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Constructing uniform 2-factorizations via row-sum matrices: Solutions to the Hamilton-Waterloo problem

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In this paper, we formally introduce the concept of a row-sum matrix over an arbitrary group G. When G is cyclic, these types of matrices have been widely used to build uniform 2-factorizations of small Cayley graphs (or, Cayley subgraphs of blown-up cycles), which themselves factorize complete (equipartite) graphs.

Here, we construct row-sum matrices over a class of non-abelian groups, the generalized dihedral groups, and we use them to construct uniform 2-factorizations that solve infinitely many open cases of the Hamilton-Waterloo problem, thus filling up large parts of the gaps in the spectrum of orders for which such factorizations are known to exist.

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1. Introduction

In this paper we denote by $L = [^{\alpha_1}a_1, \dots, {}^{\alpha_\ell}a_\ell]$ the multiset containing $\alpha_i \geq 0$ copies of the element a_i, for each $i \in \{1, \dots, \ell\}$. Note that the a_is need not be distinct. We will call such a multiset a *list*, even though order is not important, as we will be dealing extensively with so called v-lists, which are multisets as above where $\alpha_1+\alpha_2+\cdots+\alpha_\ell = v$ (see Section 2).

Given a simple graph G, we denote by $V(G)$ and $E(G)$ its sets of vertices and edges, respectively. As usual, we denote by C_ℓ a *cycle* of length ℓ (briefly, an ℓ-cycle), and by $(x_0, x_1, \dots, x_{\ell-1})$ the ℓ-cycle with edges $x_0x_1, x_1x_2, \dots, x_{\ell-1}x_0$. A *factor* of G is a spanning subgraph F of G; when F is i-regular, we speak of an i-factor. In particular, a 1-factor (resp. a 2-factor) of G is a vertex-disjoint union of edges (cycles) whose vertices cover $V(G)$. A 2-factor F of G containing only cycles of length ℓ will be called a *C_ℓ-factor* or *uniform factor*.

By K_v^* we mean the *complete graph* K_v on v vertices when v is odd and $K_v - I$, that is, K_v minus the edges of the 1-factor I, when v is even. Also, by $K_t[z]$ we denote the *complete equipartite graph* with t parts of size $z \geq 1$. Note that $K_t[1] = K_t$.

A *2-factorization* of a simple graph G is a set $\mathcal{G}$ of 2-factors of G whose edge sets partition $E(G)$. It is well known that G has a 2-factorization if and only if it is regular of even degree. However, if we require the factors of $\mathcal{G}$ to have a specific structure then the problem becomes much harder. For example, the existence of a 2-factorization of G into copies of a given 2-factor F is an open problem even when $G = K_v^*$. This is the well-known *Oberwolfach Problem*, originally posed by Ringel in 1967 for odd v. A survey of the most relevant results on this problem, updated to 2006, can be found in [13, Section VI.12]. For more recent results we refer the reader to [12].

A factorization of the simple graph G into copies of a C_ℓ-factor is briefly called a *C_ℓ-factorization* or *uniform factorization* of G. The problem of factoring K_v^* into copies of a uniform 2-factor, that is, the uniform Oberwolfach Problem, has been solved [1,2,18,25].

Theorem 1.1 *([1,2,18,25]). Let $v, \ell \geq 3$ be integers. There is a C_ℓ-factorization of K_v^* if and only if $\ell \mid v$, except that there is no C_3-factorization of K_6^* or K_{12}^*.*

A similar result when $G = K_t[z]$ has more recently been obtained in [22,23].

Theorem 1.2 *([22,23]). Let ℓ, t and z be positive integers with $\ell \geq 3$. There exists a C_ℓ-factorization of $K_t[z]$ if and only if $\ell \mid tz$, $(t-1)z$ is even, further ℓ is even when $t = 2$, and $(\ell, t, z) \notin \{(3,3,2), (3,6,2), (3,3,6), (6,2,6)\}$.*

We may generalize this problem to the *Generalized Oberwolfach Problem*, denoted $\mathrm{GOP}(G; \mathcal{R})$, where $\mathcal{R} = \{^{\alpha_1}R_1, \dots, {}^{\alpha_t}R_t\}$ is a list of 2-factors of G, where each R_i is repeated α_i times (with α_i a positive integer) and the R_i are pairwise non-isomorphic.

The Generalized Oberwolfach Problem then requires that the edges of G be factored into a union of α_i copies of R_i, $1 \leq i \leq t$. If each R_i is uniform, with cycles of length a_i, we speak of GOP$(G; [^{\alpha_1} a_1, \ldots, ^{\alpha_t} a_t])$. Since the R_i are factors and every edge of G is in one of the factors R_i, this requires that G is regular with each vertex having degree $2\sum_{i=1}^{t} \alpha_i$ and that if R_i is a C_{a_i}-factor, then a_i divides the order of G. Despite recent probabilistic results which show eventual existence, these results are non-constructive and give no lower bounds for their implementation and so this problem remains wide open; see [10] for more details on the Generalized Oberwolfach Problem.

When $\mathcal{R} = \{^{\alpha} R_1,\ ^{\beta} R_2\}$, then GOP$(G; \mathcal{R})$ represents the most studied variant of the Oberwolfach Problem, known as the *Hamilton-Waterloo Problem*, and denoted by HWP$(G; R_1, R_2; \alpha, \beta)$, or HWP$(v; R_1, R_2; \alpha, \beta)$ when G is K_v^*. This problem asks for a factorization of G into α copies of R_1 and β copies of R_2. In the case where R_1 and R_2 are a C_M-factor and C_N-factor, respectively, we refer to HWP$(G; M, N; \alpha, \beta)$, or HWP$(v; M, N; \alpha, \beta)$ when G is K_v^*, and speak of the uniform Hamilton-Waterloo Problem. Clearly, when $\alpha = 0$ or $\beta = 0$ we obtain the uniform Oberwolfach problem which is completely solved (Theorem 1.1). Therefore, from now on we will assume that both α and β are positive integers. Well-known obvious necessary conditions for the solvability of HWP$(G; M, N; \alpha, \beta)$ are given in the following theorem.

Theorem 1.3. *Let G be a graph of order v, and let M, N, α and β be non-negative integers. In order for a solution of* HWP$(G; M, N; \alpha, \beta)$ *to exist, M and N must be divisors of v greater than 2, and G must be regular of degree $2(\alpha + \beta)$.*

We are interested in constructing solutions to the uniform Hamilton-Waterloo Problem. We point out that this case (as well as the general problem) is still open, and this is quite surprising considering that the equivalent problem of factoring K_v^* into uniform factors (the uniform OP) was solved in the nineties (see Theorem 1.1).

For more details and some history on the problem, we refer the reader to [8]. That paper and [7] deal with the case where both M and N are odd positive integers and provide an almost complete solution to HWP$(v; M, N; \alpha, \beta)$ for odd v. If M and N are both even, then HWP$(v; M, N; \alpha, \beta)$ has a solution whenever the necessary conditions of Theorem 1.3 hold, except possibly when $\alpha = 1$ or $\beta = 1$; $\beta = 3$, $v \equiv 2 \pmod 4$ and $\gcd(M, N) = 2$; or $v = MN/\gcd(M, N) \equiv 2 \pmod 4$ [5,11]. However, the problem is completely solved when M and N are even and M is a divisor of N [6]. The case where M and N have different parities is the most challenging. Indeed, the only case where $M \not\equiv N \pmod 2$ that has been completely solved is when $(M, N) = (3, 4)$ [4,14,24,27]. The only other cases which have been considered are when M is a divisor of N [3,9,21]; $M = 4$ [19,24]; $M = 8$ [26]; and when M and N are not coprime, M is odd, $N = 2^k n$ and 4^k divides v [20]. However, possible exceptions remain in all of these cases. The following theorem summarizes the results in [5–9,11,20].

Theorem 1.4 *([5–9,11,20]). Let* $3 \leq M < N$*, let* v *be such that* $\mathrm{lcm}(M,N)$ *divides* v*, and let* α *and* β *be positive integers with* $\alpha+\beta = \lfloor \frac{v-1}{2} \rfloor$*. Then there is a solution to* $\mathrm{HWP}(v;M,N;\alpha,\beta)$ *when:*

1. M, N *are odd, except possibly when at least one of the following holds:*
   (a) $\beta = 1$*;*
   (b) $\beta = 3$ *and* $M \nmid N$*;*
   (c) $\alpha = 1$*,* $M \nmid N$ *and* $MN \nmid v$*;*
   (d) $v = u \cdot \mathrm{lcm}(M,N)$ *for* $u \in \{1,2,4\}$*;*
   (e) $(v,M) \in \{(6N,3),(18N,3\gcd(M,N))\}$*;*
   (f) v *is even and* $(M,N,\beta) = (5,7,5)$*.*
2. M, N *are even, except possibly when* $M \nmid N$ *and, in addition, at least one of the following holds:*
   (a) $\beta = 1$*;*
   (b) $\beta = 3$*,* $v \equiv 2 \pmod 4$ *and* $\gcd(M,N) = 2$*;*
   (c) $\alpha = 1$ *and at least one of the following holds:* $8 \mid MN$*,* $\frac{MN}{4} \nmid v$*, or* $v = \frac{MN}{2}$*;*
   (d) $v = \mathrm{lcm}(M,N) \equiv 2 \pmod 4$*, and* α *and* β *are odd.*
3. M *is odd, and* $N = 2^k n$ *with* $k \geq 1$ *and* n *odd, and either:*
   (a) M *divides* n*,* $v > 6N > 36N$ *and* $\beta \geq 3$*; or*
   (b) $\gcd(M,n) \geq 3$*,* 4^k *divides* v*,* $\frac{v}{4^k\,\mathrm{lcm}(M,n)} \geq 3$*, and* $1 \notin \{\alpha,\beta\}$*.*

The results in [8,9,20] were obtained using solutions to $\mathrm{HWP}(C_g[u],M,N,\alpha,\beta)$, where $C_g[u]$ is the graph obtained from C_g by replacing every vertex in the cycle with u copies of it. In other words, it is the graph with vertices of the form $x_{i,j}$, $0 \leq i \leq g-1$, $0 \leq j \leq u-1$, and edges of the form $x_{i,a}x_{i+1,b}$ with addition done modulo u, and $0 \leq a,b \leq u-1$.

In this paper, we make further progress when M and N are not coprime in two regards. On the one hand, we improve the result for the case when M and N have different parities, changing the condition that 4^k divides v into the condition that 2^{k+2} divides v. On the other hand, our results put no restrictions on α and β, covering the difficult case when M and N have the same parity and $1 \in \{\alpha,\beta\}$ that was previously left open. More precisely, our main result is the following.

Theorem 1.5. *Let* v*,* M *and* N *be integers greater than 3, and let* $\ell = \mathrm{lcm}(M,N)$*. A solution to* $\mathrm{HWP}(v;M,N;\alpha,\beta)$ *exists if and only if* $\alpha+\beta = \lfloor \frac{v-1}{2} \rfloor$*,* $M \mid v$ *and* $N \mid v$*, except possibly when*

1. $\gcd(M,N) \in \{1,2\}$*;*
2. 4 *does not divide* v/ℓ*;*
3. $v/4\ell \in \{1,2\}$*;*
4. $v = 16\ell$ *and* $\gcd(M,N)$ *is odd;*

5. $v = 24\ell$ *and* $\gcd(M, N) = 3$.

In Section 2 we introduce the concept of row-sum matrices and prove some preliminary results. In particular, we show how to use said matrices to obtain solutions to $\mathrm{HWP}(C_g[u], M, N, \alpha, \beta)$. Next, in Section 3 we prove the existence of the matrices we need for our main result. Finally, in Section 4 we complete the proof of Theorem 1.5.

2. Preliminary results

Recall that given a group Γ, and an integer v, a v-list of Γ is a list $\Delta = [\delta_1, \ldots, \delta_v]$ of v (not necessarily distinct) elements of Γ. Given an integer g, set $g\Delta = [g\delta_1, \ldots, g\delta_v]$. It will be helpful to refer to the list $\omega(\Delta)$ of element orders associated to Δ, defined as follows: $\omega(\Delta) = [\omega(\delta_1), \ldots, \omega(\delta_v)]$, where each $\omega(\delta_i)$ is the order of the element δ_i of the group Γ, and hence a divisor of the order of Γ.

Throughout the paper we make use of groups written in additive notation, although they are not necessarily abelian.

2.1. Δ-permutations

Let Γ be an arbitrary group of order v, and let Δ be a v-list of Γ. We say that a permutation φ of Γ is a Δ-permutation if the following condition holds:

$$[\varphi(a_1) - a_1, \varphi(a_2) - a_2, \ldots, \varphi(a_v) - a_v] = \Delta, \tag{1}$$

where $\{a_1, a_2, \ldots, a_v\} = \Gamma$.

Remark 2.1. Given any fixed $x \in \Gamma$ and $\delta \in \Delta$, we can assume that $\varphi(x) - x = \delta$. Otherwise, take $a_j \in \Gamma$ such that $\varphi(a_j) - a_j = \delta$, and set $b_i = a_i - a_j + x$ and $\phi(b_i) = \varphi(a_i) - a_j + x$, for every $i = 1, \ldots, v$. Clearly, $\Gamma = \{b_1, b_2, \ldots, b_v\}$ and ϕ is a Δ-permutation of Γ, since $\phi(b_i) - b_i = \varphi(a_i) - a_i$ for each $i = 1, \ldots, v$. Also, $b_j = x$ and $\phi(x) - x = \varphi(a_j) - a_j = \delta$.

Given an arbitrary v-list Δ of an abelian group G of order v a necessary condition for a Δ-permutation of G to exist is that $\sum \Delta = 0$, where $\sum \Delta$ denotes the sum of the elements in Δ. M. Hall [16] proved that this condition is also sufficient.

Theorem 2.2 *([16]). Let G be an abelian group of order v, and let Δ be a v-list of G. There exists a Δ-permutation of G if and only if $\sum \Delta = 0$.*

The following special Δ-permutations will be useful in the constructions of Section 4.

Theorem 2.3. *Let m and n be positive integers with m odd. Then there exists a Δ-permutation ψ of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ such that*

1. $\Delta = \left[{}^{1}(0,0)\right] \cup \left[{}^{1}\gamma \mid \gamma \in \mathbb{Z}_m \times \mathbb{Z}_{2n}, \gamma \neq (0,n)\right]$,
2. ψ *fixes* $(0,0)$ *and* $\left(-\frac{m-1}{2}, \left\lfloor\frac{n+1}{2}\right\rfloor + \frac{m-1}{2}n\right)$.

Proof. Assume that $V(K_{2mn}) = \mathbb{Z}_m \times \mathbb{Z}_{2n}$. We are going to construct a suitable matching H of K_{2mn} with $mn-1$ edges. We leave to the reader the check that the permutation that swaps every pair of adjacent vertices in H and fixes the remaining two is the desired Δ-permutation of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$.

We consider the following matching of K_{2n},

$$F = \left\{\{j,-j\} \,\middle|\, j = 1,\ldots,\left\lfloor\tfrac{n-1}{2}\right\rfloor\right\} \cup \left\{\{j,-j+1\} \,\middle|\, j = \left\lfloor\tfrac{n+3}{2}\right\rfloor,\ldots,n\right\},$$

and note that $V(F) = \mathbb{Z}_{2n} \setminus \{0,u\}$ with $u = \left\lfloor\frac{n+1}{2}\right\rfloor$.

For every non-negative integer i, we lift F to a matching $F(i)$ with vertex-set $\{\pm i\} \times (\mathbb{Z}_{2n} \setminus \{0,u\})$ defined as follows:

$$F(i) = \big\{\{(x,y_1),(-x,y_2)\} \,\big|\, x = \pm i, \{y_1,y_2\} \in E(F)\big\}.$$

Consider also the following two matchings of K_{2mn}:

$$F' = \left\{\{(i,in),-(i,in)\} \,\middle|\, i = 1,\ldots,\tfrac{m-1}{2}\right\},$$
$$F'' = \left\{\{(-i+1,u+in+n),(i,u+in)\} \,\middle|\, i = 1,\ldots,\tfrac{m-1}{2}\right\},$$

and set

$$H = \bigcup_{i=0}^{(m-1)/2} \big(F(i) + (0,in)\big) \cup F' \cup F''.$$

It is not difficult to check that H is a matching of K_{2mn} with $mn-1$ edges, missing the vertices $(0,0)$ and $\left(-\frac{m-1}{2}, u + \frac{m-1}{2}n\right)$. □

Theorem 2.4. *Let $m \geq 1$ and $n \geq 3$ be odd integers. Then there exists a Δ-permutation ψ of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ such that*

1. $\Delta = \left[{}^{2mn-6}(1,0),\ {}^{3}(2,0),\ {}^{1}(0,2),\ {}^{1}(0,n-2),\ {}^{1}(0,n)\right]$,
2. $\psi(0,0) = (0,n), \psi(0,n) = (0,n+2)$.

Proof. Let ψ be the permutation of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ defined as follows

$$\psi(0,0) = (0,n), \quad \psi(0,n) = (0,n+2), \quad \psi(0,n+2) = (0,0),$$

and for every $z \in \mathbb{Z}_m \times \mathbb{Z}_{2n} \setminus \{(0,0),(0,n),(0,n+2)\}$, let

$$\psi(z) = \begin{cases} z + (2,0) & \text{if } z \in \{(-1,0), (-1,n), (-1,n+2)\}, \\ z + (1,0) & \text{otherwise.} \end{cases}$$

One can check that ψ is the desired permutation. $\square$

2.2. Row-sum matrices and 2-factorizations of $C_g[n]$

Let Γ be a group, and let $S \subset \Gamma$. Also, let Σ be an $|S|$-list of elements of Γ. A *row-sum matrix* $\mathrm{RSM}_\Gamma(S, g; \Sigma)$ is an $|S| \times g$ matrix, whose $g \geq 2$ columns are permutations of S such that the list of (left-to-right) row-sums is Σ. We write $\mathrm{RSM}_\Gamma(S, g; \omega(\Sigma))$ whenever we are just interested in the list $\omega(\Sigma)$ of orders of the row-sums. Notice that a permutation φ of $\Gamma = \{a_1, \ldots, a_v\}$ is a Σ-permutation if and only if the matrix

$$\begin{bmatrix} \varphi(a_1) & -a_1 \\ \vdots & \vdots \\ \varphi(a_v) & -a_v \end{bmatrix}$$

is an $\mathrm{RSM}_\Gamma(\Gamma, 2; \Sigma)$.

Row-sum matrices are useful to build factorizations of suitable Cayley subgraphs of $C_g[n]$. More precisely, we denote by $C_g[\Gamma, S]$ ($g \geq 3$) the graph with point set $\mathbb{Z}_g \times \Gamma$ and edges $(i, x)(i + 1, d + x)$, $i \in \mathbb{Z}_g$, $x \in \Gamma$ and $d \in S$. In other words, $C_g[\Gamma, S] = \mathrm{cay}(\mathbb{Z}_g \times \Gamma, \{1\} \times S)$; hence, it is $2|S|$-regular. It is straightforward to see that if Γ has order n, then $C_g[n] \cong C_g[\Gamma, \Gamma]$; hence, $C_g[\Gamma, S]$ is a subgraph of $C_g[n]$.

The following result, proven in [10, Theorem 2.1] when Γ has odd order, shows that row-sum matrices can be used to build factorizations of $C_g[\Gamma, S]$.

Theorem 2.5. *If there exists an* $\mathrm{RSM}_\Gamma(S, g; \Sigma)$*, then* $\mathrm{GOP}(C_g[\Gamma, S]; g\omega(\Sigma))$ *has a solution.*

We skip the proof of Theorem 2.5 when $|\Gamma|$ is even, as it is identical to the odd case.

We now show that row-sum matrices can be easily extended over the columns whenever S is closed under taking negatives (that is, $S = -S$) or when $S = \Gamma$ has a *complete mapping*. We recall that a complete mapping of Γ is a permutation π of Γ such that $\rho(x) = x + \pi(x)$ is also a permutation.

Theorem 2.6. *If there is an* $\mathrm{RSM}_\Gamma(S, g; \Sigma)$*, then there exists an* $\mathrm{RSM}_\Gamma(S, g + i; \Sigma)$ *in each of the following cases:*

1. *$S = -S$ and $i \geq 2$ is even, or*
2. *$S = \Gamma$ has a complete mapping and $i \geq 1$.*

Proof. Let A be an $\mathrm{RSM}_\Gamma(S,g;\Sigma)$. We first assume that $S=-S$ and $i\geq 2$ is even. The existence of an $\mathrm{RSM}_\Gamma(S,g+i;\Sigma)$ was essentially proven in [10, Theorem 2.1] as follows: it is enough to extend A by adding $i/2$ copies of an $\mathrm{RSM}_\Gamma(S,2;\Sigma')$, say B, where Σ' is the $|S|$-list of zeros (i.e., each row of B sums to 0). Clearly, B can be easily built by choosing its second column to be the negative of the first one, which must be a permutation of S by assumption.

Now let $i\geq 1$, assume that $S=\Gamma$ has a complete mapping π, and set $\rho(x)=x+\pi(x)$. In view of the first part of the proof, it suffices to show the existence of an $\mathrm{RSM}_\Gamma(S,g+1;\Sigma)$. To do so, it is enough to replace each element of the last column of A, say y, with the pair $(x,\pi(x))$ where $x=\rho^{-1}(y)$. □

Throughout the paper we will only deal with solvable groups, and we will make use of a result by Hall and Paige who proved that a finite solvable group Γ has a complete mapping if and only if its 2-Sylow subgroup is either trivial or non-cyclic. This result was then extended to an arbitrary group (see [15]).

Theorem 2.7 *([17]). A finite solvable group Γ has a complete mapping if and only if its 2-Sylow subgroup is either trivial or non-cyclic.*

This, combined with Theorems 2.5 and 2.6 means that when $S=\Gamma$ is as in Theorem 2.7, it is enough to construct row-sum matrices with 3 columns.

Corollary 2.8. *Let Γ be a solvable group of order n whose 2-Sylow subgroup is either trivial or non-cyclic. If there is an $\mathrm{RSM}_\Gamma(\Gamma,3;\Sigma)$, then there exist*

1. *an $\mathrm{RSM}_\Gamma(\Gamma,g;\Sigma)$, and*
2. *a solution to $GOP(C_g[n];g\omega(\Sigma))$,*

for every $g\geq 3$.

We end this section by constructing row-sum matrices over an abelian group having a given list of associated row-sum orders.

Theorem 2.9. *Let $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_{2^\ell}\times\mathbb{Z}_{mn}$ with n odd and $\ell\geq 1$. Then, there is an $\mathrm{RSM}_\Gamma(\Gamma,g;[^{2\gamma}m,\ ^{2\delta}2^k n])$ whenever $\gamma+\delta=2^\ell mn$, $g\geq 3$ and $0\leq k\leq\ell$.*

Proof. Since $\ell\geq 1$, the Sylow 2-subgroup of Γ is non-cyclic. Hence, by Corollary 2.8, it is enough to prove the assertion for $g=3$. Set

$$\Delta=\left[{}^{\gamma}(0,0,n),\ {}^{\gamma}(0,0,-n),\ {}^{\delta}(0,2^{\ell-k},m),\ {}^{\delta}(0,-2^{\ell-k},-m)\right].$$

Since $\sum\Delta=0=\sum\Gamma$, by Theorem 2.2, there are a Δ-permutation φ of Γ and a Γ-permutation ψ of Γ. Now consider the $2^{\ell+1}mn\times 3$ matrix A whose rows, indexed over Γ, have the following form

$$A_x = \big(-\psi(x) \quad x \quad \varphi(\psi(x) - x) \big).$$

One can easily check that each column of A is a permutation of Γ, and

$$\big[\sum A_x \;\big|\; x \in \Gamma \big] = \Delta.$$

Since Δ contains 2γ elements of order m and 2δ elements of order $2^k n$, the assertion follows. $\square$

2.3. Matrices over a generalized dihedral group

From now on, g, m and n will denote positive integers with $g \geq 3$, m and n odd, and $G = \mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n}$ with $k \geq 0$. Notice that we allow both m and n to be equal to 1. In the case $m = 1$ this means that we work with $\mathbb{Z}_1 \times \mathbb{Z}_{2^{k+1}n} \simeq \mathbb{Z}_{2^{k+1}n}$. Recall that the coordinatewise multiplication by any element $x \in G$ is a homomorphism of the group G; in particular, the multiplication by $\epsilon = (-1, -1)$ is a group automorphism, and its order is 2 since $\epsilon^2 = (1, 1)$. Therefore, we can define the semidirect product $G \rtimes \mathbb{Z}_2$ whose underlying set is $G \times \mathbb{Z}_2$, and the group operation, still denoted by $+$, is defined as follows:

$$(x, \tau) + (x', \tau') = (x + \epsilon^\tau x', \tau + \tau').$$

It is not difficult to check that $G \rtimes \mathbb{Z}_2 \simeq Dih(\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n})$ is the generalized dihedral group over $\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n}$.

From now on, $\Gamma = G \rtimes \mathbb{Z}_2$ and for every subset $S \subseteq \Gamma$, we simply write $C_g[S]$ in place of $C_g[\Gamma, S]$. Note that $2\Gamma \simeq 2G = \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$ and $|2\Gamma| = 2^k mn$. Consider the interval $I = \{0, \ldots, 2^k n - 1\}$ and the function $\rho : 2\mathbb{Z}_{2^{k+1}n} \to I$ with $\rho(y)$ being defined by $2\rho(y) = y$. Similarly, for every $x \in \mathbb{Z}_m$ (m odd) we denote by $x/2$ the unique element of $\mathbb{Z}_m$ such that $2(x/2) = x$. Now let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, where each φ_h is a permutation of $\mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, and define the following five bijections:

$$a_h : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (2\mathbb{Z}_{2^{k+1}n} + 1),\ (x, y) \mapsto \varphi_h(x, y) + (0, 1),$$

for $h \in \{1, 2, 3\}$, and

$$b : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (-I),\ (x, y) \mapsto (-x/2, -\rho(y)),$$
$$c : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (I + 1),\ (x, y) \mapsto (x/2, \rho(y) + 1).$$

Finally, for every $(x, y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, let $A(\varphi, (x, y))$ and $A'(\varphi, (x, y))$ be the 3×3 matrices with entries from Γ defined as follows:

$$A(\varphi,(x,y)) = \begin{bmatrix} (a_1(x,y),0) & (b(x,y),1) & (c(x,y),1) \\ (c(x,y),1) & (a_2(x,y),0) & (b(x,y),1) \\ (b(x,y),1) & (c(x,y),1) & (a_3(x,y),0) \end{bmatrix}$$

$$A'(\varphi,(x,y)) = \begin{bmatrix} (a_1(x,y),0) & (c(x,y),1) & (b(x,y),1) \\ (c(x,y),1) & (b(x,y),1) & (a_2(x,y),0) \\ (b(x,y),1) & (a_3(x,y),0) & (c(x,y),1) \end{bmatrix}.$$

Note that $A'(\varphi,(x,y))$ is obtained from $A(\varphi,(x,y))$ by swapping columns 2 and 3. From now on, given a matrix M over an arbitrary group, $\sum M_h$ represents the (left-to-right) sum of the h-th row of M denoted by M_h.

Lemma 2.10. *For every $(x,y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, we have that*

$$\sum A(\varphi,(x,y))_h = \begin{cases} \big((\varphi_h(x,y)-(x,y)),0\big) & \text{if } h=1,3, \\ \big((x,y)-\varphi_h(x,y)),0\big) & \text{if } h=2, \end{cases}$$

$$\sum A'(\varphi,(x,y))_h = \begin{cases} \big((\varphi_h(x,y)+(x,y)),0\big)+(0,2,0) & \text{if } h=1,2, \\ -\big((\varphi_h(x,y)+(x,y)),0\big)-(0,2,0) & \text{if } h=3. \end{cases}$$

Proof. Note that $\sum A(\varphi,(x,y))_h = (-1)^h(\sigma(x,y),0)$, where $\sigma = c-b-a_h$, for every $h=1,2,3$. Also,

$$\begin{aligned} \sigma(x,y) &= (x/2,\rho(y)+1)-(-x/2,-\rho(y))-(\varphi_h(x,y)+(0,1)) \\ &= (x,2\rho(y))-\varphi_h(x,y) = (x,y)-\varphi_h(x,y). \end{aligned}$$

The result for $A'(\varphi,(x,y))$ is similar. □

3. Row-sum matrices over a generalized dihedral group

In this section, we build row-sum matrices over the group $\Gamma \simeq Dih(\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n})$ (defined in Section 2.3) and prove the following.

Theorem 3.1. *Let $g \geq 3$, $k \geq 0$, and let $m,n \geq 1$ be odd integers. Then, an* $\mathrm{RSM}_\Gamma(\Gamma, g; [{}^{\alpha}m,\ {}^{\beta}2^k n])$ *exists if and only if $\alpha+\beta = 2^{k+2}mn$ except possibly in the following cases*

1. *$k \geq 2$ and $\beta = 0$, or*
2. *$k=1$ and $\alpha,\beta \in \{0,2,4\}$.*

When $g=3$, its proof is given in Sections 3.1, 3.2 and 3.3 which respectively deal with the cases $k \geq 2$, $k=1$ and $k=0$. It mostly relies on Theorem 2.9, and the following

Theorems 3.2 and 3.3. Considering that Γ is a solvable group whose 2-Sylow subgroup is non-cyclic, the general case where $g \geq 4$ will follow from Corollary 2.8.

Recall that m, n are positive integers, and m and n odd.

Theorem 3.2. *An* $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 2f+3; [{}^{\alpha}m,\ {}^{\beta}2^k n])$ *exists when* $f \geq 0$, $\alpha \neq 1$, $\beta \neq 1$, *and* $\alpha + \beta = 3 \cdot 2^k mn$.

Proof. Considering that $\Gamma \setminus 2\Gamma = -(\Gamma \setminus 2\Gamma)$, by Theorem 2.6 it is enough to prove the assertion when $f = 0$. We split the proof into two cases, $k \geq 1$ and $k = 0$.

Assume that $k \geq 1$. We first show that an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3; [{}^{\alpha}m,\ {}^{\beta}2^k n])$ exists when α and β are even, with $0 \leq \alpha, \beta \leq 3 \cdot 2^k mn$. Letting $\alpha = 2a$ and $\beta = 2b$, and recalling that $|\Gamma \setminus 2\Gamma| = 3 \cdot 2^k mn$, we can write

$$a = \textstyle\sum_{h=1}^{3} a_h \ \text{ and } \ b = \sum_{h=1}^{3} b_h,$$

where $2(a_h + b_h) = 2^k mn = |2\Gamma|$, for every $h \in \{1, 2, 3\}$. By Theorem 2.2, there exists a Δ_h-permutation φ_h of $\mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, where

$$\Delta_h = \left[{}^{a_h}(1,0),\ {}^{a_h}(-1,0),\ {}^{b_h}(0,2),\ {}^{b_h}(0,-2)\right].$$

Notice that $\omega(\Delta_h) = [{}^{2a_h}m,\ {}^{2b_h}2^k n]$, for every $h \in \{1, 2, 3\}$. Let $A = [A(\varphi, (x, y))]$ be the column block-matrix whose blocks are the matrices $A(\varphi, (x, y))$ for $(x, y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$. Note that A is a $(3 \cdot 2^k mn) \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1)\ \cup\ \omega(\Delta_2)\ \cup\ \omega(\Delta_3) = \left[{}^{2a}m,\ {}^{2b}2^k n\right],$$

and the result follows.

We can now show the existence of an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3; [{}^{\alpha}m,\ {}^{\beta}2^k n])$ when $k \geq 1$ and α and β are odd, with $3 \leq \alpha, \beta \leq 3 \cdot 2^k mn$. Let A be the matrix built above with $2a = \alpha + 3 \geq 6$ and $2b = \beta - 3$. Since $2a \geq 6$, we can take $a_1, a_2, a_3 \geq 1$. Also, by Remark 2.1, we can assume that for $z = (\frac{m-1}{2}, 0)$ we have $\varphi_h(z) = z + (1, 0)$, for every $h = 1, 2, 3$. We denote by A' the $(3 \cdot 2^k mn) \times 3$ matrix that we obtain by replacing the block $A(\varphi, z)$ of A with $A'(\varphi, z)$. Clearly, the columns of A' are still permutations of $\Gamma \setminus 2\Gamma$. Also, note that by Lemma 2.10, each $\sum A(\varphi, z)_h$ has order m, whereas $\sum A'(\varphi, z)_h = \pm(0, 2, 0)$ has order $2^k n$, for every $h = 1, 2, 3$. Therefore, denoting by $L_{A'}$ the list of row sums of A', we have that

$$\omega(L_{A'}) = [{}^{2a-3}m,\ {}^{2b+3}2^k n] = [{}^{\alpha}m,\ {}^{\beta}2^k n].$$

It is left to deal with the case where $k = 0$. Since $\alpha, \beta \neq 1$, we can write

$$\alpha = \sum_{h=1}^{3} \alpha_h \ \text{ and } \ \beta = \sum_{h=1}^{3} \beta_h,$$

where $\alpha_h+\beta_h=mn=|2\Gamma|$ and $\alpha_h,\beta_h\neq 1$, for every $h\in\{1,2,3\}$. Considering that mn is odd, α_h and β_h have distinct parities; hence we can further write

$$(\alpha_h,\beta_h)=\begin{cases}(2a_h+1,2b_h) & \text{if } \alpha_h \text{ is odd,}\\ (2a_h,2b_h+1) & \text{if } \alpha_h \text{ is even,}\end{cases}$$

for each $h\in\{1,2,3\}$. Notice in particular that $\alpha_h,\beta_h\neq 1$ implies that $a_h\geq 1$ when α_h is odd and $b_h\geq 1$ when α_h is even. Thus, by Theorem 2.2, there exists a Δ_h-permutation φ_h of $\mathbb{Z}_m\times 2\mathbb{Z}_{2n}$, where

$$\Delta_h=\begin{cases}\left[\,^{1}(2,0),\ ^{a_h-1}(1,0),\ ^{a_h+1}(-1,0),\ ^{b_h}(0,2),\ ^{b_h}(0,-2)\right] & \text{if } \alpha_h \text{ is odd,}\\ \left[\,^{a_h}(1,0),\ ^{a_h}(-1,0),\ ^{1}(0,4),\ ^{b_h-1}(0,2),\ ^{b_h+1}(0,-2)\right] & \text{if } \alpha_h \text{ is even.}\end{cases}$$

Notice that $\omega((0,4))=n$ as $k=0$. Thus $\omega(\Delta_h)=\left[\,^{\alpha_h}m,\ ^{\beta_h}n\right]$, for every $h\in\{1,2,3\}$. As with the previous cases, let $A=[A(\varphi,(x,y))]$ be the column block-matrix whose blocks are the matrices $A(\varphi,(x,y))$ for $(x,y)\in\mathbb{Z}_m\times 2\mathbb{Z}_{2n}$. Note that A is a $(3\cdot 2mn)\times 3$ matrix whose columns are permutations of $\Gamma\setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A)=\omega(\Delta_1)\ \cup\ \omega(\Delta_2)\ \cup\ \omega(\Delta_3)=\left[\,^{\alpha}m,\ ^{\beta}n\right],$$

and the result follows. □

Theorem 3.3. *An* $\mathrm{RSM}_\Gamma(2\Gamma,2f+3;[\,^{\alpha}m,\ ^{\beta}2^kn])$ *exists whenever* $f\geq 0$, $\alpha+\beta=2^kmn$ *and one of the following conditions holds:*

1. $k=0$, $\alpha\neq 1$ *and* $\beta\neq 1$, *or*
2. $k=1$ *and* $\alpha,\beta\geq 3$ *are odd, or*
3. $k\geq 2$, α,β *are even and* $\beta\geq 2$.

Proof. Recall that $2\Gamma=2G\times\{0\}$ and $2G=\mathbb{Z}_m\times 2\mathbb{Z}_{2^{k+1}n}$. Let

$$(\alpha,\beta)=\begin{cases}(2a+3,2b+3) & \text{if } k=0,1, \text{ and } \alpha,\beta \text{ are odd,}\\ (2a,2b) & \text{if } k\neq 1, \text{ and } \alpha,\beta \text{ are even.}\end{cases}$$

By Theorems 2.2, 2.3 and Remark 2.1, there are Λ_i-permutations ψ_i of $2G$, with $i\in\{1,2\}$, such that

1. $\Lambda_1=\begin{cases}2G & \text{if } k=0,\\ \left[\,^{1}(0,0)\right]\ \cup\ \left[\,^{1}z \mid z\in 2G, z\neq(0,2^kn)\right] & \text{if } k\geq 1;\end{cases}$
2. $\Lambda_2=\begin{cases}\left[\,^{1}(2,0),\ ^{a}(1,0),\ ^{a+2}(-1,0),\ ^{1}(0,4),\ ^{b}(0,2),\ ^{b+2}(0,-2)\right] & \text{if } k=1,\\ \left[\,^{a}(1,0),\ ^{a}(-1,0),\ ^{b}(0,2),\ ^{b}(0,-2)\right] & \text{if } k\geq 2;\end{cases}$

3. if $k \geq 1$, then $\psi_2(0, 2^k n) = (0, 2^k n) + \begin{cases} (0,4) & \text{if } k = 1, \\ (0,-2) & \text{if } k \geq 2. \end{cases}$

Since $(0,0) \in \Lambda_1$, there exists a pair $\bar{z} \in 2G$ such that $\psi_1(\bar{z}) = \bar{z}$. Let B denote the $2^k mn \times 3$ matrix (with entries from 2Γ) whose rows B_z, indexed over $2G$, are defined as follows: $B_z = [(z, 0) \quad (-\psi_1(z), 0) \quad (\psi_2(w), 0)]$ where

$$w = \begin{cases} (0, 2^k n) & \text{if } z = \bar{z} \text{ and } k \neq 0, \\ \psi_1(z) - z & \text{otherwise.} \end{cases}$$

Note that the columns of B are permutations of 2Γ. Also, one can check that for the list L_B of row sums of B we have

$$\omega(L_B) = [\,^{\alpha}m,\ ^{\beta}2^k n].$$

Therefore, B is an $\mathrm{RSM}_\Gamma(2\Gamma, 3; [^{\alpha}m,\ ^{\beta}2^k n])$. Considering that $2\Gamma = -2\Gamma$, the result follows by applying Theorem 2.6 to B. □

In the following, we prove Theorem 3.1 when $g = 3$. The case $g \geq 4$ will follow from Corollary 2.8.

3.1. The proof of Theorem 3.1 when $g = 3$ and $k \geq 2$

Let α and β be non-negative integers such that $\alpha + \beta = 2^{k+2}mn$. First, we assume that both $\alpha \neq 1$ and $\beta \notin \{0, 1, 3\}$. Let $\beta_1 = 2^k mn - \alpha_1$ and let $\beta_2 = 3 \cdot 2^k mn - \alpha_2$ where α_1 and α_2 are defined as follows:

$$\alpha_1 = \begin{cases} \min\left\{2^k mn - 2, 2\left\lfloor \frac{\alpha}{2} \right\rfloor\right\} & \text{if } \alpha \geq 2^k mn + 1, \\ \alpha & \text{if } \alpha \leq 2^k mn \text{ is even,} \\ \alpha - 3 & \text{if } 3 \leq \alpha < 2^k mn \text{ is odd,} \end{cases}$$

$$\alpha_2 = \alpha - \alpha_1.$$

Clearly, α_1 and β_1 are even, hence Theorem 3.3 guarantees the existence of an $\mathrm{RSM}_\Gamma(2\Gamma, 3, [\,^{\alpha_1}m,\ ^{\beta_1}2^k n])$, say B. Furthermore, one can check that $\alpha_2 \neq 1$ and $\beta_2 \neq 1$. Hence, by Theorem 3.2 there is an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, g, [\,^{\alpha_2}m,\ ^{\beta_2}2^k n])$, say A. Therefore, $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is an $\mathrm{RSM}_\Gamma(\Gamma, 3; [^{\alpha}m,\ ^{\beta}2^k n])$.

It is left to deal with the cases $\alpha = 1$ and $\beta = 1, 3$.

Case 1: $\alpha = 1$. Let $\alpha_1, \alpha_2 = 0$, $\beta_1 = 3 \cdot 2^k mn$, and $\beta_2 = 2^k mn$. Further, let $C = \begin{bmatrix} A \\ B \end{bmatrix}$ where A is the $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3; [^{\alpha_1}m, ^{\beta_1}2^k n]$ built in Theorem 3.2, and B is the $\mathrm{RSM}_\Gamma(2\Gamma, 3; [^{\alpha_2}m, ^{\beta_2}2^k n])$ built in Theorem 3.3.

It follows that for the list L_C of row sums of C we have

$$\omega(L_C) = [\,{}^{2^{k+2}mn}2^k n]. \tag{2}$$

By Remark 2.1, we can assume that the permutations φ_1 and φ_3 used to define A in Theorem 3.2 satisfy the condition $\varphi_h(z') = z' - (0,2) = (1,0)$, with $z' = (1,2)$, hence

$$a_h(z') = \varphi_h(z') + (0,1) = (1,1),$$

for $h = 1$ or 3. Furthermore, since $\Lambda_2 = \left[{}^{2^{k-1}nm}(0,2),\ {}^{2^{k-1}nm}(0,-2)\right]$, we can assume that $\psi_2(0,0) = (0,2)$. Consider the following matrices:

$$U = \begin{bmatrix} A(\varphi, z')_1 \\ B_{(0,2)} \\ A(\varphi, z')_3 \end{bmatrix} = \begin{bmatrix} (a_1(z'),0) & (b(z'),1) & (c(z'),1) \\ (0,2,0) & (0,-2,0) & (0,2,0) \\ (b(z'),1) & (c(z'),1) & (a_3(z'),0) \end{bmatrix}$$

$$U' = \begin{bmatrix} (0,2,0) & (c(z'),1) & (c(z'),1) \\ (a_1(z'),0) & (0,-2,0) & (a_3(z'),0) \\ (b(z'),1) & (b(z'),1) & (0,2,0) \end{bmatrix}.$$

Note that U is a submatrix of C, while each column of U' is a permutation of the corresponding column of U. Therefore, by replacing the block U of C with U', we obtain a new $2^{k+2}mn \cdot 3$ matrix C' whose columns are still permutations of Γ. Denote by L' the list of row sums of C'. Taking into account (2) and considering that

$$\sum U_h = \sum U'_j = \pm(0,2,0) \text{ has order } 2^k n \text{ for } h = 1,2,3 \text{ and } j = 1,3, \text{ and}$$
$$\sum U'_2 = (2,0,0) \text{ has order } m,$$

we have $\omega(L_{C'}) = [\,{}^{1}m,\ {}^{2^{k+2}mn-1}2^k n]$.

Case 2: $\beta = 1,3$. Let $\beta_1 = 0$, $\beta_2 = \beta + 1$, $\alpha_1 = 3 \cdot 2^k mn$, and $\alpha_2 = 2^k mn - (\beta + 1)$. Further, let $C = \begin{bmatrix} A \\ B \end{bmatrix}$ where A is the $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3; [\,{}^{\alpha_1}m, {}^{\beta_1}2^k n]$ built in Theorem 3.2, and B is the $\mathrm{RSM}_\Gamma(2\Gamma, 3; [\,{}^{\alpha_2}m, {}^{\beta_2}2^k n])$ built in Theorem 3.3.

It follows that for the list L_C of row sums of C we have

$$\omega(L_C) = [\,{}^{2^{k+2}mn-\beta-1}m,\ {}^{\beta+1}2^k n]. \tag{3}$$

By Remark 2.1, we can assume that the permutations φ_1 and φ_2 (used in Theorem 3.2 to define A) satisfy the condition $\varphi_h(z') = z' + (1,0) = (1, 2^{k-1}n)$, with $z' = (0, 2^{k-1}n)$, hence

$$a_h(z') = \varphi_h(z') + (0,1) = (1, 2^{k-1}n + 1),$$

for $h = 1$ or 2.

Again, by Remark 2.1, we can assume that the permutation ψ_1 (used in Theorem 3.3 to define B) fixes $\overline{z} = (1,0)$.

Consider the following matrices:

$$U = \begin{bmatrix} A(\varphi, z')_1 \\ A(\varphi, z')_2 \\ B_{\overline{z}} \end{bmatrix} = \begin{bmatrix} (a_1(z'),0) & (b(z'),1) & (c(z'),1) \\ (c(z'),1) & (a_2(z'),0) & (b(z'),1) \\ (1,0,0) & -(1,0,0) & (0,2^k n-2,0) \end{bmatrix}$$

$$U' = \begin{bmatrix} (1,0,0) & (b(z'),1) & (b(z'),1) \\ (c(z'),1) & -(1,0,0) & (c(z'),1) \\ (a_1(z'),0) & (a_2(z'),0) & (0,2^k n-2,0) \end{bmatrix}.$$

Note that U is a submatrix of C, while each column of U' is a permutation of the corresponding column of U. Therefore, by replacing the block U of C with U', we obtain a new $2^{k+2}mn \times 3$ matrix C' whose columns are still permutations of Γ. Denote by L' the list of row sums of C'. Taking into account (3) and considering that

$$\begin{aligned} &\sum U'_h = \sum U_j = \pm(1,0,0) \text{ has order } m, \text{ for } h,j = 1,2, \\ &\sum U_3 = (0, 2^k n-2, 0) \text{ has order } 2^k n, \text{ and} \\ &\sum U'_3 = (2,0,0) \text{ has order } m, \end{aligned}$$

we have $\omega(L_{C'}) = [\,^{2^{k+2}mn-\beta}m,\ ^{\beta}2^k n]$. In both cases, C' is the desired $\mathrm{RSM}_\Gamma(\Gamma, 3; [\,^{\alpha}m,\ ^{\beta}2^k n])$.

3.2. The proof of Theorem 3.1 when $g = 3$ and $k = 1$

Let α and β be non-negative integers such that $\alpha + \beta = 2^{k+2}mn = 8mn$. We first deal with the case where $\alpha, \beta \notin \{0,1,2,4\}$, and let $\beta_1 = 2mn - \alpha_1$ $\beta_2 = 6mn - \alpha_2$, where α_1 and α_2 are defined as follows:

$$(\alpha_1, \alpha_2) = \begin{cases} (3, \alpha - 3) & \text{if } 3 \le \alpha \le 6mn + 3, \\ (2mn-3, \alpha - 2mn + 3) & \text{if } 6mn + 4 \le \alpha \le 8mn - 3. \end{cases}$$

Clearly, $\alpha_1 \ge 3$ and $\beta_1 \ge 3$ are odd, hence Theorem 3.3 guarantees the existence of an $\mathrm{RSM}_\Gamma(2\Gamma, 3, [\,^{\alpha_1}m,\ ^{\beta_1}2n])$, say B. Furthermore, $\alpha_2 \ne 1$ and $\beta_2 \ne 1$, hence Theorem 3.2 provides an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3, [\,^{\alpha_2}m,\ ^{\beta_2}2n])$, say A. Therefore, $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is an $\mathrm{RSM}_\Gamma(\Gamma, 3; [\,^{\alpha}m,\ ^{\beta}2n])$.

The case $\beta = 1$ is dealt with in Theorems 3.4 and 3.5, while the case $\alpha = 1$ is proven in Theorems 3.6 and 3.7.

Theorem 3.4. *Let $m \geq 1$ and $n \geq 3$ be odd integers and let $k = 1$. Then an $\mathrm{RSM}_\Gamma(\Gamma, 3; [{}^{8mn-1}m,\ {}^{1}2n])$ exists.*

Proof. Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_{4n}$.

By Theorem 2.2, there exists a Δ_h-permutation φ_h of $2G$, where $\Delta_h = \left[{}^{2mn}(1,0)\right]$, for every $h \in \{1,2,3\}$. Let $A = [A(\varphi, z)]$ be the row block-matrix whose blocks are the 3×3 matrices $A(\varphi, z)$ for $z \in 2G$. Note that A is a $6mn \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega(\left[{}^{6mn}(1,0,0)\right]) = \left[{}^{6mn}m\right].$$

By Theorems 2.3 and 2.4, there are Λ_i-permutation ψ_i of $2G$, with $i \in \{1,2\}$, such that

1. $\Lambda_1 = \left[{}^{1}(0,0)\right] \cup \left[{}^{1}\gamma \mid \gamma \in 2G, \gamma \neq (0,2n)\right]$,
2. $\Lambda_2 = \left[{}^{2mn-6}(1,0),\ {}^{3}(2,0),\ {}^{1}(0,4),\ {}^{1}(0,2n-4),\ {}^{1}(0,2n)\right]$,
3. ψ_1 fixes $(0,0)$ and $\overline{\gamma} = \left(-\frac{m-1}{2}, mn+1\right)$,
4. $\psi_2(0,0) = (0,2n), \psi_2(0,2n) = (0,2n+4)$.

Let B denote the $2^k mn \times 3$ matrix whose rows B_γ, indexed over $2G$, are defined as follows: $B_\gamma = [(\gamma,0) \quad (-\psi_1(\gamma),0) \quad (\psi_2(\delta),0)]$ where

$$\delta = \begin{cases} (0,2n) & \text{if } \gamma = (0,0), \\ \psi_1(\gamma) - \gamma & \text{otherwise.} \end{cases}$$

Note that the columns of B are permutations of 2Γ. Also, one can check that for the list L_B of row sums of B we have

$$L_B = \left[{}^{2mn-6}(1,0,0),\ {}^{3}(2,0,0),\ {}^{1}(0,2n+4,0),\ {}^{1}(0,2n-4,0),\ {}^{1}(0,2n,0)\right]$$

hence, $\omega(L_B) = [{}^{2mn-3}m,\ {}^{2}2n,\ {}^{1}2]$. It follows that each column of $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is a permutation of Γ. Clearly, the list of row sums of C is $L_C = L_A \cup L_B$ and $\omega(L_C) = [{}^{8mn-3}m,\ {}^{2}2n,\ {}^{1}2]$.

Now, let $\mu \in \{\pm 1\}$ such that $\mu \equiv m \pmod 4$, and take the following four elements γ_i of $2G$, whose second entry depends on μ:

$$\gamma_1 = \begin{cases} \left(\frac{m-1}{2}, 3n-1\right), & \text{if } m = 1 \text{ and } n = 3, \\ \left(\frac{m-1}{2}, n+\mu-2\right), & \text{otherwise,} \end{cases}$$

$$\gamma_2 = \begin{cases} \left(-\frac{m-1}{2}, 3n-1\right), & \text{if } m = 1 \text{ and } n = 3, \\ \left(-\frac{m-1}{2}, n+\mu-2\right), & \text{otherwise,} \end{cases}$$

$$\gamma_3 = \left(-\frac{m-1}{4}, (3+\mu)n - \mu - 1\right),$$

$$\gamma_4 = \left(\frac{m-1}{4}, (3-\mu)n - \mu - 3\right).$$

Consider the submatrix $U = \begin{bmatrix} S \\ T \end{bmatrix}$ of $C = \begin{bmatrix} A \\ B \end{bmatrix}$, where S and T are submatrices of A and B, respectively:

$$S = \begin{bmatrix} (a_1(\gamma_1), 0) & (b(\gamma_1), 1) & (c(\gamma_1), 1) \\ (c(\gamma_2), 1) & (a_2(\gamma_2), 0) & (b(\gamma_2), 1) \\ (a_1(\gamma_3), 0) & (b(\gamma_3), 1) & (c(\gamma_3), 1) \\ (c(\gamma_4), 1) & (a_2(\gamma_4), 0) & (b(\gamma_4), 1) \end{bmatrix},$$

$$T = \begin{bmatrix} (0,0,0) & (0,0,0) & (0, 2n+4, 0) \\ (\bar{\gamma}, 0) & -(\bar{\gamma}, 0) & (0, 2n, 0) \end{bmatrix}.$$

Considering that $\gamma_1 \neq \gamma_3$ and $\gamma_2 \neq \gamma_4$, then S is well-defined, that is, S contains four distinct rows of A. We denote by C' the matrix obtained from C by replacing U with the matrix U' defined below

$$U' = \begin{bmatrix} (0,0,0) & (b(\gamma_1), 1) & (b(\gamma_2), 1) \\ (c(\gamma_2), 1) & (0,0,0) & (c(\gamma_1), 1) \\ (\bar{\gamma}, 0) & (b(\gamma_3), 1) & (b(\gamma_4), 1) \\ (c(\gamma_4), 1) & -(\bar{\gamma}, 0) & (c(\gamma_3), 1) \\ (a_1(\gamma_1), 0) & (a_2(\gamma_2), 0) & (0, 2n-2\mu+2, 0) \\ (a_1(\gamma_3), 0) & (a_2(\gamma_4), 0) & (0, 2n+2\mu+2, 0) \end{bmatrix}.$$

Note that each column of U' is a permutation of the corresponding column of U. Therefore, each column of C' is a permutation of Γ.

Considering that

$$\sum S_h \text{ has order } m, \text{ for every } h = 1, \ldots, 4,$$

$$\sum T_1 = (0, 2n+4, 0) \text{ has order } 2n,$$

$$\sum T_2 = (0, 2n, 0) \text{ has order } 2, \text{ and}$$

$$\sum U'_h \text{ has order } m, \text{ for every } h = 1, \ldots, 6$$

and denoting by $L_{C'}$ the list of row sums of C', we have that $\omega(L_{C'}) = [\,^{8mn-1}m,\ ^1 2n]$. Hence, C' is the desired RSM. $\square$

Theorem 3.5. *Let $m \geq 1$ be an odd integer and let $k = n = 1$. Then an* $\mathrm{RSM}_\Gamma(\Gamma, 3; [^{8m-1}m,\ ^1 2])$ *exists.*

Proof. Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_4$. Also, $\Gamma = (\Gamma \setminus 2\Gamma) \cup 2\Gamma$. By Theorem 3.2, there exists an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3; [^{6m}m])$. Therefore, it is left to show that an $\mathrm{RSM}_\Gamma(2\Gamma, 3; [^{2m-1}m,\ ^1 2])$ exists.

By Theorem 2.2 and Remark 2.1, there are Λ_i-permutations ψ_i of 2Γ, with $i \in \{1, 2\}$, such that

1. $\Lambda_1 = \left[\,^1(0,0)\right] \cup \left[^1 z \mid z \in 2G, z \neq (0,2)\right]$, with $\psi_1(0,0) = (0,0)$, and
2. $\Lambda_2 = \left[\,^{2m-6}(1,0),\ ^3(2,0),\ ^2(0,2),\ ^1(0,0)\right]$.

It is easy to see that ψ_2 can be chosen so that it fixes $(2,0)$ and swaps $(0,0)$ and $(0,2)$.

Denote by z_0, z_1 the elements of $2G$, with $z_0 \neq (0,0)$, such that

$$\psi_1(z_0) = z_0, \text{ and } \psi_1(z_1) - z_1 = (2,0),$$

and let B denote the $2^k mn \times 3$ matrix (with entries from 2Γ) whose rows B_z, indexed over $2G$, are defined as follows: $B_z = [(z,0) \quad (-\psi_1(z),0) \quad (\psi_2(w),0)]$ where

$$w = \begin{cases} (2,0) & \text{if } z = z_0, \\ (0,2) & \text{if } z = z_1, \\ \psi_1(z) - z & \text{otherwise.} \end{cases}$$

One can check that the columns of B are permutations of 2Γ, and for the list L_B of row sums of B we have

$$\omega(L_B) = [\,^{2m-1}m,\ ^1 2].$$

Hence, B is the desired RSM. $\square$

Theorem 3.6. *Let $m \geq 1$ and $n \geq 3$ be odd integers, and let $k = 1$. Then an* $\mathrm{RSM}_\Gamma(\Gamma, 3; [^1 m,\ ^{8mn-1}2n])$ *exists.*

Proof. Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_{4n}$.

By Theorem 2.2, there exists a Δ_h-permutation φ_h of $2G$, where $\Delta_h = \left[\,^{2mn}(0,2)\right]$, for every $h \in \{1,2,3\}$. Let $A = [A(\varphi, z)]$ be the row block-matrix whose blocks are the

3×3 matrices $A(\varphi, z)$ for $z \in 2G$. Note that A is a $6mn \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega(\left[{}^{6mn}(0,2,0)\right]) = \left[{}^{6mn}2n\right].$$

By Theorems 2.3 and 2.2, there are Λ_i-permutations ψ_i of $2G$, with $i \in \{1,2\}$, such that

1. $\Lambda_1 = \left[{}^{1}(0,0)\right] \cup \left[{}^{1}\gamma \mid \gamma \in 2G, \gamma \neq (0,2n)\right]$,
2. $\Lambda_2 = \left[{}^{2mn}(0,2)\right]$,
3. ψ_1 fixes $(0,0)$ and $\overline{\gamma} = \left(-\frac{m-1}{2}, mn+1\right)$.

Let B denote the $2^k mn \times 3$ matrix whose rows B_γ, indexed over $2G$, are defined as follows: $B_\gamma = [(\gamma, 0) \quad (-\psi_1(\gamma), 0) \quad (\psi_2(\delta), 0)]$ where

$$\delta = \begin{cases} (0,2n) & \text{if } \gamma = (0,0), \\ \psi_1(\gamma) - \gamma & \text{otherwise.} \end{cases}$$

Note that the columns of B are permutations of 2Γ. Also, one can check that for the list L_B of row sums of B we have

$$L_B = \left[{}^{2mn-1}(0,2,0),\ {}^{1}(0,2n+2,0)\right],$$

hence, $\omega(L_B) = [{}^{2mn-1}2n,\ {}^{1}u]$, where u is the order of $2n+2$ in $2\mathbb{Z}_{4n}$. It follows that each column of $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is a permutation of Γ. Clearly, the list of row sums of C is $L_C = L_A \cup L_B$ and $\omega(L_C) = [{}^{8mn-1}2n,\ {}^{1}u]$.

Now, take the following four elements γ_i of $2G$:

$$\begin{aligned}
\gamma_1 &= (1, 2n-2), \\
\gamma_2 &= (1, 2n-6), \\
\gamma_3 &= \begin{cases} \left(-\frac{m-1}{2}, 3n-3\right), & \text{if } mn \equiv 1 \pmod 4, \\ \left(-\frac{m-1}{2}, 3n-5\right), & \text{if } mn \equiv 3 \pmod 4, \end{cases} \\
\gamma_4 &= \begin{cases} \left(\frac{m-1}{2}, 3n-3\right), & \text{if } mn \equiv 1 \pmod 4, \\ \left(\frac{m-1}{2}, 3n-1\right), & \text{if } mn \equiv 3 \pmod 4. \end{cases}
\end{aligned}$$

Consider the submatrix $U = \begin{bmatrix} S \\ T \end{bmatrix}$ of $C = \begin{bmatrix} A \\ B \end{bmatrix}$, where S and T are submatrices of A and B, respectively:

$$S = \begin{bmatrix} (a_1(\gamma_1), 0) & (b(\gamma_1), 1) & (c(\gamma_1), 1) \\ (c(\gamma_2), 1) & (a_2(\gamma_2), 0) & (b(\gamma_2), 1) \\ (a_1(\gamma_3), 0) & (b(\gamma_3), 1) & (c(\gamma_3), 1) \\ (c(\gamma_4), 1) & (a_2(\gamma_4), 0) & (b(\gamma_4), 1) \end{bmatrix},$$

$$T = \begin{bmatrix} (0,0,0) & (0,0,0) & (0, 2n+2, 0) \\ (\bar{\gamma}, 0) & -(\bar{\gamma}, 0) & (0,2,0) \end{bmatrix}.$$

Considering that $\gamma_1 \neq \gamma_3$ and $\gamma_2 \neq \gamma_4$, then S contains four distinct rows of A. We denote by C' the matrix obtained from C by replacing U with the matrix U' defined below

$$U' = \begin{bmatrix} (0,0,0) & (b(\gamma_1), 1) & (b(\gamma_2), 1) \\ (c(\gamma_2), 1) & (0,0,0) & (c(\gamma_1), 1) \\ (\bar{\gamma}, 0) & (b(\gamma_3), 1) & (b(\gamma_4), 1) \\ (c(\gamma_4), 1) & -(\bar{\gamma}, 0) & (c(\gamma_3), 1) \\ (a_1(\gamma_3), 0) & (a_2(\gamma_4), 0) & (0, 2n+2, 0) \\ (a_1(\gamma_1), 0) & (a_2(\gamma_2), 0) & (0,2,0) \end{bmatrix}.$$

Note that each column of U' is a permutation of the corresponding column of U. Therefore, each column of C' is a permutation of Γ.

Considering that

$$\sum S_h \text{ has order } 2n, \text{ for every } h = 1, \ldots, 4,$$
$$\sum T_1 = (0, 2n+2, 0) \text{ has order } u,$$
$$\sum T_2 = (0,2,0) \text{ has order } 2n, \text{ and}$$
$$\sum U'_h \text{ has order } 2n, \text{ for every } h = 1, \ldots, 5$$
$$\sum U'_6 \text{ has order } m,$$

and denoting by $L_{C'}$ the list of row sums of C', we have that $\omega(L_{C'}) = [\,^{8mn-1}2n,\ ^{1}m]$. Hence, C' is the desired RSM. □

Theorem 3.7. *Let $m \geq 1$ be an odd integer and let $k = n = 1$. Then an* $\mathrm{RSM}_\Gamma(\Gamma, 3; [^{1}m, {}^{8m-1}2])$ *exists.*

Proof. Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_4$. Also, $C_g[\Gamma \setminus 2\Gamma]$ and $C_g[2\Gamma]$ decompose $C_g[8m]$.

By Theorem 2.2, there exists a Δ_h-permutation φ_h of $2G$, where $\Delta_h = \left[{}^{2mn}(0,2)\right]$, for every $h \in \{1,2,3\}$. Let $A = [A(\varphi, z)]$ be the row block-matrix whose blocks are the 3×3 matrices $A(\varphi, z)$ for $z \in 2G$. Note that A is a $6mn \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega(\left[{}^{6m}(0,2,0)\right]) = \left[{}^{6m}2\right].$$

By Theorem 2.2 and Remark 2.1, there are Λ_i-permutations ψ_i of $2G$, with $i \in \{1,2\}$, such that

1. $\Lambda_1 = \left[{}^{1}(0,0)\right] \cup \left[{}^{1}z \mid z \in 2G, z \neq (0,2)\right]$, with $\psi(0,0) = (0,0)$, and
2. $\Lambda_2 = \left[{}^{2m}(0,2)\right]$.

Let B denote the $2^k m \times 3$ matrix whose rows B_γ, indexed over $2G$, are defined as follows: $B_\gamma = [(\gamma, 0) \quad (-\psi_1(\gamma), 0) \quad (\psi_2(\delta), 0)]$ where

$$\delta = \begin{cases} (0,2) & \text{if } \gamma = (0,0), \\ \psi_1(\gamma) - \gamma & \text{otherwise.} \end{cases}$$

Note that the columns of B are permutations of 2Γ. Also, one can check that for the list L_B of row sums of B we have

$$L_B = \left[{}^{2m-1}(0,2,0),\ {}^{1}(0,0,0)\right],$$

hence, $\omega(L_B) = [{}^{2m-1}2,\ {}^{1}1]$. It follows that each column of $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is a permutation of Γ. Clearly, the list of row sums of C is $L_C = L_A \cup L_B$ and $\omega(L_C) = [{}^{8m-1}2,\ {}^{1}1]$.

Set $\gamma_1 = (1,0)$ and $\gamma_2 = (1,2)$, and note that each φ_i swaps γ_1 and γ_2. Consider the submatrix $U = \begin{bmatrix} S \\ T \end{bmatrix}$ of $C = \begin{bmatrix} A \\ B \end{bmatrix}$, where S and T are submatrices of A and B, respectively defined as follows:

$$S = \begin{bmatrix} (a_1(\gamma_1), 0) & (b(\gamma_1), 1) & (c(\gamma_1), 1) \\ (c(\gamma_2), 1) & (a_2(\gamma_2), 0) & (b(\gamma_2), 1) \end{bmatrix},$$

$$T = \begin{bmatrix} (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}.$$

We denote by C' the matrix obtained from C by replacing U with the matrix U' defined below

$$U' = \begin{bmatrix} (a_1(\gamma_1),0) & (a_2(\gamma_2),0) & (0,0,0) \\ (c(\gamma_2),1) & (b(\gamma_1),1) & (b(\gamma_2),1) \\ (0,0,0) & (0,0,0) & (c(\gamma_1),1) \end{bmatrix}.$$

Note that each column of U' is a permutation of the corresponding column of U. Therefore, each column of C' is a permutation of Γ.

Considering that

$$\begin{aligned} &\sum S_h \text{ has order } 2, \text{ for } h = 1,2, \\ &\sum T = (0,0,0) \text{ has order } 1, \\ &\sum U'_1 = (2,0,0) \text{ has order } m, \\ &\sum U'_h \text{ has order } 2, \text{ for } h = 2,3 \end{aligned}$$

and denoting by $L_{C'}$ the list of row sums of C', we have that $\omega(L_{C'}) = [\,^{8m-1}2,\ ^1m]$. Hence, C' is the desired RSM. □

3.3. The proof of Theorem 3.1 when $g = 3$ and $k = 0$

Let α and β be non-negative integers such that $\alpha + \beta = 4mn$. First, we assume that both $\alpha \neq 1$ and $\beta \neq 1$ and let $\beta_1 = mn - \alpha_1$ $\beta_2 = 3mn - \alpha_2$ where α_1 and α_2 are defined as follows:

$$\alpha_1 = \begin{cases} mn & \text{if } mn + 2 \leq \alpha \leq 4mn - 2 \text{ or } \alpha = 4mn, \\ \alpha - 4 & \text{if } \alpha = mn + 1, \\ \alpha - 2 & \text{if } \alpha = mn - 1, \\ \alpha & \text{if } 2 \leq \alpha \leq mn - 2 \text{ or } \alpha \in \{0, mn\}, \end{cases}$$

$$\alpha_2 = \alpha - \alpha_1.$$

Since $\alpha_1 \neq 1$ and $\beta_1 \neq 1$, Theorem 3.3 guarantees the existence of an $\mathrm{RSM}_\Gamma(2\Gamma, 3, [\,^{\alpha_1}m,\ ^{\beta_1}n])$. Furthermore, one can check that $\alpha_2 \neq 1$ and $\beta_2 \neq 1$. Hence, by Theorem 3.2 there is an $\mathrm{RSM}_\Gamma(\Gamma \setminus 2\Gamma, 3, [\,^{\alpha_2}m,\ ^{\beta_2}n])$. Therefore, $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is an $\mathrm{RSM}_\Gamma(\Gamma, 3; [\,^{\alpha}m,\ ^{\beta}n])$.

It is then left to deal with the case $\alpha = 1$; indeed, since both m and n are odd, the case $\beta = 1$ can be obtained by exchanging the roles of m and n.

Let $\alpha = 1$. By Theorem 2.2, there exists a Δ_h-permutation φ_h of $\mathbb{Z}_m \times 2\mathbb{Z}_{2n}$, with

$$\Delta_h = \left[\,^{(mn+1)/2}(0,2),\ ^{(mn-3)/2}(0,-2), (0,-4)\right],$$

where $\omega(\Delta_h) = [\,^{mn}n]$, for every $h \in \{1,2,3,4\}$. Now let $A = [A(\varphi,(x,y))]$ be the column block-matrix whose blocks are the matrices $A(\varphi,(x,y))$ for $(x,y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2n}$, where

$\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Note that A is a $(3 \cdot 2^k mn) \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting L_A be the list of row-sums of A, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1) \;\cup\; \omega(\Delta_2) \;\cup\; \omega(\Delta_3) = \left[\,{}^{3mn}n\right].$$

Now let B be the matrix whose rows are indexed over $2G$ such that

$$B_{(x,y)} = \begin{bmatrix} (x,y,0) & (-2x,-2y,0) & (\varphi_4(x,y),0) \end{bmatrix}.$$

Notice that each column of B is a permutation of $2G$ and $\sum B(x,y) = (\varphi_4(x,y) - (x,y),0)$. Hence, letting L_B be the list of row-sums of B we have that $\omega(L_B) = \omega(\Delta_4) = [\,{}^{mn}n]$. Therefore, the matrix $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is an $\mathrm{RSM}_\Gamma(\Gamma, 3; [\,{}^{4mn}n])$.

By Remark 2.1, we can assume that the permutations φ_2 and φ_3 used to define A satisfy the condition $\varphi_h(1,2) = (1,0)$, hence

$$a_h(1,2) = \varphi_h(1,2) + (0,1) = (1,1),$$

for $h = 2$ or 3. Furthermore, we can assume that

$$B_{(0,-2)} = \begin{bmatrix} (0,-2,0) & (0,4,0) & (0,-4,0) \end{bmatrix}.$$

Now, consider the following matrices:

$$U = \begin{bmatrix} B_{(0,-2)} \\ A(\varphi,(1,2))_2 \\ A(\varphi,(1,2))_3 \end{bmatrix} = \begin{bmatrix} (0,-2,0) & (0,4,0) & (0,-4,0) \\ (c(1,2),1) & (a_2(1,2),0) & (b(1,2),1) \\ (b(1,2),1) & (c(1,2),1) & (a_3(1,2),0) \end{bmatrix}$$

$$U' = \begin{bmatrix} (0,-2,0) & (a_2(1,2),0) & (a_3(1,2),0) \\ (b(1,2),1) & (0,4,0) & (b(1,2),1) \\ (c(1,2),1) & (c(1,2),1) & (0,-4,0) \end{bmatrix}$$

$$= \begin{bmatrix} (0,-2,0) & (1,1,0) & (1,1,0) \\ (b(1,2),1) & (0,4,0) & (b(1,2),1) \\ (c(1,2),1) & (c(1,2),1) & (0,-4,0) \end{bmatrix}.$$

Notice that $\sum U'_1 = (2,0,0)$ and $\sum U'_h = (0,-4,0)$ for $h = 2,3$. Note that U is a submatrix of C, while each column of U' is a permutation of the corresponding column of U. Therefore, by replacing the block U of C with U', we obtain a new $4mn \times 3$ matrix C' whose columns are still permutations of Γ. Denoting by L' the list of row sums of C', and taking into account the values $\sum U'_h$, we have that $\omega(L_{C'}) = [\,{}^{1}m,\, {}^{4mn-1}n]$. Therefore, C' is the desired $\mathrm{RSM}_\Gamma(\Gamma, 3; [\,{}^{1}m,\, {}^{4mn-1}n])$.

4. The proof of Theorem 1.5

Theorem 4.1. *Let $g \geq 3$, $k \geq 0$, and let $m, n \geq 1$ be odd integers. Then* $\mathrm{HWP}(C_g[2^{k+2}mn]; gm, 2^k gn; \alpha, \beta)$ *has a solution if and only if* $\alpha + \beta = 2^{k+2}mn$.

Proof. Since $C_g[2^{k+2}mn]$ is regular of degree $2^{k+3}mn$, the necessity of the condition $\alpha+\beta = 2^{k+2}mn$ follows from Theorem 1.3. The converse is a straightforward consequence of Theorems 2.5, 2.9, and 3.1. $\square$

We are now ready to prove the main result of this paper.

Theorem 1.5. *Let v, M and N be integers greater than 3, and let $\ell = \mathrm{lcm}(M, N)$. A solution to* $\mathrm{HWP}(v; M, N; \alpha, \beta)$ *exists if and only if* $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$, $M \mid v$ *and* $N \mid v$, *except possibly when*

1. $\gcd(M, N) \in \{1, 2\}$*;*
2. 4 *does not divide* v/ℓ*;*
3. $v/4\ell \in \{1, 2\}$*;*
4. $v = 16\ell$ *and* $\gcd(M, N)$ *is odd;*
5. $v = 24\ell$ *and* $\gcd(M, N) = 3$.

Proof. Let $g = \gcd(M, N) > 2$. We may assume that $M = gm$ and $N = 2^k gn$, where both m and n are odd positive integers and $k \geq 0$; hence, $\ell = \mathrm{lcm}(M, N) = 2^k gmn$. By assumption, we also have that $v = 4\ell s$ with $s \geq 3$, g is even when $s = 4$, and $(g, s) \neq (3, 6)$.

Let $(t, \epsilon) = (s/2, 2)$ if $s \geq 6$ is even, otherwise set $(t, \epsilon) = (s, 1)$. We start by factorizing K_v into two graphs G_0 and G_1, where G_0 is the vertex disjoint union of t copies of $K_{4\ell\epsilon}$, while $G_1 \simeq K_t[4\ell\epsilon]$. Also, set $(\alpha_0, \beta_0) = (2\ell\epsilon - 1, 0)$ if $\alpha \geq 2\ell\epsilon - 1$; otherwise, set $(\alpha_0, \beta_0) = (0, 2\ell\epsilon - 1)$. In either case set $(\alpha_1, \beta_1) = (\alpha, \beta) - (\alpha_0, \beta_0)$.

By Theorem 1.2 there is a C_g-factorization of $K_t[g\epsilon]$, hence there exists a $C_g[4\ell/g]$-factorization of $K_t[g\epsilon][4\ell/g] \simeq K_t[4\ell\epsilon] \simeq G_1$. Note that $4\ell/g = 2^{k+2}mn$, therefore by applying Theorem 4.1 to each component of every $C_g[4\ell/g]$-factor, we obtain a solution to $\mathrm{HWP}(G_1; gm, 2^k n; \alpha_1, \beta_1)$. By adding a solution to $\mathrm{HWP}(G_0; gm, 2^k n; \alpha_0, \beta_0)$ we obtain the assertion. $\square$

Declaration of competing interest

The authors have no conflict of interest to declare.

Data availability

No data was used for the research described in the article.

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