Received: 22 July 2021 | Revised: 17 January 2022 | Accepted: 21 January 2022

DOI: 10.1002/jcd.21829

RESEARCH ARTICLE

WILEY

Vertex-regular 1-factorizations in infinite graphs

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Funding information
INdAM-GNSAGA

Abstract

The existence of 1-factorizations of an infinite complete equipartite graph $K_m[n]$ (with m parts of size n) admitting a vertex-regular automorphism group G is known only when $n = 1$ and m is countable (i.e., for countable complete graphs) and, in addition, G is a finitely generated abelian group G of order m. In this paper, we show that a vertex-regular 1-factorization of $K_m[n]$ under the group G exists if and only if G has a subgroup H of order n whose index in G is m. Furthermore, we provide a sufficient condition for an infinite Cayley graph to have a regular 1-factorization. Finally, we construct 1-factorizations that contain a given subfactorization, both having a vertex-regular automorphism group.

KEYWORDS
infinite Cayley graph, regular 1-factorization, subfactorization

MATHEMATICAL SUBJECT CLASSIFICATION
05C63, 05C70

1 | INTRODUCTION

In this paper, we deal with graphs, finite or infinite, which are simple and with no loops. Given a graph Λ, we denote by $V(\Lambda)$ the set of vertices, and by $E(\Lambda)$ the set of edges of Λ, and refer to the cardinality $|V(\Lambda)|$ of $V(\Lambda)$ as the order of Λ. As usual, we use the notation K_V for the complete graph whose vertex set is V. Also, we denote by $K_m[n]$ the complete equipartite graph on mn vertices partitioned into m parts of size n; two vertices of $K_m[n]$ are adjacent if and only if they belong to distinct parts. Clearly $K_m[1]$ is isomorphic to the complete graph K_m of order m. Note that the parameters m and n are both allowed to be infinite cardinals.

A graph is r-regular if each of its vertices has r edges incident with it. A 1-regular graph is then the vertex disjoint union of edges. A subgraph Γ of a graph Λ such that $V(\Gamma) = V(\Lambda)$ is called a factor of Λ; equivalently, a factor of Λ is a subgraph obtained by edge deletions only. A 1-regular factor is simply called a 1-factor.

A decomposition of a graph Λ is a set $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_n\}$ of subgraphs of Λ whose edge-sets partition $E(\Lambda)$. If each Γ_i is a factor (resp., 1-factor), we speak of a factorization (resp., 1-factorization) of Λ.

An automorphism of a graph Λ is a bijection α of $V(\Lambda)$ such that $\alpha(\Lambda) = \Lambda$, where $\alpha(\Lambda)$ is the graph obtained from Λ by replacing each vertex, say x, with $\alpha(x)$. An automorphism of a decomposition $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_n\}$ of a graph Λ is a bijection β of $V(\Lambda)$ such that $\beta(\Gamma_i) \in \mathcal{G}$ for every $\Gamma_i \in \mathcal{G}$. It follows that $\beta(\Lambda) = \Lambda$, hence β is necessarily an automorphism of the graph Λ. If Λ (resp., $\mathcal{G}$) has an automorphism group isomorphic to G that acts sharply transitively on $V(\Lambda)$, we say that Λ (resp., $\mathcal{G}$) is vertex-regular under G, or simply G-regular.

Although regular 1-factorizations have been widely studied for finite complete graphs [1–4,6,8,9,11,12], very little is known in the infinite case. In [5], the authors construct a G-regular 1-factorization of a countable complete graph for every finitely generated abelian infinite group G. A complementary result has been obtained in [7] where it is shown that there exists a G-regular 1-factorization of the complete graph K_G for every infinite group G—not necessarily countable—with no involutions (i.e., elements of order 2). It is worth pointing out that [7] provides a much more general result concerning the existence of a G-regular factorization of complete graphs of every infinite order. As far as we know, there is no other paper dealing with vertex-regular 1-factorizations of infinite graphs.

In this paper, we completely characterize the existence of G-regular 1-factorizations of the complete equipartite graph $K_m[n]$, for every infinite group G. More precisely, we prove the following.

Theorem 1.1. *Let G be an infinite group. There exists a G-regular 1-factorization of $K_m[n]$ if and only if G has a subgroup H of size n whose index in G is m.*

As a consequence, we show the existence of a G-regular 1-factorization of K_G for every infinite group G.

More generally, if Λ has a G-regular 1-factorization, then Λ is G-regular itself. In other words, Λ is necessarily a Cayley graph on G, and in Section 2 we recall some basic notions and well-known results on Cayley graphs and vertex-regular decompositions.

In Section 3 we provide a sufficient condition for an infinite Cayley graph on G to have a G-regular 1-factorization: Theorem 3.1. Complete (equipartite) graphs are Cayley graphs and in Section 4 we essentially prove that they satisfy the assumption of Theorem 3.1, thus proving Theorem 1.1. Finally, in Section 5, we construct vertex-regular 1-factorizations with given regular subfactorizations.

2 | PRELIMINARY NOTIONS

In this section, we recall some known facts on Cayley graphs and graph decompositions with a regular automorphism group on the vertex set. Throughout the paper, we denote groups in additive notation, even though they are not necessarily abelian.

Given a group G, a subset S of $G \setminus \{0\}$ such that $S = -S$ is called a connection set. The Cayley graph on G with connection set S is the simple graph $Cay[G : S]$ having G as its vertex set and such that two vertices x and y are adjacent if and only if $x - y \in S$. Note that, if $S = G\setminus\{0\}$ then $Cay[G : S]$ is the complete graph whose vertex set is G. More generally, if H is a subgroup of G of index m and order n, then $Cay[G : G\setminus H]$ is isomorphic to $K_m[n]$; indeed, two vertices are adjacent if and only if they belong to distinct right cosets of H, which therefore represent the m parts of $Cay[G : S]$, each of size n.

Given a graph Γ with vertices in G, the right translate of Γ by an element $g \in G$ is the graph $\Gamma + g$ obtained from Γ by replacing each of its vertices, say x, with $x + g$; clearly, $\Gamma + g$ is isomorphic to Γ. It is known that if Γ is a Cayley graph on G, then $\Gamma + g = \Gamma$ for every $g \in G$. Also, the group of right translations of Γ under the action of G (which we recall to be isomorphic to G) is a vertex-regular automorphism group of Γ. The following theorem shows that the graphs with a vertex-regular automorphism group are exactly the Cayley graphs.

Theorem 2.1 (Sabidussi [13]). *A graph Λ is G-regular if and only if Λ is isomorphic to a Cayley graph on G.*

Recalling that an automorphism of a decomposition of Λ is also an automorphism of the graph Λ, we have the following corollary.

Corollary 2.2. *If there exists a G-regular decomposition of Λ, then Λ is isomorphic to a Cayley graph on G.*

Therefore, we focus on regular 1-factorizations of Cayley graphs and recall two efficient methods to construct them.

Let $\Lambda = Cay[G : S]$, let $\{S_1, ..., S_n\}$ be a partition of S into connection sets (i.e., $S_i = -S_i$) and set $\Gamma_i = Cay[G : S_i]$, for every $i = 1, ..., n$. Clearly, $\mathcal{G} = \{\Gamma_i | i = 1, ..., n\}$ is a decomposition of Λ. Also, each Γ_i is a factor of Λ and it is fixed by the right translations induced by the elements of G. Therefore, $\mathcal{G}$ is a G-regular factorization of Λ. If $S = \{s_1, ..., s_n\}$ contains only involutions and $S_i = \{s_i\}$ for every $i = 1, ..., n$, then each Γ_i is a 1-factor of Λ and $\mathcal{G}$ is a G-regular 1-factorization of Λ. Denoting by $I(G)$ the set of all involutions of G, we then have the following.

Proposition 2.3. *$Cay[G : S]$ has a G-regular 1-factorization whenever $S \subseteq I(G)$.*

Another way of constructing G-regular 1-factorizations relies on the concept of a difference family. Given a graph Γ with vertices in G, the list of differences of Γ is the multiset $\Delta(\Gamma)$ of the differences $x - y$ and $y - x$ between every two adjacent vertices x and y of Γ.

Proposition 2.4. *Let Γ be a 1-factor of K_G. If each element of $\Delta\Gamma$ has multiplicity 1, then $\{\Gamma + g | g \in G\}$ is a G-regular 1-factorization of $Cay[G : \Delta\Gamma]$.*

Note that if $\Delta\Gamma$ contains an involution, then it must appear with even multiplicity.

3 | 1-FACTORIZATIONS OF INFINITE CAYLEY GRAPHS

In this section, we provide a sufficient condition for a Cayley graph on the group G to have a G-regular 1-factorization. More precisely, we prove the following. We recall that $I(G)$ denotes the set of all involutions of G.

Theorem 3.1. *Let G be an infinite group and let $\Gamma = Cay[G : S]$. If $|S \setminus I(G)|$ is either 0 or $|G|$, then there exists a G-regular 1-factorization of $Cay[G : S]$.*

Considering that $Cay[G : S \cap I(G)]$ has a G-regular 1-factorization $\mathcal{G}_1$ by Proposition 2.3, it is enough to prove that $Cay[G : S \setminus I(G)]$ has a G-regular 1-factorization $\mathcal{G}_2$. Indeed, one can easily check that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a G-regular 1-factorization of $\Gamma = Cay[G : S]$. Our strategy is to construct, by transfinite induction, a 1-factor Γ such that $\Delta\Gamma = S \setminus I(G)$ whenever $|S \setminus I(G)| = |G|$ (Proposition 3.6), and then apply Proposition 2.4.

We first introduce some set-theoretical notions. We will work within the Zermelo–Frankel axiomatic system with the Axiom of Choice in the form of the Well-Ordering Theorem. We recall the definition of a well-order.

Definition 3.2. A well-order $\prec$ on a set X is a total order on X with the property that every nonempty subset of X has a least element.

In particular, the following theorem is equivalent to the Axiom of Choice.

Theorem 3.3 (Well-ordering). *Every set X admits a well-order $\prec$.*

Given an element $x \in X$, we define the section $X_{\prec x}$ associated to it by

$$X_{\prec x} = \{y \in X : y \prec x\}.$$

Corollary 3.4 (See Costa [7]). *Every set X admits a well-order $\prec$ such that the cardinality of any section is smaller than $|X|$.*

We recall now that well-orderings allow proofs by induction.

Theorem 3.5 (Transfinite induction). *Let X be a set with a well-order $\prec$ and let P_x denote a property for each $x \in X$. Set $0 = \min X$ and assume that:*

- P_0 is true, and
- for every $x \in X$, if P_y holds for every $y \in X_{\prec x}$, then P_x holds.

Then P_x is true for every $x \in X$.

We are now ready to build a G-regular 1-factorization of $Cay[G : U]$ whenever U has the same cardinality as G and contains no involutions.

Proposition 3.6. *Let $\Gamma = Cay[G : U]$ where G is an infinite group. If $U \cap I(G) = \varnothing$ and $|U| = |G|$, then Γ has a G-regular 1-factorization.*

Proof. We endow G with a well-ordering $\prec$ that satisfies the property of Corollary 3.4 and such that $0 = \min_{\prec} G$. To prove the assertion, it is enough to build, by transfinite induction, an ascending chain of 1-regular graphs Γ_g, $g \in G$ (i.e., Γ_h is a subgraph of Γ_g whenever $h \prec g$) each of which satisfies the following conditions:

(1_g) Γ_g is either finite or $|V(\Gamma_g)| \leq |G_{<g}|$;
(2_g) $g \in V(\Gamma_g)$, and $g \in \Delta\Gamma_g$ whenever $g \in U$;
(3_g) $\Delta\Gamma_g \subset U$.

Indeed, one can easily see that $\Gamma := \bigcup_{g\in G} \Gamma_g$ is a 1-factor of K_G such that $\Delta\Gamma = U$. The result then follows from Proposition 2.4.

Base case. Take $z \in U$ and let Γ_0 be the edge $\{0, z\}$. This graph clearly satisfies properties 1_0, 2_0, and 3_0: indeed, Γ_0 is finite, $0 \in V(\Gamma_0)$ and $\Delta\Gamma_0 = \{\pm z\} \subset U$.

Inductive step. Assume that there exists a graph Γ_h that satisfies properties 1_h, 2_h, and 3_h for every $h \prec g$, and set $\Gamma_{<g} := \bigcup_{h<g} \Gamma_h$. Due to properties 1_h, 2_h, and 3_h, $\Gamma_{<g}$ is a 1-regular graph such that:

1. $\Gamma_{<g}$ is either finite or $|V(\Gamma_{<g})| \leq |G_{<g}|$;
2. for every $h \prec g$, $h \in V(\Gamma_{<g})$, and $h \in \Delta\Gamma_{<g}$ whenever $h \in U$;
3. $\Delta\Gamma_{<g} \subset U$.

Note that $|\Delta\Gamma_{<g}| = |V(\Gamma_{<g})| \leq |G_{<g}|$, and $|G_{<g}| < |G|$ by Corollary 3.4. Therefore, letting $H = V(\Gamma_{<g}) \cup (\Delta\Gamma_{<g} + g)$, we have that $|H| < |G|$. Since by assumption $|U| = |G|$, then $|U + g| = |G|$, hence $(U + g)\backslash H$ is nonempty.

Take $z \in (U + g)\backslash H$. Clearly, $z \notin V(\Gamma_{<g})$ and $z - g \in U\backslash\Delta\Gamma_{<g}$. If $g \in V(\Gamma_{<g})$, we set $\Gamma' = \Gamma_{<g}$, otherwise Γ' is obtained by adding to $\Gamma_{<g}$ the edge $\{g, z\}$.

Now, if $g \notin U$ or $g \in \Delta(\Gamma')$, we set $\Gamma_g = \Gamma'$. Otherwise, due to property 1) of $\Gamma_{<g}$, the set $H' = V(\Gamma') \cup (-g + V(\Gamma'))$ has cardinality smaller than $|G|$, hence $G\backslash H'$ is nonempty. Then we can take $y \in G\backslash H'$. Clearly, both y and $g + y$ do not belong to $V(\Gamma')$, therefore Γ_g is obtained by adding to Γ' the edge $\{y, g + y\}$. □

4 | 1-FACTORIZATIONS OF COMPLETE (EQUIPARTITE) INFINITE GRAPHS

In this section, we prove the main result of this paper, Theorem 1.1. We recall that the complete equipartite graph $K_m[n]$ is isomorphic to the Cayley graph $Cay[G : G\backslash H]$ where G is any group of order mn and H is any subgroup of G of order n whose index in G is m. Because of Theorem 3.1, it is enough to show that $G\backslash(H \cup I(G))$ has the same cardinality as G. This is the content of Theorem 4.2 whose proof relies on elementary group theory.

For the reader's convenience, we recall some basic results concerning groups and refer to [10] for the standard notions and definitions.

Let x and g be elements of a group G. Then $x^g = -g + x + g$ is called the conjugate of x by g, and the set $x^G = \{x^g : g \in G\}$ of all conjugates of x is called the G-orbit of x. Note that x and x^g have the same order; also, x and g commute if and only if $x = x^g$. We recall that the centralizer of x is the subgroup $C(x)$ of G consisting of all group elements that commute with g, that is,

$$C(x) := \{g : x^g = x\}.$$

The number $|G : H|$ of right (left) cosets of the subgroup H in G is called the index of H in G. It is very well known that

$$|G : C(x)| = |x^G|.$$

Considering that G is the union of all right (resp., left) cosets of H, we also have that

$$|G| = |G : H||H|.$$

Lemma 4.1. *If G is an infinite group and $G\backslash(I(G) \cup \{0\})$ is nonempty, then* $|G\backslash(I(G) \cup \{0\})| = |G|$.

Proof. Suppose that $U = G\backslash(I(G) \cup \{0\})$ is nonempty and let $x \in U$. We assume for a contradiction that $|U| < |G|$.

Set $J = C(x) \cap I(G)$. Since $x \in U$, then $x + j \in U$ for every $j \in J$ (otherwise, $x + j \in I(G)$ for some $j \in J$, hence $0 = 2(x + j) = 2x + 2j = 2x$, contradicting the assumption that x is not an involution). In other words, $x + J \subseteq U$, hence $|J| = |x + J| \leq |U| < |G|$. Since $C(x) = J \cup (C(x) \cap U) \cup \{0\}$, we have that

$$|C(x)| < |G|. \tag{1}$$

Since the conjugacy preserves the order of an element, we have that $x^G \subseteq U$, hence

$$|G : C(x)| = |x^G| < |G|. \tag{2}$$

By conditions 1 and 2, we obtain the following contradiction: $|G| = |G : C(x)| \cdot |C(x)| < |G|^2 = |G|$. Therefore, $|U| = |G|$. □

We can now prove the following result.

Theorem 4.2. *Let G be an infinite group and let H be a subgroup of G. If $U = G\backslash(I(G) \cup H)$ is nonempty, then $|U| = |G|$.*

Proof. Let $U = G\backslash(I(G) \cup H)$. By Lemma 4.1, $G\backslash(I(G) \cup \{0\})$ is empty or it has the same cardinality as G. Hence, if $|H| < |G|$, then $|U| = 0$ or $|G|$.

It is left to consider the case $|H| = |G|$. We assume that $|U| < |H|$ and show that U is necessarily empty. Note that every right coset of H must contain some involution (otherwise, U would contain a right coset of H, which has the same cardinality as $|H|$). Therefore, denoting by $J = I(G) \cap (G\backslash H)$ the set of all involutions of $G\backslash H$, we have that

$$\text{each right coset of } H, \text{ except for } H, \text{ is of the form } H + j, \text{ with } j \in J. \tag{3}$$

For each involution $j \in J$, let $H_j \subseteq H$ be the set defined as follows:

$$h \in H_j \text{ if and only if } 2(h + j) = 0. \tag{4}$$

In other words,

$$h \in H_j \text{ if and only if } h^j = -h. \tag{5}$$

Note that $\langle H_j \rangle = H$, where $\langle H_j \rangle$ is the group generated by H_j, for every $j \in J$. Indeed, by (4) and recalling that $|U| < |H|$, we have

$$|H \backslash H_j| = |(H \backslash H_j) + j| = |(H + j) \cap U| < |H|,$$

and hence $|H_j| = |H|$.

Considering that each coset of $\langle H_j \rangle$ in H, different from $\langle H_j \rangle$, has the same cardinality as $\langle H_j \rangle$ and it is contained in $H \backslash \langle H_j \rangle \subset H \backslash H_j$, it follows that $\langle H_j \rangle$ has no cosets in H other than itself, that is, $\langle H_j \rangle = H$.

We now show that H is abelian. Let $j \in J$, $h \in H_j$, and set $U^* = H_j \cap (U - j - h)$ and $H_{j,h} = H_j \backslash U^*$. Clearly, $|U^*| \leq |U| < |H|$, hence $|H_{j,h}| = |H|$ and

$$\langle H_{j,h} \rangle = H.$$

Also, if $x \in H_{j,h}$, then $(x + h) + j \notin U \cup H$, hence $(x + h) + j \in I(G)$. By (4), $x + h \in H_j$. By (5), for every $x \in H_{j,h} \subseteq H_j$ we have that:

$$-h - x = (x + h)^j = x^j + h^j = -x - h,$$

that is, $h + x = x + h$. Then, all the elements of $\langle H_{j,h} \rangle = H$ commute with every $h \in H_j$. This means that the elements of H_j commute with each other, and since they generate H, we have that H is abelian.

Since $H = \langle H_j \rangle$, for every $h \in H$ and $j \in J$ we have that $h^j = -h$, hence $H = H_j$, and by (4) we have that $H + j$ contains only involutions. Then by (3), all right cosets of H, except for H, contain no element of order greater than 2, that is, U is empty. □

We are now ready to prove the main result of this paper, whose statement is recalled in the following.

Theorem 1.1. *Let G be an infinite group. There exists a G-regular 1-factorization of $K_m[n]$ if and only if G has a subgroup H of size n whose index in G is m.*

Proof. Let G be an infinite group and let H be a subgroup of G of size n whose index in G is m. By Theorem 4.2, the set $G \backslash (I(G) \cup H)$ is either empty or has the same size as G. Therefore, by applying Theorem 3.1 with $S = G \backslash H$, we obtain the existence of a G-regular 1-factorization of $Cay[G : S]$. Clearly, $Cay[G : S]$ is isomorphic to $K_m[n]$.

Conversely, assume there is a G-regular 1-factorization $\mathcal{G}$ of $K_m[n]$. By Corollary 2.2, $K_m[n]$ is isomorphic to $Cay[G : S]$ for some connection set S of G. Considering that $Cay[G : S]$ contains no edge of the form $\{0, h\}$ for every $h \in H = G \backslash S$, it follows that H represents a part (of size n) of the equipartite complete graph $Cay[G : S]$. We are going to prove that H is a subgroup of G. If $x, y \in H$ and $y - x \in S$, then $\{x, y\}$ would be an edge of $Cay[G : S]$, contradicting the fact that H is a part of $Cay[G : S]$. Therefore, $y - x \in H$, for every $x, y \in H$, that is, H is a subgroup of G. □

By taking $n = 1$ in Theorem 1.1, we obtain the following corollary.

Corollary 4.3. *There exists a G-regular 1-factorization of K_G for every infinite group G.*

5 | 1-FACTORIZATIONS WITH SUBFACTORIZATIONS

In this section, given an H-regular 1-factorization $\mathcal{H}$ of $K_{m'}[n']$, and a group G containing H, we provide conditions on G, m, and n that guarantee the existence of a G-regular 1-factorization $\mathcal{G}$ of $K_m[n]$ that contains $\mathcal{H}$ as a subfactorization. This means that for every pair of 1-factors $(F, \Gamma) \in \mathcal{H} \times \mathcal{G}$, either $F \subseteq \Gamma$ or $F \cap \Gamma$ is empty. When speaking of an H-regular subfactorization of $\mathcal{G}$, it is understood that both G and H act on the related 1-factorizations by right translation.

Given two cardinals, m' and m, we write $m'|m$ whenever m is infinite and $m' \leq m$, or m is finite and m' is a divisor of m. In the former case, we set $m/m' = m$. We notice that, similarly to the finite case, we have that $(m/m') \cdot m' = m$. This convention allows us to consider the case where one parameter between m and n (which define the equipartite complete graph $K_m[n]$) is finite.

To ease the notation, given a direct product of groups $G = G_1 \times H$, we denote by G_1 and H the subgroups $G_1 \times \{0\}$ and $\{0\} \times H$ of G, respectively. In other words, we consider G as the direct inner product of its two trivially intersecting subgroups G_1 and H.

Lemma 5.1. *Let $\mathcal{H}$ be an H-regular 1-factorization of $Cay[H : H \backslash A]$, with $A \subset H$, and set $G = G_1 \times H$ for some group G_1. Then there exists a G-regular 1-factorization of $Cay[G : H \backslash A]$ containing $\mathcal{H}$ as a subfactorization.*

Proof. Let $\mathcal{H}^* = \{F^* : F \in \mathcal{H}\}$ be the set of 1-factors of K_G obtained from those in $\mathcal{H}$ as follows:

$$F^* = \bigcup_{x \in G_1} (F + x).$$

Clearly, $\mathcal{H}^*$ is a 1-factorization of $Cay[G : H \backslash A]$. To prove that it is G-regular, it is enough to check that, for every $F^* \in \mathcal{H}^*$, $g \in G_1$ and $h \in H$, $F^* + (g + h) \in \mathcal{H}^*$. Note that

$$F^* + g + h = \bigcup_{x \in G_1} (F + x + g + h) = \bigcup_{g' \in G_1} (F + g' + h).$$

Recalling that $\mathcal{H}$ is regular under the action of H by right translation, we have that $F + h = F' \in \mathcal{H}$ which implies

$$\bigcup_{g' \in G_1} (F + g' + h) = \bigcup_{g' \in G_1} (F' + g') = (F')^* \in \mathcal{H}^*.$$

The assertion follows. □

Theorem 5.2. *Let $\mathcal{H}$ be an H-regular 1-factorization of $K_{m'}[n']$. Also, let m and n be cardinals such that mn is infinite, $m'|m$ and $n'|n$. Then, there exists a regular 1-factorization of $K_m[n]$ containing $\mathcal{H}$ as a subfactorization.*

Proof. Let $\mathcal{H}$ be a nonempty H-regular 1-factorization of $K_{m'}[n']$. Up to isomorphism, we can assume that $K_{m'}[n'] = Cay[H : H\backslash A]$ where $|H| = m'n'$ and $|A| = n'$, and that $\mathcal{H}$ is H-regular under the action by right translation, that is, for every $F \in \mathcal{H}$ and $h \in H$, we have that $F + h \in \mathcal{H}$.

Let G_1 and L_1 be groups of order m/m' and n/n', respectively. Also, set $L = L_1 \times A$ and $G = G_1 \times L_1 \times H$. Since at least one between m and n is infinite, then

$$|G_1 \times L_1| = (m/m')(n/n') = \max(m/m', n/n') = \max(m, n) = mn = |G|.$$

Denoting by U the set of non-involutions of $(G_1 \times L_1)\backslash\{(0, 0)\}$ and assuming that $|U| > 0$, by Lemma 4.1 we have that $|U| = |G_1 \times L_1| = |G|$. Therefore, $U \times (H\backslash A)$ is a set of non-involutions, of cardinality $|G|$, contained in $G\backslash(H \cup L)$. Similarly, if G_1 is not trivial (i.e., $|G_1| > 1$) and $(G_1 \times L_1)\backslash\{(0, 0)\}$ contains only involutions, we denote by U the set of non-involutions of $H\backslash\{0\}$. In this case, if $|U| > 0$, then $((G_1\backslash\{0\}) \times L_1) \times U$ has the same cardinality as G, and it contains only elements of order greater than 2 belonging to $G\backslash(H \cup L)$. Finally, if G_1 is trivial and $L_1\backslash\{0\}$ contains only involutions, we define U as the set of non-involutions of $H\backslash A$. Here, if $|U| > 0$, then $((G_1 \times L_1)\backslash\{(0, 0)\}) \times U$ has the same cardinality as G, and it contains only elements of order greater than 2 that appear in $G\backslash(H \cup L)$.

Now, if $G\backslash(H \cup L)$ contains some non-involutions, we fall in one of the previous three cases. Then the number of its elements of order greater than 2 is $|G|$. Hence, by Theorem 3.1, there is a G-regular 1-factorization $\mathcal{F}_1$ of $Cay[G : G\backslash(H \cup L)]$. Moreover, due to Lemma 5.1, there also exists a G-regular 1-factorization $\mathcal{F}_2$ of $Cay[G : H\backslash A]$. Considering that $G\backslash(H \cup L)$ and $H\backslash L = H\backslash A$ partition $G\backslash L$, it follows that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a G-regular 1-factorization of $Cay[G : G\backslash L] = K_m[n]$ containing $\mathcal{H}$ as a subfactorization. □

As a corollary, we obtain the following.

Corollary 5.3. *Let $\mathcal{H}$ be a regular 1-factorization of $K_{m'}$. Then, given an infinite cardinal m, there exists a regular 1-factorization of K_m that admits $\mathcal{H}$ as a subfactorization if and only if $m'|m$.*

ACKNOWLEDGMENTS

The authors were partially supported by INdAM–GNSAGA.

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How to cite this article: S. Costa and T. Traetta, *Vertex-regular 1-factorizations in infinite graphs*, J. Combin. Des. (2022), 30, 354–363. https://doi.org/10.1002/jcd.21829