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RESEARCH

Existence and Exponential Decay for a Contact Problem Between Two Dissipative Beams

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Abstract
We deal with the Signorini contact problem between two Timoshenko beams. In this work we use the theory of semigroups to show the existence of solutions that decay uniformly to zero. This method is new and more effective than the widely used energy method. This is because in particular we obtain uniform decay of the solutions to zero for any boundary condition. A second important point is that we can take advantage of stabilization results of others linear dynamic systems with different dissipative mechanisms and apply them through our method for Contact Problems (see Sect. 4). Finally, thanks to Lipschitzian perturbations we can generalize the Signorini problem to more general semi linear problems in a simple way (see Sect. 4.3).

Keywords Timoshenko beams · Contact problem · Semilinear problem · Asymptotic behaviour

Mathematics Subject Classification 35Q74 · 35B40 · 74K10 · 35B35 · 74H40

1 Introduction

This work is focused on the mechanical evolution of two dissipative Timoshenko beams in unilateral contact across a joint with clearance. The area-centers of gravity of beams in their (stress free and isothermal) reference configurations are given by the intervals $I_1 := (0, \ell_*)$ and $I_2 := (\ell_*, \ell)$, respectively. Let $0 < T \leq \infty$. We denote by $\varphi_1 = \varphi_1(x,t) : I_1 \times (0,T) \to \mathbb{R}$ the transverse displacement (vertical deflection) of the cross section at $x \in I_1$ and at time $t \in (0,T)$, by $\varphi_2 = \varphi_2(x,t) : I_2 \times (0,T) \to \mathbb{R}$ the transverse displacement (vertical deflection) of the cross section at $x \in I_2$ and at time $t \in (0,T)$. Supposing that plane

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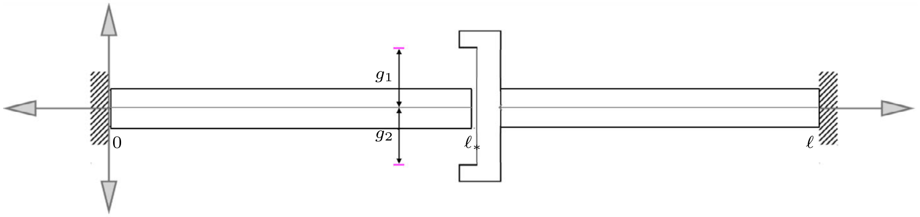

Fig. 1 The two beams and the joint at $x=\ell_*$ with clearance $g=g_1+g_2$

cross sections remain plane, the angles of rotation of a cross section are defined respectively by $\psi_1=\psi_1(x,t): I_1\times(0,T)\to\mathbb{R}$ and $\psi_2=\psi_2(x,t): I_2\times(0,T)\to\mathbb{R}$. The physical setting is represented by Fig. 1.

We describe the evolution of the system, under consideration, by the following equations (for details, see e.g. [9, 16, 18]),

$$\begin{aligned}&\rho_1\varphi_{i,tt}-S_{i,x}+\gamma_{1,i}\varphi_{i,t}=0, && (x,t)\in I_i\times(0,T),\\ &\rho_2\psi_{i,tt}-M_{i,x}+S_i+\gamma_{2,i}\psi_{i,t}=0, && (x,t)\in I_i\times(0,T),\end{aligned}\tag{1.1}$$

where $\gamma_{1,i}$ and $\gamma_{2,i}$ are real non negative functions defined in I_i with $i=1,2$. Here the coefficients represent: $\rho_1=\rho A$ the mass density, $\rho_2=\rho I$ the moment of mass inertia, $k=\kappa GA$ the shear modulus of elasticity, $b=EI$ the rigidity coefficient of cross-section, where E is the Young modulus, G is the modulus of rigidity, κ is the transversal shear factor, and I is the moment of inertia. Functions S and M stand for the shear force and the bending moment, respectively. Subscripts x and t represent partial derivatives with respect to x and t. Henceforth, unless stated otherwise, the index i, as a subscript of the variables involved, will always take the values 1 and 2. The initial conditions are given by

$$\begin{aligned}&\varphi_i(x,0)=\varphi_{i0}(x), && \varphi_{i,t}(x,0)=\varphi_{i1}(x), && \forall x\in I_i,\\ &\psi_i(x,0)=\psi_{i0}(x), && \psi_{i,t}(x,0)=\psi_{i1}(x), && \forall x\in I_i,\end{aligned}\tag{1.2}$$

for some given functions $\varphi_{i0},\varphi_{i1},\psi_{i0},\psi_{i1}: I_i\to\mathbb{R}$. In addition, we suppose that, at $x=0$, $x=\ell_*$ and $x=\ell$,

$$\begin{aligned}&\varphi_1(0,t)=0, && \psi_1(0,t)=\psi_1(\ell_*,t)=0, && \text{in } (0,+\infty),\\ &\varphi_2(\ell,t)=0, && \psi_2(\ell_*,t)=\psi_2(\ell,t)=0, && \text{in } (0,+\infty).\end{aligned}\tag{1.3}$$

The joint at $x=\ell_*$ is modeled with the Signorini non penetration condition (see, e.g., [11]). In particular, the joint with gap g is asymmetrical so that $g=g_1+g_2$, where $g_1>0$ and $g_2>0$ are, respectively, the upper and lower clearance, when the system is at rest. Then, at the point ℓ_* of the beam is assumed to move vertically only between two stops, namely

$$\varphi_2(\ell_*,t)-g_1\le\varphi_1(\ell_*,t)\le\varphi_2(\ell_*,t)+g_2,\quad 0\le t\le T.\tag{1.4}$$

This condition assures that the displacement at $x=\ell_*$ is constrained between the stops g_1 and g_2. The stress at this point if given by

$$S_1(\ell_*,t)=S_2(\ell_*,t):=S(\ell_*,t).$$

Moreover, we prescrive the condition

$$-S(\ell_*,t) \in \partial\chi_{\varphi_2(\ell_*,t)}(\varphi_1(\ell_*,t)), \tag{1.5}$$

where $\partial\chi_v(u)$ denotes the subdifferential of the indicator function $\chi_v(u)$,

$$\chi_v(u) = \begin{cases} 0 & \text{if} \quad v - g_2 < u < v + g_1, \\ \infty & \qquad\qquad \text{otherwise,} \end{cases}$$

namely

$$\partial\chi_v(u) = \begin{cases} (-\infty,0) & \text{if} \qquad u = v - g_2, \\ \quad 0 & \text{if} \quad v - g_2 < u < v + g_1, \\ (0,+\infty) & \text{if} \qquad u = v + g_1. \end{cases}$$

Let us spend a few words on the condition expressed by the condition above. When

$$\varphi_2(\ell_*,t) - g_2 < \varphi_1(\ell_*,t) < \varphi_2(\ell_*,t) + g_1,$$

is verified, there is no contact, the ends at $x = \ell_*$ are free, and $S(\ell_*,t) = 0$. On the other hand, when

$$\varphi_2(\ell_*,t) - g_2 = \varphi_1(\ell_*,t) \quad \text{or} \quad \varphi_1(\ell_*,t) = \varphi_2(\ell_*,t) + g_1,$$

the ends at $x = \ell_*$ are in contact. More precisely, when the contact occurs at the lower end, relations $\varphi_2(\ell_*,t) - g_2 = \varphi_1(\ell_*,t)$ and $S(\ell_*,t) \geq 0$ hold; when the contact takes place at the upper end, relations $\varphi_2(\ell_*,t) = \varphi_1(\ell_*,t) - g_1$ and $S(\ell_*,t) \leq 0$ are verified.

This manuscript engages and develops within the study of the existence and asymptotic behavior of solutions associated with contact problems between two beams. The system specified by (1.1)–(1.5), and the questions related, can be regarded as an extension to the viscoelastic case of the problem studied in [10]. Dynamic models for vibrations transmission across joints are of considerable interest in various industrial settings and in many applications. In most articles currently present in the literature, the Signorini contact problem has been analyzed in a weak sense, namely by considering an approximate version of the Signorini problem via the introduction of a normal compliance condition as regularization of the Signorini condition. The exponential stability of a solution as time goes to infinity is obtained in the approximate framework: the exponential decay for the approximate solution is found by introducing a suitable Lyapunov functional and by using the multiplier method. Then, by weak lower semicontinuity arguments, the exponential decay is achieved for a solution to the original problem.

In this paper we follow a new and different approach. We consider the linear Timoshenko model coupled to a dynamic boundary condition defined by an ordinary differential equation (hybrid system), the coupling is defined through a parameter ϵ which we will then approximate to zero, see the system (2.3) below. We use semigroup theory to show the well-posedness of the problem, as well as the exponential stability of the corresponding model. We arrive at the problem of contact with normal compliance condition through a Lipschitzian perturbation.

Finally, by setting $\epsilon \to 0$ the dynamic boundary condition becomes static and due to the characteristics of the chosen Liptchitzian perturbation (see the last equation in system (3.7)), we arrive at the Signorini conditions which proves the existence of solution to problem (1.1)–(1.5). This procedure is possible thanks to the observability inequalities that Timoshenko model possesses. We believe that this method is more efficient than the usual penalty method (see [4, 11, 15] and the references contained therein) because in this way we obtain more general results about the asymptotic behavior of the solution. In particular, we show that the boundary conditions of the model do not play any important role in the test of asymptotic behavior. This means that the decay result can be proved for any boundary condition, different from the results obtained in [4, 6, 7, 15] where boundary conditions played an important role in the proof of exponential decay.

The remaining part of this paper is organized as follows. In Sect. 2 we show the well possedness of the linear hybrid model. In Sect. 3 we find the main result of this paper: the existence of a solution to Signorini problem, which decays exponentially as the linear semigroup. Finally, in Sect. 4 we give some applications of our result.

2 The Hybrid Linear Model as a Compact Perturbation

To fix ideas, we consider the viscoelastic constitutive law of Kelvin-Voigt type

$$S_i := \kappa_i(\varphi_{i,x} + \psi_i) + \widetilde{\kappa}_i(x)(\varphi_{i,xt} + \psi_{i,t}), \quad M_i := b_i\psi_{i,x} + \widetilde{b}_i(x)\psi_{i,xt}, \tag{2.1}$$

with $\widetilde{\kappa}_i$ and $\widetilde{b}_i$ nonnegative functions characterizing the viscosity of the beam. Moreover, we assume that both have the same support, namely

$$\operatorname{supp}(\widetilde{\kappa}_i) = \operatorname{supp}(\widetilde{b}_i) \subset I_i. \tag{2.2}$$

In general, we assume that the viscous component is effective over sub intervals of I_i. We will denote as I_E any subinterval of I_i where the viscosity is not efective, that is $\widetilde{\kappa}_i = \widetilde{b}_i = 0$ over I_E.

To apply the semigroup theory to study the Signorini problem, let's start by considering two uncoupled linear hybrid models, one defined over the interval $I_1 := (0, \ell_*)$ and the other one over the interval $I_2 := (\ell_*, \ell)$, approaching the penalized problem associated to (1.1)–(1.5), and given by

$$\begin{aligned}
&\rho_1\,\varphi_{i,tt}(x,t) - S_{i,x}(x,t) + \gamma_{1,i}\varphi_{i,t}(x,t) = 0 && \text{in } I_i \times (0,+\infty),\\
&\rho_2\,\psi_{i,tt}(x,t) - M_{i,x}(x,t) + S_i(x,t) + \gamma_{2,i}\psi_{i,t}(x,t) = 0 && \text{in } I_i \times (0,+\infty),\\
&\epsilon v_{i,tt}(t) + \epsilon v_{i,t}(t) + \epsilon v_i(t) + (-1)^{i+1} S_i(\ell_*,t) = 0 && \text{in } (0,+\infty),
\end{aligned} \tag{2.3}$$

satisfying the boundary conditions over $I_1 =]0, \ell_*[$ and $I_2 =]\ell_*, \ell[$

$$\begin{aligned}
&\varphi_1(0,t) = 0, \quad \varphi_1(\ell_*,t) = v_1(t), \quad \psi_1(0,t) = \psi_1(\ell_*,t) = 0, \quad \text{in } (0,+\infty),\\
&\varphi_2(\ell_*,t) = 0, \quad \varphi_2(\ell,t) = v_2(t), \quad \psi_2(\ell_*,t) = \psi_2(\ell,t) = 0, \quad \text{in } (0,+\infty),
\end{aligned} \tag{2.4}$$

and verifying the initial conditions (1.2) together with

$$v_i(0) = v_{i,0}, \quad v_{i,t}(0) = v_{i,1}, \tag{2.5}$$

for some given numbers $v_{i,0}$ and $v_{i,1} \in \mathbb{C}$.

The dynamic boundary condition $(2.3)_3$ can be interpreted as a beam rigidly attached at the end $x = \ell_*$ to a tip body that models a sealed container with a granular material, for example sand. This granular material dampens the movement of the system by internal friction (for details, see [2, 3, 14]).

The phase space of our problem is

$$\mathcal{H}_i = V_0^i \times L^2(I_i) \times H_0^1(I_i) \times L^2(I_i) \times \mathbb{C}^2,$$

where

$$V_0^1 = \left\{w \in H^1(I_1) : \ w(0) = 0\right\} \quad \text{and} \quad V_0^2 = \left\{w \in H^1(I_2) : \ w(\ell) = 0\right\}.$$

Denoting by $U_i = (\varphi_i, \Phi_i, \psi_i, \Psi_i, v_i, V_i)^\top$, we define the norm

$$\|U_i\|^2_{\mathcal{H}_i} = \int_{I_i} \left(\kappa_i |\varphi_{i,x} + \psi_i|^2 + \rho_1 |\Phi_i|^2 + b_i |\psi_{i,x}|^2 + \rho_2 |\Psi_i|^2\right) dx + \epsilon |v_i|^2 + \epsilon |V_i|^2.$$

Equations (2.3) are uncoupled and independent one to the other for $i = 1, 2$. For sake of simplicity, in what follows we remove the subindex i from the variables. Denoting by $B^\top$ the transpose of a matrix B and $\Phi = \varphi_t$, $\Psi = \psi_t$ and $V = v_t$ we have

$$U(t) = (\varphi(t), \Phi(t), \psi(t), \Psi(t), v(t), V(t))^\top := (\mathcal{U}, \mathcal{V})^\top,$$

where $\mathcal{U} := (\varphi(t), \Phi(t), \psi(t), \Psi(t))^\top$ and $\mathcal{V} := (v(t), V(t))^\top$. Hence, system (2.3) can be written as a linear ODE in $\mathcal{H}_i$ of the form

$$\frac{d}{dt} U(t) = \mathcal{A}_i \, U(t), \tag{2.6}$$

where the domain $\mathcal{D}(\mathcal{A}_i)$ of the linear operator $\mathcal{A}_i : D(\mathcal{A}_i) \subset \mathcal{H}_i \to \mathcal{H}_i$ is given by

$$\mathcal{D}(\mathcal{A}_i) = \left\{U \in \mathcal{H}_i : \ S_i, \ M_i \in H^1(I_i), \ (\Phi, \Psi) \in V_0^i \times H_0^1(I_i)\right\},$$

and

$$\mathcal{A}_i U = \begin{bmatrix} \Phi \\ \dfrac{1}{\rho_1} S_{i,x} - \dfrac{\gamma_{1,i}}{\rho_1} \Phi_i \\ \Psi \\ \dfrac{1}{\rho_2} M_{i,x} - \dfrac{1}{\rho_2} S_i - \dfrac{\gamma_{2,i}}{\rho_2} \Psi_i \\ V \\ -V - v - \frac{1}{\epsilon}(-1)^{i+1} S_i(\ell_*) \end{bmatrix}. \tag{2.7}$$

According to Lumer-Phillips Theorem (see, e.g., [12, Theorem 1.2.4] or [17, Theorem 1.4.3]), the operator $\mathcal{A}_i$ is the infinitesimal generator of a contraction semigroup

$$\mathcal{T}_i(t) := e^{t\mathcal{A}_i} : \mathcal{H}_i \to \mathcal{H}_i.$$

In particular, $\mathcal{A}_i$ is dissipative. Indeed, for every $U \in \mathcal{D}(\mathcal{A}_1)$,

$$\begin{aligned}\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} = &\int_0^{\ell_*} \kappa_1(\Phi_x+\Psi)\overline{(\varphi_x+\psi)}\,dx + S_1(\ell_*)\overline{\Phi(\ell_*)}\\ &-\int_0^{\ell_*} S_1\overline{(\Phi_x+\Psi)}\,dx + \int_0^{\ell_*} b_1\Psi_x\overline{\psi_x}\,dx\\ &-\int_0^{\ell_*} M_1\overline{\Psi_x}\,dx + \epsilon V\overline{v} - \epsilon|V|^2 - \epsilon v\overline{V} - S_1(\ell_*)\overline{\Phi(\ell_*)}\\ &-\int_0^{\ell_*}\big(\gamma_1|\Phi|^2+\gamma_2|\Psi|^2\big)\,dx.\end{aligned}$$

Here we used that $V = \Phi(\ell_*)$. For the constitutive law (2.1), we get

$$\begin{aligned}\mathrm{Re}\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} = &-\int_0^{\ell_*}\big(\widetilde{\kappa}_1(x)|\Phi_x+\Psi|^2 + \widetilde{b}_1(x)|\Psi_x|^2\big)\,dx\\ &-\int_0^{\ell_*}\big(\gamma_1|\Phi|^2+\gamma_2|\Psi|^2\big)\,dx - \epsilon|V|^2.\end{aligned} \tag{2.8}$$

Similarly for $\mathcal{A}_2$. Hence we have

$$\mathrm{Re}\langle \mathcal{A}_i U, U\rangle_{\mathcal{H}_i} \le -\epsilon|V|^2 \le 0. \tag{2.9}$$

For any initial datum $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, v_0, v_1)^\top \in \mathcal{H}_i$ the solution to (2.6) is denoted by

$$U(t) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), v(t), V(t))^\top = \mathcal{T}_i(t)\,U_0. \tag{2.10}$$

Considering the resolvent equation

$$i\lambda U - \mathcal{A}_i U = F, \tag{2.11}$$

and taking inner product with U over the phase space $\mathcal{H}_i$, we get

$$\int_0^{\ell_*}\big(\widetilde{\kappa}|\Phi_x+\Psi|^2 + \widetilde{b}|\Psi_x|^2\big)\,dx + \int_0^{\ell_*}\big(\gamma_1|\Phi|^2+\gamma_2|\Psi|^2\big)\,dx + \epsilon|V|^2 = \mathrm{Re}\langle U(t), F(t)\rangle_{\mathcal{H}_i}. \tag{2.12}$$

In particular, we have

$$\epsilon|V|^2 \le \mathrm{Re}\langle U(t), F(t)\rangle_{\mathcal{H}_i}. \tag{2.13}$$

In this section we will make a comparison between the hybrid model and the non hybrid Timoshenko model given by

$$\begin{aligned}&\rho_1\,\varphi_{i,tt} - S_{i,x} + \gamma_{1,i}\varphi_{i,t} = 0 &&\text{in } I_i\times(0,+\infty),\\ &\rho_2\,\psi_{i,tt} - M_{i,x} + S_i + \gamma_{2,i}\psi_{i,t} = 0 &&\text{in } I_i\times(0,+\infty),\end{aligned} \tag{2.14}$$

for $i = 1, 2$, satisfying the boundary conditions

$$\varphi_1(0,t) = S_1(\ell_*,t) = 0, \quad \psi_1(0,t) = \psi_1(\ell_*,t) = 0, \quad \text{in } (0,+\infty), \tag{2.15}$$

$$S_2(\ell_*,t)=\varphi_2(\ell,t)=0,\quad \psi_2(\ell_*,t)=\psi_2(\ell,t)=0,\quad \text{in } (0,+\infty). \tag{2.16}$$

Let us denote the infinitesimal generator of system (2.14)–(2.16) by $\mathcal{A}_{i,T}$ where

$$\mathcal{A}_{i,T}\mathcal{U}=\begin{bmatrix} \Phi \\ \frac{1}{\rho_1}S_{i,x}-\frac{\gamma_{1,i}}{\rho_1}\varphi_{i,t} \\ \Psi \\ \frac{1}{\rho_2}M_{i,x}-\frac{1}{\rho_2}S_i-\frac{\gamma_{2,i}}{\rho_2}\psi_{i,t} \end{bmatrix}. \tag{2.17}$$

The phase space we consider for the above model is

$$\mathbf{H}_i=V_0^i\times L^2(I_i)\times H_0^1(I_i)\times L^2(I_i).$$

Hence the domain $\mathcal{D}(\mathcal{A}_{i,T})$ of the linear operator $\mathcal{A}_{i,T}:D(\mathcal{A}_{i,T})\subset\mathbf{H}_i\to\mathbf{H}_i$ is given by

$$\mathcal{D}(\mathcal{A}_{i,T})=\left\{\mathcal{U}\in\mathbf{H}_i:\ S_i,\ M_i\in H^1(I_i),\ (\Phi,\Psi)\in V_0^i\times H_0^1(I_i)\right\}.$$

Similarly as the hybrid model, we have

$$\mathrm{Re}\langle\mathcal{A}_{i,T}U,U\rangle_{\mathcal{H}_1}=-\int_0^{\ell_*}\left(\widetilde{\kappa}(x)|\Phi_x+\Psi|^2+\widetilde{b}(x)|\Psi_x|^2\right)dx-\int_0^{\ell_*}\left(\gamma_1|\Phi|^2+\gamma_2|\Psi|^2\right)dx, \tag{2.18}$$

with $\gamma_1,\gamma_2\geq 0$. Under this notations we get that system (2.14)–(2.16) can be written as

$$\frac{d}{dt}\mathcal{U}(t)=\mathcal{A}_{i,T}\mathcal{U}(t). \tag{2.19}$$

Let us denote by $\mathbf{T}_i=e^{t\mathcal{A}_{i,T}}$ the semigroup associated to system (2.14). The main objective of this section is to show that the semigroup $\mathbf{T}_i$ is exponentially stable if and only the semigroup $\mathcal{T}_i$ is also exponentially stable. This means that the dissipation produced by the ODE in (2.3)$_3$, of the hybrid model, is not relevant. Let us introduce the space

$$\widetilde{\mathbf{H}}_i=\mathbf{H}_i\times\{0\}\times\{0\},$$

intended as the extended phase space. Let us denote by Π_i the projection of $\mathcal{H}_i$ onto $\widetilde{\mathbf{H}}_i$:

$$\Pi_i(\varphi,\Phi,\psi,\Psi,v,V)=(\varphi,\Phi,\psi,\Psi,0,0).$$

Let us decompose the infinitesimal generator $\mathcal{A}_i$ in the following way

$$\mathcal{A}_i:=\begin{pmatrix}\mathcal{A}_{i,T} & \mathbf{0}_{4\times 2}\\ B & K\end{pmatrix} \tag{2.20}$$

with

$$B=\begin{pmatrix}0 & 0 & 0 & 0\\ \frac{\kappa}{\epsilon}\boldsymbol{\gamma}_1 & 0 & -\frac{\kappa}{\epsilon}\boldsymbol{\gamma}_0 & 0\end{pmatrix},\quad K=\begin{pmatrix}0 & I\\ -I & -I\end{pmatrix}$$

where $\boldsymbol{\gamma}_1\varphi = \varphi_x(\ell_*)$ and $\boldsymbol{\gamma}_0\psi = \psi(\ell_*)$. Hence, recalling that $U := (\mathcal{U}, \mathcal{V})^\top$, where $\mathcal{U} := (\varphi, \Phi, \psi, \Psi)$ and $\mathcal{V} := (v, V)$, we get

$$\mathcal{A}_i U = \begin{pmatrix} \mathcal{A}_{i,T}\mathcal{U} \\ B\mathcal{U} + K\mathcal{V} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{i,T}\mathcal{U} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K\mathcal{V} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ B\mathcal{U} \end{pmatrix}, \quad \forall U \in D(\mathcal{A}_i).$$

Under the above conditions we can state the following Lemma:

Lemma 2.1 *The difference $\mathcal{T}_i(t) - \mathbf{T}_i(t)\Pi$ is a compact operator over $\mathcal{H}_i$. Hence the corresponding essential types $\omega_{ess}(\mathcal{T}_i)$ and $\omega_{ess}(\mathbf{T}_i(t)\Pi)$ are equal.*

Proof Note that the solution of $U_t - \mathcal{A}_i U = 0$, $U(0) = U_0$ can be written as

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}_t = \begin{pmatrix} \mathcal{A}_{i,T}\mathcal{U} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K\mathcal{V} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ B\mathcal{U} \end{pmatrix}$$

with $U_0 = (\mathcal{U}_0, \mathcal{V}_0)^\top$ which implies that

$$\mathcal{U} = e^{tA_{i,T}}\mathcal{U}_0, \quad \text{and} \quad \mathcal{V} = e^{tK}\mathcal{V}_0 + \int_0^t e^{(t-s)K} B\mathcal{U}(s)\, ds.$$

Therefore

$$U(t) - \begin{pmatrix} e^{tA_{i,T}}\mathcal{U}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ e^{tK}\mathcal{V}_0 + \int_0^t e^{(t-s)K} B\mathcal{U}(s)\, ds \end{pmatrix}.$$

Note that the right hand side of the above equation is a compact operator, therefore

$$[\mathcal{T}_i(t) - \mathbf{T}_i(t)\Pi]$$

is a compact operator. So our conclusion follows. □

Remark 2.1 Let us denote by $\mathbf{A}$ the operator $\mathcal{A}_i$ or $\mathcal{A}_{i,T}$, then resolvent operator $(\mu I - \mathbf{A})^{-1}$ is not compact.

Here we consider $\gamma_{1,i} = 0$ and $\gamma_{2,i} = 0$. We will show that the spectrum of $\mathbf{A}$, $\sigma(\mathbf{A})$, does not contain only eigenvalues. In fact, let us consider $\mu \in \mathbb{R}$, then the resolvent equation can be written as

$$\begin{aligned}
&\mu\varphi - \Phi = f_1, \\
&\mu\rho_1\Phi - \kappa(\varphi_x + \psi)_x - \widetilde{\kappa}(\Phi_x + \Psi)_x = \rho_1 f_2, \\
&\mu\psi - \Psi = f_3, \\
&\mu\rho_2\Psi - b\psi_{xx} - \widetilde{b}\Psi_{xx} + \kappa(\varphi_x + \psi) + \widetilde{\kappa}(\Phi_x + \Psi) = \rho_2 f_4,
\end{aligned}$$

where we assume that κ, $\widetilde{\kappa}$, b and $\widetilde{b}$ are positive constant. Taking $f_1 = f_3 = 0$, the above system can be written as

$$\mu^2\rho_1\varphi - (\kappa + \mu\widetilde{\kappa})(\varphi_x + \psi)_x = \rho_1 f_2, \tag{2.21}$$

$$\mu^2\rho_2\psi - (b + \mu\widetilde{b})\psi_{xx} + (\kappa + \mu\widetilde{\kappa})(\varphi_x + \psi) = \rho_2 f_4. \tag{2.22}$$

Let us consider the numbers $\mu_1 = -\kappa/\widetilde{\kappa}$ and $\mu_2 = -b/\widetilde{b}$ none of them belong to the resolvent set of $\mathbf{A}$. In fact from (2.21) we get

$$\mu_1^2 \rho_1 \varphi = \rho_1 f_2 \in L^2(0, \ell_*). \tag{2.23}$$

On the other hand, if $\mu_1 \in \rho(\mathbf{A})$ the corresponding solution U must satisfy $U \in D(\mathbf{A})$ which in particular means that $\varphi \in H^1(0, \ell_*)$, which is contradictory to (2.23).

Similarly, if $\mu_2 \in \rho(\mathbf{A})$ using (2.21) and (2.22) we get

$$\mu_2^2 \rho_2 \psi_x = \rho_2 f_{4,x} - \rho_1 \mu_2^2 \varphi + \rho_1 f_2 \in H^{-1}(0, \ell_*). \tag{2.24}$$

But U must be in $D(\mathbf{A})$ which in particular means that $\psi \in H_0^1(0, \ell_*)$. This is contradictory to (2.24). Hence $\mu_2 \notin \rho(\mathbf{A})$.

Finally we will show that one of them is not an eigenvalue. Here $f_2 = f_4 = 0$, let us suppose that $\mu_2 < \mu_1$. We will show that μ_1 is not an eigenvalue. Note that (2.21) implies that $\varphi = 0$ and (2.22) can be written as

$$\mu_1^2 \rho_2 \psi + \widetilde{b}(\mu_2 - \mu_1)\psi_{xx} = 0.$$

Multiplying the above equation by ψ, integrating by parts we get

$$\int_0^{\ell_*} \left[\mu_1^2 \rho_2 |\psi|^2 + \widetilde{b}(\mu_1 - \mu_2)|\psi_x|^2\right] dx = 0.$$

So we have that $\psi = 0$ which implies that $U = 0$ that is a contradiction.

If $\mu_1 < \mu_2$, then we have that μ_2 is not an eigenfunction. This because system (2.21)–(2.22) can be written as

$$\mu_2^2 \rho_1 \varphi + \widetilde{\kappa}(\mu_1 - \mu_2)(\varphi_x + \psi)_x = 0,$$
$$\mu_2^2 \rho_2 \psi - \widetilde{\kappa}(\mu_1 - \mu_2)(\varphi_x + \psi) = 0.$$

From the above equations we get

$$\int_0^{\ell_*} \left[\mu_1^2 \rho_1 |\varphi|^2 + \mu_1^2 \rho_2 |\psi|^2 + \widetilde{\kappa}(\mu_2 - \mu_1)|\varphi_x + \psi|^2\right] dx = 0.$$

So we have that $U = 0$ which is a contradiction. The same result holds when $\mu_1 = \mu_2$. Then our conclusion follows.

Lemma 2.2 *Let us denote by* $\mathbf{A}$ *the operator* $\mathcal{A}_i$ *or* $\mathcal{A}_{i,T}$. *If* $i\mathbb{R} \not\subset \varrho(\mathbf{A})$ *then there exists* $0 \neq \sigma \in \mathbb{R}$ *such that* $i\sigma U - \mathbf{A}U = 0$.

Proof Let us denote by

$$\mathcal{N} = \{s \in \mathbb{R}^+ : \,]-is, is[\subset \rho(\mathcal{A})\}.$$

It is easy to see that $0 \in \rho(\mathbf{A})$, so we have $\mathcal{N} \neq \varnothing$. Putting $\sigma = \sup \mathcal{N}$ we have two possibilities. First $\sigma = +\infty$, which implies that $i\mathbb{R} \subseteq \rho(\mathbf{A})$, and that $0 < \sigma$ finite. We will reason by contradiction. Let us suppose that $\sigma < \infty$. Then, exists a sequence $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to \sigma < \infty$ and

$$\|(i\lambda_n I - \mathbf{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty.$$

Hence, there exists a sequence $\{f_n\} \subseteq \mathcal{H}$ verifying $\|f_n\|_{\mathcal{H}} = 1$ and $\|(i\lambda_n I - \mathbf{A})^{-1} f_n\|_{\mathcal{H}} \to \infty$. Denoting by

$$\tilde{U}_n = (i\lambda_n I - \mathbf{A})^{-1} f_n \quad \Rightarrow \quad f_n = i\lambda_n \tilde{U}_n - \mathbf{A}\tilde{U}_n$$

and $U_n = \dfrac{\tilde{U}_n}{\|\tilde{U}_n\|_{\mathcal{H}}}$, $F_n = \dfrac{f_n}{\|\tilde{U}_n\|_{\mathcal{H}}}$ we conclude thet U_n verifies $\|U_n\|_{\mathcal{H}} = 1$ and

$$i\lambda_n U_n - \mathbf{A}U_n = F_n \to 0 \quad \Rightarrow \quad -\mathrm{Re}(\mathbf{A}U_n, U_n) = (F_n, U_n) \to 0$$

Since $\|\mathbf{A}U_n\|_{\mathcal{H}} \leq C$ using (2.1) or (2.18) we have that over the viscoelastic component $I_v = \mathrm{supp}(\widetilde{\kappa}) = \mathrm{supp}(\widetilde{b})$ the sequences Φ_n, Ψ_n converges strongly, that is

$$\Phi_n, \Psi_n \to 0 \quad \text{strong in} \quad H^1(I_v). \tag{2.25}$$

Since $\lambda_n \to \sigma < \infty$ the above convergence implies that

$$\varphi_n, \psi_n \to 0 \quad \text{strong in} \quad H^1(I_v). \tag{2.26}$$

Since U_n is bounded in $D(\mathbf{A})$ we conclude that φ_n, ψ_n are bounded in $H^2(]0, \ell_*[\backslash I_v)$. In particular this means that there exists a subsequence of φ_n and ψ_n, we still denote in the same way, such that

$$(\varphi_n, \Phi_n, \psi_n, \Psi_n) \to (\varphi, \Phi, \psi, \Psi) \quad \text{strong in} \quad [H^1(]0, \ell_*[\backslash I_v) \times H^1(]0, \ell_*[\backslash I_v)]^2. \tag{2.27}$$

Therefore from convergences (2.25), (2.26) and (2.27) there exists a subsequence of U_n, we still denote in the same ways such that

$$U_n \quad \to \quad U, \quad \text{strong in} \quad \mathcal{H}.$$

Hence $\|U\|_{\mathcal{H}} = 1$. Moreover, because of $\mathbf{A}U_n = i\lambda_n U_n - F_n$, we have $\mathbf{A}U_n$ converges strongly in $\mathcal{H}$. Since $\mathbf{A}$ is closed, we conclude that U verifies

$$i\sigma U - \mathbf{A}U = 0, \tag{2.28}$$

from where our conclusion follows. □

To show the equivalence of the exponential stability between $\mathcal{T}_i(t)$ and $\mathbf{T}_i(t)$ we apply the following result

Theorem 2.1 *Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions on Banach space. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathbb{A}) \quad \textit{and} \quad \omega_{ess}(S(t)) < 0, \tag{2.29}$$

where $\omega_{ess}(S(t))$ is the essential growth bound of the semigroup $S(t)$.

Proof Here we use [8, Corollary 2.11] establishing that the type ω of the semigroup $e^{\mathbb{A}t}$ verifies

$$\omega = \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\}, \tag{2.30}$$

where $\omega_\sigma(\mathbb{A})$ is the upper bound of the spectrum of $\mathbb{A}$. Moreover, for any $c > \omega_{ess}$, the set $\mathcal{I}_c := \sigma(\mathbb{A}) \cap \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \geq c\}$ is finite.

Let us suppose that (2.29) is valid. Since the essential type of the semigroup ω_{ess} is negative, identity (2.30) states that the type of the semigroup will be negative provided $\omega_\sigma(\mathbb{A}) < 0$.

If $\omega_\sigma(\mathbb{A}) \leq \omega_{ess}$ then we have nothing to prove. Let us suppose that $\omega_\sigma(\mathbb{A}) > \omega_{ess}$. From (2.29) and Hille-Yosida Theorem we have $\overline{\mathbb{C}_+} \subset \varrho(\mathbb{A})$, hence $\omega_\sigma(\mathbb{A}) \leq 0$. On the other hand $\mathcal{I}_{\omega_{ess}+\delta}$ is finite for $\delta > 0$ verifying $\omega_{ess} + \delta < 0$ and $\omega_{ess} + \delta < \omega_\sigma(\mathbb{A})$. Therefore we have

$$\omega_\sigma(\mathbb{A}) = \sup \mathrm{Re}\,\sigma(\mathbb{A}) = \sup \mathrm{Re}\,\mathcal{I}_{\omega_{ess}+\delta} < 0.$$

Hence, the sufficient condition follows.

Reciprocally, let us suppose that the semigroup $S(t)$ is exponentially stable, in particular it goes to zero. Then, by [5, Theorem 1.1] we have that $i\mathbb{R} \subset \varrho(\mathbb{A})$. Moreover, since the type ω verifies (2.30), we have that

$$\omega_{ess} \leq \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\} = \omega < 0.$$

Then, our conclusion follows. □

Remark 2.2 The above characterization is valid for any Banach space.

Theorem 2.2 *The semigroup $\mathbf{T}_i$ is exponentially stable if and only if the semigroup $\mathcal{T}_i(t) = e^{\mathcal{A}_i t}$ associated to the hybrid system (2.3) also is exponentially stable.*

Proof Let us suppose that $\mathbf{T}_i$ is exponentially stable. First note that $i\mathbb{R} \subset \varrho(\mathcal{A}_i)$. In fact, let us suppose the contrary, then by Lemma 2.2 we have that there exists $\sigma \in \mathbb{R}$ such that

$$\langle \mathcal{A}_i U, U\rangle_{\mathcal{H}_i} = i\sigma\|U\|^2 \quad \Rightarrow \quad \mathrm{Re}\langle \mathcal{A}_i U, U\rangle_{\mathcal{H}_i} = 0.$$

Using (2.9) we find

$$0 \leq -\epsilon|V|^2 \quad \Rightarrow \quad v = V = 0.$$

By $(2.3)_3$ we get

$$S_i(\ell_*) = 0.$$

Therefore the eigenvector $U = (\mathcal{U}, \mathcal{V})^\top = (\mathcal{U}, \mathbf{0})^\top$ verifies (2.15)–(2.16). Then, we have

$$\mathcal{U} \in D(\mathcal{A}_{i,T}) \quad \text{and} \quad \mathcal{A}_{i,T}\mathcal{U} = i\lambda\mathcal{U}.$$

This implies that $i\lambda \in \sigma(\mathcal{A}_{i,T})$ which is not possible because $\mathbf{T}_i$ is exponentially stable. This contradiction comes from assuming that $i\mathbb{R} \not\subset \varrho(\mathcal{A}_i)$. Therefore $i\mathbb{R} \subset \varrho(\mathcal{A}_i)$. From Lemma 2.1 we have that $\omega_{\mathrm{ess}}(\mathcal{T}_i) = \omega_{\mathrm{ess}}(\mathbf{T}_i(t)\Pi) < 0$ then Theorem 2.1 implies the exponential stability of $\mathcal{T}_i(t) = e^{\mathcal{A}_i t}$.

Finally, let us suppose that $\mathcal{T}_i(t) = e^{\mathcal{A}_i t}$ is exponentially stable. In particular, we have $i\mathbb{R} \subset \varrho(\mathcal{A}_i)$, and as before by Theorem 2.1 it is enough to show the strong stability of $\mathcal{A}_{i,T}$. By contradiction if $i\mathbb{R} \not\subset \varrho(\mathcal{A}_{i,T})$, by Lemma 2.2 there exists $\mathcal{U} \neq 0$ such that $\mathcal{A}_{i,T}\mathcal{U} = i\sigma\mathcal{U}$ then the vector $U = (\mathcal{U}, 0, 0)$ must be an imaginary eigenvector of $\mathcal{A}_i$. But this is a contradiction. Then, our conclusion follows. □

3 The Signorini Problem

In what follows we prove the well posedness of an abstract semilinear problem and we study, under suitable conditions the asymptotic behavior of the solutions. So, we introduce a local Lipschitz function $\mathcal{F}$ defined over a Hilbert space $\mathcal{H}$. We suppose that for any ball $B_R = \{W \in \mathcal{H} : \ \|W\|_{\mathcal{H}} \le R\}$, there exists a function globally of Lipschitz $\widetilde{\mathcal{F}_R}$ such that

$$\mathcal{F}(0) = 0, \quad \mathcal{F}(U) = \widetilde{\mathcal{F}_R}(U), \quad \forall U \in B_R, \tag{3.1}$$

and additionally, that there exists a positive constant K_0 such that

$$\int_0^t \langle \widetilde{\mathcal{F}_R}(U(s)), U(s)\rangle_{\mathcal{H}}\, ds \le K_0 \|U(0)\|^2_{\mathcal{H}}, \qquad \forall U \in C([0,T];\mathcal{H}). \tag{3.2}$$

Under these conditions, we present

Theorem 3.1 *Let $\{T(t)\}_{t\ge 0}$ be a C_0 semigroup of contraction, exponentially stable semigroup with infinitesimal generator $\mathbb{A}$ over the phase space $\mathcal{H}$. Let $\mathcal{F}$ locally Lipschitz on $\mathcal{H}$ satisfying conditions* (3.1) *and* (3.2)*. Then there exists a global solution to*

$$U_t - \mathbb{A}U = \mathcal{F}(U), \quad U(0) = U_0 \in \mathcal{H}, \tag{3.3}$$

that decays exponentially.

Proof By hypotheses, there exist positive constants c_0 and γ such that $\|T(t)\| \le c_0 e^{-\gamma t}$, and $\widetilde{\mathcal{F}_R}$ globally Lipschitz with Lipschitz constant K_0 verifying conditions (3.1) and (3.2). Let us consider the following space:

$$E_\mu = \left\{ V \in L^\infty(0,\infty;\mathcal{H});\ \ t \mapsto e^{-\mu t}\|V(s)\| \in L^\infty(\mathbb{R}) \right\}.$$

Using standard fixed point arguments we can show that there exists only one global solution to

$$U^R_t - \mathbb{A}U^R = \widetilde{\mathcal{F}_R}(U^R), \quad U^R(0) = U_0 \in \mathcal{H}. \tag{3.4}$$

Multiplying the above equation by U^R we get that

$$\frac{1}{2}\frac{d}{dt}\|U^R(t)\|^2_{\mathcal{H}} - \langle \mathbb{A}U^R, U^R\rangle_{\mathcal{H}} = \langle \widetilde{\mathcal{F}_R}(U^R), U^R\rangle_{\mathcal{H}}.$$

Since the semigroup is contractive, its infinitesimal generator is dissipative, therefore

$$\|U^R(t)\|^2_{\mathcal{H}} \le \|U_0\|^2_{\mathcal{H}} + 2\int_0^t \langle \widetilde{\mathcal{F}_R}(U^R), U^R\rangle_{\mathcal{H}}\, dt.$$

Using (3.2) we get

$$\|U^R(t)\|^2_{\mathcal{H}} \le (1 + 2K_0)\|U_0\|^2_{\mathcal{H}}.$$

Note that for $R > (1 + 2K_0)\|U_0\|^2_{\mathcal{H}}$, we have that

$$\widetilde{\mathcal{F}_R}(V) = \mathcal{F}(V), \quad \forall\ \|V\|_{\mathcal{H}} \le R.$$

In particular,

$$\widetilde{\mathcal{F}_R}(U^R(t)) = \mathcal{F}(U^R(t)).$$

This means that U^R is also solution of system (3.3) and because of the uniqueness we conclude that $U^R = U$. To show the exponential stability to system (3.3), it is enough to show the exponential decay to system (3.4). To do that, we use fixed points arguments. Let us consider

$$\mathcal{T}(V) = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}_R}(V(s))\,ds.$$

Note that $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$ for δ small, with $\gamma - \delta > 0$. In fact, for any $V \in E_{\gamma-\delta}$ we have

$$\begin{aligned}
\|\mathcal{T}(V)\|_{\mathcal{H}} &\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \int_0^t \|\widetilde{\mathcal{F}_R}(V(s))\|_{\mathcal{H}} e^{-\gamma(t-s)}\,ds,\\
&\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 \int_0^t \|V(s)\|_{\mathcal{H}} e^{-\gamma(t-s)}\,ds,\\
&\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 e^{-\gamma t} \int_0^t e^{\delta s}\,ds \sup_{s\in[0,t]} \left\{ e^{(\gamma-\delta)s} \|V(s)\|_{\mathcal{H}} \right\},\\
&\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \frac{K_0 C}{\delta} e^{-(\gamma-\delta)t}.
\end{aligned}$$

Hence $\mathcal{T}(V) \in E_{\gamma-\delta}$. Using standard arguments we show that $\mathcal{T}^n$ satisfies

$$\|\mathcal{T}^n(W_1) - \mathcal{T}^n(W_2)\| \leq \frac{(k_1 t)^n}{n!} \|W_1 - W_2\|_{\mathcal{H}},$$

with $k_1 \in \mathbb{R}$. Therefore we have a unique fixed point satisfying

$$\mathcal{T}^n(U) = U = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}_R}(U(s))\,ds,$$

that is U is a solution of (3.4), and since $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$, then the solution decays exponentially. □

3.1 The Linear Model

To apply Theorem 3.1 for the linear model given by problem (2.3)–(2.5) let us introduce the functions

$$\widehat{\varphi}(x,t) = \begin{cases} \varphi_1(x,t) & \text{if} \quad x \in I_1 = (0,\ell_*),\\ \varphi_2(x,t) & \text{if} \quad x \in I_2 = (\ell_*,\ell), \end{cases} \qquad \widehat{\psi}(x,t) = \begin{cases} \psi_1(x,t) & \text{if} \quad x \in I_1 = (0,\ell_*),\\ \psi_2(x,t) & \text{if} \quad x \in I_2 = (\ell_*,\ell). \end{cases}$$

Let us denote by $\mathcal{H}$ the phase space given by

$$\mathcal{H} = V_* \times L^2(I_1 \cup I_2) \times H_0^1(I_1 \cup I_2) \times L^2(I_1 \cup I_2) \times \mathbb{R}^4.$$

Then, for any $U_0 \in \mathcal{H}$ let us introduce the semigroup

$$\mathcal{T}(t)U_0 = \left(\widehat{\varphi}(t), \widehat{\varphi}_t(t), \widehat{\psi}(t), \widehat{\psi}_t(t), v_1(t), v_{1,t}(t), v_2(t), v_{2,t}(t)\right)^\top. \tag{3.5}$$

It is easy to verify that $\mathcal{T}(t)$ is a contraction semigroup over $\mathcal{H}$. Moreover its infinitesimal generator $\mathbb{A}$ is given by

$$\mathbb{A}U = \begin{bmatrix} \widehat{\Phi} \\ \dfrac{1}{\rho_1}\widehat{S}_x - \dfrac{\gamma_1}{\rho_1}\widehat{\Phi} \\ \widehat{\Psi} \\ \dfrac{1}{\rho_2}\widehat{M}_x - \dfrac{1}{\rho_2}\widehat{S} - \dfrac{\gamma_2}{\rho_2}\widehat{\Psi} \\ V_1 \\ -V_1 - v_1 - \frac{1}{\epsilon}S_1(\ell_*) \\ V_2 \\ -V_2 - v_2 + \frac{1}{\epsilon}S_2(\ell_*) \end{bmatrix}, \quad \text{with} \quad \begin{aligned} \widehat{S}(x,t) &= \begin{cases} S_1(x,t) & \text{if} \quad x \in I_1, \\ S_2(x,t) & \text{if} \quad x \in I_2, \end{cases} \\ \widehat{M}(x,t) &= \begin{cases} M_1(x,t) & \text{if} \quad x \in I_1, \\ M_2(x,t) & \text{if} \quad x \in I_2, \end{cases} \end{aligned} \tag{3.6}$$

where the domain $\mathcal{D}(\mathbb{A})$ of the linear operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\begin{aligned} \mathcal{D}(\mathbb{A}) = \big\{ U \in \mathcal{H} : \ & \widehat{\varphi}, \widehat{\psi} \in H^2(I_1 \cup I_2), \ (\widehat{\Phi}, \widehat{\Psi}) \in V_0 \times H_0^1(I_1 \cup I_2), \\ & \text{verifying } \varphi_i(\ell_*) = v_i \text{ and (2.4)} \big\}. \end{aligned}$$

Under the above conditions we have

Theorem 3.2 *Let us suppose that the semigroups* $\mathbf{T}_i$, $i = 1, 2$ *are exponentially stable, then the semigroup defined in* (3.5)–(3.6) *is exponentially stable.*

Proof Immediate consequence of Theorem 2.2. □

3.2 The Semilinear Model

Let us consider the semilinear system

$$\begin{aligned} & \rho_1 \widehat{\varphi}^{\epsilon}_{tt} - \widehat{S}^{\epsilon}_x + \gamma_1 \widehat{\varphi}_t = 0, \quad \text{in } I_1 \cup I_2 \times (0, \infty), \\ & \rho_2 \widehat{\psi}^{\epsilon}_{tt} - \widehat{M}^{\epsilon}_x + \widehat{S}^{\epsilon} + \gamma_2 \widehat{\psi}_t = 0, \quad \text{in } I_1 \cup I_2 \times (0, \infty), \\ & \epsilon v^{\epsilon}_{i,tt} + \epsilon v^{\epsilon}_{i,t} + \epsilon v^{\epsilon}_i + (-1)^{i+1} S^{\epsilon}_i(\ell_*, t) \\ & \quad = \frac{(-1)^i}{\epsilon} \left[(v^{\epsilon}_1 - v^{\epsilon}_2 - g_1)^+ - (v^{\epsilon}_2 - v^{\epsilon}_1 - g_2)^+ \right] \quad \text{in } (0, \infty), \end{aligned} \tag{3.7}$$

for $i = 1, 2$. Note that the above system is now coupled by the dynamic boundary condition $(3.7)_3$. The above system can be written as

$$U_t - \mathcal{A}U = \mathcal{F}(U), \qquad U(0) = U_0,$$

where $\mathcal{A}$ is given by (2.7) and $\mathcal{F}$ is given by

$$\mathcal{F}(U) = (0, 0, 0, 0, 0, f(v_1, v_2), 0, -f(v_1, v_2))^{\top}, \tag{3.8}$$

with $f(v_1, v_2) = -\frac{1}{\epsilon^2}\left[(v_1^\epsilon - v_2^\epsilon - g_1)^+ - (v_2^\epsilon - v_1^\epsilon - g_2)^+\right]$. Note that $\mathcal{F}$ is a Lipschitz function verifying hypothesis (3.1)–(3.2) and does not depend on i. In fact, $\mathcal{F}(0) = 0$. Moreover,

$$\begin{aligned}
&\int_0^t \langle \mathcal{F}(U(s)), U(s)\rangle_{\mathcal{H}}\, ds \\
&\quad = -\int_0^t \frac{1}{\epsilon^2}\left[(v_1^\epsilon - v_2^\epsilon - g_1)^+ - (v_2^\epsilon - v_1^\epsilon - g_2)^+\right](v_{1,t} - v_{2,t})\, ds \\
&\quad = -\frac{1}{2\epsilon^2}\int_0^t \frac{d}{dt}\left[|(v_1^\epsilon - v_2^\epsilon - g_1)^+|^2 + |(v_2^\epsilon - v_1^\epsilon - g_2)^+|^2\right] ds \\
&\quad \le \frac{1}{2\epsilon^2}\left[|(v_1^\epsilon(0) - v_2^\epsilon(0) - g_1)^+|^2 + |(v_2^\epsilon(0) - v_1^\epsilon(0) - g_2)^+|^2\right].
\end{aligned} \tag{3.9}$$

Remark 3.1 If the initial data verifies

$$\varphi_{0,2}(\ell_*) - g_1 \le \varphi_{0,1}(\ell_*) \le \varphi_{0,2}(\ell_*) + g_2,$$

then the right hand side of the (3.9) vanishes.

Theorem 3.3 *The semilinear semigroup defined by system* (3.7) *is exponentially stable.*

Proof It is a direct consequence of Theorem 3.1. ☐

Next we show the energy inequality

Lemma 3.1 *The solution of system* (3.7) *satisfies*

$$E(t) \le E(0), \tag{3.10}$$

where

$$2E(t) = \int_0^\ell \left[\rho_1|\widehat{\varphi}_t^\epsilon|^2 + \rho_2|\widehat{\psi}_t^\epsilon|^2 + k|\widehat{\varphi}_x^\epsilon + \widehat{\psi}^\epsilon|^2 + b|\widehat{\psi}_x^\epsilon|^2\right]dx + \frac{1}{\epsilon}\mathcal{N}(t) + \sum_{i=1}^2 \epsilon|v_{i,t}^\epsilon|^2 + \epsilon|v_i^\epsilon|^2,$$

and

$$\mathcal{N}(t) := |(v_1^\epsilon - v_2^\epsilon - g_1)^+|^2 + |(v_2^\epsilon - v_1^\epsilon - g_2)^+|^2.$$

Proof Multiplying equation $(3.7)_1$ by $\widehat{\varphi}_t$, equation $(3.7)_2$ by $\widehat{\psi}_t$, and equation $(3.7)_3$ by $v_{i,t}$, summing up the product result our conclusion follows. ☐

Let us introduce the functional

$$\mathcal{I}_i(x,t) = \frac{1}{2}\left[\rho_1|\varphi_{i,t}(x,t)|^2 + \kappa_i|\varphi_{i,x}(x,t)|^2 + \rho_2|\psi_{i,t}(x,t)|^2 + b_i|\psi_{i,x}(x,t)|^2\right].$$

Under the above notations we have

Lemma 3.2 *Let* $]\alpha, \beta[$ *be an interval where* $\widetilde{\kappa}_i = \widetilde{b}_i = 0$, *then the solution of system* (3.7) *satisfies*

$$\int_0^T [\mathcal{I}_i(\alpha, s) + \mathcal{I}_i(\beta, s)]\, ds \le cT E(0).$$

Proof Over the interval $]\alpha,\beta[$ where $\widetilde{\kappa}_i=\widetilde{b}_i=0$, system (3.7) can be written as

$$\begin{aligned} \rho_1\varphi^\epsilon_{i,tt}-\kappa_i\varphi^\epsilon_{i,xx} &= \kappa_i\psi^\epsilon_{i,x}-\gamma_1\varphi_{i,t}, && \text{in }]\alpha,\beta[\times(0,\infty),\\ \rho_2\psi^\epsilon_{i,tt}-b_i\psi^\epsilon_{i,xx} &= -\kappa_i(\varphi^\epsilon_{i,x}+\psi^\epsilon_i)-\gamma_2\psi_{i,t}, && \text{in }]\alpha,\beta[\times(0,\infty). \end{aligned} \tag{3.11}$$

Let us denote by $q=x-\frac{\alpha+\beta}{2}$. Multiplying equation $(3.11)_1$ by $q\varphi_{i,x}$ and equation $(3.11)_2$ by $q\psi_{i,x}$ we get

$$\frac{d}{dt}\int_\alpha^\beta \rho_1 q\varphi^\epsilon_{i,t}\varphi^\epsilon_{i,x}dx-\frac{1}{2}\int_\alpha^\beta q\frac{d}{dx}\left(\rho_1|\varphi^\epsilon_{i,t}|^2+\kappa|\varphi^\epsilon_{i,x}|^2\right)dx=J_1. \tag{3.12}$$

Similarly we get

$$\frac{d}{dt}\int_\alpha^\beta \rho_2\psi^\epsilon_{i,t}q\psi^\epsilon_{i,x}dx-\frac{1}{2}\int_\alpha^\beta q\frac{d}{dx}\left(\rho_2|\psi^\epsilon_{i,t}|^2+b_i|\psi^\epsilon_{i,t}|^2\right)dx=J_2 \tag{3.13}$$

where

$$J_1=\int_\alpha^\beta q(\kappa_i\psi^\epsilon_{i,x}-\gamma_1\varphi^\epsilon_{i,t})\varphi^\epsilon_{i,x}dx,\quad J_2=-\int_\alpha^\beta\left[\kappa_i(\varphi^\epsilon_{i,x}+\psi^\epsilon_i)+\gamma_2\psi_{i,t}\right]q\psi^\epsilon_{i,x}dx.$$

Summing up identities (3.12) and (3.13) performing integrations by parts and integrating overt $[0,T]$ we get

$$\begin{aligned} &\frac{\beta-\alpha}{2}\int_0^T[\mathcal{I}_i(\alpha,s)+\mathcal{I}_i(\beta,s)]\,ds\\ &\quad=\int_0^T\int_\alpha^\beta[\mathcal{I}_1(x,s)+\mathcal{I}_2(x,s)]\,dxds+\int_0^T[J_1(s)+J_2(s)]ds+\mathcal{X}|_0^T \end{aligned} \tag{3.14}$$

where

$$\mathcal{X}=\int_\alpha^\beta\rho_1 q\varphi^\epsilon_{i,t}\varphi^\epsilon_{i,x}dx+\int_\alpha^\beta\rho_2\psi^\epsilon_{i,t}q\psi^\epsilon_{i,x}dx.$$

Using Lemma 3.1 we get

$$\begin{aligned} &\int_0^T\int_\alpha^\beta[\mathcal{I}_1(x,s)+\mathcal{I}_2(x,s)]\,dxds\le cTE(0),\\ &\int_0^T|J_1(s)+J_2(s)|ds\le cTE(0),\quad \left|\mathcal{X}|_0^T\right|\le cE(0). \end{aligned}$$

Substitution of the above inequalities into (3.14), our result follows. □

Let us introduce the convex set

$$\mathcal{K}=\{(u,w)\in H^1(0,T)\times H^1(0,T),\ \ w(t)-g_2\le u(t)\le g_1+w(t)\}.$$

With these notations we have

Theorem 3.4 *For any initial data* $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i) \in \mathcal{H}_i$ *such that*

$$\varphi_{0,2}(\ell_*) - g_1 \le \varphi_{0,1}(\ell_*) \le \varphi_{0,2}(\ell_*) + g_2,$$

there exists a weak solution to Signorini problem (1.1)–(1.4) *which decays as established in Theorem* 3.3.

Proof From Theorem 3.1 we have that there exists only one solution to system (3.7). Let $]\ell_* - \delta, \ell_*[$ be an interval where $\widetilde{\kappa}_i = \widetilde{b}_i = 0$, using Lemma 3.1 and Lemma 3.2, we get

$$\mathcal{I}_\epsilon(\ell_*, t) \quad \text{uniformly bounded in} \quad L^2(0, T), \tag{3.15}$$

which means that the first order energy is uniformly bounded for any $\epsilon > 0$. This implies that $(v_1^\epsilon, v_2^\epsilon)$ strongly converges to $(v_1, v_2) \in \mathcal{K}$. Standard procedures implies that the solution of system (3.7) converges in the distributional sense to system (1.1). It remains only to show that conditions (1.4) holds. First note that

$$\lim_{\epsilon \to 0} S_1^\epsilon(\ell_*, t) = \lim_{\epsilon \to 0} S_2^\epsilon(\ell_*, t).$$

In fact, from $(3.7)_3$ we get

$$\begin{aligned} &\epsilon v_{1,tt}^\epsilon + \epsilon v_{1,t}^\epsilon + \epsilon v_1^\epsilon + S_1^\epsilon(\ell_*, t) \\ &\quad = -\frac{1}{\epsilon}\left[(v_1^\epsilon - v_2^\epsilon - g_1)^+ - (v_2^\epsilon - v_1^\epsilon - g_2)^+\right] \quad \text{in } (0, \infty), \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\epsilon v_{2,tt}^\epsilon + \epsilon v_{2,t}^\epsilon + \epsilon v_2^\epsilon - S_2^\epsilon(\ell_*, t) \\ &\quad = \frac{1}{\epsilon}\left[(v_1^\epsilon - v_2^\epsilon - g_1)^+ - (v_2^\epsilon - v_1^\epsilon - g_2)^+\right] \quad \text{in } (0, \infty). \end{aligned} \tag{3.17}$$

Summing up the above equations we get

$$\epsilon(v_{1,tt}^\epsilon + v_{2,tt}^\epsilon) + \epsilon(v_{1,t}^\epsilon + v_{2,t}^\epsilon) + \epsilon(v_1^\epsilon + v_2^\epsilon) + (S_1^\epsilon(\ell_*, t) - S_1^\epsilon(\ell_*, t)) = 0 \quad \text{in } (0, \infty).$$

Then for any $\eta \in C_0^\infty(\mathbb{R}_+)$ we get

$$\int_{\mathbb{R}_+} (S_1^\epsilon(\ell_*, t) - S_1^\epsilon(\ell_*, t))\eta \, dt = -\int_{\mathbb{R}_+} \epsilon(v_1^\epsilon + v_2^\epsilon)(\eta_{tt} - \eta_t + \eta) dt \to 0.$$

So we denote by $S(\ell_*, t) = \lim_{\epsilon \to 0} S_1^\epsilon(\ell_*, t) = \lim_{\epsilon \to 0} S_2^\epsilon(\ell_*, t)$. Finally, we use the observability inequality in Theorem 3.2, and we get that $\widehat{\varphi}_t^\epsilon(\ell, t)$ and $S^\epsilon(\ell, t)$ are bounded in $L^2(0, T)$, so is v_{tt}. Using $(3.7)_3$ we obtain

$$\begin{aligned} &\int_0^T \left[\epsilon v_{1,tt}^\epsilon + \epsilon v_{1,t}^\epsilon + \epsilon v_1^\epsilon + S_1^\epsilon(\ell_*, t)\right][u - v_1^\epsilon] \, dt \\ &\quad = -\frac{1}{\epsilon}\int_0^T \left[(v_1^\epsilon - v_2^\epsilon - g_1)^+ - (v_2^\epsilon - v_1^\epsilon - g_2)^+\right][u - v_1^\epsilon] \, dt, \end{aligned}$$

for any $u \in \mathcal{K}$. It is no difficult to see that

$$\lim_{\epsilon \to 0} \int_0^T \left(\epsilon v_{tt}^\epsilon + \epsilon v_t^\epsilon + \epsilon v^\epsilon\right)[u - v_1^\epsilon] \, dt = 0.$$

In fact, from (3.7)$_3$ ϵv^ϵ_{tt} is bounded for any $\epsilon > 0$ (by a constant depending on ϵ) in $L^2(0,T)$. From (3.15) $v^\epsilon_{i,t}$ is also uniformly bounded in $L^2(0,T)$. Therefore $v^\epsilon_{i,t}$ is a continuous function, uniformly bounded in $L^\infty(0,T)$. Making an integration by parts we find

$$\int_0^T \epsilon v^\epsilon_{i,tt}[u - v^\epsilon_1]\,dt = \epsilon\; v^\epsilon_{i,t}[u - v^\epsilon_1]\big|_0^T - \int_0^T \epsilon v^\epsilon_{i,t}[u_t - v^\epsilon_{1,t}]\,dt \quad \to \quad 0.$$

Hence,

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T S^\epsilon_1(\ell,t)[u - v^\epsilon_1]\,dt\\ &\qquad = \lim_{\epsilon\to 0}\int_0^T -\frac{1}{\epsilon}\left[(v^\epsilon_1 - v^\epsilon_2 - g_1)^+ - (v^\epsilon_2 - v^\epsilon_1 - g_2)^+\right][u - v^\epsilon_1]\,dt.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^T (v^\epsilon_1 - v^\epsilon_2 - g_1)^+[u - v^\epsilon_1]\,dt\\ &\quad = \int_0^T (v^\epsilon_1 - v^\epsilon_2 - g_1)^+[u - v^\epsilon_2 - g_1]\,dt - \int_0^T (v^\epsilon_1 - v^\epsilon_2 - g_1)^+(v^\epsilon_1 - v^\epsilon_2 - g_1)\,dt\\ &\quad = \int_0^T (v^\epsilon_1 - v^\epsilon_2 - g_1)^+[u - v^\epsilon_2 - g_1]\,dt - \int_0^T |(v^\epsilon_1 - v^\epsilon_2 - g_1)^+|^2\,dt \le 0,\end{aligned}$$

for all $u \le v^\epsilon_2 + g_1$. Similarly we find

$$-\int_0^T [(v^\epsilon_2 - v^\epsilon_1 - g_2)^+[u - v^\epsilon_1]\,dt \le 0,$$

for all $u \ge v^\epsilon_2 - g_2$. Therefore, from the last two inequalities we get

$$\int_0^T \frac{1}{\epsilon}\Big[(v^\epsilon_1 - v^\epsilon_2 - g_1)^+ - (v^\epsilon_2 - v^\epsilon_1 - g_2)^+\Big][u - v^\epsilon_1]\,dt \le 0, \quad \forall \epsilon > 0,$$

for any $u \in \mathcal{K}$ such that $v^\epsilon_2 - g_2 \le u \le v^\epsilon_2 + g_1$. Taking the limit $\epsilon \to 0$ we get

$$\int_0^T S_1(\ell_*, t)[u - v_1]\,dt \ge 0, \quad \forall (u, v_2) \in \mathcal{K}. \tag{3.18}$$

Using the same above procedure to equation (3.17) we get

$$-\int_0^T S_2(\ell_*, t)[w - v_2]\,dt \ge 0, \quad \forall (v_1, w) \in \mathcal{K}. \tag{3.19}$$

From relations (3.18)–(3.19) and since $S_1(\ell_*, t) = S_2(\ell_*, t)$, we get (1.5).

In case of $\ell_* \in \operatorname{supp}(\widetilde{\kappa}_i)$ we have that φ^ϵ_t is bounded in $L^2(0,T;H^1(]\ell_* - \delta, \ell_*[))$. So the sequences $\varphi^\epsilon_t(\ell^*, t)$ converges strongly in $C(0,T)$ as $\epsilon \to 0$. Then, the above procedure is also valid. Hence, the proof of the existence is now complete. To show the asymptotic behavior, we get

$$E(t, \varphi^\epsilon, \psi^\epsilon) \le E(0, \varphi^\epsilon, \psi^\epsilon)e^{-\gamma t}.$$

Integrating over $[t_1, t_2]$ and applying the semicontinuity of the norm, we obtain the exponential stability of a solution of the Signorini problem. ☐

Remark 3.2 The uniqueness of the solution to Signorini problem (1.1)–(1.4) remains an open question.

4 Applications

In this section we present some applications of exponential stability for the contact problem between two dissipative Timoshenko beams. For that, we use known results of exponential stability for linear models of Timoshenko beams and extend it first to the hybrid models, using Theorem 2.2, then we apply Theorem 3.1 and Theorem 3.2 to show that there is a solution to the Signorini contact problem between two Timoshenko beams that decay exponentially to zero.

4.1 Continuous and Discontinuous Viscoelastic Constitutive Law

In [1] the authors studied the oscillations of a beam of length ℓ, configurated over the interval $]0, \ell[$, and splitted in three components: an elastic part I_E, without dissipative mechanism acting over it, and two viscous parts, one of them with a continuous constitutive law we denote as I_C and the other, I_D with discontinuous (discontinuity of the first kind at the border of I_D) constitutive law. This components are positioned over the intervals $I_1 =]0, \ell_0[$, $I_2 =]\ell_0, \ell_1[$, $I_3 =]\ell_1, \ell[$. We denote by $\widetilde{I} = I_1 \cup I_2 \cup I_3$. Under this conditions the constitutive law are given by

$$S = \kappa(\varphi_x + \psi) + \widetilde{\kappa}(\varphi_{xt} + \psi_t), \quad M = b\psi_x + \widetilde{b}\psi_{xt}, \tag{4.1}$$

with $\tilde{\kappa}$ and $\tilde{b}$ functions of the following type:

$$\widetilde{\kappa} = \kappa_0 + \kappa_1, \qquad \widetilde{b} = b_0 + b_1, \tag{4.2}$$

where κ_0 and b_0 are discontinuous functions over $]0, \ell[$ vanishing out side of I_D. Instead κ_1 and b_1 are $C^1(0, \ell)$ functions vanishing out side of I_C, verifying

$$|b_1'(x)|^2 \le c|b_1(x)|, \quad |\kappa_1'(x)|^2 \le c|\kappa_1(x)|, \tag{4.3}$$

and the existence of certain positive constants C_1, C_2 such that

$$C_1\kappa_1 \le b_1 \le C_2\kappa_1. \tag{4.4}$$

Typical examples for the function $\tilde{\kappa} = \kappa_0 + \kappa_1$ are given in Figs. 2 and 3 ($\tilde{b}$ is similar).

Taking $S_1 = S$, $M_1 = M$ defined over $]0, \ell^*[$ and $S_2 = S$, $M_2 = M$ defined over $]\ell^*, \ell[$, under the above conditions the authors proved (see [1]).

Theorem 4.1 *The semigroup $\mathbf{T}_i$ associated to the linear Timoshenko system* (2.14)–(2.16) *is exponentially stable if the viscous discontinuous part I_D is not in the center of the beam, provided* (4.3) *and* (4.4) *holds.*

Using our approach we extend the above result to the Signorini contact between two Timoshenko beams.

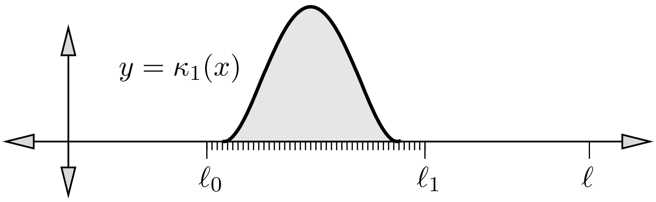

Fig. 2 An example of function κ_1

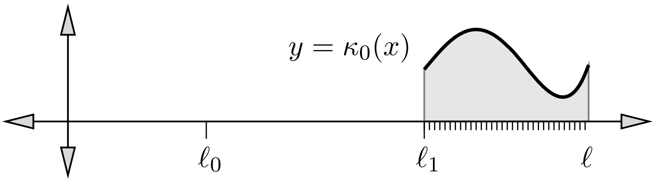

Fig. 3 An example of function κ_0

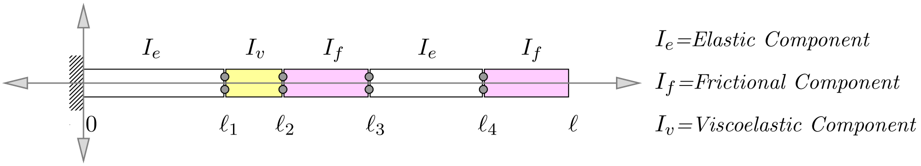

Fig. 4 An example of five-components beam

Theorem 4.2 *Assume that the constitutive law of the Signorini problem* (1.1)–(1.5) *with* $\gamma_{1,i}=\gamma_{2,i}=0$ *are given by* (4.1) *and* (4.2). *Then, for any* $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i) \in \mathcal{H}_i$ *such that*

$$\varphi_{0,2}(\ell_*) - g_1 \le \varphi_{0,1}(\ell_*) \le \varphi_{0,2}(\ell_*) + g_2,$$

there exists a weak solution which decays exponentially, provided that conditions (4.3) *and* (4.4) *hold and the discontinuous viscous component* I_D *is neither in the center of the first beam configured on* $]0, \ell^*[$ *nor in the center of the second beam configured on* $]\ell^*, \ell[$.

The above problem can be generalized to N-components discussed in the next subsection.

4.2 The Viscoelastic Beam with N-Components

In [13] the authors consider the transmission problem of a Timoshenko beam of length ℓ composed by N components, each of them can be of three different types of materials: elastic, viscoelastic or a material with a frictional damping mechanism as illustrated in Fig. 4, for $N=5$.

Unlike the case studied in [1] and described in Sect. 4.1, here only viscous components with discontinuous constitutive law are considered.

Let us decompose the interval $I=[0,\ell]$ into N subintervals, $[0,\ell]=\bigcup_{i=1}^{n}\overline{I_i}$ such that

$$I_i =]\ell_{i-1}, \ell_i[\quad \text{for} \quad i=1,2,\ldots,N, \qquad \text{with } \ell_0=0,\ \ell_N=\ell.$$

Over each interval I_i is configured one type of material. We denote by I_v, I_e or I_f the subinterval where the viscoelastic component, elastic component or the component with frictional mechanism is configured, respectively. In Fig. 4 the intervals I_1 and I_4 are of type I_e, elastic components, $I_2 =]\ell_1, \ell_2[$ is of viscoelastic type I_v and so on. Let us denote by $\widetilde{I}$ the set

$$\widetilde{I} = \bigcup_{i=1}^{n} I_i \; =]0, \ell[\setminus \{\ell_0, \ell_1, \dots, \ell_N\}.$$

$\widetilde{I}$ is a disconnected open set. The classical linear Timoshenko system given by

$$\varrho_1 \varphi_{tt}(x,t) - S_x(x,t) = -\gamma_1(x)\varphi_t(x,t), \quad \text{in} \quad \widetilde{I} \times \mathbb{R}_+, \tag{4.5}$$

$$\varrho_2 \psi_{tt}(x,t) - M_x(x,t) + S(x,t) = -\gamma_2(x)\psi_t(x,t), \quad \text{in} \quad \widetilde{I} \times \mathbb{R}_+, \tag{4.6}$$

where γ_1, γ_2 are positive only on the intervals I_f, vanishing over I_v and I_e. Here, we consider the following Dirichlet boundary conditions:

$$\varphi(0,t) = \varphi(\ell,t) = \psi(0,t) = \psi(\ell,t) = 0, \tag{4.7}$$

and the initial conditions

$$\varphi(x,\,0) = \varphi_0(x), \quad \psi(x,\,0) = \psi_0(x), \qquad \varphi_t(x,\,0) = \varphi_1(x), \quad \psi_t(x,\,0) = \psi_1(x). \tag{4.8}$$

The constitutive equations are given by

$$S(\varphi_x,\,\psi) = \kappa\,(\varphi_x + \psi) + \kappa_0\,(\varphi_{xt} + \psi_t), \qquad M(\psi) = b\,\psi_x + b_0\,\psi_{xt}. \tag{4.9}$$

We denote by b_0 and κ_0, positive functions which characterize the viscosity over I_v, vanishing over $I_e \cup I_f$. Therefore the elastic coefficients are discontinuous at the points where different materials are fitted. This characterizes the transmission problem. Hence the functions $\kappa,\ \kappa_0,\ b,\ b_0,\ \gamma_1,\ \gamma_2 : [0,\ell] \to \mathbb{R}$ are such that its restrictions to I_i, $i = 1, \dots, N$, are C^1 functions, with bounded discontinuities at the nodes ℓ_i, $i = 1, \dots, N-1$. But even so, the stress as well as the bending moment must satisfy the laws of action and reaction at each point, therefore we have that any strong solutions of the problem must verify $\varphi,\ \psi,\ S,\ M \in H^1(0,\ell)$, which in particular implies the transmission conditions at the interface points ℓ_i:

$$\varphi(\ell_i^-) = \varphi(\ell_i^+), \quad S(\ell_i^-) = S(\ell_i^+), \qquad \psi(\ell_i^-) = \psi(\ell_i^+), \quad M(\ell_i^-) = M(\ell_i^+), \tag{4.10}$$

for $i = 1, \dots N-1$. A typical example of a function $y = \kappa_0(x)$ is given in Fig. 5.

A similar graph would hold for function b_0. The frictional mechanism is characterized by the functions $y = \gamma_i(x)$, with $i = 1, 2$, for the same example is given as Fig. 6.

The authors establish in [13] the following result:

Theorem 4.3 *The transmission problem* (4.5)–(4.10) ($N \geqslant 2$) *is exponentially stable if and only if any elastic part of the beam is connected with at least one component with frictional damping mechanisms. Otherwise the system is polynomially stable, with a rate of decay of the order* t^{-2}.

Taking $S_1 = S$, $M_1 = M$ defined over $]0, \ell^*[$ and $S_2 = S$, $M_2 = M$ defined over $]\ell^*, \ell[$, under the above conditions the authors proved

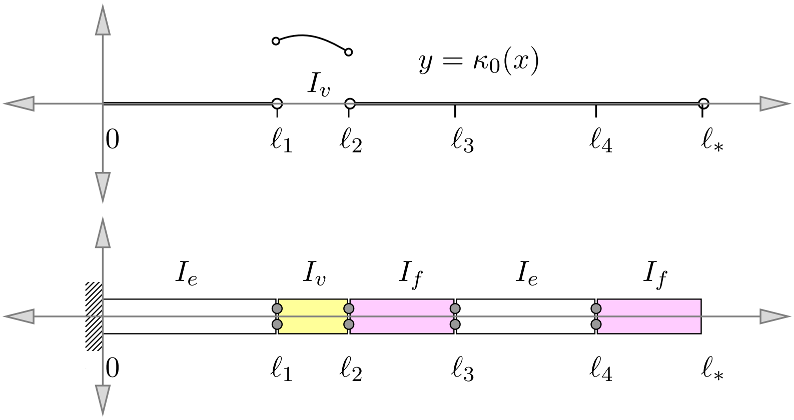

Fig. 5 An example of a function $y = \kappa_0(x)$ for a five-components beam

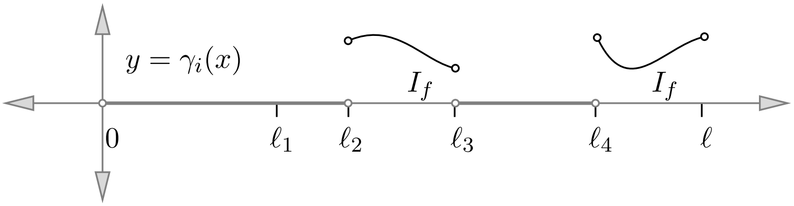

Fig. 6 An example of the functions $y = \gamma_i(x)$, with $i = 1, 2$

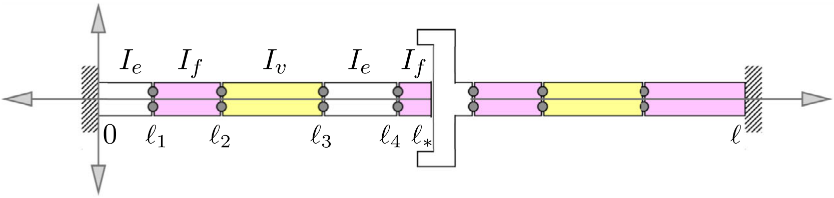

Fig. 7 Signorini contact between two beams

Theorem 4.4 *The semigroup* $\mathbf{T}_i$ *associated to the linear Timoshenko system* (4.5)–(4.10) *is exponentially stable if and only if any elastic part of the beam is connected with at least one component with frictional damping mechanisms.*

Therefore using our approach we extend the result in [13] to the Signorini contact problem between to Timoshenko beams (see Fig. 7).

Theorem 4.5 *Assume that the constitutive law of the Signorini problem* (1.1)–(1.5) *with* $\gamma_{1,i} \geq 0$ *and* $\gamma_{2,i} \geq 0$ *are positive only on the intervals* I_f*, vanishing over* I_v *and* I_e*. Then for any* $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i) \in \mathcal{H}_i$ *such that*

$$\varphi_{0,2}(\ell_*) - g_1 \leq \varphi_{0,1}(\ell_*) \leq \varphi_{0,2}(\ell_*) + g_2,$$

there exists a weak solution which decays exponentially, provided any elastic part of the beam is connected with at least one component with frictional damping mechanisms.

4.3 More General Semi Linear Problem

Theorem 3.4 and Theorem 4.5 can be easily extended to the semi linear Signorini problem

$$\begin{aligned} &\rho_1\varphi_{i,tt} - S_{i,x} + \gamma_{1,i}\varphi_{i,t} + \mu_{1,i}\varphi_i|\varphi_i|^{\alpha} = 0, && (x,t) \in I_i \times (0,T), \\ &\rho_2\psi_{i,tt} - M_{i,x} + S_i + \gamma_{2,i}\psi_{i,t} + \mu_{2,i}\psi_i|\psi_i|^{\beta} = 0, && (x,t) \in I_i \times (0,T), \end{aligned} \tag{4.11}$$

verifying conditions (1.2)–(1.5).

Theorem 4.6 *Under the same hypothesis of Theorem 3.4 or Theorem 4.5, there is at least one solution to Signorini problem (4.11) verifying conditions (1.2)–(1.5) that decays exponentially to zero.*

Proof Here we use Theorem 3.3 for the function

$$\mathcal{F}(U) = (0, -\mu_1\widehat{\varphi}^{\epsilon}|\widehat{\varphi}^{\epsilon}|^{\alpha}, 0, -\mu_2\widehat{\psi}^{\epsilon}|\widehat{\psi}^{\epsilon}|^{\beta}, 0, f(v_1, v_2), 0, -f(v_1, v_2))^{\top},$$

instead of such given in (3.8). Note that $\mathcal{F}(0) = 0$. Using the mean value theorem to $g(s) = |s|^{\alpha}s$ we obtain the inequality

$$\Big|s|s|^{\alpha} - r|r|^{\alpha}\Big| \leq (\alpha + 1)(|s|^{\alpha} + |r|^{\alpha})|s - r|.$$

Taking the norm in $\mathcal{H}$ and since φ_i^{ϵ} and ψ_i^{ϵ} belong to $H^1(0,\ell) \subset L^{\infty}(0,\ell)$, then we get

$$\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\|_{\mathcal{H}} \leq C\|U_1 - U_2\|_{\mathcal{H}}.$$

Therefore, $\mathcal{F}$ is locally Lipschtiz. Since

$$\begin{aligned} (\mathcal{F}U, U)_{\mathcal{H}} = &-\frac{d}{dt}\int_0^{\ell}\left(\frac{\mu_1}{1+\alpha}|\widehat{\varphi}^{\epsilon}|^{\alpha+2} + \frac{\mu_2}{1+\beta}|\widehat{\psi}^{\epsilon}|^{\beta+2}\right)dx \\ &-\frac{1}{2\epsilon^2}\int_0^t \frac{d}{dt}\left[|(v_1^{\epsilon} - v_2^{\epsilon} - g_1)^+|^2 + |(v_2^{\epsilon} - v_1^{\epsilon} - g_2)^+|^2\right] ds \end{aligned}$$

then

$$\begin{aligned} \int_0^t (\mathcal{F}U, U)_{\mathcal{H}} \leq &\int_0^{\ell}\left(\frac{\mu_1}{1+\alpha}|\widehat{\varphi}^{\epsilon}(0)|^{\alpha+2} + \frac{\mu_2}{1+\beta}|\widehat{\psi}^{\epsilon}(0)|^{\beta+2}\right)dx \\ &+\frac{1}{2\epsilon^2}\left[|(v_1^{\epsilon}(0) - v_2^{\epsilon}(0) - g_1)^+|^2 + |(v_2^{\epsilon}(0) - v_1^{\epsilon}(0) - g_2)^+|^2\right]. \end{aligned}$$

Thus, there exists a positive constant c_0 such that

$$\int_0^t (\mathcal{F}U, U)_{\mathcal{H}} \leq c_0\|U\|_{\mathcal{H}}^2.$$

Note that, for this function, there exists the cut-off function

$$f_{1,R_2} = \begin{cases} \mu_1 x|x|^{\alpha} & \text{if } x \leq R_2, \\ \mu_1 x|R_2|^{\alpha} & \text{if } x \geq R_2, \end{cases} \qquad f_{2,R_2} = \begin{cases} \mu_2 x|x|^{\beta} & \text{if } x \leq R_2, \\ \mu_2 x|R_2|^{\beta} & \text{if } x \geq R_2. \end{cases}$$

It is not difficult to check that

$$\widetilde{\mathcal{F}}_{R_2} = (0, -f_{1,R_2}, 0, -f_{2,R_2}, 0, f(v_1, v_2), 0, -f(v_1, v_2))^\top$$

is globally Lipschtiz. Using Theorem 3.3, Theorem 4.2 and Theorem 4.5 our conclusion follows. ☐

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Declarations

Competing interests The authors declare no competing interests.

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