

Tight globally simple nonzero sum Heffter arrays and biembeddings

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Abstract

Square relative nonzero sum Heffter arrays, denoted by $NH_t(n; k)$, have been introduced as a variant of the classical concept of Heffter array. An $NH_t(n; k)$ is an $n \times n$ partially filled array with elements in \mathbb{Z}_v , where $v = 2nk + t$, whose rows and whose columns contain k filled cells, such that the sum of the elements in every row and column is different from 0 (modulo v) and, for every $x \in \mathbb{Z}_v$ not belonging to the subgroup of order t , either x or $-x$ appears in the array. In this paper we give direct constructions of square nonzero sum Heffter arrays with no empty cells, $NH_t(n; n)$, for every n odd, when t is a divisor of n and when $t \in \{2, 2n, n^2, 2n^2\}$. The constructed arrays have also the very restrictive property of being “globally simple”; this allows us to get new orthogonal path decompositions and new biembeddings of complete multipartite graphs.

KEYWORDS

biembedding, complete multipartite graph, Heffter array, orthogonal cyclic path decomposition

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1 | INTRODUCTION

An $m \times n$ partially filled (p.f.) array on a set Ω is an $m \times n$ matrix whose elements belong to Ω and where some cells can be empty. In [1], Archdeacon introduced a class of p.f. arrays on a cyclic group, called *Heffter arrays*, which have been extensively studied since they have a vast

variety of applications and connections with other well known problems and concepts. Such arrays have been generalized as follows, see [13].

Definition 1.1. Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, where t divides $\frac{2nk}{\lambda}$, and let J be the subgroup of \mathbb{Z}_v of order t . A λ -fold Heffter array A over \mathbb{Z}_v relative to J , denoted by ${}^\lambda H_t(m, n; h, k)$, is an $m \times n$ p.f. array with elements in \mathbb{Z}_v such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b) the multiset $\{\pm x \mid x \in A\}$ contains each element of $\mathbb{Z}_v \setminus J$ exactly λ times;
- (c) the elements in every row and column sum to 0 (in \mathbb{Z}_v).

In the following, by t admissible we mean that t divides $\frac{2nk}{\lambda}$. If $t = \lambda = 1$ one finds again the concept introduced in [1]. Also if $t = 1$ or $\lambda = 1$, then it is omitted, for example, a ${}^1 H_1(m, n; h, k)$ is simply denoted by $H(m, n; h, k)$. Note that $nk = mh$, so if the array is square then $h = k$ and by ${}^\lambda H_t(n; k)$ one means a ${}^\lambda H_t(n, n; k, k)$. The most important result on Heffter arrays with $\lambda = t = 1$ is the following one, see [2, 4, 16].

Theorem 1.2. For every $n \geq k \geq 3$ there exists an $H(n; k)$.

We have to emphasize that Heffter arrays are not only combinatorial objects interesting per se, in fact there are several papers in which they are investigated to get face 2-colorable embeddings, namely biembeddings, see [1, 5, 6, 11–15]. For variants and applications of Heffter arrays, and some related topics see [20].

In a very recent paper, see [10], Costa, Della Fiore and Pasotti introduce a variant of Heffter arrays, that they call *nonzero sum Heffter arrays* since they require that all the properties of Heffter arrays are satisfied except that on row and column sums which now have to be all different from zero.

Definition 1.3. Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, where t divides $\frac{2nk}{\lambda}$, and let J be the subgroup of \mathbb{Z}_v of order t . A λ -fold nonzero sum Heffter array A over \mathbb{Z}_v relative to J , denoted by ${}^\lambda NH_t(m, n; h, k)$, is an $m \times n$ p.f. array with elements in \mathbb{Z}_v such that:

- (a₁) each row contains h filled cells and each column contains k filled cells;
- (b₁) the multiset $\{\pm x \mid x \in A\}$ contains each element of $\mathbb{Z}_v \setminus J$ exactly λ times;
- (c₁) the sum of the elements in every row and column is different from 0 (in \mathbb{Z}_v).

Also this variant is related to several other topics, in fact in [10] the authors explain its connection to a famous conjecture by Alspach, to orthogonal path decompositions and to biembeddings. We point out that nonzero sums Heffter arrays are also strongly related to magic rectangles, see for instance [17]. In [10], using a probabilistic approach, the authors provide a complete (nonconstructive) solution for an $NH_t(m, n; h, k)$ for any admissible value of the parameters. Moreover, they construct square nonzero sum Heffter arrays with empty cells and rectangular ones with no empty cells that satisfy the very restrictive property of being “globally simple” when $\lambda = t = 1$. The case of λ -fold nonzero sum Heffter arrays with $\lambda > 1$ has been

investigated in [8]. Here the authors consider a generic finite group G and prove that there exists a λ -fold nonzero sum Heffter array over G relative to J whenever the trivial necessary conditions are satisfied and $|G| \geq 41$. Moreover, this value can be decreased to 29 when the array does not contain empty cells. Other results on relative nonzero sum Heffter arrays can be found in [6, 18].

Also in this paper we focus on nonzero sum Heffter arrays having the additional property of being “globally simple,” here we take $\lambda = 1$, $t > 1$ and we suppose that the array is *tight*, that is with no empty cells. In fact, after having introduced some basic definitions in Section 2, in Section 3 we consider totally filled square nonzero sum Heffter arrays. In particular we present several classes of globally simple $NH_t(n; n)$ for n odd which lead to a complete solution to the existence problem for every n prime and admissible t , and for every odd n and t divisor of n . Finally in Sections 4 and 5 we explain how relative nonzero sum Heffter arrays can be used to construct cyclic path decompositions of the complete multipartite graph and biembeddings of such decompositions into an orientable surface, respectively. As a consequence, thanks to the existence results described in Section 3, we provide new classes of such decompositions and biembeddings.

2 | SIMPLE ORDERINGS

Let a, b be two integers and suppose $a \leq b$, by $[a, b]$ we denote the set $\{a, a + 1, a + 2, \dots, b\}$. Also, given a set $S = \{a_1, a_2, \dots, a_k\}$ and an integer x , by $S + x$ we mean $\{a_1 + x, a_2 + x, \dots, a_k + x\}$.

Given a finite subset T of an abelian group G with $\sum_{t \in T} t \neq 0$ and an ordering $\omega = (t_1, t_2, \dots, t_k)$ of the elements of T , let $s_i = \sum_{j=1}^i t_j$, for any $i \in [1, k]$, be the i th partial sum of ω and let $S(\omega) = \{s_1, \dots, s_k\}$ be the unordered list of the partial sums of T computed with respect to ω . The ordering ω is *simple* if all the partial sums are nonzero and pairwise distinct or, in other words, if there is no proper subsequence of ω that sums to 0. Note that if ω is a simple ordering so is $\omega^{-1} = (t_k, t_{k-1}, \dots, t_1)$.

If A is an $m \times n$ p.f. array, its rows and columns are denoted by R_1, \dots, R_m and by C_1, \dots, C_n , respectively. By $\mathcal{E}(A)$ we mean the unordered list of the elements of the filled cells of A . Analogously, by $\mathcal{E}(R_i)$ and $\mathcal{E}(C_j)$ we denote the unordered lists of elements of the i th row and of the j th column, respectively, of A . By ω_{R_i} and ω_{C_j} we mean a simple ordering of the i th row and of the j th column, respectively. If A is a nonzero sum Heffter array, by a *natural ordering* of a row (column) of A we mean the ordering from left to right (from top to bottom) starting from a given element of such a row (column). Also, A is said

- *simple* if each row and each column of A admits a simple ordering;
- *globally simple* if each row and each column of A admits a simple natural ordering.

Remark 2.1. Note that there is an important difference in the definition of the property of being globally simple for Heffter arrays (see [3, 11]) and for nonzero sum Heffter arrays. In fact for the first ones each row and each column has exactly one natural ordering, while if we have a nonzero sum Heffter array the number of natural orderings of a row (column) is equal to the number of its filled cells, since it depends on the starting element. Suppose for instance that $R = (1, 10, 4, -11)$ is a row of an $NH(4; 4)$. If we compute its partial sums with respect to the natural ordering starting from 1 we get $S(R) = (1, 11, 15, 4)$, and hence the ordering is simple, while if we compute its partial sums with respect to the natural ordering starting from 4 we get $S(R) = (4, -7, -6, 4)$

and hence the ordering is not simple. So it is clear that when we compute the partial sums of a row or of a column we have always to indicate the starting element. In the following, given an array $A = (a_{i,j})$, with a little abuse of notation, by $S(R_i)$ ($S(C_j)$) we will denote the unordered list of the partial sums of the row R_i (the column C_j) with respect to the natural ordering calculated starting from the element $a_{i,1}$ ($a_{1,j}$).

Remark 2.2. If a row or a column admits a simple natural ordering, then all its partial sums are different from zero and hence, in particular, also the sum of all its elements is different from 0. Hence in our constructions of globally simple nonzero sum Heffter arrays first we prove only that conditions (a_1) and (b_1) of Definition 1.3 are satisfied. Then we show that each row and each column admits a simple natural ordering, obtaining as a consequence that also condition (c_1) is satisfied.

Now we can explain the connection between nonzero sum Heffter arrays and the following conjecture proposed by Alspach.

Conjecture 2.3. *Let $T \in \mathbb{Z}_v \setminus \{0\}$ with $\sum_{t \in T} t \neq 0$, then there exists a simple ordering of T .*

Clearly, if it holds, then every nonzero sum Heffter array is simple. Several partial results on Conjecture 2.3 have been obtained, see [9] and the references therein.

3 | CONSTRUCTIONS OF GLOBALLY SIMPLE $NH_t(n; n)$, n ODD

In this section, we consider tight square relative nonzero sum Heffter arrays, namely $NH_t(n; n)$; in details for every odd $n \geq 1$ we present direct constructions of a globally simple $NH_t(n; n)$ for $t \in \{2, 2n, n^2, 2n^2\}$ and whenever t divides n . In particular, this will allow us to give a complete solution to the existence problem of a globally simple $NH_t(n; n)$ for every prime n and every admissible t .

To present these constructions we have to introduce the concept of *support* of an array, of a row and of a column. Given a relative nonzero sum Heffter array $NH_t(n; n)$, say A , and viewed its elements as integers in $\pm\left\{1, \dots, \left\lfloor \frac{2n^2+t}{2} \right\rfloor\right\}$ the *support* of A , denoted by $\text{supp}(A)$, is defined to be the set of absolute values of the elements contained in A . Analogously one can define the support of a row and of a column of A . Note that if A is an $NH_t(n; n)$, then its elements belong to \mathbb{Z}_{2n^2+t} and we have to avoid the elements of the subgroup of \mathbb{Z}_{2n^2+t} of order t , denoted by $\frac{2n^2+t}{t}\mathbb{Z}_{2n^2+t}$.

The following lemma will play a fundamental role in showing that the arrays we are going to construct are globally simple. Its proof is trivial and hence it is left to the reader.

Lemma 3.1. *Let a, b and g be elements in an abelian group G , and let ℓ be a positive integer. Consider the orderings $\omega = (a, b, a + g, b - g, a + 2g, b - 2g, \dots, b - (\ell - 1)g, a + \ell g)$ and $\nu = (a, b, a + g, b - g, a + 2g, b - 2g, \dots, a + \ell g, b - \ell g)$. Then:*

(1) *the unordered list of the partial sums of the ordering ω is*

$$S(\omega) = \{k(a + b) : 1 \leq k \leq \ell\} \cup \{a + k(a + b + g) : 0 \leq k \leq \ell\};$$

(2) the unordered list of the partial sums of the ordering ω^{-1} is

$$S(\omega^{-1}) = \{k(a + b + g) : 1 \leq k \leq \ell\} \cup \{a + \ell g + k(a + b) : 0 \leq k \leq \ell\};$$

(3) the unordered list of the partial sums of the ordering ν is

$$S(\nu) = \{k(a + b) : 1 \leq k \leq \ell + 1\} \cup \{a + k(a + b + g) : 0 \leq k \leq \ell\};$$

(4) the unordered list of the partial sums of the ordering ν^{-1} is

$$S(\nu^{-1}) = \{k(a + b) : 1 \leq k \leq \ell + 1\} \cup \{b - \ell g + k(a + b + g) : 0 \leq k \leq \ell\}.$$

We point out that all the following proofs are constructive, in fact we give direct constructions of a globally simple $NH_t(n; n)$, for suitable values of n and t , and we show that all the required properties are satisfied, keeping in mind Remarks 2.1 and 2.2.

3.1 | Globally simple $NH_2(n; n)$

The arrays described in the following proposition can be found in [10, Theorem 5.5], where a construction of a globally simple $NH(m, n; n, m)$ for every $m, n \geq 1$ is given. Here, we show that for every integer $n = m$ the same array is also a globally simple $NH_2(n; n)$. We underline that in this construction we do not require n odd.

Proposition 3.2. *For every integer $n \geq 1$, there exists a globally simple $NH_2(n; n)$.*

Proof. Let $n \geq 1$ be an integer, and consider the $n \times n$ matrix $A = (a_{i,j})$ with elements in \mathbb{Z}_{2n^2+2} whose (i, j) th entry is:

$$a_{i,j} = \varepsilon [j + (i - 1)n],$$

where $\varepsilon = 1$ if $i \equiv j \pmod{2}$, and $\varepsilon = -1$ otherwise. Since the array is tight and, as shown in [10], $\text{supp}(A) = [1, n^2]$, conditions (a₁) and (b₁) of Definition 1.3 are satisfied.

It now remains to check that the array satisfies condition (c₁) of Definition 1.3 and it is globally simple. As proved in [10], the unordered list of the partial sums of the j th column of A , for $j \in [1, n]$, is $S(C_j) = X_j \cup Y_j$, where:

$$X_j = \left\{ (-1)^j kn : k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right] \right\},$$

$$Y_j = \left\{ (-1)^{j+1} (j + kn) : k \in \left[0, \left\lfloor \frac{n-1}{2} \right\rfloor \right] \right\}.$$

It can be easily seen that X_j and Y_j are disjoint sets modulo $2n^2 + 2$, containing only nonzero elements.

Now, for every $i \in [1, n]$ the unordered list of partial sums of the i th row of A is $S(R_i) = X_i \cup Y_i$, where:

$$X_i = \left\{ (-1)^i k : k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right] \right\},$$

$$Y_i = \left\{ (-1)^{i+1} (1 + (i-1)n + k) : k \in \left[0, \left\lfloor \frac{n-1}{2} \right\rfloor \right] \right\}.$$

As before, it can be seen that X_i and Y_i are disjoint sets modulo $2n^2 + 2$, and that no element in these sets is zero.

Hence, for every integer $n \geq 1$ the array is globally simple, so, in particular, every row and column has nonzero sum, proving that also condition (c₁) of Definition 1.3 is satisfied. \square

Remark 3.3. From the globally simple array A shown in Proposition 3.2, the total sum of the elements of each row and of each column can be easily found, and it depends on the parity of n . For every odd integer $n \geq 1$, considering the elements of Y_j and of Y_i obtained for $k = \frac{n-1}{2}$, we get that for every $i, j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left(j + \frac{n-1}{2} n \right);$$

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i+1} \left((i-1)n + \frac{n+1}{2} \right).$$

Analogously, for every even integer $n \geq 1$, considering the elements of X_j and of X_i obtained when $k = \frac{n}{2}$, we have that for every $i, j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^j \frac{n^2}{2};$$

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^i \frac{n}{2}.$$

Example 3.4. Following the proof of Proposition 3.2, we obtain the $NH_2(11; 11)$ below, whose elements belong to \mathbb{Z}_{244} :

| | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|
| 1 | -2 | 3 | -4 | 5 | -6 | 7 | -8 | 9 | -10 | 11 |
| -12 | 13 | -14 | 15 | -16 | 17 | -18 | 19 | -20 | 21 | -22 |
| 23 | -24 | 25 | -26 | 27 | -28 | 29 | -30 | 31 | -32 | 33 |
| -34 | 35 | -36 | 37 | -38 | 39 | -40 | 41 | -42 | 43 | -44 |
| 45 | -46 | 47 | -48 | 49 | -50 | 51 | -52 | 53 | -54 | 55 |
| -56 | 57 | -58 | 59 | -60 | 61 | -62 | 63 | -64 | 65 | -66 |
| 67 | -68 | 69 | -70 | 71 | -72 | 73 | -74 | 75 | -76 | 77 |
| -78 | 79 | -80 | 81 | -82 | 83 | -84 | 85 | -86 | 87 | -88 |
| 89 | -90 | 91 | -92 | 93 | -94 | 95 | -96 | 97 | -98 | 99 |
| -100 | 101 | -102 | 103 | -104 | 105 | -106 | 107 | -108 | 109 | -110 |
| 111 | -112 | 113 | -114 | 115 | -116 | 117 | -118 | 119 | -120 | 121 |

And similarly, we obtain the $NH_2(10; 10)$ below, filled with elements in \mathbb{Z}_{202} :

| | | | | | | | | | |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | -2 | 3 | -4 | 5 | -6 | 7 | -8 | 9 | -10 |
| -11 | 12 | -13 | 14 | -15 | 16 | -17 | 18 | -19 | 20 |
| 21 | -22 | 23 | -24 | 25 | -26 | 27 | -28 | 29 | -30 |
| -31 | 32 | -33 | 34 | -35 | 36 | -37 | 38 | -39 | 40 |
| 41 | -42 | 43 | -44 | 45 | -46 | 47 | -48 | 49 | -50 |
| -51 | 52 | -53 | 54 | -55 | 56 | -57 | 58 | -59 | 60 |
| 61 | -62 | 63 | -64 | 65 | -66 | 67 | -68 | 69 | -70 |
| -71 | 72 | -73 | 74 | -75 | 76 | -77 | 78 | -79 | 80 |
| 81 | -82 | 83 | -84 | 85 | -86 | 87 | -88 | 89 | -90 |
| -91 | 92 | -93 | 94 | -95 | 96 | -97 | 98 | -99 | 100 |

3.2 | Globally simple $NH_{2n}(n; n)$

Proposition 3.5. *For every odd integer $n \geq 1$ there exists a globally simple $NH_{2n}(n; n)$.*

Proof. Let $n \geq 1$ be odd and consider the $n \times n$ array A with elements in \mathbb{Z}_{2n^2+2n} whose (i, j) th entry is:

$$a_{i,j} = \begin{cases} i + (n + 1)(j - 1) & \text{for } i \equiv j \pmod{2}, \\ n^2 + n - i - (n + 1)(j - 1) & \text{for } i \not\equiv j \pmod{2}. \end{cases}$$

A has no empty cells, thus condition (a_1) of Definition 1.3 is trivially satisfied. It can then be verified that, for $j \in \left[1, \frac{n-1}{2}\right]$, the support of the columns is:

$$\text{supp}(C_j \cup C_{n-j+1}) = [1 + (n + 1)(j - 1), n + (n + 1)(j - 1)] \cup [n^2 - (n + 1)(j - 1), n^2 + n - 1 - (n + 1)(j - 1)]$$

and that

$$\text{supp}\left(C_{\frac{n+1}{2}}\right) = \left[1 + \frac{n^2 - 1}{2}, n + \frac{n^2 - 1}{2}\right],$$

from which we obtain that $\text{supp}(A) = [1, n^2 + n] \setminus \{n + 1, 2(n + 1), \dots, n(n + 1)\}$.

We now examine the partial sums of the columns of A . For any odd $j \in [1, n]$ the j th column of A is given by:

$$C_j = (1 + (n + 1)(j - 1), n^2 + n - 2 - (n + 1)(j - 1), 3 + (n + 1)(j - 1), n^2 + n - 4 - (n + 1)(j - 1), \dots, n^2 + n - (n - 1) - (n + 1)(j - 1), n + (n + 1)(j - 1)).$$

Define now:

$$X = \left\{ k(n^2 + n - 1) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 1 + (n+1)(j-1) + k(n^2 + n + 1) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 3.1(1) with $a = 1 + (n+1)(j-1)$, $b = n^2 + n - 2 - (n+1)(j-1)$, $g = 2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any column C_j , with j odd, it results $S(C_j) = X \cup Y_j$. We notice that in \mathbb{Z}_{2n^2+2n} both X and Y_j are sets not containing zero. It remains to check that X and Y_j are disjoint. Assume by contradiction that there exist $k_1 \in \left[1, \frac{n-1}{2} \right]$, $k_2 \in \left[0, \frac{n-1}{2} \right]$ and an odd j such that:

$$k_1(n^2 + n - 1) \equiv 1 + (n+1)(j-1) + k_2(n^2 + n + 1) \pmod{2n^2 + 2n}.$$

It is easy to see that this would imply:

$$k_1 + k_2 + 1 \equiv 0 \pmod{n+1}.$$

However, by the hypothesis on k_1 and k_2 we have $1 \leq k_1 + k_2 \leq n-1$, so we obtain a contradiction. Then, the partial sums of C_j , j odd, are pairwise distinct, and nonzero.

It can be checked that for any even $j \in [1, n]$ the j th column of A is:

$$C_j = (n^2 + n - 1 - (n+1)(j-1), 2 + (n+1)(j-1), n^2 + n - 3 - (n+1)(j-1), 4 + (n+1)(j-1), \dots, n-1 + (n+1)(j-1), n^2 - (n+1)(j-1)).$$

Now, consider:

$$X = \left\{ k(n^2 + n + 1) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ n^2 - 2 - (n+1)(j-2) + k(n^2 + n - 1) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 3.1(1) with $a = n^2 + n - 1 - (n+1)(j-1)$, $b = (n+1)(j-1) + 2$, $g = -2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any column C_j , with j even, we have $S(C_j) = X \cup Y_j$. Reasoning as before, one can see that X and Y_j are disjoint sets not containing zero, hence the partial sums are pairwise distinct and nonzero.

Now we consider the partial sums of the rows. It is easy to see that for any odd $i \in [1, n]$ the i th row of the matrix is:

$$R_i = (i, n^2 - 1 - i, i + 2(n+1), n^2 - 1 - i - 2(n+1), i + 4(n+1), \dots, n^2 - 1 - i - (n-3)(n+1), i + (n-1)(n+1)).$$

Then, set:

$$X = \left\{ k(n^2 - 1) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_i = \left\{ i + k(n^2 + 2n + 1) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 3.1(1) with $a = i, b = n^2 - 1 - i, g = 2n + 2,$ and $\ell = \frac{n-1}{2},$ it can be seen that for any row R_i, i odd, it results $S(R_i) = X \cup Y_i.$ Since we are working in $\mathbb{Z}_{2n^2+2n},$ we can rewrite Y_i as:

$$Y_i = \left\{ i + k(1 - n^2) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

Now, we have to show that $S(R_i)$ is a set and that $0 \notin S(R_i).$ We first notice that in \mathbb{Z}_{2n^2+2n} both X and Y_i are sets not containing zero. By way of contradiction, assume that for some $k_1 \in \left[1, \frac{n-1}{2} \right], k_2 \in \left[0, \frac{n-1}{2} \right]$ and odd i we have:

$$k_1(n^2 - 1) \equiv i + k_2(1 - n^2) \pmod{2n^2 + 2n}.$$

This would imply that:

$$(k_1 + k_2)(n^2 - 1) \equiv (k_1 + k_2)(n - 1)(n + 1) \equiv i \pmod{2n^2 + 2n}.$$

It would then follow that i is a multiple of $n + 1;$ however, $i \in [1, n].$ We have then proven that all the partial sums are distinct and nonzero.

Finally, it is not hard to see that for any even $i \in [1, n]$ the i th row of A is:

$$R_i = (n^2 + n - i, i + (n + 1), n^2 + n - i - 2(n + 1), i + 3(n + 1), \dots, i + (n - 2)(n + 1), n^2 + n - i - (n + 1)(n - 1)).$$

Now, let:

$$X = \left\{ k(n^2 + 2n + 1) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_i = \left\{ n^2 + n - i + k(n^2 - 1) : \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 3.1(1) with $a = n^2 + n - i, b = n + 1 + i, g = -2n - 2$ and $\ell = \frac{n-1}{2},$ it can be seen that for any row R_i, i even, we have $S(R_i) = X \cup Y_i.$ Since, as before, we are working in $\mathbb{Z}_{2n^2+2n},$ we can rewrite X as:

$$X = \left\{ k(1 - n^2) : k \in \left[1, \frac{n-1}{2} \right] \right\}.$$

With an argument similar to the previous one we can show that X and Y_i are disjoint sets not containing zero, proving that the rows of A admit a simple natural ordering. This concludes the proof. \square

Remark 3.6. Let A be the globally simple array described in Proposition 3.5. For every odd $j \in [1, n]$ looking at the element of the corresponding Y_j obtained for $k = \frac{n-1}{2}$ we get

$$\sum_{a_{i,j} \in C_j} a_{i,j} = j(n + 1) + \frac{n^3 - 2n - 1}{2} = (n + 1) \left(\frac{n^2 - n - 1}{2} + j \right);$$

while for every even $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = -j(n + 1) + \frac{n^3 + 2n^2 + 2n + 1}{2} = (n + 1) \left(\frac{n^2 + n + 1}{2} - j \right).$$

It can be seen that the previous expressions can be written in a more compact form, obtaining that for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (n + 1) \left(\frac{n^2}{2} + (-1)^j \frac{n + 1 - 2j}{2} \right).$$

Finally, for every $i \in [1, n]$, considering the element of the corresponding Y_i with $k = \frac{n-1}{2}$ we get

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{n^2 + n}{2} + (-1)^i \left(\frac{n^3 + 1}{2} - i \right).$$

Example 3.7. Following the proof of Proposition 3.5, we obtain the $NH_{22}(11; 11)$ below, whose elements belong to \mathbb{Z}_{264} :

| | | | | | | | | | | |
|-----|-----|-----|----|----|----|----|----|-----|-----|-----|
| 1 | 119 | 25 | 95 | 49 | 71 | 73 | 47 | 97 | 23 | 121 |
| 130 | 14 | 106 | 38 | 82 | 62 | 58 | 86 | 34 | 110 | 10 |
| 3 | 117 | 27 | 93 | 51 | 69 | 75 | 45 | 99 | 21 | 123 |
| 128 | 16 | 104 | 40 | 80 | 64 | 56 | 88 | 32 | 112 | 8 |
| 5 | 115 | 29 | 91 | 53 | 67 | 77 | 43 | 101 | 19 | 125 |
| 126 | 18 | 102 | 42 | 78 | 66 | 54 | 90 | 30 | 114 | 6 |
| 7 | 113 | 31 | 89 | 55 | 65 | 79 | 41 | 103 | 17 | 127 |
| 124 | 20 | 100 | 44 | 76 | 68 | 52 | 92 | 28 | 116 | 4 |
| 9 | 111 | 33 | 87 | 57 | 63 | 81 | 39 | 105 | 15 | 129 |
| 122 | 22 | 98 | 46 | 74 | 70 | 50 | 94 | 26 | 118 | 2 |
| 11 | 109 | 35 | 85 | 59 | 61 | 83 | 37 | 107 | 13 | 131 |

3.3 | Globally simple $NH_{n^2}(n; n)$

Proposition 3.8. For every odd integer $n \geq 1$, there exists a globally simple $NH_{n^2}(n; n)$.

Proof. Let $n \geq 1$ be an odd integer, and consider the $n \times n$ matrix A with elements in \mathbb{Z}_{3n^2} whose (i, j) th entry is:

$$a_{i,j} = \begin{cases} \varepsilon(3n(j-1) + 3(i-1) + 1) & \text{for } j \in \left[1, \frac{n+1}{2}\right], \\ \varepsilon(3nj - 3i + 1) & \text{for } j \in \left[\frac{n+3}{2}, n\right], \end{cases}$$

where $\varepsilon = 1$ if $i \equiv j \pmod{2}$ and $\varepsilon = -1$ otherwise. Condition (a₁) of Definition 1.3 is trivially verified. It is easy to see that for $j \in \left[1, \frac{n-1}{2}\right]$:

$$\text{supp}(C_j \cup C_{n-j+1}) = [1 + 3n(j-1), 3nj - 1] \setminus \{3k : k \in [n(j-1) + 1, nj - 1]\},$$

also,

$$\text{supp}\left(C_{\frac{n+1}{2}}\right) = \left[1 + 3n\frac{n-1}{2}, \frac{3n^2-1}{2}\right] \setminus \left\{3k : k \in \left[\frac{n^2-n+2}{2}, \frac{n^2-1}{2}\right]\right\}.$$

It then follows that condition (b₁) of Definition 1.3 is satisfied.

We now need to check that the property of being globally simple holds. It is easy to see that for any odd $j \in \left[1, \frac{n+1}{2}\right]$, the j th column of A is:

$$C_j = (3n(j-1) + 1, -3n(j-1) - 4, 3n(j-1) + 7, -3n(j-1) - 10, \dots, -3n(j-1) - 3(n-2) - 1, 3n(j-1) + 3(n-1) + 1).$$

Define now:

$$X = \left\{-3k : k \in \left[1, \frac{n-1}{2}\right]\right\},$$

$$Y_j = \left\{3n(j-1) + 3k + 1 : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1), with $a = 3n(j-1) + 1$, $b = -3n(j-1) - 4$, $g = 6$, and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. It can then be seen that X and Y_j are sets not containing 0. Note also that X and Y_j are trivially disjoint for every odd $j \in \left[1, \frac{n+1}{2}\right]$, as their elements belong to different equivalence classes modulo 3.

Now, for every odd $j \in \left[\frac{n+3}{2}, n\right]$, the j th column of A is:

$$C_j = (3nj - 2, -3nj + 5, 3nj - 8, -3nj + 11, \dots, -3nj + 3(n-1) - 1, 3nj - 3n + 1).$$

Define now:

$$X = \left\{3k : k \in \left[1, \frac{n-1}{2}\right]\right\},$$

$$Y_j = \left\{3nj - 2 - 3k : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1) with $a = 3nj - 2$, $b = -3nj + 5$, $g = -6$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. Clearly, X and Y_j are sets, containing only nonzero elements. By following the same argument to the previous one, we have that they are disjoint for every odd $j \in \left[\frac{n+3}{2}, n\right]$.

It can be verified that for every even $j \in \left[1, \frac{n+1}{2}\right]$ the j th column of A is:

$$C_j = (-3n(j-1) - 1, 3n(j-1) + 4, -3n(j-1) - 7, 3n(j-1) + 10, \dots, \\ 3n(j-1) + 3(n-2) + 1, -3n(j-1) - 3(n-1) - 1).$$

Define:

$$X = \left\{3k : k \in \left[1, \frac{n-1}{2}\right]\right\}, \\ Y_j = \left\{-3n(j-1) - 1 - 3k : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1) with $a = -3n(j-1) - 1$, $b = 3n(j-1) + 4$, $g = -6$, and $\ell = \frac{n-1}{2}$ it is easy to see that $S(C_j) = X \cup Y_j$. Moreover, X and Y_j are sets not containing zero that are disjoint for every even $j \in \left[1, \frac{n+1}{2}\right]$ since their elements belong to different equivalence classes modulo 3.

Finally, for every even $j \in \left[\frac{n+3}{2}, n\right]$ the j th column of A is:

$$C_j = (-3nj + 2, 3nj - 5, -3nj + 8, 3nj - 11, \dots, 3nj - 3(n-1) + 1, -3nj \\ + 3n - 1).$$

Define then:

$$X = \left\{-3k : k \in \left[1, \frac{n-1}{2}\right]\right\}, \\ Y_j = \left\{2 - 3nj + 3k : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1) with $a = -3nj + 2$, $b = 3nj - 5$, $g = 6$, and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. As before, note that, X and Y_j are disjoint sets, containing only nonzero elements.

We have then proven that every column of the array admits a simple natural ordering.

Now, to check that the same property holds also for the rows, we need to distinguish two cases according to the congruence class of n modulo 4.

Case 1: $n \equiv 1 \pmod{4}$.

Let $i \in [1, n]$ be odd, it can then be checked that the i th row of A is:

$$R_i = \left(\begin{array}{c} 3i - 2, -3n - 3i + 2, 3i - 2 + 6n, -9n - 3i + 2, \dots, 3i - 2 + 3n\frac{n-1}{2}, \\ \hline \frac{n+1}{2} \text{ terms} \\ \\ 3n\frac{n-3}{2} + 3i - 1, -3n\frac{n-5}{2} - 3i + 1, 3n\frac{n-7}{2} + 3i - 1, \\ -3n\frac{n-9}{2} - 3i + 1, \dots, 3n + 3i - 1, -3i + 1 \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right).$$

Define:

$$\begin{aligned} X &= \left\{ -3nk : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ Y_i &= \left\{ 3i - 2 + 3nk : k \in \left[0, \frac{n-1}{4} \right] \right\}, \\ Z_i &= \left\{ 3i - 2 + 3n\left(k + \frac{n-1}{4}\right) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ T_i &= \left\{ 6i - 3 + 3n\frac{3n-7}{4} - 3nk : k \in \left[0, \frac{n-5}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 3.1(1) to the first $\frac{n+1}{2}$ elements of R_i , with $a = 3i - 2$, $b = -3n - 3i + 2$, $g = 6n$, and $\ell = \frac{n-1}{4}$, we obtain that $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 3.1(4) to the remaining $\frac{n-1}{2}$ elements of R_i with $a = 1 - 3i$, $b = -1 + 3n + 3i$, $g = -6n$, and $\ell = \frac{n-5}{4}$, and adding the $\frac{n+1}{2}$ th partial sum of R_i , that is $3i - 2 + 3n\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It is easy to see that these are sets not containing zero; it then remains to check that they are pairwise disjoint. Notice that some of these sets are trivially disjoint modulo $3n^2$, since their elements belong to different equivalence classes modulo 3. Hence, it suffices to check that $X \cap T_i = \emptyset$ and $Y_i \cap Z_i = \emptyset$. In the first case, assume by contradiction that there exist $k_1 \in \left[1, \frac{n-1}{4} \right]$, $k_2 \in \left[0, \frac{n-5}{4} \right]$, and i such that:

$$-3nk_1 \equiv 6i - 3 + 3n\frac{3n-7}{4} - 3nk_2 \pmod{3n^2}.$$

This can be rewritten as:

$$6i - 3 + 3n\left(\frac{3n - 7}{4} + k_1 - k_2\right) \equiv 0 \pmod{3n^2}.$$

By the hypothesis on k_1 and k_2 , the left-hand side is strictly positive in \mathbb{Z} and the element in brackets is less than or equal to $n - 2$. This fact, combined with $i \leq n$, proves that the left-hand side in \mathbb{Z} is an integer in $[1, 3n^2 - 3]$, so we get a contradiction. Now, if there exists i such that $Y_i \cap Z_i \neq \emptyset$, we would have:

$$3n\left(k_2 - k_1 + \frac{n - 1}{4}\right) \equiv 0 \pmod{3n^2}$$

with $k_1 \in \left[0, \frac{n-1}{4}\right]$ and $k_2 \in \left[1, \frac{n-1}{4}\right]$. However, the term inside brackets is strictly positive and cannot exceed $n/2 - 1$, thus the equation cannot hold.

It can be checked that for every even $i \in [1, n]$ the i th row of A is:

$$R_i = \left(\begin{array}{c} -1 - 3(i - 1), 1 + 3(i - 1) + 3n, -1 - 3(i - 1) \\ -6n, 1 + 3(i - 1) + 9n, \dots, -1 - 3(i - 1) - 3n\frac{n - 1}{2} \\ \hline \frac{n+1}{2} \text{ terms} \\ \\ 1 - 3n\frac{n - 3}{2} - 3i, -1 + 3n\frac{n - 5}{2} + 3i, 1 - 3n\frac{n - 7}{2} - 3i, -1 \\ + 3n\frac{n - 9}{2} + 3i, \dots, -1 + 3n + 3i, 1 - 3i \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right).$$

Set:

$$\begin{aligned} X &= \left\{ 3nk : k \in \left[1, \frac{n - 1}{4}\right] \right\}, \\ Y_i &= \left\{ 2 - 3i - 3nk : k \in \left[0, \frac{n - 1}{4}\right] \right\}, \\ Z_i &= \left\{ 2 - 3i - 3n\left(k + \frac{n - 1}{4}\right) : k \in \left[1, \frac{n - 1}{4}\right] \right\}, \\ T_i &= \left\{ 3 - 6i + 3n\frac{7 - 3n}{4} + 3nk : k \in \left[0, \frac{n - 5}{4}\right] \right\}. \end{aligned}$$

By applying Lemma 3.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = -1 - 3(i - 1)$, $b = 1 + 3(i - 1) + 3n$, $g = -6n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Then,

by applying Lemma 3.1(4) to the remaining $\frac{n-1}{2}$ terms of R_i , with $a = -1 + 3i$, $b = 1 - 3n - 3i$, $g = 6n$ and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ th partial sum of R_i that is $2 - 3i - 3n\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It can be seen that these are sets containing only nonzero elements, thus it remains to check that they are pairwise disjoint modulo $3n^2$. Some pairs of these sets are trivially disjoint, as their elements belong to different equivalence classes modulo 3. We then need to verify $Y_i \cap Z_i = \emptyset$ and $X \cap T_i = \emptyset$. In the first case, suppose by contradiction that:

$$3n\left(\frac{n-1}{4} + k_2 - k_1\right) \equiv 0 \pmod{3n^2}$$

for some $k_1 \in \left[0, \frac{n-1}{4}\right]$ and $k_2 \in \left[1, \frac{n-1}{4}\right]$. However, it can be seen that this cannot hold as $k_2 - k_1 \leq \frac{n-1}{4}$. In the second case, again by contradiction, suppose we have:

$$3nk_1 \equiv 3 - 6i + 3n\frac{7-3n}{4} + 3nk_2 \pmod{3n^2},$$

for some $k_1 \in \left[1, \frac{n-1}{4}\right]$, $k_2 \in \left[0, \frac{n-5}{4}\right]$ and for some even i . This is equivalent to:

$$3 - 6i + 3n\left(\frac{7-3n}{4} + k_2 - k_1\right) \equiv 0 \pmod{3n^2}.$$

Now, it can be seen that the term inside the brackets is strictly negative in \mathbb{Z} , and is bounded from below by $2 - n$. By noticing that $i < n$, it can be seen that the left-hand side is a negative integer greater than $3 - 3n^2$, so we get a contradiction.

Case 2: $n \equiv 3 \pmod{4}$.

It is not hard to see that, since we are working modulo $3n^2$, the i th row of A , for every odd $i \in [1, n]$, is:

$$R_i = \left(\begin{array}{c} 1 + 3(i-1), -1 - 3(i-1) - 3n, 1 + 3(i-1) + 6n, \\ -1 - 3(i-1) - 9n, \dots, -1 - 3(i-1) - 3n\frac{n-1}{2} \\ \hline \frac{n+1}{2} \text{ terms} \\ \\ 1 - 3n\frac{n-3}{2} - 3i, -1 + 3n\frac{n-5}{2} + 3i, 1 - 3n\frac{n-7}{2} - 3i, -1 \\ + 3n\frac{n-9}{2} + 3i, \dots, -1 + 3n + 3i, 1 - 3i \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right),$$

Define:

$$\begin{aligned} X &= \left\{ -3nk : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\ Y_i &= \left\{ 3i - 2 + 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\ Z &= \left\{ -3n \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\ T_i &= \left\{ 1 - 3i - 3n \frac{3n-5}{4} + 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}. \end{aligned}$$

Now, by applying Lemma 3.1(3) to the first $\frac{n+1}{2}$ elements of R_i , with $a = 3i - 2$, $b = -3n - 3i + 2$, $g = 6n$ and $\ell = \frac{n-3}{4}$ we obtain that $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 3.1(2) to the remaining $\frac{n-1}{2}$ elements with $a = 1 - 3i$, $b = -1 + 3n + 3i$, $g = -6n$ and $\ell = \frac{n-3}{4}$, and adding the $\frac{n+1}{2}$ th partial sum of R_i that is $-3n \frac{n+1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z \cup T_i$. It can then be easily seen that these are sets not containing any zero element, thus it remains to check that they are pairwise disjoint. By considering the same sets written modulo 3, we only need to check that $X \cap Z = \emptyset$ and $Y_i \cap T_i = \emptyset$. In the first case suppose by contradiction that:

$$-3nk_1 \equiv -3n \left(k_2 + \frac{n+1}{4} \right) \pmod{3n^2},$$

for some $k_1 \in \left[1, \frac{n+1}{4} \right]$ and $k_2 \in \left[1, \frac{n-3}{4} \right]$. Clearly, this cannot hold since $k_2 - k_1 \leq \frac{n-7}{4}$. Suppose now $Y_i \cap T_i \neq \emptyset$, namely that

$$3i - 2 + 3nk_1 \equiv 1 - 3i - 3n \frac{3n-5}{4} + 3nk_2 \pmod{3n^2}$$

for some $k_1, k_2 \in \left[0, \frac{n-3}{4} \right]$ and some odd i . Then:

$$3 - 6i - 3n \left(\frac{3n-5}{4} - k_2 + k_1 \right) \equiv 0 \pmod{3n^2}.$$

It can then be seen that in \mathbb{Z} the left-hand side is strictly negative and that the term in brackets cannot exceed $n - 2$. Since $i \in [1, n]$ the left-hand side is an integer contained in $[-3n^2 + 3, -1]$, hence we have a contradiction.

Recalling that we are working in \mathbb{Z}_{3n^2} , one can check that for every even $i \in [1, n]$ the i th row of A is:

$$R_i = \left(\begin{array}{c} -1 - 3(i - 1), 1 + 3(i - 1) + 3n, -1 - 3(i - 1) - 6n, 1 \\ + 3(i - 1) + 9n, \dots, 1 + 3(i - 1) + 3n \frac{n - 1}{2} \\ \hline \frac{n+1}{2} \text{ terms} \\ \\ -1 + 3n \frac{n - 3}{2} + 3i, 1 - 3n \frac{n - 5}{2} - 3i, -1 + 3n \frac{n - 7}{2} + 3i, 1 \\ - 3n \frac{n - 9}{2} - 3i, \dots, 1 - 3n - 3i, -1 + 3i \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right),$$

Define now:

$$\begin{aligned} X &= \left\{ 3nk : k \in \left[1, \frac{n + 1}{4} \right] \right\}, \\ Y_i &= \left\{ 2 - 3i - 3nk : k \in \left[0, \frac{n - 3}{4} \right] \right\}, \\ Z &= \left\{ 3n \left(k + \frac{n + 1}{4} \right) : k \in \left[1, \frac{n - 3}{4} \right] \right\}, \\ T_i &= \left\{ -1 + 3i + 3n \frac{3n - 5}{4} - 3nk : k \in \left[0, \frac{n - 3}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 3.1(3) to the first $\frac{n+1}{2}$ elements of R_i with $a = -1 - 3(i - 1)$, $b = 1 + 3(i - 1) + 3n$, $g = -6n$ and $\ell = \frac{n-3}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Finally, by applying Lemma 3.1(2) to the remaining $\frac{n-1}{2}$ terms, with $a = -1 + 3i$, $b = 1 - 3n - 3i$, $g = 6n$, and $\ell = \frac{n-3}{4}$, and adding the $\frac{n+1}{2}$ th partial sum of R_i , that is $3n \frac{n+1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z \cup T_i$. We can see that these are sets containing only nonzero elements, so it remains to prove that they are pairwise disjoint. As before, it suffices to verify $X \cap Z = \emptyset$ and $Y_i \cap T_i = \emptyset$. In the first case, by means of contradiction, suppose

$$3nk_1 \equiv 3n \left(k_2 + \frac{n + 1}{4} \right) \pmod{3n^2}$$

for some $k_1 \in \left[1, \frac{n+1}{4} \right]$ and $k_2 \in \left[1, \frac{n-3}{4} \right]$. This cannot hold since $k_2 - k_1 \leq \frac{n-3}{4}$. In the second case, by contradiction, assume that for some $k_1, k_2 \in \left[0, \frac{n-3}{4} \right]$ and even i it holds:

$$2 - 3i - 3nk_1 \equiv -1 + 3i + 3n \frac{3n-5}{4} - 3nk_2 \pmod{3n^2}.$$

This is equivalent to:

$$6i - 3 + 3n \left(\frac{3n-5}{4} + k_1 - k_2 \right) \equiv 0 \pmod{3n^2}.$$

It can be seen that the left-hand side is strictly positive in \mathbb{Z} ; since $i \leq n$ and $k_1 - k_2 \leq \frac{n-3}{4}$, the left-hand side is smaller than or equal to $3n^2 - 3$. Thus, the previous equation cannot hold, proving that Y_i and T_i are disjoint sets.

We then have shown that for every odd n the rows and the columns of A admit a simple natural ordering. This concludes the proof. \square

Remark 3.9. For every $j \in [1, n]$ the sum of the elements of each column of the globally simple array A described in Proposition 3.8 is the following:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j-1} \left(3nj - \frac{3n+1}{2} \right).$$

Looking at the columns it is easy to see that for every $j \in [1, n]$ and for every $k \in \left[0, \left\lfloor \frac{n-5}{4} \right\rfloor \right]$ we have:

$$a_{1+2k,j} + a_{2+2k,j} = -(a_{n-2k,j} + a_{n-2k-1,j}).$$

Hence if $n \equiv 1 \pmod{4}$ the total sum of the j th column is given by $a_{\frac{n+1}{2},j}$, while for $n \equiv 3 \pmod{4}$ the sum is given by $a_{\frac{n-1}{2},j} + a_{\frac{n+1}{2},j} + a_{\frac{n+3}{2},j}$.

Analogously, it can be seen that for every $i \in [1, n]$ and for every $k \in \left[0, \left\lfloor \frac{n-5}{4} \right\rfloor \right]$ we have:

$$a_{i,1+2k} + a_{i,2+2k} = -(a_{i,n-2k} + a_{i,n-2k-1}).$$

We then obtain that for $n \equiv 1 \pmod{4}$ the total sum of the i th row is given by $a_{i, \frac{n+1}{2}}$, while for $n \equiv 3 \pmod{4}$ the sum is given by $a_{i, \frac{n-1}{2}} + a_{i, \frac{n+1}{2}} + a_{i, \frac{n+3}{2}}$. Note also that $|a_{i, \frac{n-1}{2}} - a_{i, \frac{n+3}{2}}| = 1$.

Hence, for the rows we have that, if $n \equiv 1 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3i - 2 + 3n \frac{n-1}{2} \right) = a_{i, \frac{n+1}{2}};$$

while, if $n \equiv 3 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3(n+1-i) - 2 + 3n \frac{n-1}{2} \right) = a_{i, \frac{n+1}{2}} + (-1)^i.$$

Example 3.10. Following the proof of Proposition 3.8, we obtain the $NH_{169}(13; 13)$ below, whose elements belong to \mathbb{Z}_{507} :

| | | | | | | | | | | | | |
|-----|-----|------|------|------|------|------|------|------|------|------|-----|-----|
| 1 | -40 | 79 | -118 | 157 | -196 | 235 | 197 | -158 | 119 | -80 | 41 | -2 |
| -4 | 43 | -82 | 121 | -160 | 199 | -238 | -200 | 161 | -122 | 83 | -44 | 5 |
| 7 | -46 | 85 | -124 | 163 | -202 | 241 | 203 | -164 | 125 | -86 | 47 | -8 |
| -10 | 49 | -88 | 127 | -166 | 205 | -244 | -206 | 167 | -128 | 89 | -50 | 11 |
| 13 | -52 | 91 | -130 | 169 | -208 | 247 | 209 | -170 | 131 | -92 | 53 | -14 |
| -16 | 55 | -94 | 133 | -172 | 211 | -250 | -212 | 173 | -134 | 95 | -56 | 17 |
| 19 | -58 | 97 | -136 | 175 | -214 | 253 | 215 | -176 | 137 | -98 | 59 | -20 |
| -22 | 61 | -100 | 139 | -178 | 217 | -251 | -218 | 179 | -140 | 101 | -62 | 23 |
| 25 | -64 | 103 | -142 | 181 | -220 | -248 | 221 | -182 | 143 | -104 | 65 | -26 |
| -28 | 67 | -106 | 145 | -184 | 223 | 245 | -224 | 185 | -146 | 107 | -68 | 29 |
| 31 | -70 | 109 | -148 | 187 | -226 | -242 | 227 | -188 | 149 | -110 | 71 | -32 |
| -34 | 73 | -112 | 151 | -190 | 229 | 239 | -230 | 191 | -152 | 113 | -74 | 35 |
| 37 | -76 | 115 | -154 | 193 | -232 | -236 | 233 | -194 | 155 | -116 | 77 | -38 |

And similarly, we obtain the $NH_{121}(11; 11)$ below, filled with elements in \mathbb{Z}_{363} :

| | | | | | | | | | | |
|-----|-----|-----|------|------|------|------|------|-----|-----|-----|
| 1 | -34 | 67 | -100 | 133 | -166 | -134 | 101 | -68 | 35 | -2 |
| -4 | 37 | -70 | 103 | -136 | 169 | 137 | -104 | 71 | -38 | 5 |
| 7 | -40 | 73 | -106 | 139 | -172 | -140 | 107 | -74 | 41 | -8 |
| -10 | 43 | -76 | 109 | -142 | 175 | 143 | -110 | 77 | -44 | 11 |
| 13 | -46 | 79 | -112 | 145 | -178 | -146 | 113 | -80 | 47 | -14 |
| -16 | 49 | -82 | 115 | -148 | 181 | 149 | -116 | 83 | -50 | 17 |
| 19 | -52 | 85 | -118 | 151 | 179 | -152 | 119 | -86 | 53 | -20 |
| -22 | 55 | -88 | 121 | -154 | -176 | 155 | -122 | 89 | -56 | 23 |
| 25 | -58 | 91 | -124 | 157 | 173 | -158 | 125 | -92 | 59 | -26 |
| -28 | 61 | -94 | 127 | -160 | -170 | 161 | -128 | 95 | -62 | 29 |
| 31 | -64 | 97 | -130 | 163 | 167 | -164 | 131 | -98 | 65 | -32 |

3.4 | Globally simple $NH_{2n^2}(n; n)$

Proposition 3.11. For every odd integer $n \geq 1$, there exists a globally simple $NH_{2n^2}(n; n)$.

Proof. Let $n \geq 1$ be odd, and consider the $n \times n$ matrix A with elements in \mathbb{Z}_{4n^2} whose (i, j) th entry is:

$$a_{i,j} = \begin{cases} 2n(j - 1) + 2i - 1 & \text{for } i \equiv j \pmod{2} \text{ and } j \in \left[1, \frac{n+1}{2}\right], \\ 2n(n - j + 1) - 2i + 1 & \text{for } i \not\equiv j \pmod{2} \text{ and } j \in \left[1, \frac{n+1}{2}\right], \\ 2nj - 2i + 1 & \text{for } i \equiv j \pmod{2} \text{ and } j \in \left[\frac{n+3}{2}, n\right], \\ 2n(n - j) + 2i - 1 & \text{for } i \not\equiv j \pmod{2} \text{ and } j \in \left[\frac{n+3}{2}, n\right]. \end{cases}$$

Since A has no empty cells, condition (a₁) of Definition 1.3 is trivially satisfied.

For $j \in \left[1, \frac{n-1}{2}\right]$, one can check that $\text{supp}(C_j \cup C_{n-j+1})$ contains exactly all the odd elements of $[1 + 2n(j-1), 2nj - 1] \cup [2n^2 + 1 - 2nj, 2n^2 - 1 - 2n(j-1)]$, and that $\text{supp}(C_{\frac{n+1}{2}})$ contains exactly all the odd elements of $[n(n-1) + 1, n(n+1) - 1]$. Since we have to avoid the elements of the subgroup of order $2n^2$ of \mathbb{Z}_{4n^2} , condition (b₁) of Definition 1.3 is satisfied too.

Now we consider the partial sums of the columns of A . By the description of A it immediately follows that for every odd $j \in \left[1, \frac{n+1}{2}\right]$ we have:

$$C_j = (2n(j-1) + 1, 2n(n-j+1) - 3, 2n(j-1) + 5, 2n(n-j+1) - 7, \dots, \\ 2n(n-j+1) - 2n + 1, 2n(j-1) + 2n - 1).$$

Define now:

$$X = \left\{k(2n^2 - 2) : k \in \left[1, \frac{n-1}{2}\right]\right\}, \\ Y_j = \left\{1 + 2n(j-1) + k(2n^2 + 2) : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1) with $a = 2n(j-1) + 1$, $b = 2n(n-j+1) - 3$, $g = 4$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. It can be checked that both X and Y_j are sets, containing no zero elements, and they are trivially disjoint as their elements belong to different classes modulo 2. Similarly, it can be seen that for every odd $j \in \left[\frac{n+3}{2}, n\right]$, the j th column of A is:

$$C_j = (2nj - 1, 2n(n-j) + 3, 2nj - 5, 2n(n-j) + 7, \dots, 2n(n-j) + 2n - 3, \\ 2nj - 2n + 1).$$

Set:

$$X = \left\{k(2n^2 + 2) : k \in \left[1, \frac{n-1}{2}\right]\right\}, \\ Y_j = \left\{2nj - 1 + k(2n^2 - 2) : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 3.1(1) with $a = 2nj - 1$, $b = 2n(n-j) + 3$, $g = -4$ and $\ell = \frac{n-1}{2}$, we get $S(C_j) = X \cup Y_j$. It can then be easily verified that these are disjoint sets not containing zero.

Now, for any even $j \in \left[1, \frac{n+1}{2}\right]$ the j th column of A is:

$$C_j = (2n(n-j+1) - 1, 2n(j-1) + 3, 2n(n-j+1) - 5, 2n(j-1) + 7, \dots, \\ 2n(j-1) + 2n - 3, 2n(n-j+1) - 2n + 1).$$

Consider:

$$X = \left\{ k(2n^2 + 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2n(n-j+1) - 1 + k(2n^2 - 2) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 3.1(1) with $a = 2n(n-j+1) - 1$, $b = 2n(j-1) + 3$, $g = -4$ and $\ell = \frac{n-1}{2}$ we have $S(C_j) = X \cup Y_j$. It can be seen that X and Y_j are sets, containing exclusively nonzero elements, and as before, X and Y_j are trivially disjoint.

Similarly, it can be seen that for every even $j \in \left[\frac{n+3}{2}, n \right]$ the j th column of A is:

$$C_j = (2n(n-j) + 1, 2nj - 3, 2n(n-j) + 5, 2nj - 7, \dots, 2nj - 2n + 3, 2n(n-j) + 2n - 1).$$

By applying Lemma 3.1(1) with $a = 2n(n-j) + 1$, $b = 2nj - 3$, $g = 4$ and $\ell = \frac{n-1}{2}$, we have $S(C_j) = X \cup Y_j$, where:

$$X = \left\{ k(2n^2 - 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2n(n-j) + 1 + k(2n^2 + 2) : k \in \left[0, \frac{n-1}{2} \right] \right\},$$

which can be checked to be disjoint sets not containing zero elements.

It remains to consider the partial sums of every row. We have to split the proof into two cases according to the congruence class of n modulo 4.

Case 1: $n \equiv 1 \pmod{4}$.

For every odd $i \in [1, n]$, the i th row of A is:

$$R_i = \left(\begin{array}{c} 2i - 1, 2n(n-1) - 2i + 1, 4n + 2i - 1, 2n(n-3) - 2i + 1, \dots, \\ 2n \frac{n-1}{2} + 2i - 1 \\ \hline \frac{n+1}{2} \text{ terms} \\ \dots \\ 2n \frac{n-3}{2} + 2i - 1, 2n \frac{n+5}{2} - 2i + 1, 2n \frac{n-7}{2} + 2i - 1, 2n \frac{n+9}{2} \\ - 2i + 1, \dots, 2n + 2i - 1, 2n^2 - 2i + 1 \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right).$$

Set:

$$\begin{aligned} X &= \left\{ k(2n^2 - 2n) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ Y_i &= \left\{ 2i - 1 + k(2n^2 + 2n) : k \in \left[0, \frac{n-1}{4} \right] \right\}, \\ Z_i &= \left\{ -1 + 2i + (2n^2 + 2n) \left(k + \frac{n-1}{4} \right) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ T_i &= \left\{ 4i - 2 + \frac{n^3 + 2n^2 - 7n}{2} + k(2n^2 - 2n) : k \in \left[0, \frac{n-5}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 3.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = 2i - 1$, $b = 2n(n-1) - 2i + 1$, $g = 4n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 3.1(4) to the remaining elements with $a = 2n^2 - 2i + 1$, $b = 2n + 2i - 1$, $g = -4n$, and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ th partial sum of R_i , that is $2i - 1 + (2n^2 + 2n)\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. We can see that these are sets, containing only nonzero elements. It remains to check that they are pairwise disjoint. Considering these sets modulo 2, it is easy to see that it suffices to verify that $Y_i \cap Z_i = \emptyset$ and $X \cap T_i = \emptyset$. In the first case, assuming that there exists an element in the intersection, this would imply that for some $k_1 \in \left[0, \frac{n-1}{4} \right]$, $k_2 \in \left[1, \frac{n-1}{4} \right]$ and i odd the following holds:

$$2i - 1 + k_1(2n^2 + 2n) \equiv 2i - 1 + (2n^2 + 2n) \left(k_2 + \frac{n-1}{4} \right) \pmod{4n^2},$$

that is:

$$\left(k_2 + \frac{n-1}{4} - k_1 \right) (2n^2 + 2n) \equiv 0 \pmod{4n^2}.$$

Since we are working in \mathbb{Z}_{4n^2} , $2n^2(k_2 + \frac{n-1}{4} - k_1) \in \{0, 2n^2\}$, we get a contradiction by noticing that $k_2 + \frac{n-1}{4} - k_1 \leq \frac{n-1}{2}$.

In the second case, again by contradiction, suppose we have:

$$k_1(2n^2 - 2n) \equiv 4i - 2 + \frac{n^3 + 2n^2 - 7n}{2} + k_2(2n^2 - 2n) \pmod{4n^2},$$

for some $k_1 \in \left[1, \frac{n-1}{4} \right]$, $k_2 \in \left[0, \frac{n-5}{4} \right]$ and some odd i . This can be rewritten as

$$-2 + 4i + \frac{n^3 + 2n^2 - 7n}{2} + (k_2 - k_1)(2n^2 - 2n) \equiv 0 \pmod{4n^2}.$$

Note that, since $n \equiv 1 \pmod{4}$, we have that $n^3 = (4l + 1)n^2 \equiv n^2 \pmod{4n^2}$, hence we can rewrite the previous expression modulo $2n^2$, obtaining:

$$-2 + 4i + \frac{3n^2 - 7n}{2} - 2n(k_2 - k_1) \equiv 0 \pmod{2n^2},$$

which is equivalent to:

$$-2 + 4i + 2n\left(\frac{3n - 7}{4} - k_2 + k_1\right) \equiv 0 \pmod{2n^2}.$$

After noticing that the left-hand side is strictly positive in \mathbb{Z} , we observe that the bracket is bounded from above by $n - 2$, thus from $i \leq n$, the left-hand side seen in \mathbb{Z} is less than or equal to $2n^2 - 2$, that is a contradiction.

For every even $i \in [1, n]$ the i th row of A is:

$$R_i = \left(\begin{array}{c} 2n^2 - 2i + 1, 2n + 2i - 1, 2n(n - 2) - 2i + 1, 6n + 2i - 1, \dots, \\ 2n\frac{n + 1}{2} - 2i + 1 \\ \hline \frac{n+1}{2} \text{ terms} \end{array} \right),$$

$$\left(\begin{array}{c} 2n\frac{n + 3}{2} - 2i + 1, 2n\frac{n - 5}{2} + 2i - 1, 2n\frac{n + 7}{2} - 2i + 1, 2n\frac{n - 9}{2} \\ + 2i - 1, \dots, 2n(n - 1) - 2i + 1, 2i - 1 \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right).$$

Let now:

$$X = \left\{ k(2n^2 + 2n) : k \in \left[1, \frac{n - 1}{4}\right] \right\},$$

$$Y_i = \left\{ 2n^2 + 1 - 2i + k(2n^2 - 2n) : k \in \left[0, \frac{n - 1}{4}\right] \right\},$$

$$Z_i = \left\{ 2n^2 + 1 - 2i + (2n^2 - 2n)\left(\frac{n - 1}{4} + k\right) : k \in \left[1, \frac{n - 1}{4}\right] \right\},$$

$$T_i = \left\{ 2 - 4i + \frac{n^3 + 4n^2 + 7n}{2} + k(2n^2 + 2n) : k \in \left[0, \frac{n - 5}{4}\right] \right\}.$$

By applying Lemma 3.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = 2n^2 - 2i + 1$, $b = 2n + 2i - 1$, $g = -4n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 3.1(4) to the remaining elements with $a = 2i - 1$, $b = 2n(n - 1) - 2i + 1$, $g = 4n$ and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ th partial sum of R_i , that is $2n^2 + 1 - 2i + (2n^2 - 2n)\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It can be easily seen that these are sets containing only nonzero elements, thus it only remains to check that they have trivial intersection. As before, it suffices to check that $Y_i \cap Z_i = \emptyset$ and that $X \cap T_i = \emptyset$. In the first case, by contradiction, we would have for some $k_1 \in \left[0, \frac{n-1}{4}\right]$, $k_2 \in \left[1, \frac{n-1}{4}\right]$ and even i :

$$2n^2 + 1 - 2i + k_1(2n^2 - 2n) \equiv 2n^2 + 1 - 2i + (2n^2 - 2n)\left(\frac{n-1}{4} + k_2\right) \pmod{4n^2},$$

that is:

$$(2n^2 - 2n)\left(\frac{n-1}{4} + k_2 - k_1\right) \equiv 0 \pmod{4n^2}.$$

We then have that $2n^2(\frac{n-1}{4} + k_2 - k_1) \in \{0, 2n^2\}$ modulo $4n^2$, thus we obtain a contradiction by noticing that $\frac{n-1}{4} + k_2 - k_1$ is strictly positive and bounded by $\frac{n-1}{2}$.

In the second case, again by contradiction, we would have for some $k_1 \in [1, \frac{n-1}{4}]$, $k_2 \in [0, \frac{n-5}{4}]$ and even i that:

$$k_1(2n^2 + 2n) \equiv 2 - 4i + \frac{n^3 + 4n^2 + 7n}{2} + k_2(2n^2 + 2n) \pmod{4n^2}.$$

As before $n^3 \equiv n^2 \pmod{4n^2}$, hence we would have:

$$2 - 4i + \frac{n^2 + 7n}{2} + 2n(k_2 - k_1) \equiv 0 \pmod{2n^2}.$$

It can be seen that the left-hand side is strictly positive in \mathbb{Z} , and as $i \in [1, n]$ and $k_2 - k_1 \in [\frac{1-n}{4}, \frac{n-9}{4}]$, is bounded from below by 2 and from above by $n^2 - n - 2$, hence we have a contradiction.

Case 2: $n \equiv 3 \pmod{4}$.

It can be checked that for every odd $i \in [1, n]$ the i th row of A is:

$$R_i = \left(\begin{array}{c} 2i - 1, 2n(n - 1) - 2i + 1, 4n + 2i - 1, 2n(n - 3) - 2i + 1, \dots, \\ 2n\frac{n+1}{2} - 2i + 1 \\ \hline \frac{n+1}{2} \text{ terms} \\ \hline 2n\frac{n+3}{2} - 2i + 1, 2n\frac{n-5}{2} + 2i - 1, 2n\frac{n+7}{2} - 2i + 1, 2n\frac{n-9}{2} \\ + 2i - 1, \dots, 2n + 2i - 1, 2n^2 - 2i + 1 \\ \hline \frac{n-1}{2} \text{ terms} \end{array} \right),$$

Now, define:

$$\begin{aligned}
 X &= \left\{ k(2n^2 - 2n) : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\
 Y_i &= \left\{ 2i - 1 + k(2n^2 + 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\
 Z &= \left\{ (2n^2 - 2n) \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\
 T_i &= \left\{ 1 - 2i + \frac{n^3 + 2n^2 + 5n}{2} + k(2n^2 + 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}.
 \end{aligned}$$

As done for the case $n \equiv 1 \pmod{4}$, it is not hard to see that these are disjoint sets not containing zero. Also, applying again Lemma 3.1, one can check that $S(R_i) = X \cup Y_i \cup Z \cup T_i$.

For every even $i \in [1, n]$ the i th row of A is:

$$R_i = \left(\begin{array}{c} 2n^2 - 2i + 1, 2n + 2i - 1, 2n(n - 2) - 2i + 1, 6n + 2i - 1, \dots, \\ \frac{2n \frac{n-1}{2} + 2i - 1}{\frac{n+1}{2} \text{ terms}}, \\ \dots \\ 2n \frac{n-3}{2} + 2i - 1, 2n \frac{n+5}{2} - 2i + 1, 2n \frac{n-7}{2} + 2i - 1, 2n \frac{n+9}{2} \\ - 2i + 1, \dots, 2n(n-1) - 2i + 1, 2i - 1 \\ \frac{n-1}{2} \text{ terms} \end{array} \right).$$

Let now:

$$\begin{aligned}
 X &= \left\{ k(2n^2 + 2n) : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\
 Y_i &= \left\{ 2n^2 - 2i + 1 + k(2n^2 - 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\
 Z &= \left\{ (2n^2 + 2n) \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\
 T_i &= \left\{ 2i - 1 + \frac{n^3 + 4n^2 - 5n}{2} + k(2n^2 - 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}.
 \end{aligned}$$

It can then be seen that these are pairwise disjoint sets not containing zero, and, applying Lemma 3.1, that $S(R_i) = X \cup Y_i \cup Z \cup T_i$.

We can then conclude that for every odd n the array A is a globally simple $\text{NH}_{2n^2}(n; n)$. \square

Remark 3.12. For every odd integer n we write the total sum of every row and column of the array A constructed in the proof of Proposition 3.11. In particular, for every $j \in [1, n]$, if we take the element of the corresponding set Y_j with $k = \frac{n-1}{2}$, we get:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = n^3 + (-1)^{j-1}(2nj - n^2 - n).$$

Analogously, if $n \equiv 1 \pmod{4}$ then for every $i \in [1, n]$ if we take the element of the corresponding set T_i with $k = \frac{n-5}{4}$, we get:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(2i - n - 1);$$

if $n \equiv 3 \pmod{4}$ then for every $i \in [1, n]$ taking the element of the corresponding set T_i with $k = \frac{n-3}{4}$, we obtain:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(n + 1 - 2i).$$

Example 3.13. Following the proof of Proposition 3.11, we obtain the $\text{NH}_{338}(13; 13)$ below, whose elements belong to \mathbb{Z}_{676} :

| | | | | | | | | | | | | |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 311 | 53 | 259 | 105 | 207 | 157 | 131 | 233 | 79 | 285 | 27 | 337 |
| 335 | 29 | 283 | 81 | 231 | 133 | 179 | 205 | 107 | 257 | 55 | 309 | 3 |
| 5 | 307 | 57 | 255 | 109 | 203 | 161 | 135 | 229 | 83 | 281 | 31 | 333 |
| 331 | 33 | 279 | 85 | 227 | 137 | 175 | 201 | 111 | 253 | 59 | 305 | 7 |
| 9 | 303 | 61 | 251 | 113 | 199 | 165 | 139 | 225 | 87 | 277 | 35 | 329 |
| 327 | 37 | 275 | 89 | 223 | 141 | 171 | 197 | 115 | 249 | 63 | 301 | 11 |
| 13 | 299 | 65 | 247 | 117 | 195 | 169 | 143 | 221 | 91 | 273 | 39 | 325 |
| 323 | 41 | 271 | 93 | 219 | 145 | 167 | 193 | 119 | 245 | 67 | 297 | 15 |
| 17 | 295 | 69 | 243 | 121 | 191 | 173 | 147 | 217 | 95 | 269 | 43 | 321 |
| 319 | 45 | 267 | 97 | 215 | 149 | 163 | 189 | 123 | 241 | 71 | 293 | 19 |
| 21 | 291 | 73 | 239 | 125 | 187 | 177 | 151 | 213 | 99 | 265 | 47 | 317 |
| 315 | 49 | 263 | 101 | 211 | 153 | 159 | 185 | 127 | 237 | 75 | 289 | 23 |
| 25 | 287 | 77 | 235 | 129 | 183 | 181 | 155 | 209 | 103 | 261 | 51 | 313 |

And similarly, we obtain the $\text{NH}_{450}(15; 15)$ below, filled with elements in \mathbb{Z}_{900} :

| | | | | | | | | | | | | | | |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 419 | 61 | 359 | 121 | 299 | 181 | 239 | 269 | 151 | 329 | 91 | 389 | 31 | 449 |
| 447 | 33 | 387 | 93 | 327 | 153 | 267 | 213 | 183 | 297 | 123 | 357 | 63 | 417 | 3 |
| 5 | 415 | 65 | 355 | 125 | 295 | 185 | 235 | 265 | 155 | 325 | 95 | 385 | 35 | 445 |
| 443 | 37 | 383 | 97 | 323 | 157 | 263 | 217 | 187 | 293 | 127 | 353 | 67 | 413 | 7 |
| 9 | 411 | 69 | 351 | 129 | 291 | 189 | 231 | 261 | 159 | 321 | 99 | 381 | 39 | 441 |
| 439 | 41 | 379 | 101 | 319 | 161 | 259 | 221 | 191 | 289 | 131 | 349 | 71 | 409 | 11 |
| 13 | 407 | 73 | 347 | 133 | 287 | 193 | 227 | 257 | 163 | 317 | 103 | 377 | 43 | 437 |
| 435 | 45 | 375 | 105 | 315 | 165 | 255 | 225 | 195 | 285 | 135 | 345 | 75 | 405 | 15 |
| 17 | 403 | 77 | 343 | 137 | 283 | 197 | 223 | 253 | 167 | 313 | 107 | 373 | 47 | 433 |
| 431 | 49 | 371 | 109 | 311 | 169 | 251 | 229 | 199 | 281 | 139 | 341 | 79 | 401 | 19 |
| 21 | 399 | 81 | 339 | 141 | 279 | 201 | 219 | 249 | 171 | 309 | 111 | 369 | 51 | 429 |
| 427 | 53 | 367 | 113 | 307 | 173 | 247 | 233 | 203 | 277 | 143 | 337 | 83 | 397 | 23 |
| 25 | 395 | 85 | 335 | 145 | 275 | 205 | 215 | 245 | 175 | 305 | 115 | 365 | 55 | 425 |
| 423 | 57 | 363 | 117 | 303 | 177 | 243 | 237 | 207 | 273 | 147 | 333 | 87 | 393 | 27 |
| 29 | 391 | 89 | 331 | 149 | 271 | 209 | 211 | 241 | 179 | 301 | 119 | 361 | 59 | 421 |

3.5 | Globally simple $NH_t(n; n)$, t divides n

Theorem 3.14. *For every odd integer $n \geq 1$ and for every divisor t of n , there exists a globally simple $NH_t(n; n)$.*

Proof. Let n and t be as in the statement and set $v = 2n^2 + t$. Let H be the $n \times t$ array with elements in \mathbb{Z}_v whose columns are so defined: for j odd, set

$$C_j = \left((j-1)\frac{v}{t} + 1, j\frac{v}{t} - 2, (j-1)\frac{v}{t} + 3, j\frac{v}{t} - 4, \dots, j\frac{v}{t} - (n-1), (j-1)\frac{v}{t} + n \right),$$

for j even, set

$$C_j = \left(-(j-1)\frac{v}{t} - 1, -j\frac{v}{t} + 2, -(j-1)\frac{v}{t} - 3, -j\frac{v}{t} + 4, \dots, -j\frac{v}{t} + (n-1), -(j-1)\frac{v}{t} - n \right).$$

For $\alpha \in \left[1, \frac{n}{t} - 1\right]$, let $H_{\alpha n}$ be the matrix obtained from H by adding $\epsilon \alpha n$ to the elements of H , where $\epsilon = 1$ for the elements of H in the positions (i, j) with $i \equiv j \pmod{2}$, $\epsilon = -1$ for the elements of H in the positions (i, j) with $i \not\equiv j \pmod{2}$. Now let A be the $n \times n$ array so constructed: $\begin{bmatrix} H & -H_n & H_{2n} & -H_{3n} & H_{4n} & \dots & H_{(\frac{n}{t}-1)n} \end{bmatrix}$ where if $t = n$, $A = H$ is understood. To prove that $\text{supp}(A)$ is the required one, it is useful to observe that, since the elements belong to \mathbb{Z}_v , some columns of H can be written also in the following equivalent expression: if j odd and $j \in \left[\frac{t+3}{2}, t\right]$ then

$$C_j = \left((j-t-1)\frac{v}{t} + 1, (j-t)\frac{v}{t} - 2, (j-t-1)\frac{v}{t} + 3, (j-t)\frac{v}{t} - 4, \dots, \right. \\ \left. (j-t)\frac{v}{t} - (n-1), (j-t-1)\frac{v}{t} + n \right),$$

while if j even and $j \in \left[\frac{t+3}{2}, t \right]$ then

$$C_j = \left((t-j+1)\frac{v}{t} - 1, (t-j)\frac{v}{t} + 2, (t-j+1)\frac{v}{t} - 3, (t-j)\frac{v}{t} + 4, \dots, \right. \\ \left. (t-j)\frac{v}{t} + (n-1), (t-j+1)\frac{v}{t} - n \right).$$

Now, it is not hard to see that for any $j \in \left[0, \frac{t-3}{2} \right]$ it results

$$\text{supp}(C_{j+1} \cup C_{t-j}) = \left[j\frac{v}{t} + 1, j\frac{v}{t} + n \right] \cup \left[(j+1)\frac{v}{t} - n, (j+1)\frac{v}{t} - 1 \right]$$

also

$$\text{supp}\left(C_{\frac{t+1}{2}}\right) = \left[\frac{t-1}{2} \cdot \frac{v}{t} + 1, \frac{t-1}{2} \cdot \frac{v}{t} + n \right].$$

By the definition of the arrays $H_{\alpha n}$, it follows that

$$\text{supp}(H_{\alpha n}) = \left(\left[j\frac{v}{t} + 1, j\frac{v}{t} + n \right] + \alpha n \right) \cup \left(\left[(j+1)\frac{v}{t} - n, (j+1)\frac{v}{t} - 1 \right] - \alpha n \right) \cup \\ \cup \left(\left[\frac{t-1}{2} \cdot \frac{v}{t} + 1, \frac{t-1}{2} \cdot \frac{v}{t} + n \right] + \alpha n \right)$$

where $j \in \left[0, \frac{t-3}{2} \right]$. This implies that $\text{supp}(A) = \left[1, \frac{v-1}{2} \right] \setminus \left\{ \frac{v}{t}, 2\frac{v}{t}, \dots, \frac{t-1}{2}\frac{v}{t} \right\}$.

Now we consider the partial sums of the columns of A . If for every odd $j \in [1, t]$ we apply Lemma 3.1(1) to the j th column of A , with $a = (j-1)\frac{v}{t} + 1$, $b = j\frac{v}{t} - 2$, $g = 2$, and $\ell = \frac{n-1}{2}$, we get $S(C_j) = X_j \cup Y_j$, where:

$$X_j = \left\{ k \left[(2j-1)\frac{v}{t} - 1 \right] : k \in \left[1, \frac{n-1}{2} \right] \right\}, \\ Y_j = \left\{ (j-1)\frac{v}{t} + 1 + k \left[(2j-1)\frac{v}{t} + 1 \right] : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

It is easy to see that these are sets containing nonzero elements, and that they are disjoint as their elements belong to different classes modulo $\frac{v}{t}$.

Similarly, for every even $j \in [1, t]$ Lemma 3.1(1) can be applied to the j th column of A with $a = -(j - 1)\frac{v}{t} - 1$, $b = -j\frac{v}{t} + 2$, $g = -2$ and $\ell = \frac{n-1}{2}$, obtaining $S(C_j) = X_j \cup Y_j$, where:

$$X_j = \left\{ k \left[(1 - 2j)\frac{v}{t} + 1 \right] : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ -(j - 1)\frac{v}{t} - 1 + k \left[(1 - 2j)\frac{v}{t} - 1 \right] : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

With the very same argument it can be shown that X_j and Y_j are disjoint sets not containing zero.

One can directly check that, for $\alpha \in \left[1, \frac{n}{t} - 1 \right]$ and for every $j \in [1, t]$ the partial sums of the j th column of $H_{\alpha n}$ are given by $X_{j,\alpha} \cup Y_{j,\alpha}$ where:

$$X_{j,\alpha} = X_j \quad \text{and} \quad Y_{j,\alpha} = Y_j + (-1)^{j+1}\alpha n.$$

It can then be seen that the sets $X_{j,\alpha}$ and $Y_{j,\alpha}$ are disjoint and contain only nonzero elements.

Hence each column of A admits a simple natural ordering.

To prove that the same property holds also for the rows of A , we start by observing that the rows of the array H are the following ones. For every odd $i \in [1, n]$

$$R_i = \left(i, -\frac{v}{t} - i, 2\frac{v}{t} + i, -3\frac{v}{t} - i, \dots, (t - 3)\frac{v}{t} + i, -(t - 2)\frac{v}{t} - i, (t - 1)\frac{v}{t} + i \right),$$

and for every even $i \in [1, n]$

$$R_i = \left(\frac{v}{t} - i, -2\frac{v}{t} + i, 3\frac{v}{t} - i, -4\frac{v}{t} + i, \dots, (t - 2)\frac{v}{t} - i, -(t - 1)\frac{v}{t} + i, -i \right).$$

For every odd $i \in [1, n]$ we can apply Lemma 3.1(1) with $a = i$, $b = -\frac{v}{t} - i$, $g = 2\frac{v}{t}$ and $\ell = \frac{t-1}{2}$, obtaining $S(R_i) = X \cup Y_i$ where:

$$X = \left\{ -\frac{v}{t}k : k \in \left[1, \frac{t-1}{2} \right] \right\},$$

$$Y_i = \left\{ i + k\frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

Analogously, for every even $i \in [1, n]$ we can apply Lemma 3.1(1) with $a = \frac{v}{t} - i$, $b = -2\frac{v}{t} + i$, $g = 2\frac{v}{t}$ and $\ell = \frac{t-1}{2}$, obtaining $S(R_i) = X \cup Y_i$ where:

$$X = \left\{ -\frac{v}{t}k : k \in \left[1, \frac{t-1}{2} \right] \right\},$$

$$Y_i = \left\{ \frac{v}{t} - i + k\frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

It is immediate to see that, in both cases, X and Y_i are disjoint sets not containing zero.

One can directly check that, for $\alpha \in \left[1, \frac{n}{t} - 1\right]$ and for every $i \in [1, n]$ the partial sums of the i th row of $H_{\alpha n}$ are given by $X_\alpha \cup Y_{i,\alpha}$ where:

$$X_\alpha = X \quad \text{and} \quad Y_{i,\alpha} = Y_i + (-1)^{i+1}\alpha n.$$

The sets are disjoint and do not contain zero.

Now, let $\alpha_1, \alpha_2 \in \left[0, \frac{n}{t} - 1\right]$ be distinct integers, for any $i \in [1, n]$ and for $j \in \{1, 2\}$ denote by $R_{i,\alpha_j} = (a_{i,1+\alpha_j t}, a_{i,2+\alpha_j t}, \dots, a_{i,t+\alpha_j t})$ the elements in the i th row of A corresponding to the block $(-1)^\alpha H_{\alpha n}$, where $H_0 = H$ is understood. By $S(R_{i,\alpha_j})$ we mean the list of the partial sums of R_i corresponding to the subsequence R_{i,α_j} , that is $\{s_a | a \in [1 + \alpha_j t, t + \alpha_j t]\}$. To prove that the rows of A admit a simple natural ordering, we have to show that $S(R_{i,\alpha_1}) \cap S(R_{i,\alpha_2}) = \emptyset$ for every $i \in [1, n]$ and for every choice of $\alpha_1, \alpha_2 \in \left[0, \frac{n}{t} - 1\right]$ with $\alpha_1 \neq \alpha_2$.

As a first step, we notice that it suffices to control the aforementioned property for $|\alpha_1 - \alpha_2| = 1$. In fact, the i th row of A written modulo $\frac{v}{t}$ is:

$$R_i \equiv (-1)^{i+1} \left(\underbrace{(i, -i, i, \dots, i)}_{t \text{ terms}}, \underbrace{-(n+i), n+i, \dots, -(n+i)}_{t \text{ terms}}, \right. \\ \left. \underbrace{2n+i, -(2n+i), \dots, 2n+i}_{t \text{ terms}}, \underbrace{-(3n+i), 3n+i, \dots, -(3n+i)}_{t \text{ terms}}, \dots, \right. \\ \left. \underbrace{\left(\frac{n}{t}-1\right)n+i, -\left(\left(\frac{n}{t}-1\right)n+i\right), \dots, \left(\frac{n}{t}-1\right)n+i\right)}_{t \text{ terms}} \right),$$

hence its ordered list of partial sums, considered modulo $\frac{v}{t}$, is given by:

$$(-1)^{i+1} \left(\underbrace{(i, 0, i, \dots, i)}_{t \text{ terms}}, \underbrace{-n, i, \dots, -n}_{t \text{ terms}}, \underbrace{n+i, -n, \dots, n+i}_{t \text{ terms}}, \underbrace{-2n, n+i, \dots, -2n}_{t \text{ terms}}, \dots, \right. \\ \left. \underbrace{\frac{1}{2}\left(\frac{n}{t}-1\right)n+i, -\frac{1}{2}\left(\frac{n}{t}-1\right)n, \dots, \frac{1}{2}\left(\frac{n}{t}-1\right)n+i}_{t \text{ terms}} \right).$$

In particular, it can be seen that for every $\alpha \in \left[0, \frac{n}{t} - 1\right]$ the partial sums of $R_{i,\alpha}$ modulo $\frac{v}{t}$ are:

$$\begin{aligned} &(-1)^{i+1}\left(i + \frac{\alpha - 1}{2}\right) \quad \text{and} \quad (-1)^i \frac{\alpha + 1}{2}n \quad \text{for } \alpha \text{ odd,} \\ &(-1)^{i+1}\left(i + \frac{\alpha}{2}\right) \quad \text{and} \quad (-1)^i \frac{\alpha}{2}n \quad \text{for } \alpha \text{ even.} \end{aligned}$$

We can then conclude that for $|\alpha_1 - \alpha_2| > 1$ the partial sums of the i th row of A corresponding to R_{i,α_1} and R_{i,α_2} are trivially disjoint. In the following for any $i \in [1, n]$ and $\alpha \in \left[0, \frac{n}{t} - 1\right]$ set $c = \sum_{j=1}^{\alpha t} a_{i,j}$. Fix now an odd $i \in [1, n]$, let $\alpha \in \left[0, \frac{n}{t} - 1\right]$ be an even integer, we have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$X_\alpha = \left\{c - k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha} = \left\{c + i + \alpha n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

It is then not hard to see that $S(R_{i,\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + \left(c + i + \alpha n + \frac{t-1}{2} \frac{v}{t}\right)$, where:

$$X_{\alpha+1} = \left\{k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha+1} = \left\{-i - (\alpha + 1)n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

We have only to check that $Y_{i,\alpha} \cap \left(X_{\alpha+1} + \left(c + i + \alpha n + \frac{t-1}{2} \frac{v}{t}\right)\right) = \emptyset$. Assume by contradiction that there exist $k_1 \in \left[0, \frac{t-1}{2}\right]$, $k_2 \in \left[1, \frac{t-1}{2}\right]$ and an odd i such that:

$$c + i + \alpha n + k_1 \frac{v}{t} \equiv k_2 \frac{v}{t} + c + i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This can be rewritten as:

$$\left(\frac{t-1}{2} + k_2 - k_1\right) \frac{v}{t} \equiv 0 \pmod{v}.$$

It can then be easily seen that this equation cannot hold, as $k_2 - k_1 \in \left[\frac{3-t}{2}, \frac{t-1}{2}\right]$.

Now, let $\alpha \in \left[0, \frac{n}{t} - 1\right]$ be an odd integer; similarly, we have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$X_\alpha = \left\{c + k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha} = \left\{c - i - \alpha n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

It can now be seen that $S(R_{i,\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + \left(c - i - \alpha n - \frac{t-1}{2} \frac{v}{t}\right)$, where:

$$X_{\alpha+1} = \left\{-k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha+1} = \left\{i + (\alpha + 1)n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

As before, it is sufficient to check that $Y_{i,\alpha} \cap \left(X_{\alpha+1} + \left(c - i - \alpha n - \frac{t-1}{2} \frac{v}{t} \right) \right) = \emptyset$. Assume by contradiction that there exist $k_1 \in \left[0, \frac{t-1}{2} \right]$, $k_2 \in \left[1, \frac{t-1}{2} \right]$ and an odd i such that:

$$c - i - \alpha n - k_1 \frac{v}{t} \equiv -k_2 \frac{v}{t} + c - i - \alpha n - \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This can be rewritten as:

$$\left(\frac{t-1}{2} + k_2 - k_1 \right) \frac{v}{t} \equiv 0 \pmod{v}.$$

With the very same argument used before, it can be seen that this equation is never verified, obtaining a contradiction.

Assume now that $i \in [1, n]$ is even, and consider any even integer $\alpha \in \left[0, \frac{n}{t} - 1 \right]$. We have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$\begin{aligned} X_\alpha &= \left\{ c - k \frac{v}{t} : k \in \left[1, \frac{t-1}{2} \right] \right\}, \\ Y_{i,\alpha} &= \left\{ c + \frac{v}{t} - i - \alpha n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}. \end{aligned}$$

We have then that $S(R_{\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + \left(c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \frac{v}{t} \right)$, where:

$$\begin{aligned} X_{\alpha+1} &= \left\{ k \frac{v}{t} : k \in \left[1, \frac{t-1}{2} \right] \right\}, \\ Y_{i,\alpha+1} &= \left\{ -\frac{v}{t} + i + (\alpha + 1)n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}. \end{aligned}$$

We need to check $Y_{i,\alpha} \cap \left(X_{\alpha+1} + \left(c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \frac{v}{t} \right) \right) = \emptyset$; assume by contradiction that there exist $k_1 \in \left[0, \frac{t-1}{2} \right]$, $k_2 \in \left[1, \frac{t-1}{2} \right]$ and an even i such that:

$$c + \frac{v}{t} - i - \alpha n + k_1 \frac{v}{t} \equiv k_2 \frac{v}{t} + c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This is equivalent to:

$$\left(\frac{t-1}{2} + k_2 - k_1 \right) \frac{v}{t} \equiv 0 \pmod{v}.$$

It is easy to see that this is never the case.

With the very same argument, it can be shown that for every even i , the partial sums of $R_{i,\alpha}$ and $R_{i,\alpha+1}$ are distinct for every odd α . \square

Remark 3.15. From the globally simple array A constructed in Theorem 3.14, one can deduce the total sum of each row and of each column. Let $n, t \geq 1$ be odd integers with t dividing n , then for every $j \in [1, t]$ and every $\alpha \in [0, \frac{n}{t} - 1]$ we have:

$$\sum_{a_{i,j} \in C_{j+\alpha t}} a_{i,j} = (-1)^{\alpha+j+1} \left[(j-1) \frac{v}{t} + 1 + \alpha n + \frac{n-1}{2} \left[(2j-1) \frac{v}{t} + 1 \right] \right].$$

We now focus on the rows of A . Consider any odd $i \in [1, n]$, it can be seen that for every $\alpha \in [0, \frac{n}{t} - 1]$ we have:

$$\sum_{j \in [1+\alpha t, t+\alpha t]} a_{i,j} = (-1)^\alpha \left(i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right);$$

hence the sum of the elements in the i th row of A , for i odd, is:

$$\sum_{\alpha \in [0, \frac{n}{t}-1]} (-1)^\alpha \left(i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right).$$

Thus, it is not hard to see that for every odd $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = i + n \left(\frac{n}{t} - 1 \right) \cdot \frac{1}{2} + \frac{t-1}{2} \cdot \frac{v}{t}.$$

Similarly, it can be seen that for every even $i \in [1, n]$ and for every $\alpha \in [0, \frac{n}{t} - 1]$:

$$\sum_{j \in [1+\alpha t, t+\alpha t]} a_{i,j} = (-1)^\alpha \left(\frac{v}{t} - i - \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right).$$

From which we obtain that for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{v}{t} - i - n \left(\frac{n}{t} - 1 \right) \cdot \frac{1}{2} + \frac{t-1}{2} \cdot \frac{v}{t}.$$

Example 3.16. Below we have an $NH_{11}(11; 11)$, say A , whose elements belong to \mathbb{Z}_{253} . Following the notation of the proof of Theorem 3.14, we are in the case $t = n$ hence $A = H$.

| | | | | | | | | | | |
|----|-----|----|-----|-----|------|------|----|-----|----|-----|
| 1 | -24 | 47 | -70 | 93 | -116 | -114 | 91 | -68 | 45 | -22 |
| 21 | -44 | 67 | -90 | 113 | 117 | -94 | 71 | -48 | 25 | -2 |
| 3 | -26 | 49 | -72 | 95 | -118 | -112 | 89 | -66 | 43 | -20 |
| 19 | -42 | 65 | -88 | 111 | 119 | -96 | 73 | -50 | 27 | -4 |
| 5 | -28 | 51 | -74 | 97 | -120 | -110 | 87 | -64 | 41 | -18 |
| 17 | -40 | 63 | -86 | 109 | 121 | -98 | 75 | -52 | 29 | -6 |
| 7 | -30 | 53 | -76 | 99 | -122 | -108 | 85 | -62 | 39 | -16 |
| 15 | -38 | 61 | -84 | 107 | 123 | -100 | 77 | -54 | 31 | -8 |
| 9 | -32 | 55 | -78 | 101 | -124 | -106 | 83 | -60 | 37 | -14 |
| 13 | -36 | 59 | -82 | 105 | 125 | -102 | 79 | -56 | 33 | -10 |
| 11 | -34 | 57 | -80 | 103 | -126 | -104 | 81 | -58 | 35 | -12 |

Example 3.17. Here we take $n = 15$, $t = 3, 5$ and we follow the proof of Theorem 3.14. An $NH_3(15; 15)$, say A , has elements in \mathbb{Z}_{453} , $\frac{v}{t} = 151$ and we have

$$A = \begin{bmatrix} H & -H_{15} & H_{30} & -H_{45} & H_{60} \end{bmatrix}$$

| | | | | | | | | | | | | | | |
|-----|------|------|------|------|-----|-----|------|------|------|------|-----|----|------|-----|
| 1 | -152 | -150 | -16 | 167 | 135 | 31 | -182 | -120 | -46 | 197 | 105 | 61 | -212 | -90 |
| 149 | 153 | -2 | -134 | -168 | 17 | 119 | 183 | -32 | -104 | -198 | 47 | 89 | 213 | -62 |
| 3 | -154 | -148 | -18 | 169 | 133 | 33 | -184 | -118 | -48 | 199 | 103 | 63 | -214 | -88 |
| 147 | 155 | -4 | -132 | -170 | 19 | 117 | 185 | -34 | -102 | -200 | 49 | 87 | 215 | -64 |
| 5 | -156 | -146 | -20 | 171 | 131 | 35 | -186 | -116 | -50 | 201 | 101 | 65 | -216 | -86 |
| 145 | 157 | -6 | -130 | -172 | 21 | 115 | 187 | -36 | -100 | -202 | 51 | 85 | 217 | -66 |
| 7 | -158 | -144 | -22 | 173 | 129 | 37 | -188 | -114 | -52 | 203 | 99 | 67 | -218 | -84 |
| 143 | 159 | -8 | -128 | -174 | 23 | 113 | 189 | -38 | -98 | -204 | 53 | 83 | 219 | -68 |
| 9 | -160 | -142 | -24 | 175 | 127 | 39 | -190 | -112 | -54 | 205 | 97 | 69 | -220 | -82 |
| 141 | 161 | -10 | -126 | -176 | 25 | 111 | 191 | -40 | -96 | -206 | 55 | 81 | 221 | -70 |
| 11 | -162 | -140 | -26 | 177 | 125 | 41 | -192 | -110 | -56 | 207 | 95 | 71 | -222 | -80 |
| 139 | 163 | -12 | -124 | -178 | 27 | 109 | 193 | -42 | -94 | -208 | 57 | 79 | 223 | -72 |
| 13 | -164 | -138 | -28 | 179 | 123 | 43 | -194 | -108 | -58 | 209 | 93 | 73 | -224 | -78 |
| 137 | 165 | -14 | -122 | -180 | 29 | 107 | 195 | -44 | -92 | -210 | 59 | 77 | 225 | -74 |
| 15 | -166 | -136 | -30 | 181 | 121 | 45 | -196 | -106 | -60 | 211 | 91 | 75 | -226 | -76 |

An $NH_5(15; 15)$, say A , has elements in \mathbb{Z}_{455} , $\frac{v}{t} = 91$ and we have

$$A = \begin{bmatrix} H & -H_{15} & H_{30} \end{bmatrix}$$

| | | | | | | | | | | | | | | |
|-----|------|------|-----|-----|-----|-----|------|------|----|----|------|------|-----|-----|
| 1 | -92 | 183 | 181 | -90 | -16 | 107 | -198 | -166 | 75 | 31 | -122 | 213 | 151 | -60 |
| 89 | -180 | -184 | 93 | -2 | -74 | 165 | 199 | -108 | 17 | 59 | -150 | -214 | 123 | -32 |
| 3 | -94 | 185 | 179 | -88 | -18 | 109 | -200 | -164 | 73 | 33 | -124 | 215 | 149 | -58 |
| 87 | -178 | -186 | 95 | -4 | -72 | 163 | 201 | -110 | 19 | 57 | -148 | -216 | 125 | -34 |
| 5 | -96 | 187 | 177 | -86 | -20 | 111 | -202 | -162 | 71 | 35 | -126 | 217 | 147 | -56 |
| 85 | -176 | -188 | 97 | -6 | -70 | 161 | 203 | -112 | 21 | 55 | -146 | -218 | 127 | -36 |
| 7 | -98 | 189 | 175 | -84 | -22 | 113 | -204 | -160 | 69 | 37 | -128 | 219 | 145 | -54 |
| 83 | -174 | -190 | 99 | -8 | -68 | 159 | 205 | -114 | 23 | 53 | -144 | -220 | 129 | -38 |
| 9 | -100 | 191 | 173 | -82 | -24 | 115 | -206 | -158 | 67 | 39 | -130 | 221 | 143 | -52 |
| 139 | -172 | -192 | 101 | -10 | -66 | 157 | 207 | -116 | 25 | 51 | -142 | -222 | 131 | -40 |
| 11 | -102 | 193 | 171 | -80 | -26 | 117 | -208 | -156 | 65 | 41 | -132 | 223 | 141 | -50 |
| 79 | -170 | -194 | 103 | -12 | -64 | 155 | 209 | -118 | 27 | 49 | -140 | -224 | 133 | -42 |
| 13 | -104 | 195 | 169 | -78 | -28 | 119 | -210 | -154 | 63 | 43 | -134 | 225 | 139 | -48 |
| 77 | -168 | -196 | 105 | -14 | -62 | 153 | 211 | -120 | 29 | 47 | -138 | -226 | 135 | -44 |
| 15 | -106 | 197 | 167 | -76 | -30 | 121 | -212 | -152 | 61 | 45 | -136 | 227 | 137 | -46 |

3.6 | Globally simple $NH_t(n; n)$, for every n prime and admissible t

We start with an easy remark for $n = 2$.

Remark 3.18. Note that the existence of a globally simple $NH_t(2; 2)$ for any admissible t is trivial. Below we give an example for each possible case, that is for $t = 1, 2, 4, 8$

(the first array is both an $NH(2; 2)$ and an $NH_2(2; 2)$):

| | |
|---|---|
| 1 | 2 |
| 3 | 4 |

| | |
|---|---|
| 1 | 2 |
| 4 | 5 |

| | |
|---|---|
| 1 | 3 |
| 5 | 7 |

Now we can present a complete solution whenever n is a prime.

Theorem 3.19. *Let n be a prime. There exists a globally simple $NH_t(n; n)$ for every admissible t .*

Proof. Since n is a prime the admissible values for t are $1, 2, n, 2n, n^2, 2n^2$. For $t = 1$ the result is contained in Theorem 5.5 of [10]. For $t \neq 1$ the result follows by Remark 3.18 and by Propositions 3.2, 3.5, 3.8, 3.11 and Theorem 3.14. □

4 | CONNECTION WITH PATH-DECOMPOSITIONS

Simple nonzero sum Heffter arrays are a useful tool for constructing orthogonal path decompositions. To present this connection we introduce some notations and we recall some basic definitions. Given a graph Γ , by $V(\Gamma)$ and $E(\Gamma)$ we denote its vertex-set and its edge-set, respectively. By $K_v, K_{q \times r}$, and P_k we denote the complete graph on v vertices, the complete multipartite graph with q parts each of size r and the path of length k (with $k + 1$ distinct vertices), respectively. In the following the complete multipartite graph $K_{\frac{2nk+t}{t} \times t}$ is equivalent to the Cayley graph $\text{Cay}[\mathbb{Z}_{2nk+t} : \mathbb{Z}_{2nk+t} \setminus \frac{2nk+t}{t} \mathbb{Z}_{2nk+t}]$. We recall that given a subgraph Γ of a graph K , a Γ -decomposition of K is a set of graphs, called blocks, all isomorphic to Γ whose edges partition the edge-set of K . Also, a Γ -decomposition \mathcal{D} of a graph K is said to be cyclic if, up to isomorphisms, $V(K) = \mathbb{Z}_v$ and for any $B \in \mathcal{D}, B + 1 \in \mathcal{D}$ too. Two graph decompositions \mathcal{D} and \mathcal{D}' are said orthogonal if for any $B \in \mathcal{D}$ and any $B' \in \mathcal{D}', |E(B) \cap E(B')| \leq 1$. The connection between nonzero sum Heffter arrays and graph decompositions is explained by the following result.

Proposition 4.1 (Proposition 2.9, [10]). *Let A be an $NH_t(m, n; h, k)$ simple with respect to the orderings ω_{R_i} for $i \in [1, m]$ and ω_{C_j} for $j \in [1, n]$. Then:*

- (1) *there exists a cyclic P_h -decomposition \mathcal{D}_R of $K_{\frac{2nk+t}{t} \times t}$;*
- (2) *there exists a cyclic P_k -decomposition \mathcal{D}_C of $K_{\frac{2nk+t}{t} \times t}$;*
- (3) *the decompositions \mathcal{D}_R and \mathcal{D}_C are orthogonal.*

The base blocks of the decompositions of previous proposition are so constructed. Let ω_{R_i} be a simple ordering of the i th row R_i of A , then consider the walk $P_h^i = (0, s_{i,1}, s_{i,2}, \dots, s_{i,h})$, where by $s_{i,j}$ we denote the j th partial sum of ω_{R_i} . Since the ordering is simple, the partial sums are pairwise distinct and nonzero, hence P_h^i is a path on $h + 1$ distinct vertices. The set $\{P_h^i | i \in [1, m]\}$ is a set of base blocks of \mathcal{D}_R . Reasoning in the same way on the columns, we can construct the base blocks of the P_k -decomposition \mathcal{D}_C .

The existence results proved in the previous section allow us to get the following results on cyclic path decompositions.

Theorem 4.2. For every odd integer $n \geq 1$ and every divisor t of n , there exists a pair of orthogonal cyclic P_n -decompositions of $K_{\frac{2n^2+t}{t} \times t}$.

Proof. The result follows by Theorem 3.14 and Proposition 4.1. \square

Proposition 4.3. For every odd integer $n \geq 1$ and any $t \in \{2, 2n, n^2, 2n^2\}$ there exists a pair of orthogonal cyclic P_n -decompositions of $K_{\frac{2n^2+t}{t} \times t}$.

Proof. The result follows by Propositions 3.2, 3.5, 3.8, 3.11, and 4.1. \square

Remark 4.4. Since Proposition 3.2 holds for every integer n , the previous proposition holds also when n is even and $t = 2$.

5 | CONNECTION WITH BIEMBEDDINGS

Relative nonzero sum Heffter arrays can be used to construct biembeddings of circuit decompositions on orientable surfaces, as explained in [10]. In this section, we get new results on biembeddings obtained thanks to the constructions presented in Section 3. We start by recalling some definitions, see [19].

Definition 5.1. Let Γ be a graph endowed with the topology of a one-dimensional simplicial complex; then, an *embedding* of Γ in a surface Σ is a continuous injective mapping $\phi : \Gamma \rightarrow \Sigma$.

A connected component of $\Sigma \setminus \phi(\Gamma)$ is called *ϕ -face*. An embedding ϕ is said to be *cellular* if every ϕ -face is homeomorphic to an open disc.

Definition 5.2. A *biembedding* of two circuit decompositions \mathcal{D} and \mathcal{D}' of a simple graph Γ is a face 2-colorable embedding of Γ in which one color class is comprised of the circuits in \mathcal{D} and the other one of the circuits in \mathcal{D}' .

For every edge e of a given graph Γ , we will consider its two possible directions e^+ and e^- , and we denote by τ the involution which swaps e^+ and e^- for every e . Then, for every vertex v in Γ , a *local rotation* ρ_v is a cyclic permutation of the edges directed out of v . If we choose a local rotation for each vertex of Γ , then we obtain a rotation of the directed edges of Γ . We also recall the following result [1, 19].

Theorem 5.3. A rotation ρ on Γ is equivalent to a cellular embedding of Γ in an orientable surface. The face boundaries of the embedding corresponding to ρ are the orbits of $\rho \circ \tau$.

Moreover, by knowing the number of faces the genus g of the surface can be obtained from Euler's formula $|V| - |E| + |F| = 2 - 2g$, where V , E , and F denote the number of vertices, edges, and faces determined by the embedding in the surface, respectively.

Let $A = \text{NH}_t(m, n; h, k)$ be a relative nonzero sum Heffter array. If for any $i \in [1, m]$ and for any $j \in [1, n]$, the orderings ω_{R_i} and ω_{C_j} are simple, we define by $\omega_r = \omega_{R_1} \circ \dots \circ \omega_{R_m}$ the simple ordering for the rows and by $\omega_c = \omega_{C_1} \circ \dots \circ \omega_{C_n}$ the simple ordering for the columns.

The orderings ω_r and ω_c are said to be *compatible* if $\omega_c \circ \omega_r$ is a cycle of length $|\mathcal{E}(A)|$. From two compatible orderings a rotation on $K_{\frac{2nk+t}{t} \times t}$ can be obtained, implying by Theorem 5.3 the existence of a cellular biembedding of this graph. Cellular biembeddings obtained from Heffter arrays are also called *Archdeacon embeddings*, see for instance [6].

Since the existence results on compatible orderings of [7] do not depend on the sum of the elements of a row or of a column of the array, from Theorem 3.3 of [7] we have that an $NH_t(n; n)$ admits compatible natural orderings if and only if n is odd.

Let A be an $NH_t(m, n; h, k)$, we denote by $\mathcal{D}_{\omega_r^{-1}}$ the cyclic P_h -decomposition of $K_{\frac{2nk+t}{t} \times t}$ obtained from Proposition 4.1 with respect to the orderings $\omega_{R_i}^{-1}$, where the starting element is the last element of ω_{R_i} . We recall that, as remarked in Section 2, if $\omega_{R_i} = (a_{i,1}, a_{i,2}, \dots, a_{i,k})$ is simple then $\omega_{R_i}^{-1} = (a_{i,k}, a_{i,k-1}, \dots, a_{i,1})$ is simple too. Analogously, by \mathcal{D}_{ω_c} we mean the cyclic P_k -decomposition of $K_{\frac{2nk+t}{t} \times t}$ obtained from Proposition 4.1 with respect to the orderings ω_{C_j} .

In [10], the authors construct a decomposition of K_v into circuits starting from a cyclic path decomposition \mathcal{D} of K_v , with $V(K_v) = \mathbb{Z}_v$, as follows. Given $P = (x_0, x_1, \dots, x_k) \in \mathcal{D}$, let λ_P be the minimum positive integer such that $\lambda_P(x_k - x_0) = 0$ modulo v . Then define the circuit:

$$C_P := \cup_{i=0}^{\lambda_P-1} P + i(x_k - x_0).$$

A decomposition of K_v into circuits is given by $\mathcal{C}(\mathcal{D}) := \{C_P : P \in \mathcal{D}\}$.

Then, we report Theorem 6.4 of [10], which links cellular biembeddings with nonzero sum Heffter arrays:

Proposition 5.4. *Let A be a nonzero sum Heffter array $NH(m, n; h, k)$ that is simple with respect to the compatible orderings ω_r and ω_c . Then there exists a cellular biembedding of the circuit decompositions $\mathcal{C}(\mathcal{D}_{\omega_r^{-1}})$ and $\mathcal{C}(\mathcal{D}_{\omega_c})$ of K_{2nk+1} into an orientable surface.*

Remark 5.5. We emphasize that if in previous proposition we remove the hypothesis that the orderings are simple we get again a biembedding (as shown in Theorem 2.5 of [6]), but now the faces may not necessarily be union of paths of length h or k .

We will now see that Proposition 5.4 can be generalized to the case of a *relative* nonzero sum Heffter array.

Proposition 5.6. *Let A be a relative nonzero sum Heffter array $NH_t(m, n; h, k)$ that is simple with respect to two compatible orderings ω_r and ω_c . Then there exists a cellular biembedding of two circuit decompositions $\mathcal{C}(\mathcal{D}_{\omega_r^{-1}})$ and $\mathcal{C}(\mathcal{D}_{\omega_c})$ of $K_{\frac{2nk+t}{t} \times t}$ into an orientable surface.*

Proof. Let A be an $NH_t(m, n; h, k)$ admitting two compatible orderings ω_r and ω_c , thus $\omega_c \circ \omega_r$ is a cycle of length $|\mathcal{E}(A)|$. Let $\bar{\rho}_0$ be a permutation on $\pm\mathcal{E}(A) = \mathbb{Z}_{2nk+t} \setminus \frac{2nk+t}{t}\mathbb{Z}_{2nk+t}$ defined by:

$$\bar{\rho}_0 = \begin{cases} -\omega_r(a) & \text{if } a \in \mathcal{E}(A); \\ \omega_c(-a) & \text{if } a \in -\mathcal{E}(A); \end{cases}$$

which, as done in Theorem 6.4 of [10], can be proven to act cyclically on $\pm\mathcal{E}(A)$.

As done before, identify the complete multipartite graph $K_{\frac{2nk+t}{t} \times t}$ as the Cayley graph $\text{Cay}[\mathbb{Z}_{2nk+t} : \mathbb{Z}_{2nk+t} \setminus \frac{2nk+t}{t} \mathbb{Z}_{2nk+t}]$, that is $\text{Cay}[\mathbb{Z}_{2nk+t} : \pm \mathcal{E}(A)]$.

Define the map ρ on the oriented edges of $K_{\frac{2nk+t}{t} \times t}$ as:

$$\rho((x, x+a)) = (x, x + \bar{\rho}_0(a)).$$

It can then be seen that ρ is a rotation of $K_{\frac{2nk+t}{t} \times t}$, since $\bar{\rho}_0$ acts cyclically on $\pm \mathcal{E}(A)$. By Theorem 5.3 there exists a cellular embedding of $K_{\frac{2nk+t}{t} \times t}$ in an orientable surface; in particular, its face boundaries correspond to the orbits of $\rho \circ \tau$, where $\tau((x, x+a)) = (x+a, x)$.

Consider now the oriented edge $(x, x+a)$ with $a \in \mathcal{E}(A)$ and $x \in \mathbb{Z}_{2nk+t}$, and let C be the column of A containing a . Let λ_c be the minimum positive integer such that $\sum_{i=0}^{\lambda_c |\mathcal{E}(C)|-1} \omega_c^i(a) = 0$ in \mathbb{Z}_{2nk+t} . Similarly to Theorem 6.4 of [10] $(x, x+a)$ belongs to the face F_1 whose boundary is:

$$\left(x, x+a, x+a+\omega_c(a), \dots, x + \sum_{i=0}^{\lambda_c |\mathcal{E}(C)|-2} \omega_c^i(a) \right).$$

Moreover, notice that λ_c is the minimum positive integer such that $\lambda_c \sum_{a_{ij} \in C} a_{ij} = 0$ in \mathbb{Z}_{2nk+t} and F_1 has length $k\lambda_c$. We can also conclude that F_1 covers exactly λ_c edges of type $(y, y+a)$ for some $y \in \mathbb{Z}_{2nk+t}$. Since the total number of such edges is $2nk+t$, the column C induces $(2nk+t)/\lambda_c$ different faces.

Let now $a \notin \mathcal{E}(A)$ and R be the row containing $-a$. Consider now the oriented edge $(x, x+a)$ and denote by λ_r the minimum positive integer such that $\sum_{i=0}^{\lambda_r |\mathcal{E}(R)|-1} \omega_r^i(-a) = 0$ in \mathbb{Z}_{2nk+t} . Then, $(x, x+a)$ belongs to the face F_2 whose boundary is:

$$\left(x, x + \sum_{i=1}^{\lambda_r |\mathcal{E}(R)|-1} \omega_r^{-i}(-a), x + \sum_{i=1}^{\lambda_r |\mathcal{E}(R)|-2} \omega_r^{-i}(-a), \dots, x + \omega_r^{-1}(-a) \right).$$

Similarly to the previous case, we notice that λ_r is the minimum positive integer such that $\lambda_r \sum_{a_{ij} \in R} a_{ij} = 0$ in \mathbb{Z}_{2nk+t} and F_2 has length $h\lambda_r$. Moreover, we have that the row R induces $(2nk+t)/\lambda_r$ different faces.

In particular, we proved that any nonoriented edge $\{x, x+a\}$ of $K_{\frac{2nk+t}{t} \times t}$ belongs to the boundaries of exactly two faces: one of type F_1 and one of type F_2 , thus the embedding is 2-colorable. Finally, note that those face boundaries are circuits of the decompositions $\mathcal{C}(\mathcal{D}_{\omega_r^{-1}})$ and $\mathcal{C}(\mathcal{D}_{\omega_c})$. \square

All the nonzero sum Heffter arrays constructed in this paper, for n odd, admit compatible orderings in view of Theorem 3.3 of [7]. Now, by following Proposition 5.6, one can compute the length of each face of the biembeddings, using the total sums reported in Remarks 3.3, 3.6, 3.9, 3.12, and 3.15. In the next propositions, we consider some particular cases where the lengths of the faces have nice expressions.

We begin by considering the Archdeacon embedding obtained for $t = 2$:

Proposition 5.7. *Let $n \geq 1$ be an odd integer such that $\frac{n^2+1}{2}$ is a prime, then there exists a cellular biembedding of $K_{(n^2+1) \times 2}$ such that every face has length $4n$ or a multiple of $n\frac{n^2+1}{2}$.*

Proof. Let n be an odd positive integer and A be the $NH_2(n; n)$ constructed in Proposition 3.2; it will be enough to prove that the total sum of every row and of every column has order 4 or a multiple of $\frac{n^2+1}{2}$ modulo $2(n^2 + 1)$. By Remark 3.3, for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left(j + \frac{n-1}{2} n \right).$$

Note that for $j = \frac{n+1}{2}$ the previous expression yields $\pm \frac{n^2+1}{2}$, obtaining order 4 modulo $2(n^2 + 1)$. Since $\frac{n^2+1}{2}$ is a prime, it is easy to see that for $j \neq \frac{n+1}{2}$ the greatest common divisor between the total sum and $n^2 + 1$ is either 1 or 2. Hence the order is a multiple of $\frac{n^2+1}{2}$.

Now, again by Remark 3.3, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i+1} \left((i-1)n + \frac{n+1}{2} \right).$$

Similarly to the previous case, for $i = \frac{n+1}{2}$ we obtain order 4, while for $i \neq \frac{n+1}{2}$ the order is a multiple of $\frac{n^2+1}{2}$. □

Proposition 5.8. *Let n be an odd prime, then there exists a cellular biembedding of $K_{(n+1) \times 2n}$ such that every face has length $4n$ or a multiple of n^2 .*

Proof. Let A be the $NH_{2n}(n; n)$ described in Proposition 3.5. By Remark 3.6, we have that for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (n+1) \left(\frac{n^2}{2} + (-1)^j \frac{n+1-2j}{2} \right).$$

As n is prime and $j \in [1, n]$, it can be seen that for $j \neq \frac{n+1}{2}$ the rightmost bracket is coprime with n ; hence, the order in $\mathbb{Z}_{2n(n+1)}$ is $4n$, so we obtain an embedding where the length of the corresponding faces is $4n^2$. For $j = \frac{n+1}{2}$, the previous expression reads $n^2(n+1)/2$, obtaining order 4, thus the length of the corresponding faces is $4n$.

We will now prove that for every row of A the order of its total sum in \mathbb{Z}_{2n^2+2n} is either 4 or a multiple of n . For every odd $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{-n^3 + n^2 + n - 1 + 2i}{2}.$$

First, since $n^2 + 2n = -n^2$ in \mathbb{Z}_{2n^2+2n} , for $i = \frac{n+1}{2}$ the previous expression reads $\frac{-n^3-n^2}{2} = \frac{-n^2(n+1)}{2}$, thus obtaining order 4 and corresponding length of the faces $4n$. For $i \neq \frac{n+1}{2}$ the total sum is not a multiple of n , as $\frac{-n^3+n^2}{2}$ is a multiple of n , while $\frac{n-1+2i}{2}$ is not. Thus, for $i \neq \frac{n+1}{2}$, the order of the total sum is a multiple of n .

Finally, for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{n^3 + n^2 + n + 1 - 2i}{2}.$$

As for the previous case, it can be seen that the order of the total sum of the i th row of A is a multiple of n for $i \neq \frac{n+1}{2}$, and it is 4 for $i = \frac{n+1}{2}$. \square

Proposition 5.9. *Let n be an odd prime, then there exists a cellular biembedding of $K_{3 \times n^2}$ such that every face has length $3n^2$ or $3n^3$.*

Proof. Let A be the $\text{NH}_{n^2}(n; n)$ described in Proposition 3.8; we follow Remark 3.9 where the total sums of rows and columns are described. In particular, for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j-1} \left(3nj - \frac{3n+1}{2} \right).$$

Now, since $\frac{3n+1}{2}$ is neither a multiple of n , nor a multiple of 3, we obtain that $\sum_{a_{i,j} \in C_j} a_{i,j}$ and $3n^2$ are coprime, thus every face constructed from the columns of A has length $3n^3$.

If $n \equiv 1 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3i - 2 + 3n \frac{n-1}{2} \right).$$

It can then be seen that this term is coprime with $3n^2$ except when $3i$ is equal to $n+2$ (that implies $n \equiv 1 \pmod{3}$) or $2n+2$ (that may hold if $n \equiv 2 \pmod{3}$). Let \bar{i} denote such an index, note that $3n \sum_{a_{i,j} \in R_i} a_{i,j} = 0 \pmod{3n^2}$.

For every $n \equiv 3 \pmod{4}$ and $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3(n+1-i) - 2 + 3n \frac{n-1}{2} \right).$$

Similarly, this term is coprime with $3n^2$ except when $3i$ is equal to $n+1$ (that implies $n \equiv 2 \pmod{3}$) or $2n+1$ (that may hold if $n \equiv 1 \pmod{3}$). Let \bar{i} be such an index, then $3n \sum_{a_{i,j} \in R_i} a_{i,j} = 0 \pmod{3n^2}$.

In any case, we have proven that the length of the faces obtained by the \bar{i} th row is $3n^2$, while every other face has length $3n^3$. \square

Proposition 5.10. *Let n be an odd prime, then there exists a cellular biembedding of $K_{2 \times 2n^2}$ such that every face has length $4n$, $4n^2$, or $4n^3$.*

Proof. Let A be the $NH_{2n^2}(n; n)$ described in Proposition 3.11; we recall that the total sum of the rows and the columns of A is given in Remark 3.12. For every $j \in [1, n]$ we have:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = n^3 + (-1)^{j-1}(2nj - n^2 - n).$$

It can be seen that for $j \neq \frac{n+1}{2}$, the order of the sum in \mathbb{Z}_{4n^2} is $4n$. For $j = \frac{n+1}{2}$ the previous expression yields n^3 , which has order 4 in \mathbb{Z}_{4n^2} . Thus the corresponding faces have length $4n^2$ or $4n$.

For every $n \equiv 1 \pmod{4}$ and $i \in [1, n]$ we have:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(2i - n - 1).$$

It can be seen that for $i = \frac{n+1}{2}$ the previous expression yields n^3 , thus the corresponding faces have length $4n$. For $i \in [1, n]$, $i \neq \frac{n+1}{2}$, the previous expression gives a number coprime with $4n^2$, obtaining that the faces constructed from these rows have length $4n^3$.

For every $n \equiv 3 \pmod{4}$ and for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(n + 1 - 2i).$$

It is then an easy check to verify that this term is coprime with $4n^2$ for every $i \in [1, n]$, $i \neq \frac{n+1}{2}$, obtaining faces with length $4n^3$, while for $i = \frac{n+1}{2}$ the faces have length $4n$. \square

Proposition 5.11. *Let $n \geq 1$ be an odd integer such that $2n + 1$ is prime, then there exists a cellular biembedding of $K_{(2n+1) \times n}$ such that the length of every face is a multiple of $n(2n + 1)$.*

Proof. Let A be the $NH_n(n; n)$ obtained from Theorem 3.14. We follow Remark 3.15 where we reported the total sums of rows and columns of A , keeping in mind that $t = n$ which implies $\alpha = 0$. In particular, for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left[(j - 1)(2n + 1) + 1 + \frac{n - 1}{2} [(2j - 1)(2n + 1) + 1] \right].$$

Since we are in \mathbb{Z}_{2n^2+n} , the previous expression is equivalent to:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1}(-n^2 - n) = (-1)^j n(n + 1).$$

We can then conclude by noticing that $n + 1$ and $2n + 1$ are coprime, thus obtaining that the total sum of the elements in every column has order $2n + 1$ in \mathbb{Z}_{2n^2+n} and that the corresponding faces of the Archdeacon embedding have length $n(2n + 1)$.

Now, for every odd $i \in [1, n]$, we have:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = i + \frac{n - 1}{2}(2n + 1).$$

Since $2n + 1$ is prime, it is immediate to see that this expression and $2n + 1$ are coprime, thus the length of the corresponding faces is a multiple of $n(2n + 1)$. Similarly, for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = -i + \frac{n+1}{2}(2n+1).$$

As before, it can be seen that this expression is coprime with $2n + 1$, obtaining as length face multiples of $n(2n + 1)$. \square

Remark 5.12. Since all the nonzero sum Heffter arrays constructed in this paper have the additional property of being globally simple, the faces of the biembeddings constructed in Propositions 5.7, 5.8, 5.9, 5.10, and 5.11 are union of paths of length n .

Remark 5.13. Let A be a (globally) simple $\text{NH}_t(n; n)$, with n odd, and consider one of its rows R with one of its simple orderings (a similar reasoning can be done on the columns). Then, from Proposition 4.1, the number of edge-disjoint paths that can be constructed from R is exactly $2n^2 + t$, that is one base block and its translates under the action of \mathbb{Z}_{2n^2+t} . Moreover, if the total sum of R has order λ in \mathbb{Z}_{2n^2+t} , then the faces of the embedding associated to A and constructed from R have length λn (notice that this does not depend on the chosen simple ordering).

This property can be useful to determine some further properties of an embedding. For instance, it can be noticed that in the embedding constructed in Proposition 5.9 every face constructed from the columns of the array have maximum length, hence, from Euler's formula, the obtained biembedding will lie on a surface having large genus.

Finally, let m_r and m_c be the number of rows and columns, respectively, of A having total sum of order λ , and let $\ell = \lambda n$. Then, the total number f_ℓ of faces of the biembedding having length ℓ is:

$$f_\ell = (m_r + m_c) \frac{n(2n^2 + t)}{\ell} = (m_r + m_c) \frac{(2n^2 + t)}{\lambda}.$$

In the following table, we give the total number of faces of the embeddings constructed in Propositions 5.9 and 5.10, based on their face length.

| Proposition | t | ℓ | $m_r + m_c$ | f_ℓ |
|-------------|--------|--------|-------------|-----------|
| 5.9 | n^2 | $3n^2$ | $1 + 0$ | n |
| | | $3n^3$ | $(n-1) + n$ | $2n-1$ |
| 5.10 | $2n^2$ | $4n$ | $1 + 1$ | $2n^2$ |
| | | $4n^2$ | $0 + (n-1)$ | $n^2 - n$ |
| | | $4n^3$ | $(n-1) + 0$ | $n-1$ |

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