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On well-posedness of the plasma-vacuum interface problem with displacement current in vacuum

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Abstract. We consider the plasma-vacuum interface problem in ideal incompressible magneto-hydrodynamics. Unlike the classical statement when the vacuum magnetic field obeys the div-curl system of pre-Maxwell dynamics, to better understand the influence of the electric field in vacuum we do not neglect the displacement current in the vacuum region and consider the Maxwell equations for electric and magnetic fields. Under the necessary and sufficient stability condition for a planar interface found earlier by Trakhinin, we prove an energy a priori estimate for the linearized constant coefficient problem. The process of derivation of this estimate is based on various methods, including a secondary symmetrization of the vacuum Maxwell equations, the derivation of a hyperbolic evolutionary equation for the interface function and the construction of a degenerate Kreiss-type symmetrizer for an elliptic-hyperbolic problem for the total pressure.

1. Introduction

Plasma-vacuum interface problems are considered in the mathematical modeling of plasma confinement by magnetic fields in thermonuclear energy production (as in Tokamaks, Stellarators; see, e.g., [1, 2]). There are also important applications in astrophysics, where the plasma-vacuum interface problem can be used for modeling the motion of a star or the solar corona when magnetic fields are taken into account.

In the classical description of [1] (see also [2]) the plasma is described by the equations of ideal compressible magnetohydrodynamics (MHD) for the unknowns $v = (v_1, v_2, v_3)$, $H = (H_1, H_2, H_3)$, $q = p + \frac{1}{2}|H|^2$, p , denoting respectively the velocity, the magnetic field, the total pressure and the pressure, whereas the vacuum magnetic field \mathcal{H} obeys the so-called *pre-Maxwell dynamics* $\nabla \times \mathcal{H} = 0$, $\operatorname{div} \mathcal{H} = 0$. The elliptic system for \mathcal{H} is obtained from the Maxwell equations by neglecting the displacement current $(1/\varepsilon)\partial_t \mathcal{E}$, where the positive constant $\varepsilon \ll 1$, being the ratio between a characteristic (average) speed of the plasma flow and the speed of light in vacuum, is a natural *small parameter*, and the vacuum electric field $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ can be considered as a secondary variable that may be computed from \mathcal{H} .

In [3] a basic energy a priori estimate in Sobolev spaces for the linearized plasma-vacuum interface problem was proved under the non-collinearity condition $(H \times \mathcal{H})|_\Gamma \neq 0$ satisfied on the interface Γ for the unperturbed flow. Under this non-collinearity condition satisfied at the initial time, the well-posedness of the nonlinear problem was proved by Secchi and Trakhinin [4, 5] for the compressible MHD equations, and by Sun, Wang and Zhang [6] for incompressible MHD. For the relativistic plasma-vacuum problem, for which the displacement current in vacuum must not



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be neglected, Trakhinin had shown in [7] the possible ill-posedness in the presence of a *sufficiently strong* vacuum electric field. Since relativistic effects play a rather passive role in the analysis of [7], it was natural to expect a similar behavior for the non-relativistic problem. In fact, it was shown in [8,9] that a *sufficiently weak* vacuum electric field precludes ill-posedness and gives the well-posedness of the linearized problem.

The results in [7–9] induce a natural question: how strong the vacuum electric field has to be in order to enforce ill-posedness. The answer to this question has been recently given by Trakhinin in [10] for the incompressible plasma-vacuum problem, where a necessary and sufficient condition for the violent instability of a planar plasma-vacuum interface was obtained (the opposite of this condition is given in (9)).

In the present paper we discuss our recent result [11] for the incompressible plasma-vacuum interface problem [10]. We describe the main steps in the derivation of an energy a priori estimate for the linearized constant coefficient problem under the stability condition (9). This estimate shows the stability of the planar plasma-vacuum interface and is necessary for the future analysis towards the proof of the well-posedness of the original nonlinear problem, provided that condition (9) holds at each point of the initial interface.

2. Statement of the problem and main result

For technical simplicity, we assume that the free interface $\Gamma(t)$ has the form of a graph and the domains $\Omega^\pm(t)$ occupied by the plasma and the vacuum are unbounded:

$$\Gamma(t) = \{x_1 = \varphi(t, x')\}, \quad \Omega^\pm(t) = \{\pm(x_1 - \varphi(t, x')) > 0, x' \in \mathbb{R}^2\}, \quad x' = (x_2, x_3).$$

The plasma in the domain $\Omega^+(t)$ is assumed to be ideal and incompressible whereas in the vacuum region $\Omega^-(t)$ we do not neglect the displacement current and consider the Maxwell equations. In a dimensionless form [10] the plasma-vacuum interface problem then reads:

$$\frac{dv}{dt} - (H \cdot \nabla)H + \nabla q = 0, \quad \frac{dH}{dt} - (H \cdot \nabla)v = 0, \quad \operatorname{div} v = 0 \quad \text{in } \Omega^+(t), \quad (1)$$

$$\varepsilon \partial_t \mathcal{H} + \nabla \times \mathcal{E} = 0, \quad \varepsilon \partial_t \mathcal{E} - \nabla \times \mathcal{H} = 0 \quad \text{in } \Omega^-(t), \quad (2)$$

$$\partial_t \varphi = v_N, \quad q = \frac{1}{2} (|\mathcal{H}|^2 - |\mathcal{E}|^2), \quad \mathcal{E}_{\tau_2} = \varepsilon \mathcal{H}_3 \partial_t \varphi, \quad \mathcal{E}_{\tau_3} = -\varepsilon \mathcal{H}_2 \partial_t \varphi \quad \text{on } \Gamma(t), \quad (3)$$

where $d/dt = \partial_t + v \cdot \nabla$ is the material derivative, $v_N = v \cdot N$, $N = (1, -\partial_2 \varphi, -\partial_3 \varphi)$ and $\mathcal{E}_{\tau_i} = \mathcal{E}_i \partial_i \varphi + \mathcal{E}_i$ ($i = 2, 3$). System (1)–(3) is supplemented with suitable initial data for the unknown (U, V, φ) , where $U = (v, H)$, $V = (\mathcal{H}, \mathcal{E})$. One can show that

$$\operatorname{div} H = 0 \quad \text{in } \Omega^+(t), \quad \operatorname{div} \mathcal{H} = 0, \quad \operatorname{div} \mathcal{E} = 0 \quad \text{in } \Omega^-(t), \quad H_N = 0, \quad \mathcal{H}_N = 0 \quad \text{on } \Gamma(t), \quad (4)$$

with $H_N = H \cdot N$ and $\mathcal{H}_N = \mathcal{H} \cdot N$, are the divergence and boundary constraints on the initial data. The boundary conditions (3) are discussed, for instance, in [9,10].

Referring to [11] for full details, for the free boundary problem (1)–(3) we just formulate the associated linear constant coefficient problem

$$\begin{cases} Lv - KH + \nabla q = f_1, & LH - Kv = f_2, & \operatorname{div} v = 0, \\ \varepsilon \partial_t \mathcal{H} + \nabla \times \mathcal{E} = 0, & \varepsilon \partial_t \mathcal{E} - \nabla \times \mathcal{H} = 0 \end{cases} \quad \text{in } \Omega, \quad (5)$$

$$\begin{cases} L\varphi = v_1, & q = \widehat{\mathcal{H}}_2 \mathcal{H}_2 + \widehat{\mathcal{H}}_3 \mathcal{H}_3 - \widehat{\mathcal{E}}_1 \mathcal{E}_1, \\ \mathcal{E}_2 = \varepsilon \widehat{\mathcal{H}}_3 \partial_t \varphi - \widehat{\mathcal{E}}_1 \partial_2 \varphi, & \mathcal{E}_3 = -\varepsilon \widehat{\mathcal{H}}_2 \partial_t \varphi - \widehat{\mathcal{E}}_1 \partial_3 \varphi \end{cases} \quad \text{on } \omega, \quad (6)$$

$$(U, V, \varphi) = 0 \quad \text{for } t < 0, \quad (7)$$

for the perturbations U, V, φ and q (which are denoted by the same letters as the unknowns of the nonlinear problem) in the domain $\Omega = \{t \in \mathbb{R}, x_1 > 0, x' \in \mathbb{R}^2\}$ with the boundary $\omega = \{t \in \mathbb{R}, x_1 = 0, x' \in \mathbb{R}^2\}$, where $\nabla^- = (-\partial_1, \nabla')$ is the “reflected” gradient, $\nabla' = (\partial_2, \partial_3)$,

$$L = \partial_t + (\hat{v}' \cdot \nabla'), \quad K = (\hat{H}' \cdot \nabla'), \quad \hat{v}' = (\hat{v}_2, \hat{v}_3), \quad \hat{H}' = (\hat{H}_2, \hat{H}_3), \quad \hat{\mathcal{H}}' = (\hat{\mathcal{H}}_2, \hat{\mathcal{H}}_3),$$

and $\hat{v}_k, \hat{H}_k, \hat{\mathcal{H}}_k$ ($k = 2, 3$) and $\hat{\mathcal{E}}_1$ are some constants. Problem (5)–(7) is the result of the linearization of problem (1)–(3) about its solution

$$(U, V, \varphi) = (\hat{U}, \hat{V}, \sigma t) = (\sigma, \hat{v}', \hat{H}, \hat{\mathcal{H}}, \hat{\mathcal{E}}_1, \varepsilon \sigma \hat{\mathcal{H}}_3, -\varepsilon \sigma \hat{\mathcal{H}}_2, \sigma t),$$

with $\hat{H} = (0, \hat{H}')$, $\hat{\mathcal{H}} = (0, \hat{\mathcal{H}}')$, and several subsequent transformations (in particular, a Galileo transform and the reflection of the vacuum region into Ω) reducing the linearized problem to that with $\sigma = 0$, homogenous boundary conditions, the homogeneous *constraints* (cf. (4))

$$\operatorname{div} H = 0, \quad \operatorname{div}^- \mathcal{H} = 0, \quad \operatorname{div}^- \mathcal{E} = 0 \quad \text{in } \Omega, \quad H_1 = K\varphi, \quad \mathcal{H}_1 = \mathcal{K}\varphi \quad \text{on } \omega, \quad (8)$$

and homogeneous interior equations except those with the source terms $f_1(t, x)$ and $f_2(t, x)$ (since the Maxwell equations are not Galileo invariant, we are not allowed to set the constant $\sigma = 0$ from the very beginning; see [11] for more details), where $\mathcal{K} = (\hat{\mathcal{H}}' \cdot \nabla')$ and the “reflected” divergence $\operatorname{div}^- a = -\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3$ for any vector $a = (a_1, a_2, a_3)$. For the zero initial data (7), constraints (8) are automatically satisfied by the solutions of problem (5)–(7). We assume that the source terms f_1 and f_2 vanish in the past and postpone the case of nonzero initial data to the nonlinear analysis.

Before stating our main result we should introduce the weighted Sobolev spaces $H_\gamma^m(\Omega)$ and $H_\gamma^m(\omega)$, where $H_\gamma^0 := L_\gamma^2$, $L_\gamma^2 := e^{\gamma t} L^2$, $H_\gamma^m := e^{\gamma t} H^m$, with $\gamma \geq 1$. The spaces $H_\gamma^m(\omega)$ and $H_\gamma^m(\Omega)$ are equipped with the norms

$$\|u\|_{H_\gamma^m(\Omega)}^2 := \sum_{|\beta| \leq m} \gamma^{2(m-|\beta|)} \|\partial^\beta \bar{u}\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|v\|_{H_\gamma^m(\omega)}^2 := \sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)} \|\partial_{\tan}^\alpha \bar{v}\|_{L^2(\omega)}^2,$$

where $\bar{u} = e^{-\gamma t} u$, $\bar{v} = e^{-\gamma t} v$, $\partial_{\tan}^\alpha := \partial_t^{\alpha_0} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, with $\alpha = (\alpha_0, \alpha_2, \alpha_3) \in \mathbb{N}^3$.

Theorem 1. *For every given planar plasma-vacuum interface described by the constant solution (\hat{U}, \hat{V}) and satisfying the stability condition*

$$\mathcal{E}_1^2 < \frac{|H|^2 + |\mathcal{H}|^2 - \sqrt{(|H|^2 + |\mathcal{H}|^2)^2 - 4|H \times \mathcal{H}|^2}}{2} \quad (9)$$

(with $\mathcal{E}_1 = \hat{\mathcal{E}}_1$, $H = \hat{H}$ and $\mathcal{H} = \hat{\mathcal{H}}$), there exist constants $\varepsilon_* > 0$ and $C > 0$ such that for all $0 < \varepsilon < \varepsilon_*$, $\gamma \geq 1$, any solution $((U, V), \varphi) \in L_\gamma^2(\Omega) \times H_\gamma^1(\omega)$ of problem (5)–(7), with source term $f = (f_1, f_2) \in H_\gamma^3(\Omega)$ vanishing in the past, obeys the a priori estimate

$$\|(U, V)\|_{L_\gamma^2(\Omega)}^2 + \|(U, \mathcal{H}_1, \mathcal{E}_2, \mathcal{E}_3)|_\omega\|_{L_\gamma^2(\omega)}^2 + \|\varphi\|_{H_\gamma^1(\omega)}^2 \leq \frac{C}{\gamma^4} \|f\|_{H_\gamma^3(\Omega)}^2. \quad (10)$$

In theorem 1 the assertion about the existence of a (small) value ε_* just means that the necessary and sufficient neutral stability condition (9) found in [10] is valid in the *nonrelativistic limit* $\varepsilon \rightarrow 0$.

3. Main steps of the derivation of the energy a priori estimate

We now briefly describe the main steps in the derivation of the a priori estimate (10). We again refer to [11] for full details.

3.1. Estimate of the interior unknowns through the interface function

We first derive the following preparatory estimate of the unknowns U and V through φ (and the source term f):

$$\gamma \|(U, V)\|_{L^2_\gamma(\Omega)}^2 \leq C \left(\gamma \|\varphi\|_{H^1_\gamma(\omega)}^2 + \frac{1}{\gamma} \|f\|_{L^2_\gamma(\Omega)}^2 \right). \tag{11}$$

Here and below C is the general notation for a positive constant. The crucial role in the derivation of estimate (11) is played by the so-called secondary symmetrization of the vacuum Maxwell equations proposed in [7]. Following [7] (see also [4, 8, 9]) and using the divergences for \mathcal{H} and \mathcal{E} from (8), we equivalently rewrite the Maxwell equations as the symmetric system

$$\mathcal{B}_0 \partial_t V + \varepsilon^{-1} \sum_{j=1}^3 \mathcal{B}_j \partial_j V = 0 \quad \text{in } \Omega, \tag{12}$$

where

$$\mathcal{B}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \nu_3 & -\nu_2 \\ 0 & 1 & 0 & -\nu_3 & 0 & \nu_1 \\ 0 & 0 & 1 & \nu_2 & -\nu_1 & 0 \\ 0 & -\nu_3 & \nu_2 & 1 & 0 & 0 \\ \nu_3 & 0 & -\nu_1 & 0 & 1 & 0 \\ -\nu_2 & \nu_1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} -\nu_1 & -\nu_2 & -\nu_3 & 0 & 0 & 0 \\ -\nu_2 & \nu_1 & 0 & 0 & 0 & 1 \\ -\nu_3 & 0 & \nu_1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\nu_1 & -\nu_2 & -\nu_3 \\ 0 & 0 & -1 & -\nu_2 & \nu_1 & 0 \\ 0 & 1 & 0 & -\nu_3 & 0 & \nu_1 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} -\nu_2 & \nu_1 & 0 & 0 & 0 & 1 \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 \\ 0 & \nu_3 & -\nu_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -\nu_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \\ 1 & 0 & 0 & 0 & \nu_3 & -\nu_2 \end{pmatrix}, \quad \mathcal{B}_3 = \begin{pmatrix} -\nu_3 & 0 & \nu_1 & 0 & -1 & 0 \\ 0 & -\nu_3 & \nu_2 & 1 & 0 & 0 \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\nu_3 & 0 & \nu_1 \\ -1 & 0 & 0 & 0 & -\nu_3 & \nu_2 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \end{pmatrix},$$

and ν_i ($i = 1, 2, 3$) are arbitrary constants satisfying the hyperbolicity condition $\mathcal{B}_0 > 0$, i.e., $\nu_1^2 + \nu_2^2 + \nu_3^2 < 1$. The rest arguments towards the proof of (11) are based on the application of the standard energy method (see [11]) to the system of linearized MHD equations contained in (5) and system (12) for the choice $\nu_1 = 0$, $\nu_2 = \varepsilon \hat{\nu}_2$ and $\nu_3 = \varepsilon \hat{\nu}_3$ (clearly, the hyperbolicity condition $\mathcal{B}_0 > 0$ holds for $\varepsilon \ll 1$).

3.2. Hyperbolic evolution equation and estimate for the interface function

Following ideas of [6], we can deduce an evolution equation for the interface function φ . Omitting detailed calculations (see [11]), by using the boundary conditions (6), constraints (8) and the interior equations (5) restricted to the boundary we can finally obtain the evolution equation

$$\mathcal{L}\varphi = F \quad \text{on } \omega, \tag{13}$$

where

$$\mathcal{L} = L^2 - K^2 - \mathcal{K}^2 + \widehat{\mathcal{E}}_1^2 \Delta' - 2\varepsilon \widehat{\mathcal{E}}_1 \partial_t \mathcal{K}^\perp + \varepsilon^2 |\widehat{\mathcal{H}}'|^2 \partial_t^2, \quad F = (\partial_1 q^- - \partial_1 q + f_{1,1})|_\omega,$$

$$\Delta' = \partial_2^2 + \partial_3^2, \quad \mathcal{K}^\perp = \widehat{\mathcal{H}}_3 \partial_2 - \widehat{\mathcal{H}}_2 \partial_3, \quad q^- := \widehat{\mathcal{H}}_2 \mathcal{H}_2 + \widehat{\mathcal{H}}_3 \mathcal{H}_3 - \widehat{\mathcal{E}}_1 \mathcal{E}_1,$$

and $f_{1,1}$ is the first component of the source term f_1 . One can check that in the nonrelativistic setting $\varepsilon \ll 1$ the operator \mathcal{L} is *hyperbolic* if and only if the stability condition (9) holds. As for the wave equation, from (13) we can estimate the H_γ^1 norm of φ through the L_γ^2 norm of F . Using for $f_{1,1}|_\omega$ the trace theorem, we come to the estimate

$$\gamma \|\varphi\|_{H_\gamma^1(\omega)}^2 \leq C \left(\frac{1}{\gamma^2} \|f\|_{H_\gamma^1(\omega)}^2 + \frac{1}{\gamma} \|\partial_1 q|_\omega\|_{L_\gamma^2(\omega)}^2 + \frac{1}{\gamma} \|\partial_1 q^-|_\omega\|_{L_\gamma^2(\omega)}^2 \right). \tag{14}$$

In view of (14), for “closing” estimate (11) it remains to estimate the traces $\partial_1 q|_\omega$ and $\partial_1 q^-|_\omega$ through the source f .

3.3. Elliptic-hyperbolic problem for the total pressures

It is clear that we can derive the Poisson equation for q and the wave equation for q^- (recall that q^- is a linear combination of components the vacuum magnetic and electric fields). We can finally get the following elliptic-hyperbolic problem for $q^+ = q - \check{q}$ and q^- (see [11]):

$$\Delta q^+ = 0, \quad \varepsilon^2 \partial_t^2 q^- - \Delta q^- = 0 \quad \text{in } \Omega, \tag{15}$$

$$\Sigma^- \partial_1 q^+ + \Sigma^+ \partial_1 q^- = g_1, \quad q^+ - q^- = g_2, \quad \text{on } \omega, \tag{16}$$

where the “shift” \check{q} satisfies the elliptic problem $\{\Delta \check{q} = \text{div } f_1 \text{ in } \Omega, \check{q} = \partial_1 \check{q} + f_{1,1} \text{ on } \omega\}$, and

$$\Sigma^- = \mathcal{L} - L^2 + K^2, \quad \Sigma^+ = L^2 - K^2, \quad g_1 = \Sigma^- f_{1,1}|_\omega - \Sigma^- \partial_1 \check{q}|_\omega, \quad g_2 = -\check{q}|_\omega.$$

Note that for the “shift” \check{q} we can obtain the estimate

$$\|\check{q}|_\omega\|_{H_\gamma^m(\omega)}^2 + \|\nabla \check{q}|_\omega\|_{H_\gamma^m(\omega)}^2 \leq C \|f_1\|_{H_\gamma^{m+1}(\omega)}^2, \quad \forall m \in \mathbb{N}. \tag{17}$$

3.4. Symbolic symmetrizer for the elliptic-hyperbolic problem

Applying the Fourier–Laplace transform to (15), (16) (the Fourier transform with respect to x' and the Laplace transform with respect to t , with the dual variables $\eta' = (\eta_2, \eta_3)$ and $\tau = \gamma + i\delta$), we get the following boundary value problem [11]:

$$\frac{d}{dx_1} Y = \mathcal{A}(\tau, \eta') Y \quad \text{for } x_1 > 0, \quad \beta(\tau, \eta') Y = G \quad \text{at } x_1 = 0, \tag{18}$$

where $Y = \left(\frac{d\tilde{q}^+}{dx_1}, \eta \tilde{q}^+, \frac{d\tilde{q}^-}{dx_1}, \sigma \tilde{q}^+ \right)$, $\eta = |\eta'|$, $\sigma = \sqrt{\eta^2 + \varepsilon^2 \tau^2}$, $\mathcal{A} = \text{diag}(\mathcal{A}^+, \mathcal{A}^-)$,

$$\mathcal{A}^+ = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}, \quad \mathcal{A}^- = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \frac{\sigma^-}{\Lambda^2} & 0 & \frac{\sigma^+}{\Lambda^2} & 0 \\ 0 & \frac{\sigma}{\Lambda} & 0 & -\frac{\eta}{\Lambda} \end{pmatrix}, \quad G = \begin{pmatrix} \frac{\tilde{g}_1}{\Lambda^2} \\ \frac{\eta \sigma \tilde{g}_2}{\Lambda} \end{pmatrix},$$

$$\Lambda = \sqrt{|\tau|^2 + \eta^2}, \quad \sigma^+ = (\tau + i(\hat{v}' \cdot \eta'))^2 + (\hat{H}' \cdot \eta)^2, \quad \sigma^- = (\hat{H}' \cdot \eta')^2 - \hat{\mathcal{E}}_1^2 \eta^2 + \mathcal{O}(\varepsilon),$$

and $\tilde{h}(\tau, x_1, \eta')$ denotes the Fourier-Laplace transform of a function $h(t, x)$. Note that σ denotes the principal square root of $\eta^2 + \varepsilon^2 \tau^2$, that is the square root of positive real part for $\Re \tau > 0$, extended as a continuous function up to “boundary frequencies” $(\tau, \eta') \neq (0, 0)$ with $\Re \tau = 0$.

As in the hyperbolic theory [12], for problem (18) we can define the Lopatinski determinant $\Delta(\tau, \eta')$, and one can show that the Lopatinski determinant never vanishes for $\Re \tau > 0$ if the stability condition (9) is satisfied [10, 11]. Moreover, it follows from [10] that the equation

$\Delta(\tau, \eta') = 0$ has only simple roots $(\tau, \eta') \in \Sigma$ with $\Re\tau = 0$, provided that (9) holds, where the hemi-sphere $\Sigma = \{(\tau, \eta') \in \mathbb{C} \times \mathbb{R}^2 : |\tau|^2 + \eta'^2 = 1, \Re\tau \geq 0\}$.

A general idea of the symbolic symmetrizer for our elliptic-hyperbolic problem follows from the same lines of the analogous construction made in [13], which is inspired by the idea of Kreiss' symmetrizer [12] for hyperbolic problems. We first reduce the ODE system in (18) to a diagonal form. Then, multiplying the resulting system by a Hermitian matrix $r(\tau, \eta')$ (*symmetrizer*) and using the boundary conditions and special properties of r , we derive the estimate

$$|Y(\tau, 0, \eta')|^2 \leq \frac{C}{\gamma^2} |G|^2 \Lambda^2 \quad (19)$$

by standard “energy” arguments (see [11]). Our symmetrizer is degenerate in the sense that the uniform Lopatinski condition is violated and we have to treat the boundary points (τ_0, η'_0) of Σ where the Lopatinski condition breaks down (i.e., $\Delta(\tau_0, \eta'_0) = 0$).

Integrating (19) with respect to (δ, η') on \mathbb{R}^3 , applying Parseval's identity and using (17), we finally deduce the estimate [11]

$$\|\nabla q|_\omega\|_{L^2_\gamma(\omega)}^2 + \|\partial_1 q|_\omega\|_{L^2_\gamma(\omega)}^2 \leq \frac{C}{\gamma^2} \|f_1\|_{H^3(\Omega)}^2 \quad (20)$$

Restricting the linearized MHD equations to the boundary, by standard arguments we get the following estimate for the trace of U :

$$\gamma \|U|_\omega\|_{L^2_\gamma(\omega)}^2 \leq \frac{C}{\gamma} \left(\|\nabla q|_\omega\|_{L^2_\gamma(\omega)}^2 + \|f|_\omega\|_{L^2_\gamma(\omega)}^2 \right). \quad (21)$$

From (11), (14), (20), (21) and the last two boundary conditions in (6) we derive the desired a priori estimate (10).

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