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ANALYSIS OF FINITE ELEMENT APPROXIMATIONS OF STOKES EQUATIONS WITH NONSMOOTH DATA[*]

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Abstract. In this paper we analyze the finite element approximation of the Stokes equations with nonsmooth Dirichlet boundary data. To define the discrete solution, we first approximate the boundary datum by a smooth one and then apply a standard finite element method to the regularized problem. We prove almost optimal order error estimates for two regularization procedures in the case of general data in fractional order Sobolev spaces and for the Lagrange interpolation (with appropriate modifications at the discontinuities) for piecewise smooth data. Our results apply in particular to the classic lid-driven cavity problem, improving the error estimates obtained in Cai and Wang [*Math. Comp.*, 78 (2009), pp. 771–787]. Finally, we introduce and analyze an a posteriori error estimator. We prove its reliability and efficiency and show some numerical examples which suggest that optimal order of convergence is obtained by an adaptive procedure based on our estimator.

Key words. Stokes equations, finite elements, nonsmooth data, a posteriori error analysis

AMS subject classifications. 65N30, 65N15

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1. Introduction.

The goal of this paper is to analyze finite element approximations of the Stokes equations with nonsmooth Dirichlet boundary data. For the Laplace equation, the analogous problem has been analyzed in recent years in [5, 6].

Before explaining the problem and goals, let us introduce some notation. For s a real number, $1 \leq p \leq \infty$, and D a domain in $\mathbb{R}^d$ or its boundary or some part of it, we denote by $W^{s,p}(D)$ the Sobolev space on D and by $\|\cdot\|_{s,p,D}$ and $|\cdot|_{s,p,D}$ its norm and seminorm, respectively (see, for example, [2, 1]). As usual, we write $H^s(D) = W^{s,2}(D)$ and omit the p in the norm and seminorm when it is 2. Moreover, bold characters denote vector-valued functions and the corresponding functional spaces. The notation $(\cdot,\cdot)_D$ stands for the scalar product in $L^2(D)$ as well as for the duality pairing between a Sobolev space and its dual; when no confusion may arise, the subscript indicating the domain is dropped.

The subspace of $H^1(D)$ with zero trace on the boundary is denoted as usual by $H^1_0(D)$, while $L^2_0(D)$ is the subspace of $L^2(D)$ of functions with zero mean value.

Let $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, be a Lipschitz domain with boundary $\Gamma = \partial\Omega$, and denote by $\mathbf{n}$ the outward unit vector normal to the boundary.

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We consider the Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= \eta \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\mathbf{f}$, η, and $\mathbf{g}$ are given data. If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\eta \in L^2(\Omega)$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, and the compatibility condition

$$\int_\Gamma \mathbf{g} \cdot \mathbf{n} = \int_\Omega \eta$$

is satisfied, the existence and uniqueness of solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $p \in L^2(\Omega)/\mathbb{R}$ is a well-known result (see, for example, [22, page 31]). Moreover, the following a priori estimate holds true:

$$\|\mathbf{u}\|_{1,\Omega} + \|p\|_{L^2(\Omega)/\mathbb{R}} \le C\big(\|\mathbf{f}\|_{-1,\Omega} + \|\eta\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\big). \tag{1.2}$$

The classic analysis of finite element methods for this problem is based on the variational formulation working with the spaces $\mathbf{H}^1(\Omega)$ for the velocity $\mathbf{u}$ and $L^2(\Omega)$ for the pressure p. If $\mathbf{g} \notin \mathbf{H}^{1/2}(\Gamma)$, then the solution $\mathbf{u} \notin \mathbf{H}^1(\Omega)$, and therefore that theory cannot be applied. This situation arises in many practical situations. A typical example is the so-called lid-driven cavity problem, where Ω is a square and the boundary velocity $\mathbf{g}$ is a piecewise constant vector field which has jumps at two of the vertices and therefore does not belong to $\mathbf{H}^{1/2}(\Gamma)$. However, this example is used in many papers as a model problem to test finite element methods using some regularization of $\mathbf{g}$ (although many times how the boundary condition is treated is not clearly explained). Error estimates for this particular case were obtained in [10, 17]. In [10], the authors work with L^p-based norms and use that $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ for $1 < p < 2$. In [17], a particular regularization of the boundary datum is considered.

More generally, we will consider boundary data $\mathbf{g} \in \mathbf{L}^2(\Gamma)$ using some regularization of $\mathbf{g}$ to define the finite element approximation. In this way the a priori error analysis is separated in two parts: the error due to the regularization and that due to the discretization. We will analyze the first error in general, assuming a given approximation of $\mathbf{g}$ and considering afterward some particular regularizations that can be used in practice.

For piecewise smooth boundary data, as in the case of the lid-driven cavity problem, it is natural to use as an approximation to $\mathbf{g}$ its Lagrange interpolation at continuity points with some appropriate definition at the discontinuities. This is a particular regularization, and so we can apply our theory. We will show that this procedure produces an optimal order approximation for the lid-driven cavity problem improving, in particular, the result obtained in [10], where the order was suboptimal. Let us remark that, since in this example the solution belongs to $H^s(\Omega)$ for all $s < 1$ (see [3, 20]), the best expected order for the error in the L^2-norm using quasi-uniform meshes is $O(h)$.

In the second part of the paper we introduce and analyze an a posteriori error estimator of the residual type. We will prove that the estimator is equivalent to appropriate norms of the error. Numerical examples will show that an adaptive procedure based on our estimator produces optimal order error estimates for the lid-driven cavity problem.

Since (1.1) with $\mathbf{g} = 0$ has been already analyzed, we restrict ourselves to study the case $\mathbf{f} = 0$ and $\eta = 0$ as follows.

Problem 1.1. Given $\mathbf{g} \in \mathbf{L}^2(\Gamma)$ with

$$\int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0, \tag{1.3}$$

find $(\mathbf{u}, p)$ such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= 0 \qquad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \qquad \text{on } \Gamma. \end{aligned} \tag{1.4}$$

The existence and uniqueness of solution is known. Indeed, we have the following.

PROPOSITION 1.1. *Let Ω be a convex polygon or polyhedron, and let $\mathbf{g} \in \mathbf{L}^2(\Gamma)$ satisfy the compatibility condition* (1.3). *Then the Stokes system* (1.4) *has a unique solution $(\mathbf{u}, p) \in \mathbf{L}^2(\Omega) \times H^{-1}(\Omega)/\mathbb{R}$.*

Moreover, there exists a constant C, depending only on Ω, such that

$$\|\mathbf{u}\|_{0,\Omega} + \|p\|_{H^{-1}(\Omega)/\mathbb{R}} \le C \|\mathbf{g}\|_{0,\Gamma}. \tag{1.5}$$

Proof. The existence of solution is proved in [17] in the two-dimensional case and in [15] in the three-dimensional case. Actually, in [17] the a priori estimate is proved only for smooth solutions, but a standard density argument, as the one we use below in Proposition 2.4, can be applied to obtain the general case.

On the other hand, in [15] it is not explicitly stated that $p \in H^{-1}(\Omega)$. However, since $\mathbf{u} \in \mathbf{L}^2(\Omega)$, it follows immediately that $\nabla p \in H^{-2}(\Omega)$, from which one can get $p \in H^{-1}(\Omega)$ and (1.5) (see [17, page 317] and references therein). Let us also mention that the method used in [15] could also be applied in the two-dimensional case, as was done for the case of the Laplace equation in [23]. ∎

The rest of the paper is organized as follows. In section 2 we introduce the finite element approximation, which is based in replacing the boundary datum $\mathbf{g}$ by smooth approximations $\mathbf{g}_h$. Then we develop the a priori error analysis, which is divided in two subsections. In the first one we estimate the error between the exact solution of the original problem and the regularized one in terms of $\mathbf{g} - \mathbf{g}_h$. In the second subsection, considering some appropriate computable approximations, we analyze the error due to the finite element approximation of the regularized problem and prove a theorem which gives a bound for the total error in terms of fractional order norms of $\mathbf{g}$. Then in section 3 we consider the case of piecewise smooth data approximated by a suitable modification of the Lagrange interpolation. Section 4 deals with a posteriori error estimates. We introduce and analyze an error indicator for the regularized problem. Finally, in section 5 we present some numerical examples for the lid-driven cavity problem using two well-known stable methods: the so-called Mini element and the Hood–Taylor one.

2. Finite element approximation and a priori estimates.

In this section we introduce the finite element approximation to Problem 1.1 and prove a priori error estimates. As we have mentioned, in general the solution $\mathbf{u}$ of this problem is not in $\mathbf{H}^1(\Omega)$, and so the standard finite element formulation and analysis cannot be applied. Therefore, to define the numerical approximation, we first approximate the original problem by more regular ones and then solve these problems by standard finite elements. Consequently, our error analysis is divided into two parts that we present in the following subsections. In the first one, we analyze the error due to the regularization, while in the second one the finite element discretization error.

Given $\mathbf{g} \in \mathbf{L}^2(\Gamma)$, let $\mathbf{g}_h \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ be approximations of $\mathbf{g}$ such that

$$\int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = 0 \tag{2.1}$$

and

$$\lim_{h\to 0} \|\mathbf{g} - \mathbf{g}_h\|_{0,\Gamma} = 0. \tag{2.2}$$

Here $h > 0$ is an abstract parameter which afterward will be related to the finite element meshes. The existence of approximations satisfying the compatibility condition (2.1) is not difficult to prove. Anyway, we will construct explicit approximations later using suitable interpolations or projections.

For each h, we consider the following regularized problem: Find $\mathbf{u}(h)$ and $p(h)$ such that

$$\begin{aligned} -\Delta \mathbf{u}(h) + \nabla p(h) &= 0 \quad \text{in } \Omega,\\ \operatorname{div} \mathbf{u}(h) &= 0 \quad \text{in } \Omega,\\ \mathbf{u}(h) &= \mathbf{g}_h \quad \text{on } \Gamma. \end{aligned} \tag{2.3}$$

This problem has a unique solution which, in view of (1.2), satisfies

$$\|\mathbf{u}(h)\|_{1,\Omega} + \|p(h)\|_{L^2(\Omega)/\mathbb{R}} \le C\|\mathbf{g}_h\|_{1/2,\Gamma}. \tag{2.4}$$

The standard variational formulation of this regularized problem reads as follows: Find $\mathbf{u}(h) \in \mathbf{H}^1(\Omega)$ with $\mathbf{u}(h) = \mathbf{g}_h$ on Γ and $p(h) \in L^2_0(\Omega)$ such that

$$\begin{aligned} (\nabla \mathbf{u}(h), \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(h)) &= 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\\ (\operatorname{div} \mathbf{u}(h), q) &= 0 \quad \forall q \in L^2_0(\Omega). \end{aligned} \tag{2.5}$$

2.1. Analysis of the error due to the approximation of the boundary datum.

We will make use of the following well-known result.

PROPOSITION 2.1. *Let Ω be a convex polygonal or polyhedral domain and $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Then the system*

$$\begin{aligned} -\Delta \boldsymbol{\phi} + \nabla q &= \mathbf{f} \quad \textit{in } \Omega,\\ \operatorname{div} \boldsymbol{\phi} &= 0 \quad \textit{in } \Omega,\\ \boldsymbol{\phi} &= 0 \quad \textit{on } \Gamma \end{aligned} \tag{2.6}$$

has a unique solution $(\boldsymbol{\phi}, q) \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \times H^1(\Omega)/\mathbb{R}$, which satisfies the following a priori estimate

$$\|\boldsymbol{\phi}\|_{2,\Omega} + \|q\|_{H^1(\Omega)/\mathbb{R}} \le C\|\mathbf{f}\|_{0,\Omega}. \tag{2.7}$$

Proof. This is proved in [18, Theorem 2] for $d = 2$ and in [14, Theorem 9.20(b)] for $d = 3$. ◻

The result given in the next lemma is known, but we outline the proof in order to make explicit the dependence of the involved constant on s. We will denote by Γ_i, $1 \le i \le N_e$, the edges or faces of Γ.

LEMMA 2.2. *There exists a constant C independent of s such that, for $0 \le s < \frac{1}{2}$,*

$$\|f\|_{-s,\Gamma_i} \le \frac{C}{1-2s}\|f\|_{-s,\Gamma} \qquad \forall f \in L^2(\Gamma). \tag{2.8}$$

Proof. Given $\phi \in H^s(\Gamma_i)$, let $\widetilde{\phi}$ be its extension by 0 to Γ. Tracing constants in the proof of [19, Theorem 11.4 in Chapter 1], we can show that for $0 \le s < \frac{1}{2}$,

$$\|\widetilde{\phi}\|_{s,\Gamma} \le \frac{C}{1-2s}\|\phi\|_{s,\Gamma_i} \qquad \forall \phi \in H^s(\Gamma_i), \tag{2.9}$$

and then we have

$$\|f\|_{-s,\Gamma_i} = \sup_{0\neq\phi\in H^s(\Gamma_i)} \frac{\int_{\Gamma_i} f\phi\, ds}{\|\phi\|_{s,\Gamma_i}} = \sup_{0\neq\phi\in H^s(\Gamma_i)} \frac{\int_{\Gamma_i} f\widetilde{\phi}\, ds}{\|\widetilde{\phi}\|_{s,\Gamma}} \frac{\|\widetilde{\phi}\|_{s,\Gamma}}{\|\phi\|_{s,\Gamma_i}},$$

which yields

$$\|f\|_{-s,\Gamma_i} \le \frac{C}{1-2s} \sup_{0\neq\phi\in H^s(\Gamma_i)} \frac{\int_{\Gamma_i} f\widetilde{\phi}\, ds}{\|\widetilde{\phi}\|_{s,\Gamma}} \le \frac{C}{1-2s} \sup_{0\neq\phi\in H^s(\Gamma)} \frac{\int_{\Gamma_i} f\phi\, ds}{\|\phi\|_{s,\Gamma}},$$

that is, (2.8). ☐

Remark 2.3. The dependence of the constant in terms of s is optimal. Indeed, by duality, it is enough to see that this is true for (2.9). Consider, for example, $\phi \in H^s(0,1)$ and $\tilde{\phi}$ its extension by zero to $(-1,1)$. An elementary argument gives

$$|\tilde{\phi}|^2_{s,(-1,1)} = |\phi|^2_{s,(0,1)} + 2\int_0^1\int_{-1}^0 \frac{|\phi(x)|^2}{(x-y)^{1+2s}}dydx \ge C_1\int_0^1 \frac{|\phi(x)|^2}{x^{2s}}dx$$

with $C_1 > 0$ independent of s, and so (2.9) implies the fractional Hardy inequality

$$\|x^{-s}\phi\|_{0,(0,1)} \le \frac{C}{1-2s}\|\phi\|_{s,(0,1)},$$

and it is known that the constant in this inequality is optimal (see [8]).

In the following proposition we estimate the error between the solutions $(\mathbf{u}, p)$ of (1.4) and $(\mathbf{u}(h), p(h))$ of (2.3) in the $L^2(\Omega)$-norm for the velocity and in the $H^{-1}(\Omega)/\mathbb{R}$-norm for the pressure.

PROPOSITION 2.4. *Let Ω be a convex polygonal or polyhedral domain and $(\mathbf{u}, p)$ and $(\mathbf{u}(h), p(h))$ be the solutions of* (1.4) *and* (2.3)*, respectively. Then there exists a constant C, independent of h, such that, for $0 \le s < \frac{1}{2}$,*

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le \frac{C}{1-2s}\|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma}. \tag{2.10}$$

Proof. First, we will estimate the $L^2(\Omega)$-norm of $\mathbf{v} := \mathbf{u} - \mathbf{u}(h)$ using a duality argument. Since Ω is convex, we know from Proposition 2.1, that the solution of (2.6) with $\mathbf{f} = \mathbf{v}$ satisfies (2.7). Take h_1 another value of the parameter. Then, taking into account (2.5), we have

$$\begin{aligned}(\mathbf{u}(h_1) - \mathbf{u}(h), \mathbf{v})_\Omega &= (\mathbf{u}(h_1) - \mathbf{u}(h), -\Delta\boldsymbol{\phi} + \nabla q)_\Omega \\ &= -\left(\mathbf{g}_{h_1} - \mathbf{g}_h, \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}}\right)_\Gamma + ((\mathbf{g}_{h_1} - \mathbf{g}_h)\cdot\mathbf{n}, q)_\Gamma\,.\end{aligned} \tag{2.11}$$

Summarizing,

$$(\mathbf{u}(h_1)-\mathbf{u}(h),\mathbf{v})_\Omega = -\left(\mathbf{g}_{h_1}-\mathbf{g}_h,\frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}}\right)_\Gamma + ((\mathbf{g}_{h_1}-\mathbf{g}_h)\cdot\mathbf{n},q)_\Gamma\,. \tag{2.12}$$

Since from (1.5) and (2.2) we know that, for $h_1\to 0$,

$$\|\mathbf{u}-\mathbf{u}(h_1)\|_{0,\Omega}\le C\|\mathbf{g}-\mathbf{g}_{h_1}\|_{0,\Gamma}\to 0,$$

taking $h_1\to 0$ in (2.12), we obtain

$$(\mathbf{u}-\mathbf{u}(h),\mathbf{v})_\Omega = -\left(\mathbf{g}-\mathbf{g}_h,\frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}}\right)_\Gamma + ((\mathbf{g}-\mathbf{g}_h)\cdot\mathbf{n},q)_\Gamma\,. \tag{2.13}$$

We estimate the right-hand side in terms of $\|\mathbf{g}-\mathbf{g}_h\|_{H^{-s}(\Gamma)}$. For the second term we note that while $q\in H^{\frac12}(\Gamma)$, due to the discontinuities of $\mathbf{n}$, we cannot enssure that $q\mathbf{n}\in\mathbf{H}^{\frac12}(\Gamma)$. Therefore, with $0\le s<\frac12$, we have

$$\begin{aligned}(\mathbf{g}-\mathbf{g}_h,q\mathbf{n})_\Gamma &= \sum_{i=1}^{N_e}(\mathbf{g}-\mathbf{g}_h,q\mathbf{n})_{\Gamma_i}\le\sum_{i=1}^{N_e}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma_i}\,\|q\mathbf{n}\|_{s,\Gamma_i}\\ &\le C\left(\sum_{i=1}^{N_e}\|\mathbf{g}-\mathbf{g}_h\|^2_{-s,\Gamma_i}\right)^{\frac12}\|q\|_{1,\Omega}\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\|q\|_{1,\Omega},\end{aligned}$$

where, in the last inequality, we have used (2.8). With a similar argument, we obtain for the first term in the right-hand side of (2.13) the estimate

$$\left(\mathbf{g}-\mathbf{g}_h,\frac{\partial\boldsymbol{\phi}}{\partial\mathbf{n}}\right)_\Gamma\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\|\boldsymbol{\phi}\|_{2,\Omega}.$$

Hence, from (2.13) and the a priori estimate (2.7) we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}(h)\|^2_{0,\Omega}&=(\mathbf{u}-\mathbf{u}(h),\mathbf{v})\\ &\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\left(\|\boldsymbol{\phi}\|_{2,\Omega}+\|q\|_{1,\Omega}\right)\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\|\mathbf{u}-\mathbf{u}(h)\|_{0,\Omega},\end{aligned}$$

and so

$$\|\mathbf{u}-\mathbf{u}(h)\|_{0,\Omega}\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}. \tag{2.14}$$

Now for the error in the pressure we have

$$\begin{aligned}\|p-p(h)\|_{H^{-1}(\Omega)/\mathbb{R}}&\le C\|\nabla(p-p(h))\|_{-2,\Omega}=C\|\Delta(\mathbf{u}-\mathbf{u}(h))\|_{-2,\Omega}\\ &\le C\|\mathbf{u}-\mathbf{u}(h)\|_{0,\Omega}\le\frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma},\end{aligned}$$

which concludes the proof. ∎

Remark 2.5. The estimate of the previous proposition can be compared with [5, Lemma 2.12], where the corresponding result for the approximation of a Poisson equation with a nonsmooth Dirichlet boundary datum is considered. A constant independent of s is obtained in [5], while our estimate contains a factor $C/(1-2s)$. Indeed, we could bound the first term in the right-hand side of (2.13) exactly as in [5]. However, the slightly worse factor $C/(1-2s)$ arises due to the presence of the second term, which involves the pressure q.

2.2. Analysis of the finite element approximation error. Let $\{\mathcal{T}_h\}$, $h > 0$, be a family of meshes of Ω, which is assumed to be shape-regular, with h being the maximum diameter of the elements in $\mathcal{T}_h$. Each mesh $\mathcal{T}_h$ induces a mesh $\mathcal{T}_{\Gamma,h}$ along the boundary fitted with the edges/faces Γ_i, $i = 1, \ldots, N_e$.

We consider a family of pairs $\mathbf{V}_h = \mathbf{W}_h \cap \mathbf{H}_0^1(\Omega)$ and $Q_h \subset L_0^2(\Omega)$ of finite element spaces, with $\mathbf{W}_h \subset \mathbf{H}^1(\Omega)$, which are uniformly stable for the Stokes problem; that is, the following inf-sup condition is satisfied for some $\beta > 0$ independent of h (see, e.g., [7, Chapter 8]):

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)_{0,\Omega}}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega} \qquad \forall q_h \in Q_h, \quad \forall h > 0.$$

Moreover, we assume that

$$[\mathcal{P}_1(\mathcal{T}_h) \cap H^1(\Omega)]^d \subseteq \mathbf{W}_h, \tag{2.15}$$

where $\mathcal{P}_k(\mathcal{T}_h)$ stands for the vector space of piecewise polynomials of degree not grater than k on the mesh $\mathcal{T}_h$. In the following we shall use interpolant operators onto the discrete spaces $\mathbf{W}_h$ and Q_h. For functions $\boldsymbol{\phi} \in \mathbf{H}^2(\Omega)$, we define $\boldsymbol{\phi}^I \in \mathbf{W}_h$ as the continuous piecewise linear Lagrange interpolation of $\boldsymbol{\phi}$. The following error estimates are well known:

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}^I\|_{m,T} \leq C h^{2-m} |\boldsymbol{\phi}|_{2,T}, \quad m = 0, 1, \quad \text{for all } \boldsymbol{\phi} \in \mathbf{H}^2(\Omega). \tag{2.16}$$

Let P_0 be the L^2-projection of $L_0^2(\Omega)$ onto Q_h, we assume that there exists a positive constant independent of h such that

$$\|q - P_0 q\|_{0,T} \leq C h |q|_{1,\Omega} \quad \text{for all } q \in H^1(\Omega).$$

From now on, we assume that $\mathbf{g}_h$ is the trace of a function $E\mathbf{g}_h \in \mathbf{W}_h$; for example, it is enough to assume that $\mathbf{g}_h$ is continuous and piecewise linear. Moreover, it is known that $E\mathbf{g}_h$ can be chosen such that $\|E\mathbf{g}_h\|_{1,\Omega} \leq C \|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}$.

We consider the finite element approximation of (2.5) that reads as follows: Find $\mathbf{u}_h \in \mathbf{W}_h$ and $p_h \in Q_h$ such that $\mathbf{u}_h = \mathbf{g}_h$ on Γ and

$$\begin{aligned} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, p_h) &= 0 \qquad \forall\ \mathbf{v}_h \in \mathbf{V}_h, \\ (\operatorname{div} \mathbf{u}_h, q_h) &= 0 \qquad \forall\ q_h \in Q_h. \end{aligned} \tag{2.17}$$

By taking $\mathbf{v}_h = \mathbf{u}_h - E\mathbf{g}_h$ and $q_h = p_h$ in (2.17) and using the inf-sup condition, we obtain existence and uniqueness and the estimate

$$\|\mathbf{u}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} \leq C \|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}. \tag{2.18}$$

In the following proposition we estimate the finite element error in norms corresponding with the ones used in Proposition 2.4.

PROPOSITION 2.6. *Let* $(\mathbf{u}(h), p(h)) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ *with* $\mathbf{u}(h) = \mathbf{g}_h$ *on* Γ *and* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times Q_h$ *with* $\mathbf{u}_h = E\mathbf{g}_h + \mathbf{u}_{0h}$ *be the solutions of* (2.5) *and* (2.17), *respectively. Then we have*

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq C h \|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}. \tag{2.19}$$

Proof. Subtracting (2.17) from (2.5), we get the following error equations:

$$\begin{aligned} (\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla\mathbf{v}_h)-(\operatorname{div}\mathbf{v}_h,p(h)-p_h)&=0 \qquad \forall\mathbf{v}_h\in V_h,\\ (\operatorname{div}(\mathbf{u}(h)-\mathbf{u}_h),q_h)&=0 \qquad \forall q_h\in Q_h. \end{aligned} \tag{2.20}$$

In order to use a duality argument, we introduce the solution $(\boldsymbol{\phi},q)$ satisfying (2.6) with $\mathbf{f}=\mathbf{u}(h)-\mathbf{u}_h$. From Proposition 2.1, $\boldsymbol{\phi}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega)$ and $q\in H^1(\Omega)\cap L^2_0(\Omega)$ with the a priori estimate (2.7). We have

$$\|\mathbf{u}(h)-\mathbf{u}_h\|^2_{0,\Omega}=(\mathbf{u}(h)-\mathbf{u}_h,-\Delta\boldsymbol{\phi}+\nabla q)\,.$$

Then integration by parts, the error equations (2.20), the approximation properties (2.16) and (2.18), the fact that $\mathbf{u}(h)=\mathbf{u}_h=\mathbf{g}_h$ on the boundary, and the a priori estimates (2.4) and (2.18) give

$$\begin{aligned} \|\mathbf{u}(h)-\mathbf{u}_h\|^2_{0,\Omega}&=(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla\boldsymbol{\phi})-(\operatorname{div}(\mathbf{u}(h)-\mathbf{u}_h),q)-(\operatorname{div}\boldsymbol{\phi},p(h)-p_h)\\ &=\Big(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}^I)\Big)\\ &\qquad-\Big(\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}^I),p(h)-p_h\Big)-(\operatorname{div}(\mathbf{u}(h)-\mathbf{u}_h),q-P_0q)\\ &\le Ch\left(|\boldsymbol{\phi}|_{2,\Omega}+|q|_{1,\Omega}\right)\|\nabla(\mathbf{u}(h)-\mathbf{u}_h)\|_{0,\Omega}+Ch|\boldsymbol{\phi}|_{2,\Omega}\|p(h)-p_h\|_{0,\Omega}\\ &\le Ch\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}\left(\|\nabla\mathbf{u}(h)\|_{0,\Omega}+\|\nabla\mathbf{u}_h\|_{0,\Omega}+\|p(h)\|_{0,\Omega}+\|p_h\|_{0,\Omega}\right)\\ &\le Ch\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{g}_h\|_{\frac12,\Gamma}, \end{aligned}$$

which provides the desired estimate for the velocity field

$$\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}\le Ch\|\mathbf{g}_h\|_{\frac12,\Gamma}. \tag{2.21}$$

Let us now estimate $p(h)-p_h$. Since $p(h)-p_h\in L^2_0(\Omega)$, we have

$$\|p(h)-p_h\|_{H^{-1}(\Omega)/\mathbb{R}}=\sup_{\substack{q\in H^1_0(\Omega)\\ \int_\Omega q=0}}\frac{(p(h)-p_h,q)}{\|q\|_{1,\Omega}}. \tag{2.22}$$

Given $q\in H^1_0(\Omega)$ with $\int_\Omega q=0$, we know that there exists $\boldsymbol{\psi}\in\mathbf{H}^2_0(\Omega)$ such that [21, Theorem 1]

$$\operatorname{div}\boldsymbol{\psi}=q\quad\text{in }\Omega,\qquad \|\boldsymbol{\psi}\|_{2,\Omega}\le C\|q\|_{1,\Omega}. \tag{2.23}$$

Then using the interpolant $\boldsymbol{\psi}^I$ as in (2.16) and the error equation (2.20), we have

$$\begin{aligned} (p(h)-p_h,q)&=(p(h)-p_h,\operatorname{div}\boldsymbol{\psi})\\ &=\Big(p(h)-p_h,\operatorname{div}(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)+\left(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla\psi^I\right)\\ &=\Big(p(h)-p_h,\operatorname{div}(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)-\Big(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)\\ &\qquad+(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla\boldsymbol{\psi})\,. \end{aligned}$$

Integrating by parts the last term, we have

$$\begin{aligned} (p(h)-p_h,q)=\Big(p(h)-p_h,\operatorname{div}(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)&-\Big(\nabla(\mathbf{u}(h)-\mathbf{u}_h),\nabla(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)\\ &-(\mathbf{u}(h)-\mathbf{u}_h,\Delta\boldsymbol{\psi})\,. \end{aligned} \tag{2.24}$$

Then we obtain

$$\begin{aligned}(p(h)-p_h,q) \le C\big[h\left(\|p(h)\|_{0,\Omega}+\|p_h\|_{0,\Omega}+\|\nabla\mathbf{u}(h)\|_{0,\Omega}+\|\nabla\mathbf{u}_h\|_{0,\Omega}\right)\\ +\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}\big]\|q\|_{1,\Omega}.\end{aligned}$$

Substituting this inequality into (2.22) implies

$$\begin{aligned}\|p(h)-p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le Ch\left(\|p(h)\|_{0,\Omega}+\|p_h\|_{0,\Omega}+\|\nabla\mathbf{u}(h)\|_{0,\Omega}+\|\nabla\mathbf{u}_h\|_{0,\Omega}\right)\\ +\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}.\end{aligned}$$

Then the stability estimates (2.4) and (2.18), joint with (2.21), give

$$\|p(h)-p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le Ch\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}, \tag{2.25}$$

which together with (2.21) concludes the proof. ∎

The regularization of the boundary datum $\mathbf{g}$ could be obtained by finite element discretization. By construction of the mesh $\mathcal{T}_h$, the boundary Γ is subdivided into boundary elements fitted with the edges/faces Γ_i, $i=1,\dots,N_e$, and $\mathcal{T}_{\Gamma,h}$ denotes the mesh along the boundary. Let h_Γ be the maximum diameter of the elements in $\mathcal{T}_{\Gamma,h}$, and define the discrete space on the boundary as

$$\mathbf{G}_h = \{\mathbf{z}_h \in \mathbf{C}^0(\Gamma) : \mathbf{z}_h \in \mathcal{P}^1(E)\ \forall E \in \mathcal{T}_{\Gamma,h}\}. \tag{2.26}$$

Then the function $\mathbf{g}_h$ can be obtained either as the $\mathbf{L}^2(\Gamma)$-projection of $\mathbf{g}$ onto the space $\mathbf{G}_h$ or using the Carstensen interpolant $\mathbf{C}_h\mathbf{g}$ of $\mathbf{g}$ (see [11]) or by a suitable Lagrange interpolation; see section 3. It is straightforward to check that both the L^2-projection and the Carstensen interpolant provide approximations $\mathbf{g}_h$ of $\mathbf{g}$ which satisfy the compatibility condition (2.1), while this is not always the case for the standard Lagrange interpolation; Moreover, we can show the following regularization error estimates for $\mathbf{g}_h$ (see [4, Lemmas 2.13 and A.2]).

PROPOSITION 2.7. *Let $\mathbf{g}_h \in \mathbf{G}_h$ be either the piecewise linear Carstensen interpolant of $\mathbf{g}$ or the $L^2(\Gamma)$-projection on the continuous piecewise linear functions. Then we have*

$$\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma} \le Ch^{s+t}\|\mathbf{g}\|_{t,\Gamma} \qquad \forall \mathbf{g} \in H^t(\Gamma) s, t \in [0,1]. \tag{2.27}$$

We also have

$$\|\mathbf{g}_h\|_{t,\Gamma} \le C\|\mathbf{g}\|_{t,\Gamma} \qquad \forall \mathbf{g} \in H^t(\Gamma), t \in [0,1], \tag{2.28}$$

where, for $t>0$, it is assumed that the mesh $\mathcal{T}_{\Gamma,h}$ is quasi-uniform.

Proof. Inequality (2.27) is proved in [4, Lemma A.2] for $\mathbf{g}_h$ being the Carstensen interpolant and in [4, Remark A.3] when $\mathbf{g}_h$ is the L^2-projection on piecewise linear functions on Γ.

Inequality (2.28) for $t=0$ is also proved in [4]. For $t>0$ we can proceed as follows. Let $\Pi_h : \mathbf{H}^1(\Gamma) \to \mathbf{G}_h$ be Clément's operator. Then, if the mesh is quasi-uniform, we can use an inverse inequality and obtain

$$\begin{aligned}\|\nabla\mathbf{g}_h\|_{0,\Gamma} &\le \|\nabla\Pi_h\mathbf{g}\|_{0,\Gamma}+\|\nabla(\mathbf{g}_h-\Pi_h\mathbf{g})\|_{0,\Gamma} \le c\|\mathbf{g}\|_{1,\Gamma}+\frac{C_I}{h}\|\mathbf{g}_h-\Pi_h\mathbf{g}\|_{0,\Gamma}\\ &\le c\|\mathbf{g}\|_{1,\Gamma}+\frac{C_I}{h}\|\mathbf{g}_h-\mathbf{g}\|_{0,\Gamma}+\frac{C_I}{h}\|\mathbf{g}-\Pi_h\mathbf{g}\|_{0,\Gamma} \le C\|\mathbf{g}\|_{1,\Gamma}.\end{aligned}$$

Then, by interpolation of Sobolev spaces (see, e.g., [9, Proposition 14.1.5]), we get (2.28). ∎

The bounds (2.10) and (2.19) together with the inequalities in Proposition 2.7 give the following result.

Theorem 2.8. *Let Ω be a convex polygonal or polyhedral domain. If the family of meshes $\mathcal{T}_{\Gamma,h}$ is quasi-uniform and $\mathbf{g}_h$ is given as in Proposition* 2.7*, then, for* $0 \le t < \frac{1}{2}$ *and* $\mathbf{g} \in H^t(\Gamma)$*, we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le C\,|\log h| h^{\frac{1}{2}+t} \|\mathbf{g}\|_{t,\Gamma}. \tag{2.29}$$

Proof. From Proposition 2.4 and (2.27), we have, for $0 \le s < \frac{1}{2}$,

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le \frac{C}{1-2s}\, h^{s+t} \|\mathbf{g}\|_{t,\Gamma}.$$

Taking $s = 1/2 + 1/\log h < 1/2$ we obtain

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le C\, h^{\frac{1}{2}+t}\,|\log h| \|\mathbf{g}\|_{t,\Gamma}. \tag{2.30}$$

On the other hand, from Proposition 2.6,

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le C\, h \|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}. \tag{2.31}$$

Now, using an inverse inequality ([13, Theorem 4.1]) and (2.28), we obtain

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le C h^{t-\frac{1}{2}} \|\mathbf{g}_h\|_{t,\Gamma} \le C h^{t-\frac{1}{2}} \|\mathbf{g}\|_{t,\Gamma},$$

which, substituted in (2.31), gives

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)} \le C\, h^{\frac{1}{2}+t} \|\mathbf{g}\|_{t,\Gamma}. \tag{2.32}$$

Combining (2.30) and (2.32) we arrive at the desired estimate (2.29). ∎

3. A priori error estimates for piecewise smooth boundary data.

In this section we analyze the approximation of piecewise smooth data; in particular, our results can be applied to the lid-driven cavity problem. In practice, the most usual way to deal with the nonhomogeneous Dirichlet boundary condition is to use the Lagrange interpolation or a simple modification of it to treat discontinuities and to obtain a compatible approximation $\mathbf{g}_h$.

We shall use the following notation for the norm of $\mathbf{g}$:

$$|\!|\!|\mathbf{g}|\!|\!|_{k,\Gamma} = \left(\sum_{i=1}^{N_e} \|\mathbf{g}\|^2_{k,\Gamma_i} \right)^{\frac{1}{2}}. \tag{3.1}$$

In the following, we consider separately the case $d = 2$ or $d = 3$.

3.1. Two-dimensional case.

Let $\mathbf{g} = (g_1, g_2) : \Gamma \to \mathbb{R}^2$ be such that $\mathbf{g}|_{\Gamma_i} \in \mathbf{H}^1(\Gamma_i)$ for $i = 1, \dots, N_e$, where Γ_i are the boundary segments $\Gamma_i = [A_i, A_{i+1}]$ (with $A_{N_e+1} = A_1$) and A_i, $i = 1, \dots, N_e$ are the boundary vertices. We observe that $\mathbf{g} \in \mathbf{H}^s(\Gamma)$ with $0 \le s < \frac{1}{2}$. Indeed, let us set $\mathbf{g}_i = \mathbf{g}|_{\Gamma_i}$. Since, for $0 \le s < \frac{1}{2}$, $H^1(\Gamma_i) \subset H^s(\Gamma_i)$, we have that the extension by zero $\tilde{\mathbf{g}}_i$ of $\mathbf{g}_i \in \mathbf{H}^s(\Gamma_i)$ belongs to $\mathbf{H}^s(\Gamma)$ (see [19, Theorem 11.4 in Chapter 1]) and, thanks to (2.9),

$$\|\tilde{\mathbf{g}}_i\|_{s,\Gamma} \le \frac{C}{1-2s} \|\mathbf{g}_i\|_{s,\Gamma_i}.$$

Then $\mathbf{g} = \sum_{i=1}^{N_e} \tilde{\mathbf{g}}_i$ belongs to $\mathbf{H}^s(\Gamma)$, with

$$\|\mathbf{g}\|_{s,\Gamma} \leq \frac{C}{1-2s} \|\!|\mathbf{g}|\!\|_{1,\Gamma}. \tag{3.2}$$

We denote by B_i, $1 \leq i \leq M$, the boundary nodes of the mesh numbered consecutively and set $B_{M+1} = B_1$ (of course these nodes depend on h, but we omit this in the notation for simplicity) and $h_i = |B_{i+1} - B_i|$. In principle, we would define $\mathbf{g}_h$ as the continuous piecewise linear vector field on Γ such that $\mathbf{g}_h(B_j) = \mathbf{g}(B_j)$ if $\mathbf{g}$ is continuous in B_j and $\mathbf{g}_h(B_j) = \mathbf{g}(B_j^-)$ or $\mathbf{g}_h(B_j) = \mathbf{g}(B_j^+)$ (or some average of these two values) if not. Notice that $|\mathbf{g}_h(B_j)| \leq \|\mathbf{g}\|_{L^\infty(\Gamma)}$. However, in general, this definition does not satisfy the compatibility condition (2.1). We now show how to enforce compatibility by a simple modification.

LEMMA 3.1. *Given* $\mathbf{g} \in \mathbf{L}^2(\Gamma)$ *such that* $\mathbf{g}|_{\Gamma_i} \in \mathbf{H}^1(\Gamma_i)$ *for* $i = 1, \ldots, N_e$, *there exists a piecewise linear function* $\mathbf{g}_h$ *which is a modified Lagrange interpolant of* $\mathbf{g}$ *satisfying the compatibility condition* (2.1). *Moreover,*

$$\|\mathbf{g}_h\|_{L^\infty(\Gamma)} \leq C \|\!|\mathbf{g}|\!\|_{1,\Gamma}. \tag{3.3}$$

Proof. We modify the definition of $\mathbf{g}_h$ given above in some node B_k. For simplicity, let us choose this node different from all the vertices and their neighbors and such that h_k is comparable to h. For each j, let Γ_{B_j} be the union of the two segments of $\mathcal{T}_{\Gamma,h}$ containing B_j. Moreover, we set $\Gamma_V = \cup_{i=1}^{N_e} \Gamma_{A_i}$.

We want to define $\mathbf{g}_h(B_k)$ in such a way that

$$0 = \int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = \int_{\Gamma \setminus (\Gamma_V \cup \Gamma_{B_k})} \mathbf{g}_h \cdot \mathbf{n} + \int_{\Gamma_V} \mathbf{g}_h \cdot \mathbf{n} + \int_{\Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n}$$

or, equivalently,

$$\int_{\Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n} = -\int_{\Gamma \setminus (\Gamma_V \cup \Gamma_{B_k})} \mathbf{g}_h \cdot \mathbf{n} - \int_{\Gamma_V} \mathbf{g}_h \cdot \mathbf{n} = -\int_{\Gamma \setminus \Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n}.$$

But

$$\begin{aligned} \int_{\Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n} &= \frac{1}{2} h_{k-1} \left[\mathbf{g}(B_{k-1}) + \mathbf{g}_h(B_k)\right] \cdot \mathbf{n} + \frac{1}{2} h_k \left[\mathbf{g}_h(B_k) + \mathbf{g}(B_{k+1})\right] \cdot \mathbf{n} \\ &= \frac{1}{2} \left[h_{k-1} \mathbf{g}(B_{k-1}) + h_k \mathbf{g}(B_{k+1})\right] \cdot \mathbf{n} + \frac{1}{2}(h_{k-1} + h_k) \mathbf{g}_h(B_k) \cdot \mathbf{n}. \end{aligned}$$

We introduce

$$L_1(\mathbf{g}) = -\int_{\Gamma \setminus \Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n} - \frac{1}{2} \left[h_{k-1} \mathbf{g}(B_{k-1}) + h_k \mathbf{g}(B_{k+1})\right] \cdot \mathbf{n}.$$

Notice that the integral $\int_{\Gamma \setminus \Gamma_{B_k}} \mathbf{g}_h \cdot \mathbf{n}$ appears in the definition of L_1. Actually, $\mathbf{g}_h$ has been already defined in all the boundary nodes except for B_k using the values of $\mathbf{g}$. Hence, the notation $L_1(\mathbf{g})$ is consistent.

We define the value $\mathbf{g}_h(B_k)$ such that

$$\begin{aligned} &\frac{1}{2}(h_{k-1} + h_k) \mathbf{g}_h(B_k) \cdot \mathbf{n} = L_1(\mathbf{g}), \\ &\frac{1}{2}(h_{k-1} + h_k) \mathbf{g}_h(B_k) \cdot \mathbf{t} = 0, \end{aligned} \tag{3.4}$$

where $\mathbf{t}$ denotes the unit tangential vector on Γ. Taking into account that $\mathbf{g}$ satisfies the compatibility condition, we have

$$\begin{aligned} L_1(\mathbf{g}) = \int_{\Gamma\setminus(\Gamma_V\cup\Gamma_{B_k})} (\mathbf{g}-\mathbf{g}_h)\cdot\mathbf{n} + \int_{\Gamma_V} (\mathbf{g}-\mathbf{g}_h)\cdot\mathbf{n} + \int_{\Gamma_{B_k}} \mathbf{g}\cdot\mathbf{n} \\ - \frac{1}{2}\left[h_{k-1}\mathbf{g}(B_{k-1}) + h_k\mathbf{g}(B_{k+1})\right]\cdot\mathbf{n}. \end{aligned}$$

The first term can be bounded using standard results for interpolation errors on $\Gamma\setminus(\Gamma_V\cup\Gamma_{B_k})$. To bound the other three terms, we use that $\|\mathbf{g}\|_{L^\infty(\Gamma)} \le |\!|\!|\mathbf{g}|\!|\!|_{1,\Gamma}$ and that the length of the integration set is less than h. Then we obtain

$$|L_1(\mathbf{g})| \le Ch|\!|\!|\mathbf{g}|\!|\!|_{1,\Gamma}. \tag{3.5}$$

It is easy to check that the matrix of the system (3.4) (for $\mathbf{g}_h(B_k)$) is nonsingular and that its inverse has norm of order h^{-1} so that we have

$$|\mathbf{g}_h(B_k)| \le C|\!|\!|\mathbf{g}|\!|\!|_{1,\Gamma}, \tag{3.6}$$

where $|\mathbf{g}_h(B_k)|$ stands for the Euclidean norm of the vector $\mathbf{g}_h(B_k)$. Therefore, $\mathbf{g}_h$ is defined on the entire Γ and satisfies the compatibility condition and the bound (3.3). ☐

In the proof of the next proposition, we will use the embedding inequality for $0 \le s < \frac{1}{2}$,

$$\|\phi\|_{L^q(\Gamma)} \le C_s\|\phi\|_{s,\Gamma} \qquad \forall\phi \in H^s(\Gamma) \tag{3.7}$$

with $q = \frac{2}{1-2s}$ and

$$C_s \sim \sqrt{\frac{1}{1-2s}} \qquad \text{when } s \to \left(\frac{1}{2}\right)^-. \tag{3.8}$$

Inequality (3.7) is proved in [12, Theorem 1.1] in $\mathbb{R}$. The analogous result follows for an interval and therefore for Γ by using an extension theorem.

Proposition 3.2. *For all* $0 \le s < \frac{1}{2}$ *we have*

$$\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma} \le \frac{C}{\sqrt{1-2s}} h^{\frac{1}{2}+s} |\!|\!|\mathbf{g}|\!|\!|_{1,\Gamma}.$$

Proof. Let us set $p = \frac{2}{1+2s}$ and $q = \frac{2}{1-2s}$ its dual exponent. Using the Hölder inequality and the embedding inequality (3.7), we have

$$\begin{aligned} \|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma} &= \sup_{\phi:\|\phi\|_{s,\Gamma}=1} \int_\Gamma (\mathbf{g}-\mathbf{g}_h)\,\phi \\ &\le \sup_{\phi:|\phi\|_{s,\Gamma}=1} \|\mathbf{g}-\mathbf{g}_h\|_{L^p(\Gamma)}\|\phi\|_{L^q(\Gamma)} \le C_s\|\mathbf{g}-\mathbf{g}_h\|_{L^p(\Gamma)}. \end{aligned} \tag{3.9}$$

Since $\mathbf{g}_h$ coincides with the Lagrange interpolation of $\mathbf{g}$ on $\Gamma\setminus(\Gamma_V\cup\Gamma_{B_k})$, $|\Gamma_V\cup\Gamma_{B_k}| \le Ch$, and $1 < p < 2$, we have

$$\begin{aligned} \|\mathbf{g}-\mathbf{g}_h\|^p_{L^p(\Gamma)} &= \|\mathbf{g}-\mathbf{g}_h\|^p_{L^p(\Gamma\setminus(\Gamma_V\cup\Gamma_{B_k}))} + \|\mathbf{g}-\mathbf{g}_h\|^p_{L^p(\Gamma_V\cup\Gamma_{B_k})} \\ &\le Ch^p\|\mathbf{g}\|^p_{W^{1,p}(\Gamma\setminus(\Gamma_V\cup\Gamma_{B_k}))} + Ch\|\mathbf{g}\|^p_{L^\infty(\Gamma_V\cup\Gamma_{B_k})} \le Ch\sum_{i=1}^{N_e}\|\mathbf{g}\|^p_{H^1(\Gamma_i)}, \end{aligned}$$

which, together with (3.9), yields

$$\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma} \le C\, C_s h^{\frac{1}{p}} |||\mathbf{g}|||_{1,\Gamma}\,.$$

Using (3.8) and recalling that $p = \frac{2}{1+2s}$, we conclude the proof. ∎

In the next proposition we obtain a quasi-uniform-in-h estimate of the $H^{\frac{1}{2}}$-norm of $\mathbf{g}_h$.

PROPOSITION 3.3. *If the family of meshes $\mathcal{T}_{\Gamma,h}$ is quasi-uniform, we have*

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le C|\log h|\, |||\mathbf{g}|||_{1,\Gamma}.$$

Proof. Let $\tilde{\mathbf{g}}_h$ be the Carstensen approximation. Then, for $0 < s < 1/2$, inverse estimates imply

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le \|\mathbf{g}_h - \tilde{\mathbf{g}}_h\|_{\frac{1}{2},\Gamma} + \|\tilde{\mathbf{g}}_h\|_{\frac{1}{2},\Gamma} \le C\left(h^{-\frac{1}{2}}\|\mathbf{g}_h - \tilde{\mathbf{g}}_h\|_{0,\Gamma} + h^{s-\frac{1}{2}}\|\tilde{\mathbf{g}}_h\|_{s,\Gamma}\right),$$

and so, by (2.28) and the fact that $\mathbf{g} \in \mathbf{H}^s(\Gamma)$,

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le Ch^{-\frac{1}{2}}\left(\|\mathbf{g}_h - \mathbf{g}\|_{0,\Gamma} + \|\mathbf{g} - \tilde{\mathbf{g}}_h\|_{0,\Gamma} + h^s\|\mathbf{g}\|_{s,\Gamma}\right).$$

Using Proposition 3.2, (2.27), and (3.2), we obtain

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le C\left(|||\mathbf{g}|||_{1,\Gamma} + h^{s-\frac{1}{2}}\|\mathbf{g}\|_{s,\Gamma}\right) \le C\frac{h^{s-\frac{1}{2}}}{1-2s}|||\mathbf{g}|||_{1,\Gamma}.$$

Choosing s such that $1 - 2s = 1/|\log h|$, we conclude the proof. ∎

Remark 3.4. The quasi-uniformity assumption in the previous proposition is not essential. We have given the proof under this hypothesis for the sake of simplicity. However, for a general family of meshes, an elementary but rather technical computation using the definition of the fractional norm leads to the estimate

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le C|\log(h_{min})|\, |||\mathbf{g}|||_{1,\Gamma},$$

where h_{min} denotes the minimum meshsize of $\mathcal{T}_{\Gamma,h}$.

3.2. Three-dimensional case. We assume that the boundary Γ is composed by N_e polygonal faces Γ_i and that $\mathbf{g}|_{\Gamma_i} \in \mathbf{H}^2(\Gamma_i)$. Therefore, $\mathbf{g} \in \mathbf{L}^\infty(\Gamma)$ and $\|\mathbf{g}\|_{L^\infty(\Gamma)} \le C|||\mathbf{g}|||_{2,\Gamma}$. Moreover, we can show, as in the two-dimensional case, that $\mathbf{g} \in \mathbf{H}^s(\Gamma)$ for $0 \le s < \frac{1}{2}$ and that

$$\|\mathbf{g}\|_{s,\Gamma} \le \frac{C}{1-2s}|||\mathbf{g}|||_{2,\Gamma}. \tag{3.10}$$

Assume that we have a triangular mesh $\mathcal{T}_{\Gamma,h}$ which is quasi-uniform. A construction, similar to the one proposed here, can be made also in the case of quadrilateral quasi-uniform meshes.

As for the two-dimensional case, let $\{B_j\}$ be the set of nodes of $\mathcal{T}_{\Gamma,h}$, and define

$$E = \bigcup \{\bar{e} : e \text{ is an edge of } \Omega\}.$$

For each node $B_j \in E$, let us choose T_{B_j} an element of $\mathcal{T}_{\Gamma,h}$ such that $B_j \in \overline{T}_{B_j}$. Finally, let e_0 be a polygon contained in a face Γ_k of Ω, with $|e_0| = O(1)$, made up of sides of triangles in $\mathcal{T}_{\Gamma,h}$ and such that triangles with a vertex on e_0 do not have vertices on E. It is clear that we can take it. We denote by $\mathbf{n}_{e_0}$ the normal vector to the face Γ_k containing e_0.

LEMMA 3.5. *Given* $\mathbf{g} \in \mathbf{L}^2(\Gamma)$ *such that* $\mathbf{g}|_{\Gamma_i} \in \mathbf{H}^2(\Gamma_i)$, *where* Γ_i *for* $i = 1, \dots, N_e$, *are the faces of* Γ, *there exists a piecewise linear function* $\mathbf{g}_h \in \mathbf{G}_h$ *which is a modified Lagrange interpolant of* $\mathbf{g}$ *satisfying the compatibility condition* (2.1) *and*

$$\|\mathbf{g}_h\|_{L^\infty(\Gamma)} \le C \|\!|\mathbf{g}|\!\|_{2,\Gamma}. \tag{3.11}$$

Proof. We define the Lagrange interpolation $\mathbf{g}_h \in \mathbf{G}_h$ of $\mathbf{g}$ as the continuous piecewise linear function on $\mathcal{T}_{\Gamma,h}$ such that, for each node B_j in $\mathcal{T}_{\Gamma,h}$, we have

$$\mathbf{g}_h(B_j) = \begin{cases} \mathbf{g}(B_j) & \text{if } B_j \notin (E \cup e_0), \\ \mathbf{g}|_{T_{B_j}}(B_j) & \text{if } B_j \in E, \\ \boldsymbol{\alpha} & \text{if } B_j \in e_0, \end{cases}$$

where $\boldsymbol{\alpha}$ is a vector to be chosen in order to verify the compatibility condition (2.1).

For a set $A \subset \Gamma$, we denote by $\omega_{\Gamma,A}$ the union of the closures of the elements in $\mathcal{T}_{\Gamma,h}$ having a vertex on the closure of A. Then we impose

$$0 = \int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = \int_{\Gamma \setminus (\omega_{\Gamma,E} \cup \omega_{\Gamma,e_0})} \mathbf{g}_h \cdot \mathbf{n} + \int_{\omega_{\Gamma,E}} \mathbf{g}_h \cdot \mathbf{n} + \int_{\omega_{\Gamma,e_0}} \mathbf{g}_h \cdot \mathbf{n}.$$

Let us compute the last term. Clearly, ω_{Γ,e_0} lays on the face Γ_k with normal $\mathbf{n}_{e_0}$. Each triangle T in ω_{Γ,e_0} has $r_T \ge 1$ vertices on $\overline{e}_0$ that we denote $P_{T,1}, \dots, P_{T,r_t}$, while $P_{T,r_T+1}, \dots, P_{T,3}$ are the remaining ones. Then

$$\int_{\omega_{\Gamma,e_0}} \mathbf{g}_h \cdot \mathbf{n} = \frac{1}{3} \sum_{T \subset \omega_{\Gamma,e_0}} |T| r_T \boldsymbol{\alpha} \cdot \mathbf{n}_{e_0} + \frac{1}{3} \sum_{T \subset \omega_{\Gamma,e_0}} |T| \sum_{i=r_T+1}^{3} \mathbf{g}_h \cdot \mathbf{n}_{e_0}(P_{T,i}).$$

We require that the vector α is such that the following equality holds true:

$$\left(\frac{1}{3} \sum_{T \subset \omega_{\Gamma,e_0}} |T| r_T \right) \boldsymbol{\alpha} \cdot \mathbf{n}_{e_0} = L_1(\mathbf{g}),$$

where, taking into account that the continuous solution satisfies (1.3),

$$\begin{aligned} L_1(\mathbf{g}) := &\int_{\Gamma \setminus (\omega_{\Gamma,E} \cup \omega_{\Gamma,e_0})} (\mathbf{g} - \mathbf{g}_h) \cdot \mathbf{n} + \int_{\omega_{\Gamma,E}} (\mathbf{g} - \mathbf{g}_h) \cdot \mathbf{n} \\ &+ \int_{\omega_{\Gamma,e_0}} \mathbf{g} \cdot \mathbf{n} - \frac{1}{3} \sum_{T \subset \omega_{\Gamma,e_0}} |T| \sum_{i=r_T+1}^{3} \mathbf{g}_h \cdot \mathbf{n}_{e_0}(P_{T,i}). \end{aligned}$$

Since $|\omega_{\Gamma,E}|$ and $|\omega_{\Gamma,e_0}|$ are bounded by Ch, using interpolation error estimates, we see that

$$|L_1(\mathbf{g})| \le C\, h \|\!|\mathbf{g}|\!\|_{2,\Gamma}.$$

In order to be able to find a unique $\boldsymbol{\alpha}$, we add two conditions on the tangential components, obtaining the following system:

$$\left(\frac{1}{3} \sum_{T \subset \omega_{\Gamma,e_0}} |T| r_T \right) \boldsymbol{\alpha} \cdot \mathbf{n}_{e_0} = L_1(\mathbf{g}), \quad \boldsymbol{\alpha} \cdot \mathbf{t}_1 = 0, \quad \boldsymbol{\alpha} \cdot \mathbf{t}_2 = 0,$$

where $\mathbf{t}_1$ and $\mathbf{t_2}$ are unitary vectors that together with $\mathbf{n}_{e_0}$ form an orthogonal basis of $\mathbb{R}^3$. This is a linear system for $\boldsymbol{\alpha}$, whose nonsingular matrix M verifies $\|M^{-1}\| \leq C\frac{1}{h}$ since the mesh is quasi-uniform. Therefore, we can find $\boldsymbol{\alpha}$ such that

$$|\boldsymbol{\alpha}| \leq C|\!|\!|\mathbf{g}|\!|\!|_{2,\Gamma}. \tag{3.12}$$

This inequality, together with the definition of $\mathbf{g}_h$, gives (3.11). ∎

In the following proposition we estimate $\|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma}$. Since the best possible exponent q in the embedding inequality (3.7) depends on the dimension, the argument used in Proposition 3.2 does not give an optimal result in the case of a three dimensional domain. We can give a different argument using a Hardy type inequality. It will become clear that the same argument can be used for $d = 2$, but it gives a worse constant in terms of s than that obtained in the Proposition 3.2.

Proposition 3.6. *There exists a positive constant such that, for all* $0 \leq s < \frac{1}{2}$, *the following bound holds true:*

$$\|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} \leq \frac{C}{1-2s} h^{\frac{1}{2}+s} |\!|\!|\mathbf{g}|\!|\!|_{2,\Gamma}. \tag{3.13}$$

Proof. For each Γ_i, face of Ω, and $x \in \Gamma_i$, we denote by $d_i(x)$ the distance of x from $\partial\Gamma_i$. There exists a constant C such that, for $0 \leq s < \frac{1}{2}$ and every $\phi \in H^s(\Gamma_i)$, we have

$$\left\| \frac{\phi}{d_i^s} \right\|_{0,\Gamma_i} \leq \frac{C}{1-2s} \|\phi\|_{s,\Gamma_i}. \tag{3.14}$$

This estimate with a precise constant is proved in [8] for the half-space; by standard argument, one can show that the behavior of the constant in terms of s is the same for Lipschitz-bounded domains.

For simplicity let us assume that the polygon e_0 chosen in the construction of $\mathbf{g}_h$ is close to the boundary of Γ_k; i.e., if $x \in e_0$, then $d_k(x) \leq C_1 h$ for some constant C_1. Then, for any $\phi \in H^s(\Gamma)$,

$$\int_\Gamma (\mathbf{g} - \mathbf{g}_h)\,\phi = \sum_{i=1}^{N_e} \int_{\Gamma_i} (\mathbf{g} - \mathbf{g}_h)\,\phi \leq \sum_{i=1}^{N_e} \|(\mathbf{g} - \mathbf{g}_h) d_i^s\|_{0,\Gamma_i} \left\| \frac{\phi}{d_i^s} \right\|_{0,\Gamma_i},$$

and therefore, using (3.14), we obtain

$$\|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} = \sup_{\phi:\|\phi\|_s = 1} \int_\Gamma (\mathbf{g} - \mathbf{g}_h)\,\phi \leq \frac{C}{1-2s} \sum_{i=1}^{N_e} \|(\mathbf{g} - \mathbf{g}_h) d_i^s\|_{0,\Gamma_i}.$$

But,

$$\begin{aligned}
\|(\mathbf{g} - \mathbf{g}_h) d_i^s\|_{0,\Gamma_i}^2 &= \int_{\{x \in \Gamma_i : d_i(x) \leq C_1 h\}} (\mathbf{g} - \mathbf{g}_h)^2 d_i^{2s} + \int_{\{x \in \Gamma_i : d_i(x) > C_1 h\}} (\mathbf{g} - \mathbf{g}_h)^2 d_i^{2s} \\
&\leq C \left(h^{2s+1} \|\mathbf{g}\|_{L^\infty(\Gamma)}^2 + h^2 \|\mathbf{g}\|_{1,\Gamma_i}^2 \right) \quad \text{for } i \neq k, \\
\|(\mathbf{g} - \mathbf{g}_h) d_k^s\|_{0,\Gamma_k}^2 &\leq C \left(h^{2s+1} \|\mathbf{g}\|_{L^\infty(\Gamma)}^2 + h^{2s+1} |\!|\!|\mathbf{g}|\!|\!|_{2,\Gamma}^2 + h^2 \|\mathbf{g}\|_{1,\Gamma_k}^2 \right),
\end{aligned}$$

where for the first term we have used that $|\{x \in \Gamma_i : d_i(x) \leq C_1 h\}| \leq Ch$, that $\|\mathbf{g}_h\|_{L^\infty(\Gamma)} \leq C\|\mathbf{g}\|_{L^\infty(\Gamma)}$, and inequality (3.12), while for the second one we have used that $\mathbf{g}_h$ agrees with the Lagrange interpolation. Hence, we conclude that, for all $0 \leq s < \frac{1}{2}$, the bound (3.13) holds true. ∎

The next proposition can be proved using the same argument as in Proposition 3.3.

PROPOSITION 3.7. *If the family of meshes $\mathcal{T}_{\Gamma,h}$ is quasi-uniform, we have*

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le C|\log h|\, |||\mathbf{g}|||_{2,\Gamma}.$$

We are ready to prove the main theorem of the section.

THEOREM 3.8. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a convex polygonal or polyhedral domain. Suppose that $\mathbf{g}|_{\Gamma_i} \in \mathbf{H}^{d-1}(\Gamma_i)$ for all Γ_i and that the family of meshes $\mathcal{T}_{\Gamma,h}$ is quasi-uniform. Let $\mathbf{g}_h$ be given by the modified Lagrange interpolation of $\mathbf{g}$ introduced in Lemmas 3.1 and 3.5. Then we have, for $\Omega \subset \mathbb{R}^d$, convex polyhedral domain*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le Ch|\log h|^{\frac{d+1}{2}} |||\mathbf{g}|||_{d-1,\Gamma}.$$

Proof. From Propositions 2.4, 3.2, and 3.6, we have, for $0 \le s < \frac{1}{2}$,

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le \frac{C}{(1-2s)^{\frac{d+1}{2}}} h^{\frac{1}{2}+s} |||\mathbf{g}|||_{d-1,\Gamma}.$$

Then, taking $s = 1/2 + 1/\log h < 1/2$ yields

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le C\, h\, |\log h|^{\frac{d+1}{2}} |||\mathbf{g}|||_{d-1,\Gamma}. \tag{3.15}$$

On the other hand, from Propositions 2.6, 3.3, and 3.7, we have

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le Ch\, |\log h|\, |||\mathbf{g}|||_{d-1,\Gamma},$$

which, together with (3.15), gives the desired estimates. ☐

Remark 3.9. In view of Remark 3.4, the quasi-uniformity assumption in the previous theorem can be removed, obtaining, for a general regular family of meshes, the analogous estimates with $|\log h|$ replaced by $|\log(h_{min})|$.

4. A posteriori error estimates.

In this section we introduce the error indicator for the finite element solution of our problem and show that it provides upper and lower bounds for the discretization error of the regularized problem. From Proposition 2.4, we have that the difference between the solutions of problem (1.4) and its regularized version (2.3) is bounded by the difference between the given datum $\mathbf{g}$ and its approximation $\mathbf{g}_h$, which is a computable quantity.

We denote by $\mathcal{E}_h$ the union of the interior edges/faces of the elements of the mesh $\mathcal{T}_h$ and define

$$\mathbf{J} : \mathcal{E}_h \to \mathbb{R}^d, \qquad \mathbf{J}|_e = \mathbf{J}_e \quad \text{with } \mathbf{J}_e = \left[\!\!\left[\frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} - p_h \mathbf{n} \right]\!\!\right]_e \text{ for } e \in \mathcal{E}_h,$$

where the jump of the function r across the edge $e = T^+ \cap T^-$ is given by

$$\left[\!\!\left[\frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} - p_h \mathbf{n} \right]\!\!\right]_e = \left(\frac{\partial \mathbf{u}_h|_{T^+}}{\partial \mathbf{n}^+} - p_h|_{T^+}\mathbf{n}^+ \right) + \left(\frac{\partial \mathbf{u}_h|_{T^-}}{\partial \mathbf{n}^-} - p_h|_{T^-}\mathbf{n}^- \right)$$

if $\mathbf{n}^\pm$ denotes the exterior normal to the triangle $T^\pm$.

Then we introduce the local error indicator

$$\eta_T^2 = h_T^4 \| -\Delta \mathbf{u}_h + \nabla p_h\|_{0,T}^2 + h_T^2 \|\operatorname{div} \mathbf{u}_h\|_{0,T}^2 + \frac{1}{2} \sum_{e \subset \overline{T}} h_T^3 \|\mathbf{J}_e\|_{0,e}^2. \tag{4.1}$$

Since we want to estimate the velocity in the $\mathbf{L}^2(\Omega)$-norm and the pressure in the $H^{-1}(\Omega)/\mathbb{R}$-norm, the error indicator becomes the usual error indicator for problems with smooth boundary data multiplied by h_T^2 (see, e.g., [24, 16]). Notice that the powers of h_T in (4.1) are determined by using a duality argument, as will be shown in the next proposition, and depend on the full regularity of the solution of the adjoint problem available in the case of a convex domain. As far as we know, an L^2-error indicator is available for the Poisson problem only under the assumption that the domain is convex (see, for example, [25, Proposition 3.8, page 68]). It is possible to obtain an error estimator for the nonconvex case, but the powers in h_T depend on the biggest angle, and the efficiency cannot be proved. To find an error estimator equivalent to the L^2-norm is a very interesting problem that, as far as we know, has not yet been solved.

Proposition 4.1 (robustness). *Assume that Ω is convex. The estimator η_T introduced in* (4.1) *is robust; that is, there exists a positive constant C independent of h such that*

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{\frac{1}{2}} . \tag{4.2}$$

Proof. We start with the estimate for $\mathbf{u}(h) - \mathbf{u}_h$. In order to apply a duality argument, we consider the solution $(\boldsymbol{\phi}, q)$ of (2.6). Then, taking into account the equations (2.20) and (2.5) and the approximation estimates (2.16) and (2.17), we obtain by integration by parts:

$$\begin{aligned} \|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega}^2 &= (\mathbf{u}(h) - \mathbf{u}_h, \mathbf{u}(h) - \mathbf{u}_h) = (\mathbf{u}(h) - \mathbf{u}_h, -\Delta \boldsymbol{\phi} + \nabla q) \\ &= \Big(\nabla(\mathbf{u}(h) - \mathbf{u}_h), \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^I) \Big) \\ &\quad - (\operatorname{div}(\mathbf{u}(h) - \mathbf{u}_h), q - P_0 q) - \Big(p(h) - p_h, \operatorname{div}(\boldsymbol{\phi} - \boldsymbol{\phi}^I \Big) \\ &= - \sum_{T \in \mathcal{T}_h} \Big(-\Delta \mathbf{u}_h + \nabla p_h, \boldsymbol{\phi} - \boldsymbol{\phi}^I \Big)_T + \sum_{T \in \mathcal{T}_h} (\operatorname{div} \mathbf{u}_h, q - P_0 q)_T \\ &\quad - \sum_{e \in \mathcal{E}_h} \left(\left[\!\!\left[\frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} - p_h \mathbf{n} \right]\!\!\right], \boldsymbol{\phi} - \boldsymbol{\phi}^I \right)_e . \end{aligned} \tag{4.3}$$

Thanks to (2.16), we can write

$$\begin{aligned} \|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega}^2 &\le C \sum_{T \in \mathcal{T}_h} \|-\Delta \mathbf{u}_h + \nabla p_h\|_{0,T} \, h_T^2 |\boldsymbol{\phi}|_{2,T} \\ &\quad + C \sum_{T \in \mathcal{T}_h} \|\operatorname{div} \mathbf{u}_h\|_{0,T} h_T |q|_{1,T} + C \sum_{e \in \mathcal{E}_h} \|\mathbf{J}\|_{0,e} \, h_T^{\frac{3}{2}} \, |\boldsymbol{\phi}|_{2,\omega_e} \\ &\le C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{\frac{1}{2}} \|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega}, \end{aligned} \tag{4.4}$$

where ω_e is the union ot the elements sharing $e \in \mathcal{E}_h$. This concludes the estimate of $\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega}$.

Now we consider the error for the pressure. Since $p(h)$ and p_h have zero mean value, the definition of the H^{-1}-norm reads

$$\|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} = \sup_{\substack{q\in H_0^1(\Omega)\\ \int_\Omega q=0}} \frac{(p(h)-p_h,q)}{\|q\|_{1,\Omega}}. \tag{4.5}$$

For each $q \in H_0^1(\Omega)$ with $\int_\Omega q = 0$, we take $\boldsymbol{\psi} \in \mathbf{H}_0^2(\Omega)$ with $\operatorname{div}\psi = q$ and $\|\boldsymbol{\psi}\|_{H^2(\Omega)} \le C\|q\|_{H^1(\Omega)}$ (see (2.23)); hence,

$$\begin{aligned}(p(h)-p_h,q) &= \Big(p(h)-p_h, \operatorname{div}(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big) - \Big(\nabla(\mathbf{u}(h)-\mathbf{u}_h), \nabla(\boldsymbol{\psi}-\boldsymbol{\psi}^I)\Big)\\ &\quad - (\mathbf{u}(h)-\mathbf{u}_h, \Delta\boldsymbol{\psi})\\ &= \sum_{T\in\mathcal{T}_h}\Big(-\Delta\mathbf{u}_h + \nabla p_h, \boldsymbol{\psi}-\boldsymbol{\psi}^I\Big)_T\\ &\quad + \sum_{e\in\mathcal{E}_h}\left(\left[\!\left[\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}} - p_h\mathbf{n}\right]\!\right], \boldsymbol{\psi}-\boldsymbol{\psi}^I\right)_e - (\mathbf{u}(h)-\mathbf{u}_h, \Delta\boldsymbol{\psi}).\end{aligned}$$

Then

$$\begin{aligned}(p(h)-p_h,q) &\le C\Big[\sum_{T\in\mathcal{T}_h} h_T^4\|-\Delta\mathbf{u}_h + \nabla p_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_T^3\|\mathbf{J}\|_{0,e}^2\Big]^{\frac12}\|q\|_{1,\Omega}\\ &\quad + \|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\Omega}\|q\|_{1,\Omega}.\end{aligned}$$

The proof concludes by using the estimate (4.4) and the norm definition (4.5). ◻

In the next proposition we show that the error indicator bounds locally the error by below.

PROPOSITION 4.2 (efficiency). *For all element $T \in \mathcal{T}_h$, we have*

$$\eta_T \le C\left(\|\mathbf{u}(h)-\mathbf{u}_h\|_{0,\omega_T} + \|p(h)-p_h\|_{-1,\omega_T}\right) \tag{4.6}$$

where $\omega_T = \left\{T' \in \mathcal{T}_h : \overline{T'} \cap \overline{T} \neq \emptyset\right\}$.

Proof. We estimate the three terms of the error indicator in (4.1) separately. Given an element $T \in \mathcal{T}_h$, let us consider the function

$$b_T = \left(\prod_{i=1}^{d+1}\lambda_{i,T}\right)^2$$

with $\lambda_{i,T}, i = 1,\dots,d+1$ being the barycenter coordinate functions in T. We set

$$\mathbf{w}_T = (-\Delta\mathbf{u}_h + \nabla p_h)\, b_T.$$

Thanks to the definition of b_T we have that

$$\mathbf{w}_T = 0 \quad \text{on } \partial T, \qquad \nabla\mathbf{w}_T = 0 \quad \text{on } \partial T,$$

and, by inverse inequality,

$$\begin{aligned}\|\operatorname{div}\mathbf{w}_T\|_{1,T} &\le Ch_T^{-2}\,\|-\Delta\mathbf{u}_h + \nabla p_h\|_{0,T},\\ \|\Delta\mathbf{w}_T\|_{0,T} &\le Ch_T^{-2}\,\|-\Delta\mathbf{u}_h + \nabla p_h\|_{0,T},\end{aligned} \tag{4.7}$$

Then integration by parts gives

$$\begin{aligned} \| -\Delta \mathbf{u}_h + \nabla p_h\|^2_{0,T} &= (-\Delta \mathbf{u}_h + \nabla p_h, -\Delta \mathbf{u}_h + \nabla p_h))_T \\ &\le C \left|(-\Delta \mathbf{u}_h + \nabla p_h, \mathbf{w}_T)_T\right| \\ &= C \left|(-\Delta(\mathbf{u}_h - \mathbf{u}(h)) + \nabla(p_h - p(h)), \mathbf{w}_T)\right| \\ &= C \left|(\mathbf{u}_h - \mathbf{u}(h), \Delta \mathbf{w}_T)_T + (p_h - p(h), \operatorname{div} \mathbf{w}_T)\right| . \end{aligned} \tag{4.8}$$

Due to the definition of b_T we have that $\operatorname{div} \mathbf{w}_T \in \mathbf{H}^1_0(T)$; hence, we can use the duality between $H^{-1}(T)$ and $H^1_0(T)$, to obtain

$$\|-\Delta \mathbf{u}_h+\nabla p_h\|^2_{0,T} \le C \left(\|\mathbf{u}_h - \mathbf{u}(h)\|_{0,T} \|\Delta \mathbf{w}_T\|_{0,T} + \|\left(p_h - p(h)\right)\|_{-1,T} \|\operatorname{div} \mathbf{w}_T\|_{1,T}\right),$$

which, together with (4.7), implies

$$h_T^2 \|-\Delta \mathbf{u}_h + \nabla p_h\|_{0,T} \le C \left(\|\mathbf{u}_h - \mathbf{u}(h)\|_{0,T} + \|p_h - p(h)\|_{-1,T}\right). \tag{4.9}$$

In order to bound the second term in (4.1), let us introduce $w_T = (\operatorname{div} \mathbf{u}_h) b_T$, which satisfies

$$\|\nabla w_T\|_{0,T} \le C h_T^{-1} \|\operatorname{div} \mathbf{u}_h\|_{0,T}.$$

Hence, we obtain

$$\begin{aligned} \int_T (\operatorname{div} \mathbf{u}_h)^2 &\le C \left|\int_T (\operatorname{div} \mathbf{u}_h)\, w_T\right| = C \left|\int_T \operatorname{div}(\mathbf{u}_h - \mathbf{u}(h))\, w_T\right| \\ &= C \left|\int_T (\mathbf{u}_h - \mathbf{u}(h))\, \nabla w_T\right| \le C h_T^{-1} \|\mathbf{u}_h - \mathbf{u}(h)\|_{0,T} \|\operatorname{div} \mathbf{u}_h\|_{0,T}, \end{aligned}$$

which implies

$$h_T \|\operatorname{div} \mathbf{u}_h\|_{0,T} \le C \|\mathbf{u}_h - \mathbf{u}(h)\|_{0,T}. \tag{4.10}$$

It remains to bound the last term of the indicator involving the jumps along element interfaces in $\mathcal{T}_h$. Let $e \in \mathcal{E}_h$ be an internal edge/face, and let us suppose that there are two elements T_1 and T_2 such that $e = T_1 \cap T_2$. Let $\mathtt{v_i}$ for $i = 1, \dots, d$, be the vertices of e. We denote by $\lambda_{\mathtt{v_i},\mathtt{T_j}}$, $i = 1, \dots, d$, $j = 1, 2$, the barycentric coordinate functions for the vertex $\mathtt{v_i}$ on the triangle T_j and by ω_e the union of T_1 and T_2. Then we define the bubble function

$$b_e = \left(\prod_{i=1}^{d} \lambda_{\mathtt{v}_i,T_1} \prod_{i=1}^{d} \lambda_{\mathtt{v}_i,T_2}\right)^2.$$

Setting $\mathbf{w}_e = \mathbf{J}_e b_e$ and taking into account that the mesh is regular, it is not difficult to check that the following inequalities hold true:

$$\begin{aligned} \|\Delta \mathbf{w}_e\|_{0,\omega_e} &\le C h_e^{-\frac{3}{2}} \|\mathbf{J}_e\|_{0,e}, \\ \|\operatorname{div} \mathbf{w}_e\|_{1,\omega_e} &\le C h_e^{-\frac{3}{2}} \|\mathbf{J}_e\|_{0,e}, \\ \|\mathbf{w}_e\|_{0,\omega_e} &\le C h_e^{\frac{1}{2}} \|\mathbf{J}_e\|_{0,e}. \end{aligned} \tag{4.11}$$

There exists a positive constant C such that

$$\begin{aligned}
\frac{1}{C}\|\mathbf{J}_e\|_{0,e}^2 \le (\mathbf{J}_e^2, b_e)_e &= \left(\left[\!\!\left[\frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} - p_h\mathbf{n}\right]\!\!\right], \mathbf{w}_e\right)_e \\
&= (\nabla\mathbf{u}_h - \nabla\mathbf{u}(h), \nabla\mathbf{w}_e)_{\omega_e} - (p_h - p(h), \operatorname{div}\mathbf{w}_e)_{\omega_e} + \sum_{T\subset\omega_e}(\Delta\mathbf{u}_h - \nabla p_h, \mathbf{w}_e)_T \\
&= -(\mathbf{u}_h - \mathbf{u}(h), \Delta\mathbf{w}_e)_{\omega_e} + \left(\mathbf{u}_h - \mathbf{u}(h), \frac{\partial \mathbf{w}_e}{\partial \mathbf{n}}\right)_{\partial\omega_e} \\
&\quad - (p_h - p(h), \operatorname{div}\mathbf{w}_e)_{\omega_e} + \sum_{T\subset\omega_e}(\Delta\mathbf{u}_h - \nabla p_h, \mathbf{w}_e)_T\,.
\end{aligned}$$

Using again the fact that $\operatorname{div}\mathbf{w}_e \in H_0^1(\omega_e)$, we obtain, by multiplying times h_e^3,

$$\begin{aligned}
h_e^3\|\mathbf{J}_e\|_{0,e}^2 \le C\Big(&\|\mathbf{u}_h - \mathbf{u}(h)\|_{0,\omega_e} h_e^3\|\Delta\mathbf{w}_e\|_{0,\omega_e} + \|p_h - p(h)\|_{-1,\omega_e} h_e^3\|\operatorname{div}\mathbf{w}_e\|_{1,\omega_e} \\
&+ \sum_{T\subset\omega_e} h_e^2\|\Delta\mathbf{u}_h - \nabla p_h\|_{0,T} h_e\|\mathbf{w}_e\|_{0,T}\Big).
\end{aligned}$$

Using (4.11) and (4.9), we have

$$\begin{aligned}
h_e^3\|\mathbf{J}_e\|_{0,e}^2 &\le C\Big(\|\mathbf{u}_h - \mathbf{u}(h)\|_{0,\omega_e} + \|p_h - p(h)\|_{-1,\omega_e} \\
&\qquad + \sum_{T\subset\omega_e} h_e^2\|\Delta\mathbf{u}_h - \nabla p_h\|_{0,T}\Big) h_e^{\frac{3}{2}}\|\mathbf{J}\|_{0,e} \\
&\le \left(\|\mathbf{u}_h - \mathbf{u}(h)\|_{0,\omega_e} + \|p_h - p(h)\|_{-1,\omega_e}\right) h_e^{\frac{3}{2}}\|\mathbf{J}\|_{0,e}.
\end{aligned} \tag{4.12}$$

Taking into account the definition (4.1) of the estimator η_T, together with the estimates (4.9), (4.10), and (4.12), we obtain the desired result. ∎

5. Numerical example. On $\Omega = [0,1]^2$, we consider the lid-driven cavity flow problem (1.4) with

$$\mathbf{g}(x_1, x_2) = \begin{cases} (1,0) & \text{if } 0 < x_1 < 1 \text{ and } x_2 = 1, \\ (0,0) & \text{if } x_1 = 0 \text{ or } x_1 = 1 \text{ or } x_2 = 0. \end{cases}$$

We consider the methods:

- Mini-element: $\mathbf{V}_h = (\mathcal{P}_1^b(\mathcal{T}_h))^2 \cap \mathcal{C}^0(\bar{\Omega})^2$ and $Q_h = \mathcal{P}_1(\mathcal{T}_h) \cap \mathcal{C}^0(\bar{\Omega}) \cap L_0^2(\Omega)$;
- Hood and Taylor: $\mathbf{V}_h = \mathcal{P}_2(\mathcal{T}_h)^2 \cap \mathcal{C}^0(\bar{\Omega})^2$ and $Q_h = \mathcal{P}_1(\mathcal{T}_h) \cap \mathcal{C}^0(\bar{\Omega}) \cap L_0^2(\Omega)$,

where, if for an element T, $b_T \in \mathcal{P}_3$ is the cubic bubble function vanishing on ∂T, we set

$$\mathcal{P}_1^b(T) = \mathcal{P}_1(T) \oplus \operatorname{span}\{b_T(\cdot)\}\,.$$

We consider the variational formulation (2.17) with $\mathbf{u}_h = E\mathbf{g}_h + \mathbf{u}_{0h}$, where $\mathbf{u}_{0h} \in \mathbf{V}_h$ and $\mathbf{g}_h$ is the Lagrange interpolation of $\mathbf{g}$ on the restriction of $\mathcal{T}_h$ to $\partial\Omega$. We remark that the compatibility condition (2.1) is automatically verified.

Below for the distinct methods and different refinement strategies, we estimate the convergence errors for $\mathbf{u}$ in the $L^2(\Omega)$-norm. Since we do not know the exact solution, the $L^2(\Omega)$-error is computed as the difference between the solutions obtained at two consecutive refinements.

Table 1 shows the results obtained by uniform refinements procedures starting with a coarse mesh for Mini-element (second to fifth columns) and Hood–Taylor (sixth

TABLE 1

Schemes on uniformly refined structured meshes. The second through fifth columns correspond to the Mini-element method, and the sixth through ninth columns correspond to the Hood–Taylor method.

	Mini-element				Hood–Taylor			
# vert.	$\|\mathbf{u}\|_{0,\Omega}$	Rate	η	Rate	$\|\mathbf{u}\|_{0,\Omega}$	Rate	η	Rate
289	$5.14e-2$		1.85		$4.07e-2$		3.60	
1089	$2.59e-2$	0.52	$9.31e-1$	0.52	$2.03e-2$	0.52	1.80	0.52
4225	$1.30e-2$	0.51	$4.67e-1$	0.51	$1.02e-2$	0.51	$9.03e-1$	0.51
16641	$6.48e-3$	0.51	$2.34e-1$	0.50	$5.08e-3$	0.51	$4.52e-2$	0.51

TABLE 2

Adaptive scheme for the Mini-element method using the local estimators η_T. Parameter: $\theta = 0.5$.

Step	# vert.	$\|\mathbf{u}\|_{0,\Omega}$	Rate	η	Rate
2	84	$4.16e-2$		2.48	
4	107	$1.76e-2$	9.49	1.06	4.98
6	201	$1.09e-2$	1.17	$4.43e-1$	1.35
8	514	$4.17e-3$	0.90	$1.72e-1$	0.97
10	1197	$1.72e-3$	0.91	$7.33e-2$	1.04
12	2859	$7.41e-4$	1.15	$3.08e-2$	1.06
14	6834	$3.31e-4$	0.96	$1.28e-2$	1.00
16	15443	$1.41e-4$	1.05	$5.53e-3$	1.02

TABLE 3

Adaptive scheme for the Hood–Taylor method using the local estimators η_T. Parameter: $\theta = 0.75$.

Step	# vert.	$\|\mathbf{u}\|_{0,\Omega}$	Rate	η	Rate
2	84	$3.60e-2$		4.30	
4	102	$1.59e-2$	4.67	1.39	4.77
6	160	$4.55e-3$	1.95	$4.65e-1$	1.88
8	385	$1.45e-3$	1.10	$1.41e-1$	1.26
10	992	$4.23e-4$	1.39	$4.19e-2$	1.33
12	2583	$1.30e-4$	1.48	$1.19e-2$	1.34
14	6665	$3.91e-5$	1.35	$3.25e-3$	1.38
16	16629	$1.01e-5$	1.38	$8.97e-4$	1.39
18	40802	$2.73e-6$	1.62	$2.43e-4$	1.47

to ninth columns) methods. In the first column we report the number of vertices of the mesh, which is equivalent to the number of degrees of freedom. The next four columns refer to the Mini-element method and the remaining ones to the Hood–Taylor method. For each method, we display the approximation of the L^2-error for $\mathbf{u}$ and the total estimator η with the corresponding experimental rate of convergence, where

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2, \tag{5.1}$$

with η_T given by (4.1). We observe that, in both cases, order $\frac{1}{2}$ with respect to the number of vertices (i.e., order 1 with respect to h) is obtained for the error decay in $L^2(\Omega)$ of $\mathbf{u}$. Accordingly, the error estimator η decreases with the same order.

In Tables 2 and 3 we show the results obtained by an adaptive procedure using the a posteriori error estimator (5.1). The refinement process is standard: Given $0 < \theta < 1$, a fixed parameter, suppose that $\mathcal{T}_k$ is the mesh in the k-step. If we enumerate the triangular elements such that $\mathcal{T}_k = \{T_i : i = 1, \dots, N_{el}\}$ with $\eta_{T_i} \geq \eta_{T_{i+1}}$, let

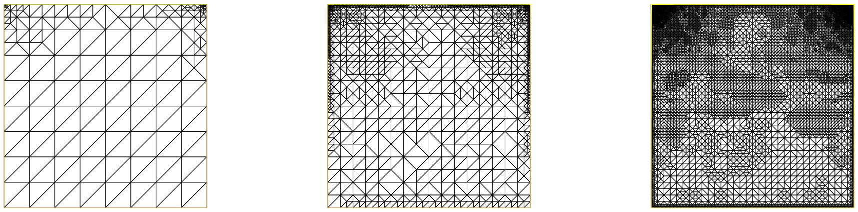

FIG. 1. *Meshes generated by the Hood–Taylor adaptive process in steps* 5, 10, *and* 15 *using the local estimators* η_T *with parameter* $\theta = 0.75$. *The starting mesh is obtained by adding a diagonal to each square in an* 8×8 *grid.*

$N_{ref,k}$ be the minimum integer such that

$$\sum_{i=1}^{N_{ref,k}} \eta_{T_i}^2 \geq \theta\, \eta^2.$$

Then the mesh for the $k+1$-step is constructed in such a way that the elements T_i, $i = 1, \dots, N_{ref,k}$ are refined. We report the $L^2(\Omega)$-error in $\mathbf{u}$, which, as before, is computed in each step as the $L^2(\Omega)$-norm of the difference between the discrete solution obtained in the current step and in the previous one of the iterative process. Similarly, the numerical orders of convergence for u and η are computed from the last two steps of the scheme.

We observe that for both Mini-element and Hood–Taylor methods, the adaptive process recovers the expected optimal order of convergence in $\mathbf{u}$, i.e., the order of convergence with respect to the number of degrees of freedom, which, for each method, is proved in the literature for regular solutions. In Figure 1 we show the initial mesh and some of the meshes obtained in the iterative process for the Hood–Taylor method.

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REFERENCES

[1] R. Adams and J. Fournier, *Sobolev Spaces*, 2nd ed., Pure Appl. Math. (Amst.) 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press, New York, 1975.

[3] H. Adjal, M. Moussaoui, and A. Zine, *Existence and regularity of solutions for Stokes systems with non-smooth boundary data in a polyhedron*, Electron. J. Differential Equations, 2017 (2017), pp. 1–10.

[4] T. Apel, S. Nicaise, and J. Pfefferer, *Discretization of the Poisson Equation with Non-Smooth Data and Emphasis on Non-Convex Domains*, 2015, https://arxiv.org/abs/1505.01229.

[5] T. Apel, S. Nicaise, and J. Pfefferer, *Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains*, Numer. Methods Partial Differential Equations, 32 (2016), pp. 1433–1454, https://doi.org/10.1002/num.22057.

[6] T. Apel, S. Nicaise, and J. Pfefferer, *Adapted numerical methods for the numerical solution of the Poisson equation with l2 boundary data in non-convex domains*, SIAM J. Numer. Anal., 55 (2017), pp. 1937–1957.

[7] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Ser. Comput. Math. 44, Springer, Heidelberg, 2013, https://doi.org/10.1007/978-3-642-36519-5.

[8] K. Bogdan and B. Dyda, *The best constant in a fractional Hardy inequality*, Math. Nachr., 284 (2011), pp. 629–638.

[9] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts Appl. Math. 15, Springer, New York, 2008, https://doi.org/10.1007/978-0-387-75934-0.

[10] Z. Cai and Y. Wang, *An error estimate for two-dimensional Stokes driven cavity flow*, Math. Comp., 78 (2009), pp. 771–787, https://doi.org/10.1090/S0025-5718-08-02177-7.

[11] C. Carstensen, *Quasi-interpolation and a posteriori error analysis in finite element methods*, ESAIM Math. Model. Numer. Anal., 33 (1999), pp. 1187–1202, https://doi.org/10.1051/m2an:1999140.

[12] A. Cotsiolis and N. K. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl., 295 (2004), pp. 225–236, https://doi.org/10.1016/j.jmaa.2004.03.034.

[13] W. Dahmen, B. Faermann, I. G. Graham, W. Hackbusch, and S. A. Sauter, *Inverse inequalities on non-quasi-uniform meshes and the application to the mortar element method*, Math. Comp., 73 (2003), pp. 1107–1138.

[14] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners.* I. *Linearized equations*, SIAM J. Math. Anal., 20 (1989), pp. 74–97, https://doi.org/10.1137/0520006.

[15] E. B. Fabes, C. E. Kenig, and G. C. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J., 57 (1988), pp. 769–793, https://doi.org/10.1215/S0012-7094-88-05734-1.

[16] M. Feischl, *Optimal Adaptivity for a Standard Finite Element Method for the Stokes Problem*, preprint, arXiv:1710.08289, 2017.

[17] M. Hamouda, R. Temam, and L. Zhang, *Modeling the lid driven flow: Theory and computation*, Int. J. Numer. Anal. Model., 14 (2017), pp. 313–341.

[18] R. B. Kellogg and J. E. Osborn, *A regularity result for the Stokes problem in a convex polygon*, J. Funct. Anal., 21 (1976), pp. 397–431.

[19] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications* Vol. I, Springer-Verlag, New York, 1972; translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[20] M. Moussaoui and A. Zine, *Existence and regularity results for the Stokes system with non smooth boundary data in a polygon*, Math. Models Methods Appl. Sci., 8 (1998), pp. 1307–1315.

[21] K. Schumacher, *Solutions to the equation* $\operatorname{div} u = f$ *in weighted Sobolev spaces*, in Parabolic and Navier-Stokes Equations, Part 2, Banach Center Publ. 81, Polish Acad. Sci. Inst. Math., Warsaw, 2008, pp. 433–440, https://doi-org.proxy.unibs.it/10.4064/bc81-0-26.

[22] R. Temam, *Navier-Stokes Equations*, revised ed., Theory and Numerical Analysis, with an appendix by F. Thomasset, Studies in Mathematics and Its Applications 2, North-Holland, Amsterdam, 1979.

[23] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), pp. 572–611.

[24] R. Verfürth, *A posteriori error estimators for the Stokes equations*, Numer. Math., 55 (1989), pp. 309–325, https://doi.org/10.1007/BF01390056.

[25] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley Teubner series, Advances in Numerical Mathematics, Wiley, New York, 1996.