\mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon, \mathsf{p}_\varepsilon)\,, \qquad \mathsf{e}_\varepsilon := e_\varepsilon \circ \mathsf{t}_\varepsilon\,, \qquad \sigma_\varepsilon := \sigma_\varepsilon \circ \mathsf{t}_\varepsilon\,, \tag{3.1b}$$

which we may assume to be defined on a fixed interval $[0,S]$, with $S := \lim_{\varepsilon\downarrow 0} \widetilde{S}_\varepsilon$ (the limit is taken along a suitable subsequence).

The rescaled solutions $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) : [0,S] \to [0,T] \times \mathbf{Q}$ satisfy the normalization condition

$$\begin{aligned}\mathsf{t}_\varepsilon'(s) + \|\mathsf{z}_\varepsilon'(s)\|_{L^1(\Omega)} + \|\mathsf{p}_\varepsilon'(s)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \sqrt{\|\mathsf{z}_\varepsilon'(s)\|^2_{L^2} + \|\mathsf{p}_\varepsilon'(s)\|^2_{L^2}} \\ \times \sqrt{\mathsf{d}^2_{\mathsf{z}}(,\mathsf{t}_\varepsilon(s))\mathsf{q}_\varepsilon(s) + \mathsf{d}^2_{\mathsf{p}}(,\mathsf{t}_\varepsilon(s))\mathsf{q}_\varepsilon(s)} \equiv 1 \qquad \text{for a.e. } s \in (0,S)\,,\end{aligned} \tag{3.2}$$

and the *reparameterized* energy-dissipation balance for every $s \in [0,S]$

$$\begin{aligned}\mathcal{E}(\mathsf{t}_\varepsilon(s), \mathsf{q}_\varepsilon(s)) &+ \int_0^s \mathcal{M}_\varepsilon(\mathsf{t}_\varepsilon(r), \mathsf{q}_\varepsilon(r), \mathsf{t}_\varepsilon'(r), \mathsf{q}_\varepsilon'(r))\,\mathrm{d}r \\ &= \mathcal{E}(\mathsf{t}_\varepsilon(0), \mathsf{q}_\varepsilon(0)) + \int_0^s \partial_t \mathcal{E}(\mathsf{t}_\varepsilon(r), \mathsf{q}_\varepsilon(r))\mathsf{t}_\varepsilon'(r)\,\mathrm{d}r\,,\end{aligned} \tag{3.3}$$

with the functional $\mathcal{M}_\varepsilon : [0,T] \times \mathbf{Q} \times (0,+\infty) \times \mathbf{Q} \to [0,+\infty]$ defined by

$$\begin{aligned}\mathcal{M}_\varepsilon(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}(z,p') + I_{\{0\}}(\|-\mathrm{D}_u\mathcal{E}(t,q)\|_{(H^1_{\mathrm{Dir}})^*}) \\ + \frac{\varepsilon}{2t'}\sqrt{\|z'\|^2_{L^2} + \|p'\|^2_{L^2}} + \frac{t'}{2\varepsilon}\sqrt{\mathsf{d}^2_{\mathsf{z}}(t,q) + \mathsf{d}^2_{\mathsf{p}}(t,q)}\,,\end{aligned} \tag{3.4}$$

where $I_{\{0\}} : \mathbb{R} \to [0,+\infty]$ is the indicator function of the singleton $\{0\}$, defined by $I_{\{0\}}(r) = 0$ if $r = 0$, and $I_{\{0\}}(r) = +\infty$ otherwise.

Remark 3.1 (on the structure of $\mathcal{M}_\varepsilon$). Recall that $\mathrm{D}_u\mathcal{E}(t,q) = -\mathrm{Div}\,\sigma(t) - F(t)$ (cf. (2.21a)). Thus, the contribution $I_{\{0\}}(\|-\mathrm{D}_u\mathcal{E}(t,q)\|_{(H^1_{\mathrm{Dir}})^*})$ encodes the fact that, for curves (t,q) along which $\int_0^S \mathcal{M}_\varepsilon(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon, \mathsf{t}_\varepsilon', \mathsf{q}_\varepsilon')\,\mathrm{d}r < +\infty$, the elastic equilibrium equation (2.27a) holds almost everywhere in $(0,S)$. In turn, the terms $\frac{\varepsilon}{2t'}\sqrt{\|z'\|^2_{L^2} + \|p'\|^2_{L^2}}$ and $\frac{t'}{2\varepsilon}\sqrt{\mathsf{d}^2_{\mathsf{z}}(t,q) + \mathsf{d}^2_{\mathsf{p}}(t,q)}$ convey the competition between the tendency of the system to be governed by *viscous* dissipation and that to relax towards the state characterized by

$$\mathsf{d}_{\mathsf{z}}(t,q) = \mathsf{d}_{\mathsf{p}}(t,q) = 0, \text{ and the elastic equilibrium } \|-\mathrm{D}_u\mathcal{E}(t,q)\|_{(H^1_{\mathrm{Dir}})^*} = 0. \tag{3.5}$$

Now in [CR21, Lemma 7.4, Remark 7.5] we have shown that conditions (3.5) occur in the *rate-independent* regime, when the displacement variable u is at elastic equilibrium and one has local stability for the damage parameter z and the plastic strain p.

Based on [MRS16b, MR23, CR21, CLR23], we expect that, up to a subsequence, the curves $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)_\varepsilon$ converge to a curve (t, q) satisfying an energy-dissipation balance akin to (3.3) and involving a *vanishing-viscosity contact potential* $\mathcal{M}_0$, which will be properly introduced (cf. (3.20) ahead) after some preliminary definitions.

3.2. Preliminary definitions and energetics for the perfectly plastic damage system. First of all, let us introduce the state space for the perfectly plastic damage system

$$\mathbf{Q}_{\mathrm{PP}} := \big\{q = (u,z,p) \in \mathrm{BD}(\Omega) \times H^{\mathrm{m}}(\Omega) \times \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}) : \\ e := \mathrm{E}(u) - p \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}),\ u \odot \mathrm{n}\,\mathcal{H}^{n-1} + p = 0 \text{ on } \Gamma_{\mathrm{Dir}}\big\}, \tag{3.6}$$

where the condition $u \odot \mathrm{n}\,\mathcal{H}^{n-1} + p = 0$ on Γ_{Dir} is a relaxation of the homogeneous Dirichlet condition $u = 0$ on Γ_{Dir}. We will consider $\mathbf{Q}_{\mathrm{PP}}$ endowed with the weak* topology of $\mathrm{BD}(\Omega)^* \times H^{\mathrm{m}}(\Omega) \times \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})^*$.

The driving energy functional for the perfectly plastic system. The energy functional is the extension of $\mathcal{E}$ from (2.20) to the space $[0,T] \times \mathbf{Q}_{\mathrm{PP}}$. To emphasize the change in the topological setup, we will use a different notation for the extended energy. Thus, we define the functional $\mathcal{E}_{\mathrm{PP}} : [0,T] \times \mathbf{Q}_{\mathrm{PP}} \to (-\infty, +\infty]$

$$\mathcal{E}_{\mathrm{PP}}(t,q) := \mathcal{Q}(z, e(t)) + \Phi(z) - \langle F(t), u + w(t)\rangle_{\mathrm{BD}(\Omega)} \tag{3.7}$$

with Φ from (2.20). Since for every $q \in \mathbf{Q}_{\mathrm{PP}}$ we have that $\mathrm{E}(u) - p \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, we have $e(t) := \mathrm{E}(u + w(t)) - p \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ for every $t \in [0,T]$; also taking into account that $F \in H^1(0,T; \mathrm{BD}(\Omega)^*)$, we find that $\mathcal{E}_{\mathrm{PP}}$ is well defined, with domain $\mathrm{dom}(\mathcal{E}_{\mathrm{PP}}) = [0,T] \times \mathscr{D}_{\mathrm{PP}}$, where

$$\mathscr{D}_{\mathrm{PP}} = \left\{ (u,z,p) \in \mathbf{Q}_{\mathrm{PP}} : \min_{x \in \overline{\Omega}} z(x) > 0 \text{ in } \overline{\Omega} \right\}.$$

Observe that for every $q \in \mathscr{D}_{\mathrm{PP}}$ the function $t \mapsto \mathcal{E}_{\mathrm{PP}}(t, \cdot)$ is in $H^1(0,T)$ and for every $(t,q) \in [0,T] \times \mathbf{Q}_{\mathrm{PP}}$ the partial time derivative $\partial_t \mathcal{E}_{\mathrm{PP}}(t,q)$ is given by (2.21a), with the duality pairings in $H^1(\Omega; \mathbb{R}^n)$ replaced by $\langle \cdot, \cdot \rangle_{\mathrm{BD}(\Omega)}$.

The stress-strain duality. More in general, along the footsteps of [DMDM06] we introduce the class of *admissible displacements and strains* associated with a function $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, that is,

$$A(w) := \{(u,e,p) \in \mathrm{BD}(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}) : \\ \mathrm{E}(v) = e + p \text{ in } \Omega,\ p = (w - v) \odot \mathrm{n}\,\mathcal{H}^{n-1} \text{ on } \Gamma_{\mathrm{Dir}}\}$$

(recall that n denotes the normal vector to $\partial\Omega$), where $\odot$ is the symmetrized tensorial product. The *space of admissible plastic strains* is

$$\Pi(\Omega) := \{p \in \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}) : \exists\, (v,w,e) \in \mathrm{BD}(\Omega) \times H^1(\mathbb{R}^n; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \\ \text{s.t. } (v,e,p) \in A(w)\}.$$

Given $\sigma \in \Sigma(\Omega)$ (cf. (2.4)), $p \in \Pi(\Omega)$, and v, e such that $(u, e, p) \in A(w)$, we define the *stress-strain duality*

$$\begin{aligned}\langle [\sigma_{\mathrm{D}} : p], \varphi \rangle := -\int_\Omega \varphi \sigma \cdot (e - \mathrm{E}(w))\,\mathrm{d}x - \int_\Omega \sigma \cdot [(v-w) \odot \nabla\varphi]\,\mathrm{d}x \\ - \int_\Omega \varphi\,(\operatorname{div}\sigma) \cdot (v-w)\,\mathrm{d}x\end{aligned} \tag{3.8}$$

for every $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$; in fact, this definition is independent of v and e. It can be checked that, for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi(\Omega)$, the duality $[\sigma_{\mathrm{D}}{:}p]$ defines a bounded Radon measure. Under these assumptions, $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ for every $r < \infty$, and there holds $\|[\sigma_{\mathrm{D}}{:}p]\|_1 \le \|\sigma_{\mathrm{D}}\|_{L^\infty} \|p\|_{L^1}$. Restricting such a measure to $\Omega \cup \Gamma_{\mathrm{Dir}}$, we set

$$\langle \sigma_{\mathrm{D}} \,|\, p \rangle := [\sigma_{\mathrm{D}}{:}p](\Omega \cup \Gamma_{\mathrm{Dir}}). \tag{3.9}$$

For later use, we record here the integration by parts formula (see [FG12] for an equivalent version)

$$\langle \sigma_{\mathrm{D}} \,|\, p \rangle = -\langle \sigma, e - \mathrm{E}(w) \rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \langle -\mathrm{Div}\,\sigma, v - w \rangle_{\mathrm{BD}(\Omega)} \tag{3.10}$$

for all $\sigma \in \Sigma(\Omega)$, $(v, e, p) \in A(w)$.

The dissipation potential for perfect plasticity. Let us emphasize that $q \in \mathbf{Q}_{\mathrm{PP}}$ means that the plastic variable p is now only a measure in $\mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$. That is why the plastic dissipation mechanism for the rate-independent damage system will be encoded by a dissipation potential that extends $\mathcal{H}(p, \cdot)$ to $\mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$ via the theory of convex functions of measures [GS64]; see also [Tem83]. Namely, we define $\mathcal{H}_{\mathrm{PP}} : \mathrm{C}^0(\overline{\Omega}; [0,1]) \times \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}) \to \mathbb{R}$:

$$\mathcal{H}_{\mathrm{PP}}(z, \pi) := \int_{\Omega\cup\Gamma_{\mathrm{Dir}}} H\left(z(x), \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\right) \mathrm{d}\mu(x),$$

where $\mu \in \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$ is a positive measure such that $\pi \ll \mu$ and $\frac{\mathrm{d}\pi}{\mathrm{d}\mu}$ is the Radon–Nikodym derivative of π with respect to μ; by the one-homogeneity of $H(z(x), \cdot)$, the definition of $\mathcal{H}_{\mathrm{PP}}$ does not depend of μ. By [AFP05, Proposition 2.37], for every $z \in \mathrm{C}^0(\overline{\Omega}; [0,1])$ the functional $p \mapsto \mathcal{H}_{\mathrm{PP}}(z, p)$ is convex and positively one-homogeneous. Moreover, for all $(z_k)_k \subset \mathrm{C}^0(\overline{\Omega}; [0,1])$ and $(\pi_k)_k \subset \mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$ such that $z_k \to z$ in $\mathrm{C}^0(\overline{\Omega})$ and $\pi_k \rightharpoonup \pi$ weakly* in $\mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$ we have that

$$\mathcal{H}_{\mathrm{PP}}(z, \pi) \le \liminf_{k\to+\infty} \mathcal{H}_{\mathrm{PP}}(z_k, \pi_k).$$

Finally, from [FG12, Proposition 3.9] it follows that

$$H\left(z, \frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p| \ge [\sigma_{\mathrm{D}} : p], \quad \text{as measures on } \Omega \cup \Gamma_{\mathrm{Dir}}, \quad \text{for all } \sigma \in \Sigma\mathrm{K}_z(\Omega), \tag{3.11}$$

where we use the notation

$$\Sigma\mathrm{K}_z(\Omega) := \{\sigma \in \Sigma(\Omega) :\ \sigma_{\mathrm{D}} \in \mathcal{K}_z(\Omega)\}. \tag{3.12}$$

In particular, we have

$$\mathcal{H}_{\mathrm{PP}}(z, p) \ge \sup_{\sigma \in \Sigma\mathrm{K}_z(\Omega)} \langle \sigma_{\mathrm{D}} \,|\, p \rangle \qquad \text{for every } p \in \Pi(\Omega), \tag{3.13}$$

to be compared with (2.17).

Slopes. We can no longer state that, for every fixed $t \in [0,T]$, the functional $\mathcal{E}_{\mathrm{PP}}(t,\cdot)$ is Gâteaux-differentiable on $\mathscr{D}_{\mathrm{PP}}$, since the "natural candidate" for $\mathrm{D}_u\mathcal{E}_{\mathrm{PP}}$, namely the term $-\mathrm{Div}\,\sigma(t) - F(t)$, need not be an element in $\mathrm{BD}(\Omega)^*$. In order to define $\mathcal{M}_0$, we will thus need a proxy for the slope term $\|-\mathrm{D}_u\mathcal{E}\|_{(H^1)^*}$ which features in (3.4). That is why we define the *slope* of $\mathcal{E}_{\mathrm{PP}}(t,\cdot)$ via

$$\mathbb{S}_{\mathsf{u}}(t,q) := \|-\mathrm{Div}\,\sigma(t) - F(t)\|_{(H^1(\Omega;\mathbb{R}^n))^*} \qquad \text{for } (t,q) \in [0,T]\times\mathscr{D}_{\mathrm{PP}}\,. \tag{3.14}$$

Observe that the above object is well defined, since for every $(t,q)\in[0,T]\times\mathscr{D}_{\mathrm{PP}}$ we in fact have $-\mathrm{Div}\,\sigma(t) - F(t) \in H^1(\Omega;\mathbb{R}^n)^*$.

Analogously, we will need to introduce a proxy for the distance term $\mathsf{d}_{\mathsf{p}}(t,q)$ from (2.25), since $\mathrm{D}_p\mathcal{E}_{\mathrm{PP}}(t,q)$ is no longer well defined as an element of $\mathcal{M}_{\mathrm{b}}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})^*$. As a surrogate, it is natural to resort to the $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$-distance of $-\sigma_{\mathrm{D}}$ from $\mathcal{K}_z(\Omega)$. Namely, we set

$$\mathbb{D}_{\mathsf{p}}(t,q) := \mathrm{dist}_{L^2(\Omega)}(-\sigma_{\mathrm{D}}, \mathcal{K}_z(\Omega))\,. \tag{3.15}$$

Finally, for notational consistency, hereafter we will use the notation

$$\mathbb{D}_{\mathsf{z}}(t,q) := \min_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_{L^2}(-\mathrm{D}_z\mathcal{E}_{\mathrm{PP}}(t,q) - \gamma) \tag{3.16}$$

in place of $\mathsf{d}_{\mathsf{z}}(t,q)$. Note that, whenever $\mathrm{D}_z\mathcal{E}_{\mathrm{PP}}(t,q)\notin L^2(\Omega)$, we have that $\mathbb{D}_{\mathsf{z}}(t,q) = +\infty$.

Later, we will resort to the following representations of $\mathbb{D}_{\mathsf{p}}$ and $\mathbb{D}_{\mathsf{z}}$.

LEMMA 3.2. *There holds for every* $(t,q)\in[0,T]\times\mathscr{D}_{\mathrm{PP}}$

(3.17a)

$$\mathbb{D}_{\mathsf{p}}(t,q) = \sup_{\substack{\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\\ \|\eta_p\|_{L^2}\le 1}} \Big(\langle\sigma_{\mathrm{D}}(t),\eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z,\eta_p)\Big)\,,$$

(3.17b)

$$\mathbb{D}_{\mathsf{z}}(t,q) = \sup_{\substack{\eta_z\in H^{\mathrm{m}}(\Omega)\\ \|\eta_z\|_{L^2}\le 1}} \Big(\langle -A_{\mathrm{m}}z - W'(z) - \frac{1}{2}\mathbf{C}'(z)e(t):e(t),\eta_z\rangle_{H^{\mathrm{m}}(\Omega)} - \mathcal{R}(\eta_z)\Big)\,.$$

Formulae (3.17) follow from a duality argument already exploited in the proof of [CR21, Lemma 3.6]; for the reader's convenience, we will revisit and generalize it in Lemma A.3 ahead, which straightforwardly implies Lemma 3.2; see also Remark A.4.

We will also rely on the following lower semicontinuity result; cf. [CR21, Proposition 7.7].

LEMMA 3.3. *Let* $(t_k)_k,\, t\subset[0,T]$, *and* $(q_k)_k\subset\mathbf{Q}$, $q\in\mathbf{Q}_{\mathrm{PP}}$, *fulfill the following as* $k\to\infty$:

$$t_k\to t,\ q_k \overset{*}{\rightharpoonup} q \ \text{in } \mathbf{Q}_{\mathrm{PP}}\,,$$
$$e(t_k) = \mathrm{E}(u_k + w(t_k)) - p_k \to e(t) = \mathrm{E}(u+w(t)) - p \ \text{in}\, L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}).$$

Then

$$\begin{cases} \mathbb{S}_{\mathsf{u}}(t,q) \le \liminf\limits_{k\to\infty} \|-\mathrm{D}_u\mathcal{E}(t_k,q_k)\|_{(H^1(\Omega;\mathbb{R}^n))^*}\,,\\ \mathbb{D}_{\mathsf{z}}(t,q) \le \liminf\limits_{k\to\infty} \mathsf{d}_{\mathsf{z}}(t_k,q_k)\,,\\ \mathbb{D}_{\mathsf{p}}(t,q) \le \liminf\limits_{k\to\infty} \mathsf{d}_{\mathsf{p}}(t_k,q_k)\,. \end{cases} \tag{3.18}$$

The vanishing-viscosity contact potential. Preliminarily, we introduce the place-holders

$$
\begin{cases} \mathcal{D}(q') := \sqrt{\|z'\|_{L^2}^2 + \|p'\|_{L^2}^2} \\ \mathcal{D}^*(t,q) := \sqrt{\mathbb{D}_{\mathsf{z}}^2(t,q) + \mathbb{D}_{\mathsf{p}}^2(t,q)} \end{cases} \qquad \text{for } (t,q) \in [0,T] \times \mathscr{D}\,. \tag{3.19}
$$

Now, whenever $\mathrm{D}_z\mathcal{E}_{\mathrm{PP}}(t,q) \notin L^2(\Omega)$, we have $\mathbb{D}_{\mathsf{z}}(t,q) = +\infty$ and hence $\mathcal{D}^*(t,q) = +\infty$. We are in a position to introduce the functional $\mathcal{M}_0 : [0,T] \times \mathbf{Q}_{\mathrm{PP}} \times [0,T] \times \mathbf{Q}_{\mathrm{PP}} \to [0,+\infty]$ by setting

$$
\mathcal{M}_0(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}_{\mathrm{PP}}(z,p') + I_{\{0\}}(\mathbb{S}_{\mathsf{u}}(t,q)) + \mathcal{M}_0^{\mathrm{red}}(t,q,t',q')\,, \tag{3.20a}
$$

where the reduced contact potential $\mathcal{M}_0^{\mathrm{red}}$ is defined on $[0,T] \times \mathbf{Q}_{\mathrm{PP}} \times [0,T] \times \mathbf{Q}_{\mathrm{PP}}$ by

$$
\mathcal{M}_0^{\mathrm{red}}(t,q,t',q') := I_{\{0\}}(\mathbb{D}_{\mathsf{z}}(t,q)) + I_{\{0\}}(\mathbb{D}_{\mathsf{p}}(t,q)) \qquad \text{if } t' > 0\,, \tag{3.20b}
$$

while for $t' = 0$ special attention needs to be paid to the case in which $\mathcal{D}^*(t,q) = +\infty$. Indeed, along the footsteps of [MR23, Definition 5.1] we set

$$
\mathcal{M}_0^{\mathrm{red}}(t,q,0,q') := \begin{cases} \mathcal{D}(q')\,\mathcal{D}^*(t,q) & \text{if } \mathcal{D}^*(t,q) < +\infty, \\ 0 & \text{if } q' = 0 \text{ and } (t,q) \in \overline{\mathrm{dom}(\mathcal{D}^*)}^{\mathrm{w}}, \\ +\infty & \text{otherwise} \end{cases} \qquad \text{if } t' = 0\,, \tag{3.20c}
$$

where $\overline{\mathrm{dom}(\mathcal{D}^*)}^{\mathrm{w}}$ is the weak closure of $\mathrm{dom}(\mathcal{D}^*) = \{(t,q) \in [0,T] \times \mathscr{D} : \mathcal{D}^*(t,q) < +\infty\}$ confined to energy sublevels, i.e.,

$$
\begin{aligned} \overline{\mathrm{dom}(\mathcal{D}^*)}^{\mathrm{w}} = \{(t,q) :\ & \exists\,(t_k,q_k)_k \subset \mathrm{dom}(\mathcal{D}^*) \text{ s.t. } t_k \to t,\ q_k \overset{*}{\rightharpoonup} q, \\ & \sup_k \mathcal{E}_{\mathrm{PP}}(t_k,q_k) < +\infty\}\,. \end{aligned}
$$

Remark 3.4 (on the structure of $\mathcal{M}_0$). We emphasize that the definition of $\mathcal{M}_0$ encodes the fact that, when $t' > 0$, in addition to the elastic equilibrium condition $\mathbb{S}_{\mathsf{u}}(t,q) = 0$ the local stability conditions $\mathbb{D}_{\mathsf{z}}(t,q) = 0$ and $\mathbb{D}_{\mathsf{p}}(t,q) = 0$ hold, which is in accord with rate-independent evolution of the system (cf. Remark 3.1). Instead, the contribution $\mathcal{D}(q')\,\mathcal{D}^*(t,q)$ to $\mathcal{M}_0(t,q,0,q')$ encodes the fact that when the system jumps (and the external time is frozen, i.e., $t' = 0$), the system may be governed by viscosity in z and p.

We now introduce the concept of *admissible parameterized curve,* tailored to describing the properties of parameterized curves $(\mathsf{t},\mathsf{q}) : [0,S] \to [0,T] \times \mathbf{Q}_{\mathrm{PP}}$ for which the term $\mathcal{D}(\mathsf{q}')\,\mathcal{D}^*(\mathsf{t},\mathsf{q})$ is well defined on a specific subset $A^\circ \subset [0,S]$. Moreover, this notion also records the fact that, in the same way as for the rate-dependent system (1.3), also for the rate-independent system (1.2) the boundary condition for the displacements is *weakly* formulated in terms of the variable $u + w$.

DEFINITION 3.5. *A curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) : [0,S] \to [0,T] \times \mathbf{Q}_{\mathrm{PP}}$ *is an* admissibile parameterized curve for the perfectly plastic system *if*

(3.21a) $\mathsf{t} \in \mathrm{AC}([0,S];[0,T])$ *is nondecreasing,*

(3.21b) $\mathsf{u} \in L^\infty\big(0,S;\mathrm{BD}(\Omega)\big)$,

(3.21c) $\mathsf{z} \in \mathrm{AC}([0,T];L^1(\Omega))$,

(3.21d) $\mathsf{p} \in \mathrm{AC}([0,T];\mathcal{M}_{\mathrm{b}}(\Omega \cup \partial_D\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))$,

(3.21e) $\mathsf{e} = \mathrm{E}(\mathsf{u}+\mathsf{w}) - \mathsf{p} \in L^\infty(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ *with* $\mathsf{w} = w{\circ}\mathsf{t}$,

(3.21f) $(\mathsf{z},\mathsf{p}) \in \mathrm{AC}_{\mathrm{loc}}(A^\circ;L^2(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))$, *where* A° *is the open set*

$$A^\circ := \{s \in (0,S) : \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0\},$$

(3.21g) t *is constant in every connected component of* A°.

We will write $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}})$.

In view of Remarks 3.1 and 3.4, hereafter we will refer to A° as the *instability set.* In what follows, we will also use the notation

$$B^\circ := [0,S] \setminus A^\circ. \tag{3.22}$$

3.3. Notion of BV solution and main result. Eventually, we are in a position to specify the concept of BV solution to the perfectly plastic system that we will obtain in the vanishing-viscosity limit.

DEFINITION 3.6. *An admissible parameterized curve* $(\mathsf{t},\mathsf{q}) : [0,S] \to [0,T]\times\mathbf{Q}_{\mathrm{PP}}$ *is a* (parameterized) balanced viscosity *(*BV*, for short) solution to the system for perfect plasticity and damage* (1.2) *if* (t,q) *fulfills the energy-dissipation balance*

$$\begin{aligned}&\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\&\quad= \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r\end{aligned} \tag{3.23}$$

for all $0 \le s \le S$*. Finally, we say that* (t,q) *is nondegenerate if it fulfills*

$$\mathsf{t}' + \|\mathsf{z}'\|_{L^2(\Omega)} + \|\mathsf{p}'\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} > 0 \qquad \text{a.e. in } (0,S).$$

Remark 3.7. Clearly, along an admissible parameterized curve the energy-dissipation balance in integral form (3.23) is *equivalent* to the *pointwise* identity

$$\mathcal{M}_0(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s)) + \partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s) \tag{3.24}$$

for almost all $s \in (0,S)$. In [CL16, CLR23] (see also [MRS16b, MR23]), relation (3.24), *combined* with the chain rule for $\mathcal{E}$, has been the starting point for deriving an additional characterization of BV solutions in terms of a system of subdifferential inclusions akin to the viscously regularized system.

However, in the present context the validity of such a chain rule along admissible curves is an open problem, due to their poor spatial regularity. That is why we are not in a position to provide further characterizations of BV solutions other than (3.24).

The first main result of this paper states the convergence of the reparameterized viscous solutions to a BV solution. Recall that $\mathsf{w} = w{\circ}\mathsf{t}$.

THEOREM 3.8. *Under Hypotheses* 1–5*, let* $(\varepsilon_k)_k \subset (0,+\infty)$ *be a null sequence and* $(q_{\varepsilon_k})_k \subset H^1(0,T;\mathbf{Q})$ *be a sequence of solutions to the Cauchy problem* (2.27)*. Let* $\mathsf{t}_{\varepsilon_k} : [0,S] \to [0,T]$ *be the time rescalings as in* (3.1b)*, and consider the reparameterized curves* $\mathsf{q}_{\varepsilon_k} := q_{\varepsilon_k}{\circ}\mathsf{t}_{\varepsilon_k}$.

Then there exist a (not relabeled) subsequence and an admissible parameterized curve $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}})$, $\mathsf{q}=(\mathsf{u},\mathsf{z},\mathsf{p})$, *such that the following convergences hold as* $k\to\infty$*:*

$$\mathsf{t}_{\varepsilon_k}(s)\to\mathsf{t}(s), \qquad \mathsf{q}_{\varepsilon_k}(s)\overset{*}{\rightharpoonup}\mathsf{q}(s) \ \textit{in } \mathbf{Q}_{\mathrm{PP}} \qquad \textit{for all } s\in[0,S], \tag{3.25}$$

and $(\mathsf{t},\mathsf{u}+\mathsf{w},\mathsf{z},\mathsf{p})$ *is a* BV *solution to the perfectly plastic damage system* (1.2) *with additional temporal regularity, namely* $\mathsf{t}\in W^{1,\infty}(0,S;[0,T])$ *and*

$$\textit{the mapping} \quad \begin{cases} \mathsf{u}:[0,S]\to \mathrm{BD}(\Omega) \ \textit{is weakly}^* \textit{ continuous},\\ \mathsf{e}:[0,S]\to L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \ \textit{is weakly continuous},\\ \mathsf{z}:[0,S]\to H^{\mathrm{m}}(\Omega) \ \textit{is weakly continuous},\\ \mathsf{p}:[0,S]\to \mathcal{M}_{\mathrm{b}}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \ \textit{is } 1\textit{-Lipschitz continuous}. \end{cases} \tag{3.26}$$

In particular, the pair (e,z) *satisfies the elastic equilibrium equation* everywhere *on* $[0,S]$, *i.e.,*

$$-\mathrm{Div}\big(\mathbf{C}(\mathsf{z}(s))\mathsf{e}(s)\big)=F(\mathsf{t}(s)) \qquad \textit{in } \mathrm{BD}(\Omega)^* \quad \textit{for all } s\in[0,S]\,. \tag{3.27}$$

Furthermore, on the instability set A° *we have the additional regularity*

$$\begin{cases} \mathsf{u}\in \mathrm{C}^0(A^\circ;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n))\,,\\ \mathsf{e}\in \mathrm{C}^0(A^\circ;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))\,,\\ \mathsf{z}\in W^{1,\infty}_{\mathrm{loc}}(A^\circ;L^2(\Omega))\,,\\ \mathsf{p}\in W^{1,\infty}_{\mathrm{loc}}(A^\circ;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))\,. \end{cases} \tag{3.28}$$

Finally, we have the following enhanced *convergences for every* $s\in[0,S]$*:*

$$\mathcal{E}(\mathsf{t}_{\varepsilon_k}(s),\mathsf{q}_{\varepsilon_k}(s)) \longrightarrow \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s)), \tag{3.29a}$$

$$\int_0^s \mathcal{M}_\varepsilon(\mathsf{t}_{\varepsilon_k}(r),\mathsf{q}_{\varepsilon_k}(r),\mathsf{t}'_{\varepsilon_k}(r),\mathsf{q}'_{\varepsilon_k}(r))\,\mathrm{d}r \longrightarrow \int_0^s \mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\,. \tag{3.29b}$$

Outline of the proof. In section 4 we will settle all the compactness properties of the sequence $(\mathsf{q}_{\varepsilon_k})_k$ and then, in Proposition 4.3 ahead, we will obtain the *upper* energy-dissipation inequality

$$\begin{aligned} &\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\ &\quad\le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r \quad \text{for all } s\in[0,S] \end{aligned} \tag{3.30}$$

via lower semicontinuity arguments. (By convention, we shall refer to (3.30) as an "upper" inequality, because there the energy at the current time, $\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s))$, is estimated from above by the energy at the initial time and by the work of the external forces.) Sections 5 and 6 will be devoted to the proof of the *lower* energy-dissipation inequality

$$\begin{aligned} &\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\ &\quad\ge \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r \quad \text{for all } s\in[0,S]\,, \end{aligned} \tag{3.31}$$

where the energy $\mathcal{E}_{\rm PP}(\mathsf{t}(s),\mathsf{q}(s))$ is estimated from below; cf. Proposition 6.1 ahead. The enhanced convergences (3.29) are then a by-product of this limiting procedure, combined with the validity of the energy-dissipation balance, via a standard argument that we choose not to detail.

In order to keep the exposition simple, we will prove the results in the particular case where the volume force f and the prescribed traction g are null, and the only external loading is w from (2.10e). Nonetheless, all the results of this paper also hold with volume and surface forces under assumptions (2.10). An extended version of the proofs with volume and surface forces is available in the preprint version of this paper [CLRpre].

4. Proof of Theorem 3.8: The upper energy-dissipation inequality. We start by collecting all a priori bounds on the sequence $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k$.

LEMMA 4.1. *There exists $C>0$ such that for every $k\in\mathbb{N}$*

$$\|\mathsf{t}_{\varepsilon_k}\|_{W^{1,\infty}(0,S)}+\|\mathsf{z}_{\varepsilon_k}\|_{W^{1,\infty}(0,S;L^1(\Omega))}+\|\mathsf{p}_{\varepsilon_k}\|_{W^{1,\infty}(0,S;L^1(\Omega;\mathbb{M}^{n\times n}_{\rm D}))}\leq C, \tag{4.1a}$$

$$\sup_{s\in(0,S)}\sqrt{\|\mathsf{z}'_{\varepsilon_k}(s)\|^2_{L^2}+\|\mathsf{p}'_{\varepsilon_k}(s)\|^2_{L^2}}\sqrt{\mathsf{d}^2_{\mathsf{z}}(,\mathsf{t}_{\varepsilon_k}(s))\mathsf{q}_{\varepsilon_k}(s)+\mathsf{d}^2_{\mathsf{p}}(,\mathsf{t}_{\varepsilon_k}(s))\mathsf{q}_{\varepsilon_k}(s)}\leq C, \tag{4.1b}$$

$$\|\mathsf{u}_{\varepsilon_k}\|_{L^\infty(0,S;\mathrm{BD}(\Omega))}+\|\mathsf{e}_{\varepsilon_k}\|_{L^\infty(0,S;L^2(\mathbb{M}^{n\times n}_{\rm sym}))}+\|\mathsf{z}_{\varepsilon_k}\|_{L^\infty(0,S;H^{\rm m}(\Omega))}\leq C\,. \tag{4.1c}$$

Proof. Estimates (4.1a) and (4.1b) obviously follow from the normalization condition (3.2). Estimate (4.1c) ensues from (2.2), (2.6b), (2.29), and (2.33); for more details we refer the reader to [CLRpre]. □

We can now settle compactness properties of the sequence $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k$. Prior to that, we recall that the convergence in the space $\mathrm{C}^0([0,S];\mathbf{X}_{\rm weak})$ is, by definition, defined by the convergence of $\mathrm{C}^0([0,S];(\mathbf{X},d_{\rm weak}))$, where the metric $d_{\rm weak}$ induces the weak topology on a closed bounded set of the reflexive space $\mathbf{X}$; the notation $\mathrm{C}^0([0,S];\mathbf{X}_{\rm weak^*})$ has the same meaning if $\mathbf{X}$ is the dual of a separable space.

Compactness. By (4.1) and by compactness results à la Ascoli and Arzelà in metric spaces [AGS08, Proposition 3.3.1], arguing as in [DDS11, Theorem 4.6] we obtain the following properties: There exist

$$\mathsf{t}\in W^{1,\infty}(0,S)\,,\quad \mathsf{z}\in W^{1,\infty}(0,S;\mathcal{M}_{\rm b}(\Omega))\,,\mathsf{p}\in W^{1,\infty}(0,S;\mathcal{M}_{\rm b}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\rm D}))$$

such that, up to subsequences, the following convergences hold as $k\to\infty$:

$$\mathsf{t}_{\varepsilon_k}\overset{*}{\rightharpoonup}\mathsf{t}\text{ in }W^{1,\infty}(0,S)\,,\qquad \mathsf{t}_{\varepsilon_k}\to\mathsf{t}\text{ in }\mathrm{C}^0([0,S])\,, \tag{4.2a}$$

$$\begin{cases}\mathsf{z}_{\varepsilon_k}\overset{*}{\rightharpoonup}\mathsf{z}\text{ in }W^{1,\infty}(0,S;\mathcal{M}_{\rm b}(\Omega))\,,\\ \mathsf{z}_{\varepsilon_k}\to\mathsf{z}\text{ in }\mathrm{C}^0([0,S];H^{\rm m}(\Omega)_{\rm weak})\,,\end{cases} \tag{4.2b}$$
$$\text{and thus }\mathsf{z}_{\varepsilon_k}(s)\rightharpoonup\mathsf{z}(s)\text{ in }H^{\rm m}(\Omega)\text{ for all }s\in[0,S]\,,$$

$$\begin{cases}\mathsf{p}_{\varepsilon_k}\overset{*}{\rightharpoonup}\mathsf{p}\text{ in }W^{1,\infty}(0,S;\mathcal{M}_{\rm b}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\rm D}))\,,\\ \mathsf{p}_{\varepsilon_k}\to\mathsf{p}\text{ in }\mathrm{C}^0([0,S];\mathcal{M}_{\rm b}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\rm D})_{\rm weak^*})\,,\end{cases} \tag{4.2c}$$
$$\text{and thus }\mathsf{p}_{\varepsilon_k}(s)\overset{*}{\rightharpoonup}\mathsf{p}(s)\text{ in }\mathcal{M}_{\rm b}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\rm D})\text{ for every }s\in[0,S].$$

Additionally, $\mathsf{z}:[0,S]\to L^1(\Omega)$ and $\mathsf{p}:[0,S]\to\mathcal{M}_{\rm b}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\rm D})$ are 1-Lipschitz continuous.

Further, there exist $\mathsf{u} \in L^\infty(0,S;\mathrm{BD}(\Omega))$ and $\mathsf{e} \in L^\infty(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ such that $\mathsf{e} = \mathrm{E}(\mathsf{u}+w\circ\mathsf{t})-\mathsf{p}$, $\mathsf{u}_{\varepsilon_k} \overset{*}{\rightharpoonup} \mathsf{u} \in L^\infty(0,S;\mathrm{BD}(\Omega))$, $\mathsf{e}_{\varepsilon_k} \overset{*}{\rightharpoonup} \mathsf{e}$ in $L^\infty(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$, and

$$\begin{cases} \mathsf{e}_{\varepsilon_k}(s_k) \rightharpoonup \mathsf{e}(s) \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \\ \mathsf{u}_{\varepsilon_k}(s_k) \overset{*}{\rightharpoonup} \mathsf{u}(s) \text{ in } \mathrm{BD}(\Omega) \end{cases} \quad \text{for all } s\in[0,S] \text{ and } (s_k)_k \text{ s.t. } s_k\to s. \tag{4.2d}$$

As a consequence, we may obtain the continuity properties (3.26) for e and u and the validity of the elastic equilibrium equation (3.27) everywhere on $[0,S]$. We have thus proven (3.25).

It remains to show that the parameterized curve (t,q) is admissible in the sense of Definition 3.5. This follows from our next result, which partly proves the statement in (3.28). The other regularity properties therein, i.e., $\mathsf{u} \in \mathrm{C}^0(A^\circ;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n))$ and $\mathsf{e} \in \mathrm{C}^0(A^\circ;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$, will be proved in Lemma 4.4 ahead.

Lemma 4.2. *In the open set A° we have*

$$(\mathsf{z},\mathsf{p}) \in W^{1,\infty}_{\mathrm{loc}}(A^\circ;L^2(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})) \tag{4.3}$$

and, up to a further subsequence,

$$\forall\,[\alpha,\beta]\subset A^\circ \ \textit{we have} \quad \begin{cases} \mathsf{z}_{\varepsilon_k} \overset{*}{\rightharpoonup} \mathsf{z} \ \textit{in } W^{1,\infty}([\alpha,\beta];L^2(\Omega)), \\ \mathsf{p}_{\varepsilon_k} \overset{*}{\rightharpoonup} \mathsf{p} \ \textit{in } W^{1,\infty}([\alpha,\beta];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})). \end{cases} \tag{4.4}$$

Proof. As we have just seen, the function $[0,S]\ni s\mapsto \mathsf{q}(s)$ is weakly* continuous in $\mathbf{Q}_{\mathrm{PP}}$. Therefore, thanks to Lemma 3.3 we have that the function $[0,S]\ni s\mapsto \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))$ is lower semicontinuous. Thus, A° is open and

$$\forall\,[\alpha,\beta]\subset A^\circ \ \exists\, c>0 \ \forall\, s\in[\alpha,\beta]: \qquad \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) \geq c > 0\,.$$

Furthermore, combining the uniform convergences from (4.2a)–(4.2d) with the lower semicontinuity properties from Lemma 3.3 we conclude that

$$\exists\,\bar{k}\in\mathbb{N} \ \forall\, k\geq\bar{k} \ \forall\, s\in[\alpha,\beta]: \qquad \sqrt{\mathsf{d}^2_{\mathsf{z}}(,\mathsf{t}_{\varepsilon_k}(s))\mathsf{q}_{\varepsilon_k}(s) + \mathsf{d}^2_{\mathsf{p}}(,\mathsf{t}_{\varepsilon_k}(s))\mathsf{q}_{\varepsilon_k}(s)} \geq c$$

and, thus, from (4.1b) we gather that

$$\exists\, C>0 \ \exists\,\bar{k}\in\mathbb{N} \ \forall\, k\geq\bar{k} \text{ for a.a. } s\in(\alpha,\beta): \qquad \sqrt{\|\mathsf{z}'_{\varepsilon_k}(s)\|^2_{L^2} + \|\mathsf{p}'_{\varepsilon_k}(s)\|^2_{L^2}} \leq C\,.$$

This entails that $\mathsf{z}\in W^{1,\infty}([\alpha,\beta];L^2(\Omega))$ and $\mathsf{p}\in W^{1,\infty}([\alpha,\beta];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))$ for every $[\alpha,\beta]\subset A^\circ$ and, up to a further subsequence (possibly depending on $[\alpha,\beta]$), convergences in (4.4) hold. Exhausting the open set A° by a countable family of intervals, with a diagonal procedure we can extract a subsequence such that convergences (4.4) hold on *any* $[\alpha,\beta]\subset A^\circ$. ☐

Proof of the upper energy-dissipation inequality. We will prove the following result.

Proposition 4.3. *The pair (t,q) satisfies the upper energy-dissipation inequality (3.30). Therefore, there exists a constant $\mathrm{M}>0$ such that*

$$\sup_{s\in[0,S]} |\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s))| \leq \mathrm{M}, \qquad \int_0^S \mathcal{M}_0(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\mathrm{d}s \leq \mathrm{M}. \tag{4.5}$$

Proof. The proof is standard and follows by passing to the limit as $k\to\infty$ in the upper energy-dissipation inequality for the viscous system, relying on classical lower semicontinuity results; cf. [MR23, section 5]. For more details we refer the reader to [CL16, Lemma 3.5], [CR21, section 7], [MR23, Proposition 7.1], [CLR23, section 5]), and the preprint version of this paper [CLRpre]. Estimate (4.5) then straightforwardly follows. ∎

From (4.5) we derive an additional estimate, (4.6) below, which will play a crucial role in the proof of Lemma 5.7 ahead. In fact, (4.6) will be derived with arguments similar to those for [DDS11, Lemmas 7.1, 7.2, 7.3], albeit adapted to handle the present case, in which the dependence of $\mathbf{C}$ on z brings about additional difficulties. In particular, we mention that in the proof of Lemma 4.4 we will resort to a consequence of the integration by parts formula (3.10), which will also be exploited in the proof of Proposition 5.1 ahead. As an immediate consequence of (4.6) we will deduce the enhanced regularity for u and e in the instability set A° stated in (3.28). Recall that $\mathsf{w}(s):=w(\mathsf{t}(s))$, as introduced above.

LEMMA 4.4. *There exists a constant $C_L>0$, only depending on γ_1 in* (2.6b)*, on the Lipschitz constant $\|\mathbf{C}\|_{\mathrm{Lip}}$ of $\mathbf{C}$, and on* M *in* (4.5)*, such that for every connected component $(a,b)\subset A^\circ$ of A° and for all $[s_1,s_2]\subset(a,b)$ there holds*

$$\|\mathsf{e}(s_2)-\mathsf{e}(s_1)\|_{L^2}\le C_L(\|\mathsf{p}(s_2)-\mathsf{p}(s_1)\|_{L^2}+\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty}). \tag{4.6}$$

As a result, we have $\mathsf{u}\in \mathrm{C}^0(A^\circ;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n))$ and $\mathsf{e}\in \mathrm{C}^0(A^\circ;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$.

Proof. From the integration by parts formula (3.10) we deduce

$$\begin{aligned}
&\langle\sigma_{\mathrm{D}}(s_2)-\sigma_{\mathrm{D}}(s_1),\mathsf{p}(s_2)-\mathsf{p}(s_1)\rangle_{L^2(\Omega)}\\
&\quad=-\langle\sigma(s_2)-\sigma(s_1),[\mathrm{E}(\mathsf{u}(s_2)+\mathsf{w}(s_2))-\mathsf{p}(s_2)]-[\mathrm{E}(\mathsf{u}(s_1)+\mathsf{w}(s_1))-\mathsf{p}(s_1)]\rangle_{L^2(\Omega)}\\
&\qquad+\langle\sigma(s_2)-\sigma(s_1),\mathrm{E}(\mathsf{w}(s_2))-\mathrm{E}(\mathsf{w}(s_1))\rangle_{L^2(\Omega)}\\
&\qquad+\langle-[\mathrm{Div}\,\sigma(s_2)-\mathrm{Div}\,\sigma(s_1)],[\mathsf{u}(s_2)+\mathsf{w}(s_2)-\mathsf{w}(s_2)]\\
&\qquad-[\mathsf{u}(s_1)+\mathsf{w}(s_1)-\mathsf{w}(s_1)]\rangle_{\mathrm{BD}(\Omega)}\\
&\quad\overset{(1)}{=}-\langle\sigma(s_2)-\sigma(s_1),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}+\langle\sigma(s_2)-\sigma(s_1),\mathrm{E}(\mathsf{w}(s_2))-\mathrm{E}(\mathsf{w}(s_1))\rangle_{L^2(\Omega)}\\
&\quad\overset{(2)}{=}-\langle\sigma(s_2)-\sigma(s_1),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}+0\,.
\end{aligned}\tag{4.7}$$

Here (1) follows from using that $\mathsf{e}=\mathrm{E}(\mathsf{u}+\mathsf{w})-\mathsf{p}$, and testing the elastic equilibrium equations (3.27), evaluated at s_1 and s_2, by $\mathsf{u}(s_2)-\mathsf{u}(s_1)$, while (2) is due to the fact that $\mathsf{t}'\equiv0$ on $[s_1,s_2]$ (recall (3.21g)), so that $\mathrm{E}(\mathsf{w}(s_2))=\mathrm{E}(\mathsf{w}(s_1))$. In turn, we have the identity

$$\begin{aligned}
&\langle\mathbf{C}(\mathsf{z}(s_2))(\mathsf{e}(s_2)-\mathsf{e}(s_1)),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}\\
&\quad+\langle\big[\mathbf{C}(\mathsf{z}(s_1))-\mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_1),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}\\
&=\langle\sigma(s_2)-\sigma(s_1),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}.
\end{aligned}\tag{4.8}$$

Combining (4.7) and (4.8) and using that $\mathbf{C}$ is (uniformly) positive definite (cf. (2.6b)), we obtain

$$\begin{aligned}
&\gamma_1\|\mathsf{e}(s_2)-\mathsf{e}(s_1)\|^2_{L^2(\Omega)}\le\Lambda_1+\Lambda_2\,,\\
&\text{with}\qquad\begin{cases}\Lambda_1=\big|\langle\big[\mathbf{C}(\mathsf{z}(s_1))-\mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_1),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)}\big|\,,\\ \Lambda_2=|\langle\sigma_{\mathrm{D}}(s_2)-\sigma_{\mathrm{D}}(s_1),\mathsf{p}(s_2)-\mathsf{p}(s_1)\rangle_{L^2(\Omega)}|\,.\end{cases}
\end{aligned}$$

Now

$$\Lambda_1 \le \|\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\|_{L^\infty(\Omega)} \|\mathsf{e}(s_1)\|_{L^2(\Omega)} \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)}$$
$$\overset{(1)}{\le} \|\mathbf{C}\|_{\mathrm{Lip}} \|\mathsf{z}(s_1)) - \mathsf{z}(s_2)\|_{L^\infty(\Omega)} \frac{\mathrm{M}}{\gamma_1} \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)},$$

where for (1) we have used the Lipschitz continuity of $\mathbf{C}$ and estimate (4.5). In order to estimate Λ_2, we use that

$$\begin{aligned}
&\|\sigma_{\mathrm{D}}(s_2) - \sigma_{\mathrm{D}}(s_1)\|_{L^2(\Omega)} \\
&\quad \le \|\mathbf{C}(\mathsf{z}(s_2))\mathsf{e}(s_2) - \mathbf{C}(\mathsf{z}(s_1))\mathsf{e}(s_1)\|_{L^2(\Omega)} \\
&\quad \le \|\mathbf{C}(\mathsf{z}(s_2))\|_{L^\infty(\Omega)} \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)} + \|\mathsf{e}(s_2)\|_{L^2(\Omega)} \|\mathbf{C}(\mathsf{z}(s_2)) - \mathbf{C}(\mathsf{z}(s_1))\|_{L^\infty(\Omega)} \\
&\quad \le \left[\|\mathbf{C}\|_{\mathrm{Lip}} \|\mathsf{z}(s_2))\|_{L^\infty(\Omega)} + \|\mathbf{C}(0)\|_{L^\infty(\Omega)}\right] \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)} \\
&\qquad + \frac{\mathrm{M}}{\gamma_1} \|\mathbf{C}\|_{\mathrm{Lip}} \|\mathsf{z}(s_2)) - \mathsf{z}(s_1)\|_{L^\infty(\Omega)}.
\end{aligned}$$

All in all, we find that

$$\begin{aligned}
&\gamma_1 \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|^2_{L^2(\Omega)} \\
&\quad \le \overline{C} \Big\{ \|\mathsf{z}(s_1)) - \mathsf{z}(s_2)\|_{L^\infty(\Omega)} \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)} \\
&\qquad + \|\mathsf{p}(s_2) - \mathsf{p}(s_1)\|_{L^2(\Omega)} \|\mathsf{e}(s_2) - \mathsf{e}(s_1)\|_{L^2(\Omega)} \\
&\qquad + \|\mathsf{z}(s_1)) - \mathsf{z}(s_2)\|_{L^\infty(\Omega)} \|\mathsf{p}(s_2) - \mathsf{p}(s_1)\|_{L^2(\Omega)} \Big\}
\end{aligned}$$

for a constant $\overline{C}$ only depending on γ_1, $\|\mathbf{C}\|_{\mathrm{Lip}}$, and M. Then (4.6) ensues. Combining (4.6) with the regularity of p ensured by (4.3) and the strong continuity of z from A° into L^∞ granted by (3.28), we deduce the claimed continuity of u and e. ∎

5. Intermediate lower energy-dissipation inequalities. In this section we lay the groundwork for the proof of Proposition 6.1 ahead, stating the validity of the lower energy-dissipation inequality (3.31) along the admissible parameterized curves $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T]\times \mathbf{Q}_{\mathrm{PP}})$ arising from the limiting procedure of section 4. In order to prove it, we will carefully revisit the techniques devised in [DDS11], also borrowing some ideas from [BFM12].

The key idea is subdividing the interval $[0,T]$ by means of suitably chosen partitions and resorting to fine approximation results of the Bochner integral via Riemann sums. We also mention that this idea has long been used in the context of the analysis of rate-independent systems, indeed for proving the "lower" energy inequality and, ultimately, the energy-dissipation balance for energetic solutions/quasi-static evolutions, dating back to, e.g., [DMFT05, FM06]; cf. also [MR15].

This is the overview of this section:

- Section 5.1 revolves around a crucial estimate, proved in Proposition 5.1 that will be at the core of our proof of the lower inequality (3.31).
- In section 5.2 we are going to outline the steps of the proof in detail. Indeed, for any fixed partition of $[0,S]$ we will distinguish three families of induced subintervals. For each type of subinterval we will obtain a suitable discrete version of (3.31).
- These discrete inequalities will be proved in sections 5.3, 5.4, and 5.5, leading to the *overall* discrete lower energy-dissipation inequality (6.1). It is in (6.1) that we will pass to the limit, as the fineness of the partition tends to zero, to conclude the lower energy-dissipation inequality (3.31).

Before delving into the above tasks, let us specify that hereafter we shall suppose that, for the parameterized curve $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T]\times \mathbf{Q}_{\mathrm{PP}})$ considered here,

$$\text{the set } F := \{s\in(0,S) :\ \mathsf{t}'(s)=0,\ \mathsf{q}'(s)=0\} \text{ has empty interior.} \tag{5.1}$$

This can be assumed without loss of generality. Indeed, suppose that $\mathsf{t}' \equiv 0$ and $\mathsf{q}' \equiv 0$ on an interval (s_*, s^*) with $0 \le s_* < s^* \le S$. Then the lower energy-dissipation inequality (3.31) trivially holds on $[s_*, s^*]$.

Let us also point out, for later use, that since for the pair (t,q) the elastic equilibrium equation (3.27) holds everywhere, by [DMDM06, Proposition 3.5] we have

$$\sigma(s)\in\Sigma(\Omega) \quad \text{for all } s\in[0,S]. \tag{5.2}$$

5.1. A crucial estimate. The cornerstone of this section is estimate (5.5) ahead. In order to state it, we need to settle some notation. For a given interval $[s_1,s_2]\subset[0,S]$ we introduce the quantities that will enter in the discrete versions of the energy-dissipation inequality, i.e.,

- the *energy/$\mathcal{H}$-dissipation variation* of the curve $(\mathsf{t},\mathsf{q}) : [0,S]\to[0,T]\times\mathbf{Q}_{\mathrm{PP}}$ on $[s_1,s_2]$:

$$\begin{aligned}\mathrm{HV}((\mathsf{t},\mathsf{q});[s_1,s_2]) := {} & \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_2), \frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right) + \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_1), \frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right)\\ & + \|\mathsf{p}(s_2)-\mathsf{p}(s_1)\|_{L^2}\frac{\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_2),\mathsf{q}(s_2))}{2},\end{aligned}$$

  where the product $\|\mathsf{p}(s_2)-\mathsf{p}(s_1)\|_{L^2}\,\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_i),\mathsf{q}(s_i))$ is set by convention equal to 0 if $\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_i),\mathsf{q}(s_i))=0$ and $\mathsf{p}(s_2)-\mathsf{p}(s_1)\notin L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$;
- the *energy/$\mathcal{R}$-dissipation variation* of (t,q) on $[s_1,s_2]$:

$$\begin{aligned}&\mathrm{RV}((\mathsf{t},\mathsf{q});[s_1,s_2])\\ &\quad := \mathcal{R}(\mathsf{z}(s_2)-\mathsf{z}(s_1)) + \|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}\frac{\mathbb{D}_{\mathsf{z}}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_{\mathsf{z}}(\mathsf{t}(s_2),\mathsf{q}(s_2))}{2}.\end{aligned}$$

The proposed names for HV and RV highlight that their definition features contributions involving both the dissipation potential and the energy.

We will also work with a version of RV augmented by the energy functional Φ from (2.20), namely

- the *augmented energy/$\mathcal{R}$-dissipation variation* of (t,q) on $[s_1,s_2]$:

$$\begin{aligned}\mathrm{ARV}((\mathsf{t},\mathsf{q});[s_1,s_2]) &:= \Phi(\mathsf{z}(s_2))-\Phi(\mathsf{z}(s_1)) + \mathrm{RV}(\mathsf{q};[s_1,s_2])\\ &\quad + K_W\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^1}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty}\\ &= \Phi(\mathsf{z}(s_2))-\Phi(\mathsf{z}(s_1)) + \mathcal{R}(\mathsf{z}(s_2)-\mathsf{z}(s_1))\\ &\quad + \|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}\frac{\mathbb{D}_{\mathsf{z}}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_{\mathsf{z}}(\mathsf{t}(s_2),\mathsf{q}(s_2))}{2}\\ &\quad + K_W\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^1}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty},\end{aligned}$$

where we have set $K_W := \|W''\|_{L^\infty(0,1)}$. Finally, estimate (5.5) will also feature

- a term approximating the work of the external forces on $[s_1,s_2]$, i.e.,

$$\mathrm{WE}((\mathsf{t},\mathsf{q});[s_1,s_2]) := \frac{1}{2}\langle\sigma(s_1)+\sigma(s_2),\mathrm{E}(\mathsf{w}(s_2)-\mathsf{w}(s_1))\rangle$$

(recall that we are now neglecting volume and surface forces). In fact, in view of Lemma A.6 in the appendix, $\mathrm{WE}((\mathsf{t},\mathsf{q});[s_1,s_2])$ turns out to be a discrete form of the external work $\int_{s_1}^{s_2}\partial_t\mathcal{E}_{\mathrm{PP}}(\mathsf{t},\mathsf{q})\,\mathsf{t}'\mathrm{d}s$.

We are now in a position to state the main result of this section, where we show a discrete form of the lower energy-dissipation inequality. Hereafter, we will use the notation

$$K_{\mathbf{C}} := \frac{\|\mathbf{C}'\|_{\mathrm{Lip}}}{\min_{z\in[m_0,1]}\mathbf{C}'(z)}\,, \qquad K_W := \|W''\|_{L^\infty(0,1)}, \tag{5.3}$$

where we have denoted by $\|\mathbf{C}'\|_{\mathrm{Lip}}$ the Lipschitz constant of $\mathbf{C}'$ and we have used for shorter notation the place-holder

$$\min_{z\in[m_0,1]}\mathbf{C}'(z) := \min_{z\in[m_0,1]}\inf_{\xi\in\mathbb{R}^{n\times n}}\mathbf{C}'(z)\xi{:}\xi\,;$$

recall that m_0 is the constant from Proposition 2.5.

Proposition 5.1. *For all $0\le s_1<s_2\le S$ we have $\mathrm{ARV}(\mathsf{q};[s_1,s_2])\ge 0$. Moreover, for $0\le s_1<s_2\le S$ such that*

$$(s_1,\, s_2\in B^\circ) \quad \textit{or} \quad (\textit{the interval } [s_1,s_2]\subset A^\circ)\,, \tag{5.4}$$

then

$$\begin{aligned}\mathrm{WE}((\mathsf{t},\mathsf{q});[s_1,s_2]) &+ \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_1),\mathsf{q}(s_1))\\ &\le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_2),\mathsf{q}(s_2)) + \mathrm{HV}(\mathsf{q};[s_1,s_2]) + \mathrm{RV}(\mathsf{q};[s_1,s_2])\\ &\quad + K_{\mathbf{C}}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty}\mathrm{ARV}(\mathsf{q};[s_1,s_2])\\ &\quad + K_W\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^1}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty}.\end{aligned} \tag{5.5}$$

Proof. Let $s_1\le s_2\in[0,S]$ be fixed. Since $\sigma(s)\in\Sigma(\Omega)$ for all $s\in[0,S]$, we may apply the integration by parts formula (3.10) with the following choices for the triple (v,e,p):

$$\begin{cases} v = (\mathsf{u}(s_2)+\mathsf{w}(s_2)) - (\mathsf{u}(s_1)+\mathsf{w}(s_1))\,,\\ e = \big(\mathrm{E}(\mathsf{u}(s_2)+\mathsf{w}(s_2)) - \mathsf{p}(s_2)\big) - \big(\mathrm{E}(\mathsf{u}(s_1)+\mathsf{w}(s_1)) - \mathsf{p}(s_1)\big)\,,\\ p = \mathsf{p}(s_2)-\mathsf{p}(s_1)\,.\end{cases}$$

This leads to

$$\begin{aligned}&\langle\sigma_{\mathrm{D}}(s_i)|\mathsf{p}(s_2)-\mathsf{p}(s_1)\rangle\\ &\quad = -\langle\sigma(s_i),[\mathrm{E}(\mathsf{u}(s_2)+\mathsf{w}(s_2))-\mathsf{p}(s_2)-\mathrm{E}(\mathsf{w}(s_2))]\\ &\qquad - [\mathrm{E}(\mathsf{u}(s_1)+\mathsf{w}(s_1))-\mathsf{p}(s_1)-\mathrm{E}(\mathsf{w}(s_1))]\rangle_{L^2(\Omega)}\\ &\qquad + \langle-\mathrm{Div}\,\sigma(s_i),[\mathsf{u}(s_2)+\mathsf{w}(s_2)-\mathsf{w}(s_2)]\\ &\qquad - [\mathsf{u}(s_1)+\mathsf{w}(s_1)-\mathsf{w}(s_1)]\rangle_{\mathrm{BD}(\Omega)}\\ &\quad \overset{(1)}{=} -\langle\sigma(s_i),\mathsf{e}(s_2)-\mathsf{e}(s_1)\rangle_{L^2(\Omega)} + \langle\sigma(s_i),\mathrm{E}(\mathsf{w}(s_2))-\mathrm{E}(\mathsf{w}(s_1))\rangle_{L^2(\Omega)}\,,\end{aligned} \tag{5.6}$$

where (1) follows from recalling that $\mathsf{e}=\mathrm{E}(\mathsf{u}+\mathsf{w})-\mathsf{p}$, and from testing the elastic equilibrium equation (3.27), evaluated at s_i, by $\mathsf{u}(s_2)-\mathsf{u}(s_1)$. Therefore, adding the relation at s_1 with that at s_2, we obtain

$$\begin{aligned}\mathrm{WE}((\mathsf{t},\mathsf{q});[s_1,s_2]) &= \left\langle\frac{1}{2}(\sigma(s_1)+\sigma(s_2)),\mathrm{E}(\mathsf{w}(s_2)-\mathsf{w}(s_1))\right\rangle\\ &= \left\langle\frac{1}{2}\left(\sigma_{\mathrm{D}}(s_1)+\sigma_{\mathrm{D}}(s_2)\right)|\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle + \left\langle\frac{1}{2}\left(\sigma(s_1)+\sigma(s_2)\right),\mathsf{e}(s_2)-\mathsf{e}(s_1)\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{5.7}$$

We now estimate from above two terms on the right-hand side of (5.7).

Claim 1: *For every $s_1 \le s_2 \in [0,S]$ there holds*

$$\left\langle \frac{1}{2}(\sigma(s_1)+\sigma(s_2)), \mathsf{e}(s_2)-\mathsf{e}(s_1) \right\rangle_{L^2} \le \mathcal{Q}(\mathsf{z}(s_2),\mathsf{e}(s_2)) - \mathcal{Q}(\mathsf{z}(s_1),\mathsf{e}(s_1)) + (1+K_{\mathbf{C}}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty})\,\mathrm{ARV}(\mathsf{q};[s_1,s_2])\,. \tag{5.8}$$

To show this, we start by observing that

$$\left\langle \frac{1}{2}(\sigma(s_1)+\sigma(s_2)), \mathsf{e}(s_2)-\mathsf{e}(s_1) \right\rangle_{L^2} = \mathcal{Q}(\mathsf{z}(s_2),\mathsf{e}(s_2)) - \mathcal{Q}(\mathsf{z}(s_1),\mathsf{e}(s_1)) + \frac{1}{2}\langle \big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_1), \mathsf{e}(s_2)\rangle_{L^2}\,. \tag{5.9}$$

Using that $\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))$ is a positive definite fourth-order tensor (cf. (2.6c)), we find that

$$\frac{1}{2}\langle \big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_1), \mathsf{e}(s_2)\rangle_{L^2} \le \frac{1}{4}\langle \big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_1), \mathsf{e}(s_1)\rangle_{L^2} + \frac{1}{4}\langle \big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_2), \mathsf{e}(s_2)\rangle_{L^2}\,. \tag{5.10}$$

We now estimate the terms $\big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_i):\mathsf{e}(s_i)$ for $i=1,2$. By the Lagrange theorem, for $i \in \{1,2\}$ there exist functions $\zeta_i:\Omega \to [0,1]$ (that we may assume are measurable) with $\mathsf{z}(s_2) \le \zeta_i \le \mathsf{z}(s_1)$ a.e. in Ω, such that for almost all $x \in \Omega$

$$\begin{aligned}&\big[\mathbf{C}(\mathsf{z}(s_1,x)) - \mathbf{C}(\mathsf{z}(s_2,x))\big]\mathsf{e}(s_i,x):\mathsf{e}(s_i,x)\\ &\quad = \mathbf{C}'(\zeta_i(x))\big(\mathsf{z}(s_1,x) - \mathsf{z}(s_2,x)\big)\mathsf{e}(s_i,x):\mathsf{e}(s_i,x)\,.\end{aligned}$$

Now we have

$$\begin{aligned}&\mathbf{C}'(\zeta_i)\big(\mathsf{z}(s_1)-\mathsf{z}(s_2)\big)\mathsf{e}(s_i):\mathsf{e}(s_i)\\ &\quad\le \Big(\mathbf{C}'(\mathsf{z}(s_i)) + \|\mathbf{C}'\|_{\mathrm{Lip}}\|\zeta_i - \mathsf{z}(s_i)\|_{L^\infty}\Big)\big(\mathsf{z}(s_1)-\mathsf{z}(s_2)\big)\mathsf{e}(s_i):\mathsf{e}(s_i)\\ &\quad\le (1+K_{\mathbf{C}}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty})\mathbf{C}'(\mathsf{z}(s_i))\big(\mathsf{z}(s_1)-\mathsf{z}(s_2)\big)\mathsf{e}(s_i):\mathsf{e}(s_i) \qquad \text{a.e. in } \Omega\,,\end{aligned}$$

where we have exploited the Lipschitz continuity of $\mathbf{C}'$ and relied on the positivity of $\big(\mathsf{z}(s_1)-\mathsf{z}(s_2)\big)$ a.e. in Ω by the unidirectionality constraint. All in all, we obtain

$$\begin{aligned}\text{r.h.s. of (5.10)} &= \sum_{i=1}^{2}\frac{1}{4}\langle \big[\mathbf{C}(\mathsf{z}(s_1)) - \mathbf{C}(\mathsf{z}(s_2))\big]\mathsf{e}(s_i), \mathsf{e}(s_i)\rangle_{L^2}\\ &\le (1+K_{\mathbf{C}}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty})\sum_{i=1}^{2} I_i\,,\\ &\quad \text{with } I_i = \int_\Omega \frac{1}{4}\mathbf{C}'(\mathsf{z}(s_i))\big(\mathsf{z}(s_1)-\mathsf{z}(s_2)\big)\mathsf{e}(s_i):\mathsf{e}(s_i)\,\mathrm{d}x\,.\end{aligned} \tag{5.11}$$

In order to estimate from above the terms I_i, we resort to the representation formula (3.17b) for $\mathbb{D}_{\mathsf{z}}(,\mathsf{t}(s_i))\mathsf{q}(s_i)$, $i=1,2$, which gives (if $\mathsf{z}(s_1) \ne \mathsf{z}(s_2)$)

$$\begin{aligned}\mathbb{D}_{\mathsf{z}}(,\mathsf{t}(s_i))\mathsf{q}(s_i) \ge{}& \frac{\langle -A_{\mathrm{m}}\mathsf{z}(s_i) - W'(\mathsf{z}(s_i)) - \frac{1}{2}\mathbf{C}'(\mathsf{z}(s_i))\mathsf{e}(s_i):\mathsf{e}(s_i), \mathsf{z}(s_2)-\mathsf{z}(s_1)\rangle_{H^{\mathrm{m}}}}{\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}}\\ &- \frac{\mathcal{R}(\mathsf{z}(s_2)-\mathsf{z}(s_1))}{\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}}\,.\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
&\frac{1}{4}\int_\Omega \mathbf{C}'(\mathsf{z}(s_1))\mathsf{e}(s_1){:}\mathsf{e}(s_1)(\mathsf{z}(s_1)-\mathsf{z}(s_2))\,\mathrm{d}x+\frac{1}{4}\int_\Omega \mathbf{C}'(\mathsf{z}(s_2))\mathsf{e}(s_2){:}\mathsf{e}(s_2)(\mathsf{z}(s_1)-\mathsf{z}(s_2))\,\mathrm{d}x\\
&\quad\le \frac{1}{2}\left(\mathbb{D}_\mathsf{z}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_\mathsf{z}(\mathsf{t}(s_2),\mathsf{q}(s_2))\right)\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}\\
&\qquad+\frac{1}{2}a_\mathrm{m}(\mathsf{z}(s_1)+\mathsf{z}(s_2),\mathsf{z}(s_2)-\mathsf{z}(s_1))\\
&\qquad+\frac{1}{2}\int_\Omega \left\{W'(\mathsf{z}(s_1))+W'(\mathsf{z}(s_2))\right\}(\mathsf{z}(s_2)-\mathsf{z}(s_1))\,\mathrm{d}x+\mathcal{R}(\mathsf{z}(s_2)-\mathsf{z}(s_1))\,.
\end{aligned}
$$

Applying the Lagrange theorem we find a function $\tilde{\zeta}_{1,2}:\Omega\to[0,1]$ (again assumed to be measurable), with $\mathsf{z}(s_2)\le\tilde{\zeta}_{1,2}\le\mathsf{z}(s_1)$ a.e. in Ω, such that $W(\mathsf{z}(s_2))-W(\mathsf{z}(s_1))=W'(\tilde{\zeta}_{1,2})(\mathsf{z}(s_1)-\mathsf{z}(s_2))$ a.e. in Ω. Thus,

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega [W'(\mathsf{z}(s_1))+W'(\mathsf{z}(s_2))](\mathsf{z}(s_2)-\mathsf{z}(s_1))\\
&\quad=\int_\Omega [W(\mathsf{z}(s_2))-W(\mathsf{z}(s_1))]\,\mathrm{d}x\\
&\qquad+\frac{1}{2}\int_\Omega \{W'(\mathsf{z}(s_1))+W'(\mathsf{z}(s_2)-2W'(\tilde{\zeta}_{1,2})\}(\mathsf{z}(s_2)-\mathsf{z}(s_1))\,\mathrm{d}x\\
&\quad\le\int_\Omega [W(\mathsf{z}(s_2))-W(\mathsf{z}(s_1))]\,\mathrm{d}x+\|W''\|_{L^\infty(0,1)}\int_\Omega(\mathsf{z}(s_1)-\mathsf{z}(s_2))^2\,\mathrm{d}x\\
&\quad\le\int_\Omega [W(\mathsf{z}(s_2))-W(\mathsf{z}(s_1))]\,\mathrm{d}x+K_W\|\mathsf{z}(s_1)-\mathsf{z}(s_2)\|_{L^1}\|\mathsf{z}(s_1)-\mathsf{z}(s_2)\|_{L^\infty}.
\end{aligned}
$$

Ultimately, we conclude that

$$
\begin{aligned}
I_1+I_2&=\frac{1}{4}\int_\Omega \mathbf{C}'(\mathsf{z}(s_1))\mathsf{e}(s_1):\mathsf{e}(s_1)(\mathsf{z}(s_1)-\mathsf{z}(s_2))\,\mathrm{d}x\\
&\quad+\frac{1}{4}\int_\Omega \mathbf{C}'(\mathsf{z}(s_2))\mathsf{e}(s_2):\mathsf{e}(s_2)(\mathsf{z}(s_1)-\mathsf{z}(s_2))\,\mathrm{d}x\\
&\le\frac{1}{2}a_\mathrm{m}(\mathsf{z}(s_2),\mathsf{z}(s_2))+\int_\Omega W(\mathsf{z}(s_2))\,\mathrm{d}x-\frac{1}{2}a_\mathrm{m}(\mathsf{z}(s_1),\mathsf{z}(s_1))-\int_\Omega W(\mathsf{z}(s_1))\,\mathrm{d}x\\
&\quad+\mathcal{R}(\mathsf{z}(s_2)-\mathsf{z}(s_1))+\frac{1}{2}\left(\mathbb{D}_\mathsf{z}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_\mathsf{z}(\mathsf{t}(s_2),\mathsf{q}(s_2))\right)\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^2}\\
&\quad+K_W\|\mathsf{z}(s_1)-\mathsf{z}(s_2)\|_{L^1}\|\mathsf{z}(s_1)-\mathsf{z}(s_2)\|_{L^\infty}\\
&=\mathrm{ARV}((\mathsf{t},\mathsf{q});[s_1,s_2])\,.
\end{aligned}
$$

Since $I_1+I_2\ge0$, we deduce in particular that $\mathrm{ARV}((\mathsf{t},\mathsf{q});[s_1,s_2])\ge0$.

Combining the previous inequality with (5.9), (5.10), and (5.11), we conclude (5.8).

Claim 2: *For every* $s_1\le s_2\in[0,S]$ *satisfying* (5.4) *there holds*

$$
\left\langle\frac{1}{2}(\sigma_\mathrm{D}(s_1)+\sigma_\mathrm{D}(s_2))|\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle\le\mathrm{HV}((\mathsf{t},\mathsf{q});[s_1,s_2])\,. \tag{5.12}
$$

Indeed, if s_1 and s_2 are in B°, then $\mathbb{D}_\mathsf{p}(\mathsf{t}(s_i),\mathsf{q}(s_i))=0$ for $i=1,2$, and hence $\sigma_\mathrm{D}(s_i)\in\mathcal{K}_{\mathsf{z}(s_i)}(\Omega)$, and, a fortiori, by (5.2) we have $\sigma(s_i)\in\Sigma\mathrm{K}_{\mathsf{z}(s_i)}(\Omega)$ (recall notation (3.12)). Therefore, by (3.13) we conclude

$$
\frac{1}{2}\langle\sigma_\mathrm{D}(s_i)|\mathsf{p}(s_2)-\mathsf{p}(s_1)\rangle\le\frac{1}{2}\mathcal{H}_\mathrm{PP}(\mathsf{z}(s_i),\mathsf{p}(s_2)-\mathsf{p}(s_1))\qquad\text{for } i=1,2,
$$

and hence

$$\left\langle \frac{1}{2}(\sigma_{\mathrm{D}}(s_1)+\sigma_{\mathrm{D}}(s_2))|\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle \le \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_2),\frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right)+\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_1),\frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right),$$

which is indeed (5.12), since $\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_i),\mathsf{q}(s_i))=0$ for $i=1,2$.

Otherwise, if $[s_1,s_2]\subset A^\circ$, then $\mathsf{p}'|_{[s_1,s_2]}$ takes values in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and thus $\mathsf{p}(s_2)-\mathsf{p}(s_1)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, so that

$$\left\langle \frac{1}{2}(\sigma_{\mathrm{D}}(s_1)+\sigma_{\mathrm{D}}(s_2)|\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle = \left\langle \frac{1}{2}(\sigma_{\mathrm{D}}(s_1)+\sigma_{\mathrm{D}}(s_2),\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle_{L^2(\Omega)}.$$

Now it follows by the representation formula (3.17a) for $\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_i),\mathsf{q}(s_i))$ that

$$\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_i),\mathsf{q}(s_i)) \ge \frac{1}{\|\mathsf{p}(s_2)-\mathsf{p}(s_1)\|_{L^2}}\Big[\langle\sigma_{\mathrm{D}}(s_i),\mathsf{p}(s_2)-\mathsf{p}(s_1)\rangle-\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s_i),\mathsf{p}(s_2)-\mathsf{p}(s_1))\Big]$$

for $i=1,2$. Therefore,

$$\begin{aligned}\left\langle \frac{1}{2}(\sigma_{\mathrm{D}}(s_1)+\sigma_{\mathrm{D}}(s_2)|\mathsf{p}(s_2)-\mathsf{p}(s_1)\right\rangle &\le \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_2),\frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right)+\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_1),\frac{\mathsf{p}(s_2)-\mathsf{p}(s_1)}{2}\right)\\ &\quad+\|\mathsf{p}(s_2)-\mathsf{p}(s_1)\|_{L^2}\frac{\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_1),\mathsf{q}(s_1))+\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s_2),\mathsf{q}(s_2))}{2},\end{aligned}$$

which is, again, (5.12).

Conclusion of the proof. We have

$$\begin{aligned}\mathrm{WE}((\mathsf{t},\mathsf{q});[s_1,s_2]) &\le \mathcal{Q}(\mathsf{z}(s_2),\mathsf{e}(s_2))-\mathcal{Q}(\mathsf{z}(s_1),\mathsf{e}(s_1))+\mathrm{HV}(\mathsf{q};[s_1,s_2])\\ &\quad+(1+K_{\mathbf{C}}\|\mathsf{z}(s_2)-\mathsf{z}(s_1)\|_{L^\infty})\,\mathrm{ARV}(\mathsf{q};[s_1,s_2]).\end{aligned} \tag{5.13}$$

It suffices to combine (5.7), (5.12), and (5.8). Then (5.5) follows by suitably rearranging some terms in (5.13). This finishes the proof. ∎

5.2. Outline of the proof of the lower energy-dissipation inequality. For given $k\in\mathbb{N}$ let us consider a partition $\mathscr{P}_k=(s_k^i)_{i=0}^{N_k}$ of the interval $[0,S]$:

$$0=s_k^0<s_k^1<\ldots<s_k^{N_{k-1}}<s_k^{N_k}=S, \quad \text{with} \quad \max_{1\le i\le N_k}(s_k^i-s_k^{i-1})\to 0. \tag{5.14}$$

Now we can choose $\mathscr{P}_k$ in such a way that

$$\mathcal{D}^*(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i))<+\infty \qquad \text{for all } i\in\{0,\ldots,N_k\}. \tag{5.15}$$

To check this claim, we observe that from (4.5) we get $\mathcal{M}_0^{\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))<+\infty$ for a.a. $s\in(0,S)$. Hence, taking into account the definition of $\mathcal{M}_0^{\mathrm{red}}$, we have that

$$\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\equiv 0 \qquad \text{for a.a. } s\in(0,S)\cap\{s\in(0,S):\ \mathsf{t}'(s)>0\}.$$

Moreover, if U is an open subset of $\tilde{F} := \{s \in [0,S] : \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) = +\infty\}$, we claim that $\mathsf{t}' = 0$ and $\mathsf{q}' = 0$ for a.a. $s \in U$, so $U = \emptyset$ by our previous assumption (5.1). Indeed, since $\mathcal{M}_0^{\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) < +\infty$ for a.a. $s \in (0,S)$, it holds that $\mathsf{t}' = \mathsf{z}' = \mathsf{p}' = 0$ for a.a. $s \in U$. By standard properties of perfect plasticity (see, e.g., [DMDM06, Theorem 3.8]) it then follows that also $\mathsf{u}' = 0$, and hence $\mathsf{q}' = 0$ for a.a. $s \in U$. Therefore, $[0,S] \setminus \tilde{F}$ has empty interior, and (5.15) ensues.

For later convenience, it is important to distinguish between the indices corresponding to consecutive partition times in the *stable set* B° from (3.22), namely

$$I_k := \{i \in \{1,\dots,N_k\} : s_k^{i-1},\, s_k^i \in B^\circ\}\,, \tag{5.16a}$$

and indices such that at least one of the corresponding consecutive partition times belongs to the instability set $A^\circ = [0,S] \setminus B^\circ$, i.e.,

$$J_k := \{1,\dots,N_k\} \setminus I_k = \{i \in \{1,\dots,N_k\} : s_k^{i-1} \in A^\circ \ \text{or} \ \ s_k^i \in A^\circ\}\,. \tag{5.16b}$$

Distinguished subintervals of the partition. Arguing as in [DDS11, Lemmas 7.7, 7.8, 8.5] and [BFM12, section 4.6], we observe that for any $i \in J_k$ we either have $(s_k^{i-1}, s_k^i) \subset A^\circ$, or $(s_k^{i-1}, s_k^i) \cap B^\circ \neq \emptyset$. We then distinguish two subfamilies of indices in J_k:

$$\hat{J}_k := \{i \in J_k :\ (s_k^{i-1}, s_k^i) \subset A^\circ\}, \qquad \check{J}_k := \{i \in J_k :\ (s_k^{i-1}, s_k^i) \cap B^\circ \neq \emptyset\}\,. \tag{5.17}$$

Next, for $i \in \check{J}_k$, let us set

$$s_k^{i-\frac{2}{3}} := \inf\{s \in B^\circ \cap (s_k^{i-1}, s_k^i)\}\,, \qquad s_k^{i-\frac{1}{3}} := \sup\{s \in B^\circ \cap (s_k^{i-1}, s_k^i)\}\,. \tag{5.18}$$

Since B° is closed, we have $s_k^{i-\frac{2}{3}}, s_k^{i-\frac{1}{3}} \in B^\circ$. Thus, we have obtained the decomposition

$$\text{for all } i \in \check{J}_k: \qquad (s_k^{i-1}, s_k^i) = (s_k^{i-1}, s_k^{i-\frac{2}{3}}) \cup [s_k^{i-\frac{2}{3}}, s_k^{i-\frac{1}{3}}] \cup (s_k^{i-\frac{1}{3}}, s_k^i)$$
$$\text{with } \begin{cases} (s_k^{i-1}, s_k^{i-\frac{2}{3}}) \subset A^\circ, & s_k^{i-\frac{2}{3}} \in B^\circ, \\ (s_k^{i-\frac{1}{3}}, s_k^i) \subset A^\circ & s_k^{i-\frac{1}{3}} \in B^\circ. \end{cases}$$

(Clearly, we have $s_k^{i-\frac{2}{3}} = s_k^{i-1}$ if $s_k^{i-1} \in B^\circ$; if, otherwise, $s_k^i \in B^\circ$, we have $s_k^{i-\frac{1}{3}} = s_k^i$.)

We now consider a refined partition consisting of $\mathscr{P}_k$ and of the nodes from (5.18). With slight abuse of notation, let us call again $\mathscr{P}_k = (s_k^i)_{i=0}^{N_k}$ the resulting partition. All in all, it is meaningful to distinguish three sets of indices:

(1) the set of indices corresponding to consecutive partition times in the stable set (including the nodes originally associated with indices in I_k, as well as the nodes from (5.18)),

$$\mathfrak{I}_k^1 := \{i \in \{1,\dots,N_k\} : s_k^{i-1},\, s_k^i \in B^\circ\}; \tag{5.19a}$$

(2) the set of indices corresponding to consecutive partition times in the instability set

$$\mathfrak{I}_k^2 := \{i \in \{1,\dots,N_k\} : s_k^{i-1},\, s_k^i \in A^\circ\}; \tag{5.19b}$$

(3) the set of all the other indices

$$\mathfrak{I}_k^3 := \{1,\dots,N_k\} \setminus \{\mathfrak{I}_k^1 \cup \mathfrak{I}_k^2\}\,. \tag{5.19c}$$

Observe that, by the previous discussion, we may suppose that for $i \in \mathfrak{I}_k^2$ the enclosed interval is "fully unstable," i.e., $[s_k^{i-1}, s_k^i] \subset A^\circ$; similarly, for $i \in \mathfrak{I}_k^3$ we have at least $(s_k^{i-1}, s_k^i) \subset A^\circ$. Let us mention in advance that the distinction between indices in $\mathfrak{I}_k^2$ and indices in $\mathfrak{I}_k^3$ is motivated by the fact that the cornerstone estimate (5.5) only holds under conditions (5.4). Therefore, it will be *directly* applicable either on intervals $[s_k^{i-1}, s_k^i]$ with $i \in \mathfrak{I}_k^1$, or on intervals $[s_k^{i-1}, s_k^i]$ with $i \in \mathfrak{I}_k^2$. For the intervals $[s_k^{i-1}, s_k^i]$ with $i \in \mathfrak{I}_k^3$ we will need to argue by approximation.

Outline of the proof of the lower inequality (3.31). We are now in a position to specify the steps in our proof of (3.31). Namely, we have the following:

[Step 1]: First, in Lemma 5.3 ahead we shall deduce from estimate (5.5) a discrete version of the lower energy-dissipation inequality on intervals whose endpoints are in $\mathfrak{I}_k^1$; cf. (5.20) below.

[Step 2]: Second, in section 5.4 we will proceed to handle the fully unstable intervals having endpoints s_k^{i-1}, s_k^i with $i \in \mathfrak{I}_k^2$, so that $[s_k^{i-1}, s_k^i] \subset A^\circ$. In that case we will derive from estimate (5.5) a discrete version of the lower energy-dissipation inequality (cf. Lemma 5.6) by resorting to suitable subpartitions.

[Step 3]: In section 5.5 we will address intervals $[s_k^{i-1}, s_k^i]$ with $i \in \mathfrak{I}_k^3$, for which in principle we only have the inclusion $(s_k^{i-1}, s_k^i) \subset A^\circ$. In this case, a discrete version of the lower energy-dissipation inequality will be proved in Lemma 5.8 by combining subpartitions with a suitable approximation argument.

[Step 4]: Eventually, in section 6 we will combine the inequalities from Lemmas 5.3, 5.6, and 5.8 to conclude an overall discrete energy-dissipation inequality for the partition $\mathscr{P}_k = (s_k^j)_{j=0}^{N_k}$. Therein, we will pass to the limit as $k \to \infty$ and finally prove the lower energy-dissipation inequality (3.31).

Remark 5.2. The present approach for the lower energy-dissipation inequality has been inspired by the corresponding analysis in [BFM12], which, in turn, borrowed ideas from [DDS11]. However, there are some remarkable differences. The first sequence of partitions in [BFM12], corresponding to that in (5.14), besides having vanishing fineness has to satisfy further conditions. Moreover, for intervals corresponding to indices in $\mathfrak{I}_k^2$ and $\mathfrak{I}_k^3$, the argument in [BFM12] exploits the fact that a chain rule is available in A°, so the desired estimates are directly obtained by an integration in time. This is essentially due to the fact that in the system from [BFM12] plasticity is not coupled with damage. Thus, the further regularity for the plastic strain in A° (absolutely continuous with values in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$) implies that all the variables are absolutely continuous, in A°, in their target spaces.

Now the presence of damage, which is absolutely continuous with values in $L^1(\Omega)$ (while its target space is $H^{\mathrm{m}}(\Omega)$), prevents us from proving absolute continuity for the whole evolution. Notice also that the enhanced estimate for the damage variable obtained for the viscous approximations as in [CR21, Proposition 4.4] cannot be repeated along jump intervals.

That is why we need to resort to a discrete approximation even in the "unstable set." This refinement of the analysis, in turn, allows us to consider a generic choice of the initial sequence of partitions as in (5.14).

5.3. Step 1. As an immediate corollary of Proposition 5.1 we have the following result.

LEMMA 5.3. *For every $i \in \mathfrak{I}_k^1$ we have*

$$\begin{aligned}&\mathrm{WE}((\mathsf{t},\mathsf{q});[s_k^{i-1},s_k^i]) + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\\ &\quad\le \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_k^{i-1}),\frac{\mathsf{p}(s_k^i)-\mathsf{p}(s_k^{i-1})}{2}\right) + \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(s_k^{i}),\frac{\mathsf{p}(s_k^i)-\mathsf{p}(s_k^{i-1})}{2}\right)\\ &\qquad + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i)) + \mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})) + \mathrm{Rem}_1([s_k^{i-1},s_k^i]),\end{aligned} \tag{5.20}$$

where the remainder term is given by

$$\mathrm{Rem}_1([s_k^{i-1},s_k^i]) = \Delta_1(s_k^{i-1},s_k^i)\,\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^\infty} \tag{5.21}$$

and we have used the place-holder

$$\begin{aligned}\Delta_1(s_k^{i-1},s_k^i) := {}& K_{\mathbf{C}}\left(\Phi(\mathsf{z}(s_k^i)) - \Phi(\mathsf{z}(s_k^{i-1})) + \mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\right)\\ &+ K_W\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^1}\left(1+K_{\mathbf{C}}\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^\infty}\right).\end{aligned} \tag{5.22}$$

Proof. This follows from Proposition 5.1 with the choices $s_1 = s_k^{i-1}$, $s_2 = s_k^i$. ∎

5.4. Step 2. With the main result of this section, Lemma 5.6, we obtain an energy-dissipation inequality for *fully unstable* intervals by resorting to suitable subpartitions. The usage of subpartitions relies on the following result, whose proof is postponed to Appendix A.1.

LEMMA 5.4. *Let $\psi : [a,b] \to (0,+\infty]$ be a lower semicontinuous function such that $\psi(a)$, $\psi(b) \in \mathbb{R}$. Then for every $\eta > 0$ there exists a partition $(r_\eta^i)_{i=0}^{N_\eta}$ of $[a,b]$ $a = r_\eta^0 < r_\eta^1 < \ldots < r_\eta^{N_\eta-1} < r_\eta^{N_\eta} = b$ such that*

$$\psi(s) \ge \frac{1}{2}\Big(\psi(r_\eta^{j-1}) + \psi(r_\eta^j)\Big) - \eta \quad \text{for all } s \in (r_\eta^{j-1}, r_\eta^j) \quad \text{and all } j \in \{1,\ldots,N_\eta\}\,. \tag{5.23}$$

Based on this result, we obtain the following estimate on intervals contained in the instability set A°.

LEMMA 5.5. *Let $[s_\#, s^\#] \subset A^\circ$ such that $\mathcal{D}^*(s_\#) \in \mathbb{R}$ and $\mathcal{D}^*(s^\#) \in \mathbb{R}$. Then for every $\eta > 0$ there exists a partition $(r_\eta^i)_{i=0}^{N_\eta}$ of $[s_\#, s^\#]$ such that*

$$\begin{aligned}&\sum_{j=1}^{N_\eta}\Bigg\{\|\mathsf{z}(r_\eta^j)-\mathsf{z}(r_\eta^{j-1})\|_{L^2}\frac{\mathbb{D}_{\mathsf{z}}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \mathbb{D}_{\mathsf{z}}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))}{2}\\ &\qquad + \|\mathsf{p}(r_\eta^j)-\mathsf{p}(r_\eta^{j-1})\|_{L^2}\frac{\mathbb{D}_{\mathsf{p}}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \mathbb{D}_{\mathsf{p}}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))}{2}\Bigg\}\\ &\quad\le \int_{s_\#}^{s^\#}\mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\,\mathrm{d}r\\ &\qquad + \eta\,\frac{1}{\min_{r\in[s_\#,s^\#]}\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))}\int_{s_\#}^{s^\#}\mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\,\mathrm{d}r\,.\end{aligned} \tag{5.24}$$

Proof. We apply Lemma 5.4 choosing $[a,b] = [s_\#, s^\#]$ and $\psi = \mathcal{D}^*(\mathsf{t}(\cdot),\mathsf{q}(\cdot))|_{[s_\#,s^\#]}$, and thus for every $\eta > 0$ we obtain a partition $(r_\eta^j)_{j=0}^{N_\eta}$ of $[s_\#, s^\#]$ such that for all $j \in \{1,\ldots,N_\eta\}$

(5.25)

$$\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r)) \geq \frac{1}{2}\Big(\mathcal{D}^*(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \mathcal{D}^*(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))\Big) - \eta \quad \text{for all } r \in (r_\eta^{j-1}, r_\eta^j)\,.$$

Now let us use the place-holders

$$\mathsf{Z} := \|\mathsf{z}(r_\eta^j) - \mathsf{z}(r_\eta^{j-1})\|_{L^2}\,, \quad \mathsf{M_Z} := \frac{\mathbb{D}_\mathsf{z}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \mathbb{D}_\mathsf{z}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))}{2}\,,$$
$$\mathsf{P} := \|\mathsf{p}(r_\eta^j) - \mathsf{p}(r_\eta^{j-1})\|_{L^2}\,, \quad \mathsf{M_P} := \frac{\mathbb{D}_\mathsf{p}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \mathbb{D}_\mathsf{p}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))}{2}\,.$$

For each $j \in \{1,\dots,N_\eta\}$ we estimate $\mathsf{Z}\cdot\mathsf{M_Z} + \mathsf{P}\cdot\mathsf{M_P} \leq \|(\mathsf{Z},\mathsf{P})\|\cdot\|(\mathsf{M_Z},\mathsf{M_P})\|$ by the Cauchy–Schwarz inequality. Now

$$\begin{aligned}\|(\mathsf{Z},\mathsf{P})\| &= \sqrt{\|\mathsf{z}(r_\eta^j)-\mathsf{z}(r_\eta^{j-1})\|_{L^2}^2 + \|\mathsf{p}(r_\eta^j)-\mathsf{p}(r_\eta^{j-1})\|_{L^2}^2} = \left\| \int_{r_\eta^{j-1}}^{r_\eta^j} (\mathsf{z}'(r),\mathsf{p}'(r))\,\mathrm{d}r \right\|_{L^2} \\ &\leq \int_{r_\eta^{j-1}}^{r_\eta^j} \|(\mathsf{z}'(r),\mathsf{p}'(r))\|_{L^2}\,\mathrm{d}r = \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathrm{d}r\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|(\mathsf{M_Z},\mathsf{M_P})\| &= \frac{1}{2}\big\|(\mathbb{D}_\mathsf{z}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})), \mathbb{D}_\mathsf{p}(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1}))) \\ &\qquad + (\mathbb{D}_\mathsf{z}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j})), \mathbb{D}_\mathsf{p}(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j})))\big\| \\ &\leq \frac{1}{2}\mathcal{D}^*(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \frac{1}{2}\mathcal{D}^*(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j}))\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathsf{Z}\cdot\mathsf{M_Z} + \mathsf{P}\cdot\mathsf{M_P} &\leq \int_{r_\eta^{j-1}}^{r_\eta^j} \left\{ \frac{1}{2}\mathcal{D}^*(\mathsf{t}(r_\eta^{j-1}),\mathsf{q}(r_\eta^{j-1})) + \frac{1}{2}\mathcal{D}^*(\mathsf{t}(r_\eta^{j}),\mathsf{q}(r_\eta^{j})) \right\} \mathcal{D}(\mathsf{q}'(r))\,\mathrm{d}r \\ &\leq \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\,\mathrm{d}r + \eta \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathrm{d}r\,,\end{aligned}$$

where the last inequality follows from (5.25). Finally, we observe that

$$\eta \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathrm{d}r \leq \eta \frac{1}{\min_{r\in[s_\#,s^\#]} \mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))} \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\,\mathrm{d}r\,,$$

where the latter estimate ensues from the upper inequality (3.30). We thus conclude estimate (5.24) upon adding over the index $j = 1,\dots,N_\eta$. ☐

Eventually, combining the key estimate (5.5) with Lemma 5.5 we obtain the counterpart to Lemma 5.3 for indices in $\mathfrak{I}_k^2$. We emphasize that now, in the right-hand side of (5.26), the "energy-dissipation" term $\int \mathcal{D}(\mathsf{q}')\,\mathcal{D}^*(\mathsf{t},\mathsf{q})\,\mathrm{d}s$ appears, whose presence records the fact that $[s_k^{i-1}, s_k^i]$ is an "unstable interval."

LEMMA 5.6. *For every* $i \in \mathfrak{I}_k^2$ *and for every* $\eta > 0$ *there exists a partition* $(r_{\eta,i}^j)_{j=0}^{M_{\eta,i}}$ *of the interval* $[s_k^{i-1}, s_k^i]$ *such that*

$$
\begin{aligned}
&\sum_{j=1}^{M_{\eta,i}} \mathrm{WE}((\mathsf{t},\mathsf{q});[r^i_{\eta,}j-1,r^i_{\eta,}j]) + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\\
&\quad\le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i)) + \mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} \mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\mathrm{d}s\\
&\qquad + \sum_{j=1}^{M_{\eta,i}} \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(r^i_{\eta,}j-1), \frac{\mathsf{p}(r^i_{\eta,}j)-\mathsf{p}(r^i_{\eta,}j-1)}{2}\right)\\
&\qquad + \sum_{j=1}^{M_{\eta,i}} \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(r^i_{\eta,}j), \frac{\mathsf{p}(r^i_{\eta,}j)-\mathsf{p}(r^i_{\eta,}j-1)}{2}\right)\\
&\qquad + \eta\,\frac{\mathrm{M}}{\min_{s\in[s_k^{i-1},s_k^i]}\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))} + \mathrm{Rem}_2([s_k^{i-1},s_k^i])
\end{aligned}
\tag{5.26}
$$

with the constant $\mathrm{M}>0$ *from estimates* (4.5)*, where the remainder term is now given by*

$$
\mathrm{Rem}_2([s_k^{i-1},s_k^i]) = \sum_{j=1}^{M_{\eta,i}} \Delta(r^i_{\eta,}j-1,r^i_{\eta,}j)\|\mathsf{z}(r^i_{\eta,}j)-\mathsf{z}(r^i_{\eta,}j-1)\|_{L^\infty}
\tag{5.27}
$$

with Δ *from* (5.22).

Proof. We apply Lemma 5.5 with $s_\# = s_k^{i-1}$ and $s^\# = s_k^i$ and obtain a partition $(r^j_{\eta,i})_{j=0}^{M_{\eta,i}}$ of the interval $[s_k^{i-1},s_k^i]$ for which (5.24) holds. Since $[s_k^{i-1},s_k^i]\subset A^\circ$, we have $[r^{j-1}_{\eta,i},r^j_{\eta,i}]\subset A^\circ$ for all $j\in\{1,\dots,M_{\eta},i\}$. Therefore, we may apply estimate (5.5) with $s_1 = r^{j-1}_{\eta,i}$ and $s_2 = r^j_{\eta,i}$. Summing (5.5) over the index j, combining it with (5.24), and observing that

$$
\begin{aligned}
\sum_{j=1}^{M_{\eta,i}} \int_{r_\eta^{j-1}}^{r_\eta^j} \mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\mathrm{d}r &= \int_{s_k^{i-1}}^{s_k^i} \mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\mathrm{d}r\\
&\le \int_{(0,S)\cap A^\circ} \mathcal{D}(\mathsf{q}'(r))\,\mathcal{D}^*(\mathsf{t}(r),\mathsf{q}(r))\mathrm{d}r \le \mathrm{M}
\end{aligned}
$$

by estimates (4.5), we arrive at (5.26). ∎

5.5. Step 3. Again, we aim to prove a discrete version of the energy-dissipation inequality on the interval $[s_k^{i-1},s_k^i]$. Since in this case we only have, in principle, that (s_k^{i-1},s_k^i) is a connected component of A° but, possibly, either $s_k^{i-1}\notin A^\circ$ or $s_k^i\notin A^\circ$, we need to devise an approximation argument in order to reproduce the situation of intervals contained in A° *with their closure.* We will rely on the following technical result: essentially, it ensures that if one of the endpoints of a connected component of A° does not belong to A°, it is in any case possible to approximate it "in energy" with points from within A°.

LEMMA 5.7. *Let* $(a,b)\subset A^\circ$ *be a connected component of* A°. *Then there exist sequences* (a_n), $(b_n)_n\subset A^\circ$ *with* $a\le a_n<b_n\le b$, *with* $a_n\downarrow a$ *decreasingly and* $b_n\uparrow b$ *increasingly as* $n\to\infty$, *such that*

$$
\begin{cases}
\sigma(a_n)\to\sigma(a) \ \textit{strongly in}\, L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}),\\
\Phi(\mathsf{z}(a_n))\to\Phi(\mathsf{z}(a)),\\
\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(a_n),\mathsf{q}(a_n))\to\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(a),\mathsf{q}(a));
\end{cases}
\tag{5.28a}
$$

$$
\begin{cases} \sigma(b_n) \to \sigma(b) \textit{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}), \\ \Phi(\mathsf{z}(b_n)) \to \Phi(\mathsf{z}(b)), \\ \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \to \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b),\mathsf{q}(b))\,. \end{cases} \tag{5.28b}
$$

Proof. To fix ideas, let us detail the construction of the sequence $(b_n)_n$ (analogous arguments give the construction of $(a_n)_n$). Clearly, if $b\in A^\circ$, then it is sufficient to take $b_n\equiv b$. Suppose thus that $b\notin A^\circ$, so that $\mathcal{D}^*(\mathsf{t}(b),\mathsf{q}(b))=0$.

We split the proof of (5.28b) in four claims; the first two settle the convergence for the stresses because, indeed, we will show that for *any* $(b_n)_n$ with $b_n\uparrow b$ there holds $\sigma(b_n)\to\sigma(b)$ *strongly* in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$.

Claim 1: *the properties*

$$
\lim_{s\uparrow b}\sigma(s)=\sigma(b) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \tag{5.29}
$$

and

$$
\lim_{s\uparrow b}\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s),\mathsf{q}(s))=0 \tag{5.30}
$$

are equivalent.

In order to show that (5.29) implies (5.30), we first of all observe that, since $\mathsf{z}:[0,S]\to H^{\mathrm{m}}(\Omega)$ is continuous w.r.t. the weak topology of $H^{\mathrm{m}}(\Omega)$ by (3.26), there holds $\mathsf{z}(s)\rightharpoonup\mathsf{z}(b)$ in $H^{\mathrm{m}}(\Omega)$, and thus $\mathsf{z}(s)\to\mathsf{z}(b)$ in $L^\infty(\Omega)$. By Remark 2.3 we have that

$$
\sup_{x\in\overline{\Omega}} d_{\mathscr{H}}(K(\mathsf{z}(s,x)),K(\mathsf{z}(b,x))) \longrightarrow 0 \text{ as } s\uparrow b\,. \tag{5.31}
$$

Therefore,

$$
\begin{aligned}
\mathbb{D}_{\mathsf{p}}(\mathsf{t}(s),\mathsf{q}(s)) &= \mathrm{dist}_{L^2(\Omega)}(-\sigma_{\mathrm{D}}(s),\mathcal{K}_{\mathsf{z}(s)}(\Omega)) \\
&\longrightarrow \mathrm{dist}_{L^2(\Omega)}(-\sigma_{\mathrm{D}}(b),\mathcal{K}_{\mathsf{z}(b)}(\Omega)) = \mathbb{D}_{\mathsf{p}}(\mathsf{t}(b),\mathsf{q}(b))=0
\end{aligned}
$$

as $s\uparrow b$ (recall that $\mathcal{D}^*(\mathsf{t}(b),\mathsf{q}(b))=0$).

Conversely, suppose (5.30) and let us fix an arbitrary sequence $(s_k)_k$ with $s_k\uparrow b$. Since $\mathsf{e}:[0,S]\to L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ is weakly continuous by (3.26), we have that $\mathsf{e}(s_k)\rightharpoonup\mathsf{e}(b)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$; we combine this with the fact that $\mathsf{z}(s_k)\rightharpoonup\mathsf{z}(b)$ in $H^{\mathrm{m}}(\Omega)$ and conclude that $\sigma(s_k)=\mathbf{C}(\mathsf{z}(s_k))\mathsf{e}(s_k)\rightharpoonup\mathbf{C}(\mathsf{z}(b))\mathsf{e}(b)=\sigma(b)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. We split the proof of the convergence

$$
\sigma(s_k)\to\sigma(b) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \tag{5.32}
$$

in three steps.

Step 1: We apply Proposition A.1 from Appendix A.2 with the choices $u_k:=\mathsf{u}(s_k)\subset \mathrm{BD}(\Omega)$, $u=\mathsf{u}(b)$, $z_k:=\mathsf{z}(s_k)\subset H^{\mathrm{m}}(\Omega)$, $z=\mathsf{z}(b)$, $p_k:=\mathsf{p}(s_k)\subset L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, and $p=\mathsf{p}(b)$: indeed, $\mathsf{u}:[0,S]\to\mathrm{BD}(\Omega)$, $\mathsf{z}:[0,S]\to H^{\mathrm{m}}(\Omega)$ are weakly* continuous (cf. again (3.26)), whereas $p:[0,S]\to\mathcal{M}_{\mathrm{b}}(\Omega\cup\partial_D\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ is 1-Lipschitz continuous. Therefore, we conclude that the sequence $(\nabla\mathsf{u}(s_k))_k$ is Cauchy w.r.t. convergence in measure. Then $(\mathrm{E}(\mathsf{u}(s_k)))_k$ is Cauchy w.r.t. convergence in measure. Now $(\mathsf{p}(s_k))_k\subset L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ is itself a Cauchy sequence, as $\|\mathsf{p}(s_k)-\mathsf{p}(s_h)\|_{L^1(\Omega)}=\|\mathsf{p}(s_k)-\mathsf{p}(s_h)\|_{\mathcal{M}_{\mathrm{b}}(\Omega)}$ for all $k,h\in\mathbb{N}$. Thus, the sequence $(\mathsf{e}(s_k))_k$ is Cauchy w.r.t. convergence in measure, and so is $\sigma(s_k)=\mathbf{C}(\mathsf{z}(s_k))\mathsf{e}(s_k)$.

Step 2: We consider the decomposition

$$\sigma_{\mathrm{D}}(s_k) = \pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)) + \sigma_{\mathrm{D}}(s_k) - \pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)), \tag{5.33}$$

where $\pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k))$ denotes the projection of $\sigma_{\mathrm{D}}(s_k)$ onto $\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)$. It follows from (5.30) that $\sigma_{\mathrm{D}}(s_k) - \pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)) \longrightarrow 0$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$. Taking into account that $(\sigma_{\mathrm{D}}(s_k))_k$ is a Cauchy sequence w.r.t. convergence in measure by Step 1, from (5.33) we conclude that $(\pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)))_k$ is a Cauchy sequence w.r.t. convergence in measure. Up to a not relabeled subsequence, $(\pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)))_k$ converges a.e. in Ω to some limit function $\overline{\pi}$. Using (5.31), it is not difficult to identify $\overline{\pi}$ as $\pi_{\mathcal{K}_{\mathsf{z}(b)}(\Omega)}(\sigma_{\mathrm{D}}(b))$. In turn, from (2.8) (cf. also Remark 2.3) we gather that $\pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)) \in B_{\overline{R}}(0)$ for all $k \in \mathbb{N}$, and hence the sequence $(\pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)))_k$ is bounded in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$. Therefore,

$$\begin{aligned} \pi_{\mathcal{K}_{\mathsf{z}(s_k)}(\Omega)}(\sigma_{\mathrm{D}}(s_k)) &\longrightarrow \pi_{\mathcal{K}_{\mathsf{z}(b)}(\Omega)}(\sigma_{\mathrm{D}}(b)) \\ &\overset{(*)}{=} \sigma_{\mathrm{D}}(b) \quad \text{in } L^p(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \text{ for all } 1 \le p < \infty\,, \end{aligned} \tag{5.34}$$

where (*) ensues from the fact that $\mathcal{D}^*(\mathsf{t}(b),\mathsf{q}(b)) = 0$. Combining (5.34) with (5.33), we ultimately conclude that

$$\sigma_{\mathrm{D}}(s_k) \to \sigma_{\mathrm{D}}(b) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}). \tag{5.35}$$

Step 3: Since $\mathrm{Div}(\sigma(s_k)) \to \mathrm{Div}(\sigma(b))$, by the weak convergence $\sigma(s_k) \rightharpoonup \sigma(b)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and by (5.35), we may apply Lemma A.5 ahead to deduce the strong convergence (5.32).

Claim 2: *Equations* (5.29) *and* (5.30) *hold true.*

In fact, assuming by contradiction that (5.30) is false (and then also (5.29) by Claim 1), we have in particular

$$\liminf_{s\uparrow b} \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0,$$

so there exist $c \in (a,b)$ and $\eta > 0$ such that $\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) \ge \eta$ for every $s \in [c,b)$. From estimate (4.5) we thus conclude that

$$\int_c^b \|\mathsf{p}'(s)\|_{L^2}\,\mathrm{d}s \le \int_c^b \sqrt{\|\mathsf{z}'(s)\|_{L^2}^2 + \|\mathsf{p}'(s)\|_{L^2}^2}\,\mathrm{d}s \le \frac{\mathrm{M}}{\eta}\,.$$

Then Lemma 4.4 gives

$$\|\mathsf{e}(s) - \mathsf{e}(b)\|_{L^2} \le C_L \left(\int_s^b \|\mathsf{p}'(s)\|_{L^2}\,\mathrm{d}s + \|\mathsf{z}(b) - \mathsf{z}(s)\|_{L^\infty} \right) \qquad \text{for all } s \ge c$$

with C_L from (4.6). Therefore, we have $\mathsf{e}(s) \to \mathsf{e}(b)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and ultimately $\sigma(s) \to \sigma(b)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. But (5.29) was false by assumption. This concludes the proof of Claim 2.

Claim 3: *There exists a sequence* $(b_n)_n$ *with* $b_n \uparrow b$ *such that*

$$\Phi(\mathsf{z}(b_n)) \to \Phi(\mathsf{z}(b)). \tag{5.36}$$

We distinguish two cases.
Case 1 $\liminf_{s\uparrow b} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s)) < +\infty$.
Case 2 $\liminf_{s\uparrow b} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s)) = +\infty$.

In **Case 1** we choose the sequence $(b_n)_n$, with $b_n \uparrow b$, such that there holds $\lim_{n\to\infty} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) = \liminf_{s\uparrow b} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$.

In **Case 2** we choose a sequence $(b_n)_n$ such that $\mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \le \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$ for all $s \in [b_n, b)$ and every $n \in \mathbb{N}$. Indeed, it is enough to define b_n as the minimizer of $s \mapsto \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$ on the interval $[\max\{b_{n-1}, b-\frac{1}{n}\}, b)$ that is the minimizer over $[\max\{b_{n-1}, b-\frac{1}{n}\}, b]$ of the (lower semicontinuous) function $\tilde{\mathbb{D}}_{\mathsf{z}}(,\mathsf{t}(s))\mathsf{q}(s)$ defined as $\mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$ for $s<b$ and $\tilde{\mathbb{D}}_{\mathsf{z}}(,\mathsf{t}(b))\mathsf{q}(b) := +\infty$.

To show (5.36), recalling Proposition 5.1 we observe that for every $n \in \mathbb{N}$ there holds

$$
\begin{aligned}
0 \le \mathrm{ARV}((\mathsf{t},\mathsf{q});[b_n,b]) := {} & \Phi(\mathsf{z}(b)) - \Phi(\mathsf{z}(b_n)) + \mathcal{R}(\mathsf{z}(b)-\mathsf{z}(b_n)) \\
& + \|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^2} \frac{\mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) + \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b),\mathsf{q}(b))}{2} \\
& + K_W \|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^1} \|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^\infty}\,.
\end{aligned} \tag{5.37}
$$

Now, since $\mathsf{z}:[0,S]\to H^{\mathrm{m}}(\Omega)$ is weakly continuous, taking into account that $H^{\mathrm{m}}(\Omega) \Subset \mathrm{C}^0(\overline{\Omega})$ we clearly have that $\mathcal{R}(\mathsf{z}(b)-\mathsf{z}(b_n)) \to 0$ as $n\to\infty$. Analogously, $K_W\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^1}\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^\infty} \to 0$. Let us show that

$$
\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^2} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \to 0. \tag{5.38}
$$

In **Case 1**, since $\lim_{n\to\infty} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) = \liminf_{s\uparrow b} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s)) < +\infty$, we have that

$$
\sup_n \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) < +\infty
$$

and (5.38) follows from $\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^2} \to 0$.

In **Case 2**, from the choice of b_n we have that for every n

$$
\mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \le \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s)) \text{ for } s \in [b_n, b)
$$

and thus

$$
\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^2} \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \le \int_{b_n}^{b} \|\dot{\mathsf{z}}(s)\|_2 \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s \longrightarrow 0,
$$

since $\|\dot{\mathsf{z}}(s)\|_2 \mathbb{D}_{\mathsf{z}}(\mathsf{t}(s),\mathsf{q}(s))$ is integrable in (a,b). We notice that $\mathsf{z}(b)-\mathsf{z}(b_n) = \int_{b_n}^{b} \dot{\mathsf{z}}(s)\,\mathrm{d}s$, where the right term is a Bochner integral in $L^2(\Omega)$ and it is well defined and finite by (4.5) and since $\liminf_{s\uparrow b} \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0$. Therefore, (5.38) is proven. Clearly, we also have

$$
\|\mathsf{z}(b)-\mathsf{z}(b_n)\|_{L^2}\, \mathbb{D}_{\mathsf{z}}(\mathsf{t}(b),\mathsf{q}(b)) \longrightarrow 0.
$$

All in all, from (5.37) we conclude that $\limsup_{n\to\infty} \Phi(\mathsf{z}(b_n)) \le \Phi(\mathsf{z}(b))$. Since, by the lower semicontinuity of Φ, we also have $\liminf_{n\to\infty} \Phi(\mathsf{z}(b_n)) \ge \Phi(\mathsf{z}(b))$, the desired convergence (5.36) follows.

Claim 4: *It holds*

$$
\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) \to \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b),\mathsf{q}(b))\,. \tag{5.39}
$$

Indeed,

$$\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b_n),\mathsf{q}(b_n)) = \mathcal{Q}(\mathsf{z}(b_n),\mathsf{e}(b_n)) + \Phi(\mathsf{z}(b_n))\,.$$

Therefore, it suffices to observe that

$$\lim_{n\to\infty}\mathcal{Q}(\mathsf{z}(b_n),\mathsf{e}(b_n)) = \lim_{n\to\infty}\frac{1}{2}\int_\Omega \sigma(b_n):\mathsf{e}(b_n)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \sigma(b):\mathsf{e}(b)\,\mathrm{d}x = \mathcal{Q}(\mathsf{z}(b),\mathsf{e}(b))\,,$$

since $\sigma(b_n)\to\sigma(b)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $\mathsf{e}(b_n)\rightharpoonup\mathsf{e}(b)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ by the weak continuity of e. Then (5.39) ensues. ∎

We thus arrive at the following result.

LEMMA 5.8. *For every $i\in\mathfrak{I}^3_k$ and for every $0<\eta,\beta\ll 1$, there exist points $s_k^{i-1}\le a^i_\beta < b^i_\beta \le s^i_k$ and a partition $(\rho^j_{\eta,i})_{j=0}^{L_{\eta,i}}$ of the interval $[a^i_\beta, b^i_\beta]$ such that*

$$a^i_\beta,\, b^i_\beta\in A^\circ,\qquad a^i_\beta - s_k^{i-1}\le\frac{\beta}{2}\,,\qquad s^i_k - b^i_\beta\le\frac{\beta}{2}\,, \tag{5.40a}$$

$$\begin{aligned}&\|\mathsf{w}(a^i_\beta)-\mathsf{w}(s_k^{i-1})\|_{H^1(\Omega)} + \|\mathsf{w}(b^i_\beta)-\mathsf{w}(s^i_k)\|_{H^1(\Omega)}\\ &\quad + \|\mathsf{p}(a^i_\beta)-\mathsf{p}(s_k^{i-1})\|_{\mathcal{M}_{\mathrm{b}}(\Omega)} + \|\mathsf{p}(b^i_\beta)-\mathsf{p}(s^i_k)\|_{\mathcal{M}_{\mathrm{b}}(\Omega)} < \beta,\end{aligned} \tag{5.40b}$$

and

$$\begin{aligned}&\sum_{j=1}^{L_{\eta,i}}\mathrm{WE}((\mathsf{t},\mathsf{q});[\rho^i_{\eta,}j-1,\rho^i_{\eta,}j]) + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\\
&\quad\le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s^i_k),\mathsf{q}(s^i_k)) + \mathcal{R}(\mathsf{z}(s^i_k)-\mathsf{z}(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i}\mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s\\
&\qquad + \sum_{j=1}^{L_{\eta,i}}\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\rho^i_{\eta,}j-1),\frac{\mathsf{p}(\rho^i_{\eta,}j)-\mathsf{p}(\rho^i_{\eta,}j-1)}{2}\right)\\
&\qquad + \sum_{j=1}^{L_{\eta,i}}\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\rho^i_{\eta,}j),\frac{\mathsf{p}(\rho^i_{\eta,}j)-\mathsf{p}(\rho^i_{\eta,}j-1)}{2}\right)\\
&\qquad + \eta\,\frac{\mathrm{M}}{\min_{s\in[a^i_\beta,b^i_\beta]}\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))} + \mathrm{Rem}_2([a^i_\beta,b^i_\beta]) + \beta\,,\end{aligned} \tag{5.41}$$

where $\mathrm{M}>0$ *is from* (4.5) *and* $\mathrm{Rem}_2([a^i_\beta,b^i_\beta])$ *is defined by the right-hand side of* (5.27).

Proof. In view of Lemma 5.7, (2.10), and (3.26), for every fixed $\beta>0$ we may pick $a^i_\beta,\, b^i_\beta\in A^\circ$ such that (5.40) hold, together with

$$\begin{aligned}\left|\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(a^i_\beta),\mathsf{q}(a^i_\beta)) - \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\right| &\le\frac{\beta}{4}\,,\\ \left|\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b^i_\beta),\mathsf{q}(b^i_\beta)) - \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s^i_k),\mathsf{q}(s^i_k))\right| &\le\frac{\beta}{4}\,.\end{aligned} \tag{5.42}$$

Furthermore, recall that $\mathsf{z}:[0,S]\to H^{\mathrm{m}}(\Omega)$ is weakly continuous, so that by the compact embedding $H^{\mathrm{m}}(\Omega)\Subset L^\infty(\Omega)$ we have that $\mathsf{z}\in\mathrm{C}^0([0,S];L^\infty(\Omega))$. Therefore, we may suppose that the quantities $\|\mathsf{z}(a^i_\beta)-\mathsf{z}(s_k^{i-1})\|_{L^\infty}$ and $\|\mathsf{z}(b^i_\beta)-\mathsf{z}(s^i_k)\|_{L^\infty}$ are so small as to ensure that

$$\mathcal{R}(\mathsf{z}(b^i_\beta)-\mathsf{z}(a^i_\beta))\le\mathcal{R}(\mathsf{z}(s^i_k)-\mathsf{z}(s_k^{i-1})) + \frac{\beta}{4}\,. \tag{5.43}$$

After these preparations, we proceed as in Lemma 5.6 and with the points a^i_β and b^i_β we associate a partition $(\rho^i_{\eta,}j)_{j=1}^{L_{\eta,}i}$ of the interval $[a^i_\beta, b^i_\beta]$ such that

$$
\begin{aligned}
&\sum_{j=1}^{L_{\eta,}i} \mathrm{WE}((\mathsf{t},\mathsf{q});[\rho^i_{\eta,}j-1,\rho^i_{\eta,}j]) + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(a^i_\beta),\mathsf{q}(a^i_\beta)) \\
&\quad \le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(b^i_\beta),\mathsf{q}(b^i_\beta)) + \mathcal{R}(\mathsf{z}(b^i_\beta)-\mathsf{z}(a^i_\beta)) + \int_{a^i_\beta}^{b^i_\beta} \mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s \\
&\qquad + \sum_{j=1}^{L_{\eta,}i} \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\rho^i_{\eta,}j-1), \frac{\mathsf{p}(\rho^i_{\eta,}j)-\mathsf{p}(\rho^i_{\eta,}j-1)}{2}\right) \\
&\qquad + \sum_{j=1}^{L_{\eta,}i} \mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\rho^i_{\eta,}j), \frac{\mathsf{p}(\rho^i_{\eta,}j)-\mathsf{p}(\rho^i_{\eta,}j-1)}{2}\right) \\
&\qquad + \eta\,\frac{\mathrm{M}}{\min_{s\in[a^i_\beta,b^i_\beta]}\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))} + \mathrm{Rem}_2([a^i_\beta,b^i_\beta])\,.
\end{aligned}
$$

Then estimate (5.41) follows by (5.42), (5.43) and by observing that

$$
\int_{a^i_\beta}^{b^i_\beta} \mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s \le \int_{s^{i-1}_k}^{s^i_k} \mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s\,. \qquad \square
$$

6. Proof of Theorem 3.8: The lower energy-dissipation inequality. In this section we eventually prove the following result.

PROPOSITION 6.1. *The pair* (t,q) *satisfies the lower energy-dissipation inequality* (3.31).

Proof. We start by adding up

(1) estimate (5.20) over all indices $i\in\mathfrak{I}^1_k$;

(2) estimate (5.26), with fixed $\eta>0$, over all indices $i\in\mathfrak{I}^2_k$;

(3) estimate (5.41), with fixed η, $\beta>0$, over all indices $i\in\mathfrak{I}^3_k$.

Eventually, we will add the resulting inequalities and obtain an overall discrete version of (3.31); cf. (6.1) below. In order to write it in a compact form, let us introduce as a place-holder

$$
\begin{aligned}
\mathbf{Work}_k[\mathsf{w}] = &\sum_{i\in\mathfrak{I}^1_k} \left\langle \frac12\left(\sigma(s^{i-1}_k)+\sigma(s^i_k)\right), \mathrm{E}(\mathsf{w}(s^i_k)-\mathsf{w}(s^{i-1}_k))\right\rangle_{L^2(\Omega)} \\
&+ \sum_{i\in\mathfrak{I}^2_k}\sum_{j=1}^{M_{\eta,}i} \left\langle \frac12\left(\sigma(r^i_{\eta,}j-1)+\sigma(r^i_{\eta,}j)\right), \mathrm{E}(\mathsf{w}(r^i_{\eta,}j)-\mathsf{w}(r^i_{\eta,}j-1))\right\rangle_{L^2(\Omega)} \\
&+ \sum_{i\in\mathfrak{I}^3_k}\sum_{j=1}^{L_{\eta,}i} \left\langle \frac12\left(\sigma(\rho^i_{\eta,}j-1)+\sigma(\rho^i_{\eta,}j)\right), \mathrm{E}(\mathsf{w}(\rho^i_{\eta,}j)-\mathsf{w}(\rho^i_{\eta,}j-1))\right\rangle_{L^2(\Omega)}.
\end{aligned}
$$

Recall that we are assuming that no volume or surface forces are present. Clearly, we have

$$
\begin{aligned}
&\sum_{i\in\mathfrak{I}^1_k} \mathrm{WE}((\mathsf{t},\mathsf{q});[s^{i-1}_k,s^i_k]) + \sum_{i\in\mathfrak{I}^2_k}\sum_{j=1}^{M_{\eta,}i} \mathrm{WE}((\mathsf{t},\mathsf{q});[r^i_{\eta,}j-1,r^i_{\eta,}j]) \\
&\quad + \sum_{i\in\mathfrak{I}^3_k}\sum_{j=1}^{L_{\eta,}i} \mathrm{WE}((\mathsf{t},\mathsf{q});[\rho^i_{\eta,}j-1,\rho^i_{\eta,}j]) = \mathbf{Work}_k[\mathsf{w}]\,.
\end{aligned}
$$

Analogously, we introduce a place-holder for the terms that approximate the integral $\int_0^S \mathcal{H}_{\rm PP}(\mathsf{z},\mathsf{p}')\,\mathrm{d}s$, namely

$$\begin{aligned}\mathbf{HV}_k(\mathsf{t},\mathsf{q};[0,S]) := &\sum_{i\in\mathfrak{I}_k^1}\mathcal{H}_{\rm PP}\left(\mathsf{z}(s_k^{i-1}),\frac{\mathsf{p}(s_k^i)-\mathsf{p}(s_k^{i-1})}{2}\right)+\mathcal{H}_{\rm PP}\left(\mathsf{z}(s_k^i),\frac{\mathsf{p}(s_k^i)-\mathsf{p}(s_k^{i-1})}{2}\right)\\ &+\sum_{i\in\mathfrak{I}_k^2}\sum_{j=1}^{M_\eta,i}\mathcal{H}_{\rm PP}\left(\mathsf{z}(r_{\eta,}^i j-1),\frac{\mathsf{p}(r_{\eta,}^i j)-\mathsf{p}(r_{\eta,}^i j-1)}{2}\right)\\ &+\mathcal{H}_{\rm PP}\left(\mathsf{z}(r_{\eta,}^i j),\frac{\mathsf{p}(r_{\eta,}^i j)-\mathsf{p}(r_{\eta,}^i j-1)}{2}\right)\\ &+\sum_{i\in\mathfrak{I}_k^3}\sum_{j=1}^{L_\eta,i}\mathcal{H}_{\rm PP}\left(\mathsf{z}(\rho_{\eta,}^i j-1),\frac{\mathsf{p}(\rho_{\eta,}^i j)-\mathsf{p}(\rho_{\eta,}^i j-1)}{2}\right)\\ &+\mathcal{H}_{\rm PP}\left(\mathsf{z}(\rho_{\eta,}^i j),\frac{\mathsf{p}(\rho_{\eta,}^i j)-\mathsf{p}(\rho_{\eta,}^i j-1)}{2}\right).\end{aligned}$$

We also consider the sum of the remainder terms

$$\mathbf{Rem}_k([0,S]) := \sum_{i\in\mathfrak{I}_k^1}\mathrm{Rem}_1([s_k^{i-1},s_k^i])+\sum_{i\in\mathfrak{I}_k^2}\mathrm{Rem}_2([s_k^{i-1},s_k^i])+\sum_{i\in\mathfrak{I}_k^3}\mathrm{Rem}_2([a_\beta^i,b_\beta^i])$$

with Rem_1 and Rem_2 from (5.21) and (5.27), respectively, and where $(a_\beta^i)_{i\in\mathfrak{I}_k^3}$ and $(b_\beta^i)_{i\in\mathfrak{I}_k^3}$ are the points associated with $(s_k^{i-1})_{i\in\mathfrak{I}_k^3}$ and $(s_k^i)_{i\in\mathfrak{I}_k^3}$, respectively, as in Lemma 5.8. Furthermore, we observe that

$$\begin{aligned}&\sum_{i\in\mathfrak{I}_k^1}\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))+\sum_{i\in\mathfrak{I}_k^2}\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))+\sum_{i\in\mathfrak{I}_k^3}\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\\ &\qquad=\int_0^S\mathcal{R}(\mathsf{z}'(s))\,\mathrm{d}s\end{aligned}$$

and, analogously,

$$\begin{aligned}&\sum_{i\in\mathfrak{I}_k^1}\{\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i))-\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\}+\sum_{i\in\mathfrak{I}_k^2}\{\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i))\\ &\qquad-\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\}+\sum_{i\in\mathfrak{I}_k^3}\{\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^i),\mathsf{q}(s_k^i))-\mathcal{E}_{\rm PP}(\mathsf{t}(s_k^{i-1}),\mathsf{q}(s_k^{i-1}))\}\\ &\quad=\mathcal{E}_{\rm PP}(\mathsf{t}(S),\mathsf{q}(S))-\mathcal{E}_{\rm PP}(\mathsf{t}(0),\mathsf{q}(0)),\end{aligned}$$

while

$$\begin{aligned}&\sum_{i\in\mathfrak{I}_k^2}\int_{s_k^{i-1}}^{s_k^i}\mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s+\sum_{i\in\mathfrak{I}_k^3}\int_{s_k^{i-1}}^{s_k^i}\mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s\\ &\qquad\le\int_{(0,S)\cap A^\circ}\mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s\,.\end{aligned}$$

Adding up (5.20), (5.26), and (5.41), we eventually obtain

$$\begin{aligned}
&\mathbf{Work}_k[\mathsf{w}] + \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(0),\mathsf{q}(0)) \\
&\quad \le \mathcal{E}_{\mathrm{PP}}(\mathsf{t}(S),\mathsf{q}(S)) + \mathbf{HV}_k(\mathsf{t},\mathsf{q};[0,S]) + \int_0^S \mathcal{R}(\mathsf{z}'(s))\,\mathrm{d}s \\
&\qquad + \int_{(0,S)\cap A^\circ} \mathcal{D}(\mathsf{q}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\,\mathrm{d}s \\
&\qquad + \mathbf{Rem}_k([0,S]) + \eta \mathrm{M} \sum_{i\in\mathfrak{I}^2_k} \frac{1}{\min_{s\in[s_k^{i-1},s_k^i]} \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))} \\
&\qquad + \eta \mathrm{M} \sum_{i\in\mathfrak{I}^3_k} \frac{1}{\min_{s\in[a^i_\beta,b^i_\beta]} \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))} + \beta\,\#(\mathfrak{I}^3_k)\,,
\end{aligned} \tag{6.1}$$

where the very last term on the right-hand side derives from adding up, for each index $i\in\mathfrak{I}^3_k$, the term β on the right-hand side of (5.41). Recalling (3.21), set

$$\widetilde{M} := \sup_{s\in[0,S]} \left(\|\mathsf{s}(s)\|_{L^2(\Omega)} + \|\mathsf{u}(s)+\mathsf{w}(s)\|_{\mathrm{BD}(\Omega)} + \|\mathsf{z}(s)\|_{L^\infty(\Omega)} \right). \tag{6.2}$$

Let us take the limit in (6.1) as $\eta\to 0$, $\beta\to 0$, and $k\to+\infty$ in this order. In fact, for any fixed $k\in\mathbb{N}$ it is possible to choose $\beta=\beta(k)$ and then $\eta=\eta(\beta,k)$ in such a way to make the last three terms in the right-hand side of (6.1) arbitrarily small. More precisely, for fixed $k\in\mathbb{N}$ we choose $0<\beta\ll 1$ in such a way as to make the last term on the right-hand side of (6.1) arbitrarily small; then we choose $0<\eta\ll 1$, depending on the intervals $\mathfrak{I}^2_k$, $\mathfrak{I}^3_k$, and on the previously found β so that the third-to-last term and the second-to-last term on the right-hand sides of (6.1) are arbitrarily small.

Moreover, in the following lines we will show that also the (discrete) terms depending on k in (6.1) may be made arbitrarily close to their continuous counterparts in the lower energy-dissipation inequality (3.31) as $k\to+\infty$ for suitable $\beta=\beta(k)$.

Let us introduce the partition of $[0,S]$

$$(\mathfrak{s}^j_k)_{j=0}^{L_k} := \bigcup_{i\in\mathfrak{I}^1_k} \{s^i_k\} \cup \bigcup_{i\in\mathfrak{I}^2_k} \bigcup_{j=1}^{M_{\eta,i}} \{r^j_{\eta,i}\} \cup \bigcup_{i\in\mathfrak{I}^3_k} \left(\bigcup_{j=1}^{L_{\eta,i}} \{\varrho^j_{\eta,i}\} \cup \{s^i_k\} \right).$$

In fact, the number of nodes of this partition also depends on η, but for shorter notation we do not highlight this dependence. Alternatively, one could formally consider a suitable vanishing sequence $(\eta_k)_k$, to avoid explicit dependence on η.

Thus, we may express

$$\begin{aligned}
\mathbf{Work}_k[\mathsf{w}] = &\sum_{j=1}^{L_k} \left\langle \frac12 \left(\sigma(\mathfrak{s}^{j-1}_k)+\sigma(\mathfrak{s}^j_k)\right), \mathrm{E}(\mathsf{w}(\mathfrak{s}^j_k)-\mathsf{w}(\mathfrak{s}^{j-1}_k)) \right\rangle_{L^2(\Omega)} \\
&- \frac12 \sum_{i\in\mathfrak{I}^3_k} \Big(\langle \sigma(a^i_\beta)+\sigma(s^{i-1}_k), \mathrm{E}(\mathsf{w}(s^{i-1}_k)-\mathsf{w}(a^i_\beta))\rangle_{L^2(\Omega)} \\
&+ \langle \sigma(b^i_\beta)+\sigma(s^i_k), \mathrm{E}(\mathsf{w}(s^i_k)-\mathsf{w}(b^i_\beta))\rangle_{L^2(\Omega)} \Big),
\end{aligned}$$

so that by (5.40b), for $\widetilde{M}$ from (6.2),

$$\left| \mathbf{Work}_k[\mathsf{w}] - \sum_{j=1}^{L_k} \left\langle \frac12 \left(\sigma(\mathfrak{s}^{j-1}_k)+\sigma(\mathfrak{s}^j_k)\right), \mathrm{E}(\mathsf{w}(\mathfrak{s}^j_k)-\mathsf{w}(\mathfrak{s}^{j-1}_k)) \right\rangle_{L^2(\Omega)} \right| \le \beta\,\widetilde{M}\,\#(\mathfrak{I}^3_k)\,.$$

Now it follows from Lemma A.6 ahead that

$$\lim_{k\to\infty}\sum_{j=1}^{L_k}\langle\frac{1}{2}(\sigma(\mathfrak{s}_k^{j-1})+\sigma(\mathfrak{s}_k^{j})),\mathrm{E}(\mathsf{w}(\mathfrak{s}_k^{j})-\mathsf{w}(\mathfrak{s}_k^{j-1}))\rangle_{L^2(\Omega)}=\int_0^S\langle\sigma_k(s),\mathrm{E}\mathsf{w}'(s)\rangle\mathrm{d}s\,,$$

and therefore we ultimately have

$$\lim_{k\to\infty}\lim_{\beta\to0}\mathbf{Work}_k[\mathsf{w}]=\int_0^S\langle\sigma_k(s),\mathrm{E}\mathsf{w}'(s)\rangle\mathrm{d}s\,.$$

The convergence

$$\lim_{k\to\infty}\lim_{\beta\to0}\mathbf{HV}_k(\mathsf{t},\mathsf{q};[0,S])=\int_0^S\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s))\mathrm{d}s$$

follows from [DDS11, Lemma 8.2], noticing that (similarly as before) we may obtain

$$\Bigg|\mathbf{HV}_k(\mathsf{t},\mathsf{q};[0,S])-\frac{1}{2}\sum_{j=1}^{L_k}\Big[\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\mathfrak{s}_k^{j-1}),\mathsf{p}(\mathfrak{s}_k^{j})-\mathsf{p}(\mathfrak{s}_k^{j-1})\right)$$
$$+\mathcal{H}_{\mathrm{PP}}\left(\mathsf{z}(\mathfrak{s}_k^{j}),\mathsf{p}(\mathfrak{s}_k^{j})-\mathsf{p}(\mathfrak{s}_k^{j-1})\right)\Big]\Bigg|\le\beta\,\widetilde{M}\,\#\mathfrak{I}_k^3\,.$$

Eventually, $\mathbf{Rem}_k([0,S])\to0$ by Lemma 6.2 ahead. All in all, we conclude the lower energy-dissipation inequality (3.31). ∎

We prove the following technical lemma employed in the proof of Proposition 6.1.

LEMMA 6.2. *There exists a constant $K_{\mathrm{M}}>0$, only depending on the toughness constant κ, on the constants K_W and $K_{\mathbf{C}}$ from* (5.3), *and on* $\mathrm{M}>0$ *from* (4.5), *such that*

$$\forall\,\delta\in(0,1]\ \exists\,\bar{k}\in\mathbb{N}\ \forall\,k\ge\bar{k}:\quad \mathbf{Rem}_k([0,S])\le\delta K_{\mathrm{M}}\,. \tag{6.3}$$

Proof. Recall that $\mathsf{z}\in\mathrm{C}^0([0,S];L^\infty(\Omega))$. Therefore, for every $\delta\in(0,1]$ there exists $\bar{k}\in\mathbb{N}$ such that, for $k\ge\bar{k}$, the fineness $\max_{i=1,\dots,N_k}(s_k^i-s_k^{i-1})$ of the partition is so small that

$$\sup_{i=1,\dots,N_k}\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^\infty(\Omega)}\le\delta\,.$$

Then

(6.4)
$$\sum_{i\in\mathfrak{I}_k^1}\mathrm{Rem}_1([s_k^{i-1},s_k^i])\le\delta\sum_{i\in\mathfrak{I}_k^1}\Big(K_{\mathbf{C}}\left\{\Phi(\mathsf{z}(s_k^i))-\Phi(\mathsf{z}(s_k^{i-1}))+\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\right\}$$
$$+K_W\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^1}\left\{1+K_{\mathbf{C}}\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^\infty}\right\}\Big)$$
$$\overset{(1)}{\le}\delta\left((2C_{\mathrm{M}}+\mathrm{M})K_{\mathbf{C}}+\frac{1}{\kappa}\mathrm{M}K_W(1+K_{\mathbf{C}}\delta)\right)\doteq\delta\widehat{K}_{\mathrm{M}}$$

with $\kappa>0$ the toughness constant from (1.2d). Now for (1) we have used that

$$\sum_{i\in\mathfrak{I}_k^1}\left(\Phi(\mathsf{z}(s_k^i))-\Phi(\mathsf{z}(s_k^{i-1}))\right)\le2\sup_{s\in[0,S]}\Phi(\mathsf{z}(s))\le2C_{\mathrm{M}}$$

for some positive constant C_{M} only depending on the constant $\mathrm{M}>0$ from (4.5) and on the problem data, since $\sup_{s\in[0,S]}|\mathcal{E}_{\mathrm{PP}}(\mathsf{t}(s),\mathsf{q}(s))|\leq \mathrm{M}$. We have also used that

$$\left.\begin{array}{r}\sum_{i\in\mathfrak{I}_k^1}\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\\ \kappa\sum_{i\in\mathfrak{I}_k^1}\|\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1})\|_{L^1}\end{array}\right\}\leq\int_0^S\mathcal{R}(\mathsf{z}'(s))\mathrm{d}s\leq\mathrm{M}\,.$$

By adapting the above calculations we estimate for each $i\in\mathfrak{I}_k^2$

$$\begin{aligned}\mathrm{Rem}_2([s_k^{i-1},s_k^i]) &= \sum_{j=1}^{M_\eta,i}\Delta(r_{\eta,}^i j-1,r_{\eta,}^i j)\|\mathsf{z}(r_{\eta,}^i j)-\mathsf{z}(r_{\eta,}^i j-1)\|_{L^\infty}\\ &\leq\delta\sum_{j=1}^{M_\eta,i}\Big(K_{\mathbf{C}}\left\{\Phi(\mathsf{z}(r_{\eta,}^i j))-\Phi(\mathsf{z}(r_{\eta,}^i j-1))+\mathcal{R}(r_{\eta,}^i j)-\mathsf{z}(r_{\eta,}^i j-1))\right\}\\ &\qquad+K_W\|\mathsf{z}(r_{\eta,}^i j)-\mathsf{z}(r_{\eta,}^i j-1)\|_{L^1}\left\{1+K_{\mathbf{C}}\delta\right\}\Big)\\ &\overset{(2)}{\leq}\delta\Bigg(\Phi(\mathsf{z}(s_k^i)-\Phi(\mathsf{z}(s_k^{i-1})+\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\\ &\qquad+K_W(1+K_{\mathbf{C}}\delta)\int_{s_k^{i-1}}^{s_k^i}\|\mathsf{z}'(s)\|_{L^1(\Omega)}\mathrm{d}s\Bigg),\end{aligned}$$

where for (2) we have used that the points $(r_{\eta,}^i j)_{j=1}^{M_\eta,i}$ provide a partition of the interval $[s_k^{i-1},s_k^i]$. Therefore,

$$\begin{aligned}\sum_{i\in\mathfrak{I}_k^2}\mathrm{Rem}_2([s_k^{i-1},s_k^i])\leq\delta\sum_{i\in\mathfrak{I}_k^2}&\Bigg(\Phi(\mathsf{z}(s_k^i)-\Phi(\mathsf{z}(s_k^{i-1})+\mathcal{R}(\mathsf{z}(s_k^i)-\mathsf{z}(s_k^{i-1}))\\ &+K_W(1+K_{\mathbf{C}}\delta)\int_{s_k^{i-1}}^{s_k^i}\|\mathsf{z}'(s)\|_{L^1(\Omega)}\mathrm{d}s\Bigg)\leq\delta\widehat{K}_{\mathrm{M}}\end{aligned}\tag{6.5}$$

by the very same arguments as in the proof of (6.4). In a completely analogous way we find that

$$\sum_{i\in\mathfrak{I}_k^3}\mathrm{Rem}_2([a_\eta^i,b_\eta^i])\leq\delta\widehat{K}_{\mathrm{M}}\,.\tag{6.6}$$

Eventually, adding (6.4), (6.5), and (6.6), we conclude (6.3) with $K_{\mathrm{M}}:=3\widehat{K}_{\mathrm{M}}$. ∎

Appendix A. We collect here some technical results employed in the paper. Most proofs are omitted here for brevity; all of them can be found in [CLRpre].

A.1. Proof of Lemma 5.4. We provide here the proof of Lemma 5.4.

Step 1. Let

$$m_1:=\min_{x\in[a,b]}\psi(x)\,.$$

Since ψ attains its minimum on $[a,b]$ and is finite in a and b, we have that $m_1\in(0,+\infty)$. Then the set

$$A_1:=\{x\in[a,b]:\psi(x)\leq m_1+\eta\}$$

is not empty. Since ψ is lower semicontinuous, A_1 is a compact subset of $\mathbb{R}$; let

$$a_1 := \min A_1\,; \qquad b_1 := \max A_1\,; \qquad J_1 := \{a_1,\, b_1\}\,. \tag{A.1}$$

Let us distinguish different cases.

Case 1.I: $a_1 = a$**,** $b_1 = b$**.** If $a_1 = a$, $b_1 = b$, we set $r^0_\eta := a_1 = a$ and $r^1_\eta := b_1 = b$. We have that

$$\psi(s) \geq \frac{1}{2}\Big(\psi(r^0_\eta) + \psi(r^1_\eta)\Big) - \eta \quad \text{for every } s \in (a, b)\,,$$

so that the partition given by $J_1 = (r^j_\eta)_{j=0}^1$ satisfies (5.23) and the proof is finished.

Case 1.II: Otherwise, three subcases may occur.

Case 1.IIa: $a < a_1 \leq b_1 = b$ (notice that possibly $a_1 = b_1$). We define the function $\widehat{\psi}_1^- : [a, a_1] \to (0, +\infty]$ as

$$\widehat{\psi}_1^-(x) := \begin{cases} \psi(x), & x \in [a, a_1)\,, \\ \liminf\limits_{r \to a_1^-} \psi(r), & x = a_1\,. \end{cases} \tag{A.2}$$

Case 1.IIb: $a = a_1 \leq b_1 < b$**.** We define $\widehat{\psi}_1^+ : [b_1, b] \to (0, +\infty]$ as

$$\widehat{\psi}_1^+(x) := \begin{cases} \psi(x), & x \in (b_1, b]\,, \\ \liminf\limits_{r \to b_1^+} \psi(r), & x = b_1\,. \end{cases} \tag{A.3}$$

Case 1.IIc: $a < a_1 \leq b_1 < b$**.** We define $\widehat{\psi}_1^- : [a, a_1] \to (0, +\infty]$, $\widehat{\psi}_1^+ : [b_1, b] \to (0, +\infty]$ as in (A.2) and (A.3), respectively.

In any case, if $a_1 < b_1$, we get

$$\psi(s) \geq \frac{1}{2}\Big(\psi(a_1) + \psi(b_1)\Big) - \eta \quad \text{for } s \in (a_1, b_1)\,. \tag{A.4}$$

This conclude the discussion in Step 1.

Step 2. Let

$$m_2^\pm := \min \widehat{\psi}_1^\pm\,.$$

We notice that m_2^+, $m_2^- \geq m_1 + \eta$. Then we define

$$A_2^- := \{x \in [a, a_1] : \widehat{\psi}_1^-(x) \leq m_2^- + \eta\}\,, \quad A_2^+ := \{x \in [b_1, b] : \widehat{\psi}_1^+(x) \leq m_2^+ + \eta\}\,.$$

Similarly as in Step 1, we consider

$$a_2 := \min A_2^-\,, \qquad b_2 := \max A_2^+$$

and observe that by construction

$$\psi(s) \geq \frac{1}{2}\Big(\psi(a_2) + \psi(a_1)\Big) - \eta \quad \text{for } s \in (a_2, a_1)\,, \tag{A.5a}$$

$$\psi(s) \geq \frac{1}{2}\Big(\psi(b_1) + \psi(b_2)\Big) - \eta \quad \text{for } s \in (b_1, b_2)\,, \tag{A.5b}$$

where possibly $a_1 = a_2$, $b_1 = b_2$. We set

$$J_2 := J_1 \cup \{a_2,\, b_2\}\,.$$

As in Step 1, we distinguish different cases.

Case 2.I: $a_2 = a$**,** $b_2 = b$**.** We have that the points of the set J_2 provide a partition of $[a, b]$ satisfying in view of (A.4) and (A.5), and this finishes the proof.

Case 2.II: We again distinguish three subcases.

Case 2.IIa: $a < a_2 \le b_2 = b$**.** We define $\widehat{\psi}_2^- : [a, a_2] \to (0, +\infty]$ as

$$\widehat{\psi}_2^-(x) := \begin{cases} \psi(x), & x \in [a, a_2)\,, \\ \liminf\limits_{r \to a_2^-} \psi(r), & x = a_2\,. \end{cases} \tag{A.6}$$

Case 2.IIb: $a = a_2 \le b_2 < b$**.** We define $\widehat{\psi}_2^+ : [b_2, b] \to (0, +\infty]$ as

$$\widehat{\psi}_2^+(x) := \begin{cases} \psi(x), & x \in (b_2, b]\,, \\ \liminf\limits_{r \to b_2^+} \psi(r), & r = b_2\,. \end{cases} \tag{A.7}$$

Case 2.IIc: $a < a_2 \le b_2 < b$**.** We consider both $\widehat{\psi}_2^-$ and $\widehat{\psi}_2^+$.

This concludes the discussion of Step 2.

Step 3. We consider$m_3^\pm := \min \widehat{\psi}_3^\pm$ and accordingly set

$$\begin{aligned} &A_3^- := \{x \in [a, a_2] : \widehat{\psi}_2^-(x) \le m_2^- + \eta\}\,, \quad A_3^+ := \{x \in [b_2, b] : \widehat{\psi}_2^+(x) \le m_2^+ + \eta\}\,, \\ &a_3 := \min A_3^-\,, \qquad b_3 := \max A_3^+\,, \qquad J_3 = J_2 \cup \{a_3, b_3\}\,. \end{aligned}$$

If $a_3 = a$ and $b_3 = b$, then the proof is finished. Otherwise, similarly as in the previous steps, if $a_3 = a$ and $b_3 < b$, in the subsequent step we have still to refine the partition only "from the right" by resorting to a function $\widehat{\psi}_3^+$defined in analogy with $\widehat{\psi}_1^+$, $\widehat{\psi}_2^+$, while "from the left" the partition is fine for our purposes. Similarly, if $a_3 > a$ and $b_3 = b$, in the next step we have still to refine the partition only "from the left" by using $\widehat{\psi}_3^-$, while "from the right" the partition is fine. In this way, we arrive at the next step.

Step n. We are given the points in

$$J_{n-1} = \{a_{n-1}, \dots, a_1, b_1, \dots, b_{n-1}\}$$

with $a_{n-1} \le a_{n-2} \le \cdots \le a_1 \le b_1 \le \cdots \le b_{n-1}$. Expressing J_{n-1} as $\{\tilde r^1, \dots, \tilde r^m\}$ with $m \le 2(n-1)$ and $\tilde r^j < \tilde r^{j+1}$ for $j \in \{1, \dots, m-1\}$, it holds that

$$\psi(s) \ge \frac{1}{2}\Big(\psi(\tilde r^j) + \psi(\tilde r^{j+1})\Big) - \eta \text{ for every } s \in (\tilde r^j, \tilde r^{j+1}) \quad \text{and every } j \in \{1, \dots, m-1\}\,. \tag{A.8}$$

If $a < a_{n-1}$ we consider the function

$$\widehat{\psi}_{n-1}^-(x) := \begin{cases} \psi(x), & x \in [a, a_{n-1})\,, \\ \liminf\limits_{r \to a_{n-1}^-} \psi(r), & x = a_{n-1}\,, \end{cases}$$

and if $b_{n-1} < b$, we also introduce the function

$$\widehat{\psi}_{n-1}^+(x) := \begin{cases} \psi(x), & x \in (b_{n-1}, b], \\ \liminf\limits_{r \to b_{n-1}^+} \psi(r), & x = b_{n-1}. \end{cases}$$

Let $m_n^\pm := \min \widehat{\psi}_{n-1}^\pm$. There exist

$$m_n^\pm \ge m_{n-1}^\pm + \eta \ge \cdots \ge m_1 + (n-1)\eta\,. \tag{A.9}$$

We repeat the procedure performed in the previous step: we set

(A.10a)
$$A_n^- := \{x \in [a, a_{n-1}] : \widehat{\psi}_{n-1}^-(x) \leq m_n^- + \eta\}\,, \quad A_n^+ := \{x \in [b_{n-1}, b] : \widehat{\psi}_{n-1}^+(x) \leq m_n^+ + \eta\}\,,$$

(A.10b)
$$a_n := \min A_n^-\,, \qquad b_n := \max A_n^+\,.$$

By construction,

(A.11a)
$$\psi(s) \geq \frac{1}{2}\Big(\psi(a_n) + \psi(a_{n-1})\Big) - \eta \quad \text{for } s \in (a_n, a_{n-1})\,,$$

(A.11b)
$$\psi(s) \geq \frac{1}{2}\Big(\psi(b_{n-1}) + \psi(b_n)\Big) - \eta \quad \text{for } s \in (b_{n-1}, b_n)\,,$$

where possibly $a_{n-1} = a_n$, $b_{n-1} = b_n$.

Conclusion. The procedure can then be iterated, obtaining at each application a larger set of nodes satisfying the analogue of (5.23). In order to obtain exactly (5.23), it is enough to find n_η such that both $a_{n_\eta} = a$ and $b_{n_\eta} = b$. In fact, this can be done in a finite number of steps: since $\psi(a)$, $\psi(b) \in \mathbb{R}$, then at most $\frac{\max\{\psi(a),\psi(b)\}}{\eta} + 1$ steps are needed, in view of (A.9) and the definition of a_n, b_n (cf. (A.10)).

A.2. An elliptic regularity estimate. The following proposition uses elliptic regularity arguments as a way to derive an estimate that will ultimately enhance the compactness properties of a suitable sequence. We refer the reader to [CLRpre] for the proof, which is based on a corresponding elliptic regularity result from [DDS11, Theorem 9.1].

PROPOSITION A.1. *Let $(u_k)_k \subset H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, $(z_k)_k \subset H^{\mathrm{m}}(\Omega)$, $(p_k)_k \subset L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ fulfill $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BD}(\Omega)$, $z_k \rightharpoonup z$ in $H^{\mathrm{m}}(\Omega)$, and $p_k \to p$ in $\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, and suppose that (u_k, z_k, p_k) satisfy the equations*

$$-\mathrm{div}(\mathbf{C}(z_k)\mathrm{E}u_k) = -\mathrm{div}(\mathbf{C}(z_k)p_k) \quad \textit{in } \Omega \quad \textit{for every } k \in \mathbb{N},$$

where $\mathbf{C}(z) = \mathrm{V}(z)\mathbb{C}$, with $\mathrm{V} \in \mathrm{C}^{1,1}(\mathbb{R})$ as in (2.6d) and $\mathbb{C}$ isotropic.

Then $(\nabla u_k)_k$ is a Cauchy sequence w.r.t. the convergence in measure. In particular, there exists a not relabeled subsequence of $(u_k)_k$ such that $\nabla u_k \to \nabla u$ a.e. in Ω.

Remark A.2. In this estimate we have to resort to the structural condition $\mathbf{C}(z) = \mathrm{V}(z)\mathbb{C}$. Indeed, Proposition A.1 is a consequence of [DDS11, Theorem 9.1], which is based on an explicit representation formula for solutions to the problem of linear elasticity, which holds for $\mathbb{C}$ isotropic (in particular with constant coefficients).

A.3. An auxiliary duality result. Let $(\mathsf{X}, \mathsf{H}, \mathsf{X}^*)$ be a Hilbert triple (i.e., the Hilbert space X fulfills $\mathsf{X} \subset \mathsf{H}$ densely and continuously), and consider the extended H-norm on X^*:

$$\mathfrak{f}_{\mathsf{H}}(\xi) := \begin{cases} \|\xi\|_{\mathsf{H}} & \text{if } \xi \in \mathsf{H}, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Let $\mathsf{R} : \mathsf{H} \to [0, +\infty]$ be a positively 1-homogeneous functional, with $\mathsf{R}(0) = 0$. Consider the convex conjugate and the subdifferential of R in the X^*-X duality, i.e.,

$$\begin{aligned} &\mathsf{R}^* : \mathsf{X}^* \to [0, +\infty], \quad &&\mathsf{R}^*(w) := \sup_{v \in \mathsf{X}} \left(\langle w, v\rangle_{\mathsf{X}} - \mathsf{R}(v)\right), \\ &\partial\mathsf{R} : \mathsf{X} \rightrightarrows \mathsf{X}^*, &&\partial\mathsf{R}(v) = \{w \in \mathsf{R}^* : \mathsf{R}(\eta) - \mathsf{R}(v) \geq \langle w, \eta - v\rangle_{\mathsf{X}}\ \forall\, \eta \in \mathsf{X}\}\,. \end{aligned}$$

We have the following lemma, whose proof can be found in [CLRpre].

LEMMA A.3. *In the above setup, there holds*

$$\inf_{w\in\partial\mathsf{R}(0)} \mathfrak{f}_{\mathsf{H}}(\xi - w) = \sup\left\{ \langle\xi,\eta\rangle_{\mathsf{X}} - \mathsf{R}(\eta) :\ \eta\in\mathsf{X},\ \|\eta\|_{\mathsf{H}}\leq 1\right\}.$$

Remark A.4. Clearly, the representation formula (3.17a) for $\mathbb{D}_{\mathsf{p}}$ immediately follows with $\mathsf{X}=\mathsf{H}=L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, $\mathfrak{f}_{\mathsf{H}}=\|\cdot\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}$, and $\mathsf{R}=\mathcal{H}(z,\cdot)$. Analogously, (3.17b) for $\mathbb{D}_{\mathsf{z}}$ ensues with $\mathsf{X}=H^{\mathrm{m}}(\Omega)$, $\mathsf{H}=L^2(\Omega)$, $\mathfrak{f}_{\mathsf{H}}=\mathfrak{f}_{L^2(\Omega)}$, and $\mathsf{R}=\mathcal{R}$.

A.4. A result on the convergence of the stresses. The following lemma is inspired by [FG12, Proposition 6.1]. Its proof can be found in [CLRpre].

LEMMA A.5. *Let Ω be a Lipschitz domain and $(\sigma_k)_k\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ be such that $\sigma_k\rightharpoonup\sigma$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $\mathrm{div}\,\sigma_k\to\mathrm{div}\,\sigma$ in $L^2(\Omega;\mathbb{R}^n)$, and $(\sigma_k)_{\mathrm{D}}\to\sigma_{\mathrm{D}}$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$. Then $\sigma_k\to\sigma$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$.*

A.5. Approximation of the work of the external forces. Let $(\mathfrak{s}_k^\ell)_{\ell=0}^{\mathfrak{S}_k}\subset[0,S]$ with

$$0=\mathfrak{s}_k^0<\mathfrak{s}_k^1<\cdots<\mathfrak{s}_k^{\mathfrak{S}_k}=S \quad\text{ and }\quad \max_{\ell=1,\dots,\mathfrak{S}_k}(\mathfrak{s}_k^\ell-\mathfrak{s}_k^{\ell-1})\to 0 \text{ as } k\to\infty$$

be a generic partition of the interval $[0,S]$. The following result concerns the approximation of the integral $\int_0^S\langle\sigma,\mathrm{E}(\mathsf{w}')\rangle_{L^2(\Omega)}\,\mathrm{d}s$ that contributes to the work of the external forces. Its proof can be found in [CLRpre].

LEMMA A.6. *There holds*

$$\begin{aligned}&\lim_{k\to\infty}\sum_{\ell=1}^{\mathfrak{S}_k}\langle\sigma(\mathfrak{s}_k^{\ell-1}),\mathrm{E}(\mathsf{w}(\mathfrak{s}_k^\ell)-\mathsf{w}(\mathfrak{s}_k^{\ell-1}))\rangle_{L^2(\Omega)}\\ &\quad=\int_0^S\langle\sigma(s),\mathrm{E}\mathsf{w}'(s)\rangle_{L^2(\Omega)}\,\mathrm{d}s=\lim_{k\to\infty}\sum_{\ell=1}^{\mathfrak{S}_k}\langle\sigma(\mathfrak{s}_k^{\ell}),\mathrm{E}(\mathsf{w}(\mathfrak{s}_k^\ell)-\mathsf{w}(\mathfrak{s}_k^{\ell-1}))\rangle_{L^2(\Omega)}\,.\end{aligned}$$

In particular,

$$\lim_{k\to\infty}\sum_{\ell=1}^{\mathfrak{S}_k}\frac{1}{2}\langle\sigma(\mathfrak{s}_k^{\ell-1})+\sigma(\mathfrak{s}_k^\ell),\mathrm{E}(\mathsf{w}(\mathfrak{s}_k^\ell)-\mathsf{w}(\mathfrak{s}_k^{\ell-1}))\rangle_{L^2(\Omega)}=\int_0^S\langle\sigma(s),\mathrm{E}\mathsf{w}'(s)\rangle_{L^2(\Omega)}\,\mathrm{d}s\,.$$

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