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On the dynamics of the zeros of solutions of the Airy equation

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Abstract

We study the dependence of the zeros of solutions of the Airy equation on two parameters introduced in the equation. The parameters characterize the general solution of the equation. A system of infinitely many nonlinear evolution differential equations is obtained, displaying interesting properties.

Keywords: Airy equation; Zeros of entire functions; Value distribution; Dynamics of the zeros

1. Introduction

The Airy functions are the solution of the following linear non-autonomous second order ordinary differential equation

$$\frac{d^2y(z)}{dz^2} = zy(z). \tag{1}$$

The solutions of this equation find applications in many branches of mathematical physics, such as optics, electromagnetism, fluid mechanics and nonlinear wave propagation (see e.g. [11] or [13]). There are two independent solutions of (1) that are commonly taken as basis for the general solution. They are denoted by $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$: $\mathrm{Ai}(z)$ is the only solution of Eq. (1), up to a constant factor, exponentially decaying on the positive real axis. The function $\mathrm{Bi}(z)$ is exponentially increasing on the positive real axis. $\mathrm{Ai}(z)$ corresponds to the solution satisfying (1) with the initial conditions $y(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}$ and $y'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}$, whereas $\mathrm{Bi}(z)$ corresponds to the solution identified by the initial conditions $y(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}$ and $y'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}$ (see [11, Eqs. 9.2.3–9.2.6]).

The characterization of the behavior of the solution at infinity, or at $z = 0$, is not the only way to pick a solution out of the general solution of Eq. (1). Eq. (1) possesses entire solutions, meaning that they are holomorphic at all the finite points of the complex plane. Further the order of growth is $3/2$, not an integer, implying that any solution has an infinite number of zeros in the complex plane (see e.g. [12]). Also, the solutions of Eq. (1) possess movable zeros: if the initial conditions change, the positions of the zeros change. The position of one zero can be considered a constant of integration (see also [8]). A second constant is the global scaling factor arising from the linearity

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of Eq. (1). It is then possible to pick a solution by assigning the position of one zero and the value of the first derivative of the function at this zero: this point of view has been considered in [14]. In particular, if the solution of Eq. (1) has a zero at $z = z_0$, with z_0 an arbitrary point, by a translation $z \to z + z_0$ and a re-scaling of the independent variable, one arrives at the following equation

$$\frac{d^2\tau(z)}{dz^2} = (az + b)\tau(z) \tag{2}$$

where a and b are two arbitrary constants. A solution of Eq. (2) having a zero at $z = 0$ corresponds to a solution of Eq. (1) having a zero at $z_0 = \frac{b}{\eta^2}$, where $a = \eta^3$ [14]. To characterize the solution of Eq. (1) having a zero in $z = z_0$ then one has to look at the particular solution of Eq. (2) having a zero in $z = 0$ for an arbitrary value of b. In [14] it has been shown how, by investigating deeply in this direction, it is possible to obtain a recursion for the zeros and a characterization of their distribution in the complex plane (notice that this same idea has been shown to be useful also for the description of the properties of other transcendental functions, like the solutions of the Painlevé I equation [10]).

In this work we keep this same point of view, by extending the results obtained in [14]. In particular we will consider the dependence of the solutions of Eq. (2) and of their zeros on the values of the variables a and b. In Section 2 we will recapitulate the results obtained in [14] necessary to this work; in Section 3 we will show that the solutions of Eq. (2) satisfy two differential equations involving the parameters, the first one being just the consequence of the homogeneity of the solutions of (2) with respect to the variables a, b and z. As a consequence, it will be shown that the zeros of $\tau(z)$ satisfy a differential equation with respect to the parameters a and b. This result is the analog, for the Airy functions, of the derivatives of the Weierstrass elliptic function $\wp(z, g_2, g_3)$ with respect to the invariants g_2 and g_3. Further, a system of infinitely many nonlinear evolution differential equations for the zeros are obtained (see also [5], where recently have been obtained similar results on the zeros of entire functions). Then, we give some numerical results on the solutions of the system obtained. Finally, in the conclusions we suggest how to generalize the results given here to other entire functions of physical interest.

2. A recursion for the zeros

As discussed in the introduction, we are interested in the particular solution $S(z, a, b)$ of Eq. (2) having a zero at $z = 0$. It is known that this solution has an infinite number of other zeros approaching asymptotically the lines $\arg(z) = \pm\frac{\pi}{3}$ and $\arg(z) = \pi$ [8]. We fix the global scaling factor of the solution to be equal to 1, so that $S(z, a, b)$ is completely defined by the behavior $S(z, a, b) = z + O(z^3)$ around $z = 0$.

From the Weierstrass–Hadamard factorization theorem [12], it follows that the function $S(z, a, b)$ can be represented by the infinite products over its zeros as

$$S(z, a, b) = z \prod_{\substack{k=1 \\ \Omega_k \neq 0}} \left(1 - \frac{z}{\Omega_k}\right) e^{\frac{z}{\Omega_k}}. \tag{3}$$

The logarithmic derivative of S, i.e. the function $u(z, a, b)$ defined by

$$u(z, a, b) \doteq \frac{S'(z, a, b)}{S(z, a, b)}, \tag{4}$$

solves the Riccati equation

$$\frac{du}{dz} + u^2 = az + b. \tag{5}$$

Since S is entire, u is a meromorphic function of z. Further, from the representation (3) and (4), it follows that $u(z, a, b)$ can be represented by the sum over its poles as

$$u(z, a, b) = \frac{1}{z} + \sum_{\substack{k=1 \\ \Omega_k \neq 0}} \frac{1}{z - \Omega_k} + \frac{1}{\Omega_k}, \tag{6}$$

By expanding the solution of Eq. (5) with a pole at $z = 0$ in Laurent series, one has

$$u(z, a, b) = \frac{1}{z} + \sum_{n=1} c_n(a, b) z^n. \tag{7}$$

The coefficients c_n can be determined by the quadratic equation (5): indeed they must satisfy the recursion

$$c_{n+1} = -\frac{1}{n+3}\sum_{k=1}^{n-1} c_k c_{n-k}, \qquad c_1 = \frac{b}{3},\ c_2 = \frac{a}{4}. \tag{8}$$

From the compatibility of the two expansions for $u(z)$, (6) and (7) one gets a relation among the coefficients c_n and the zeros of the function $S(z)$

$$c_n = -\sum_{\substack{k=1\\ \Omega_k\neq 0}} \frac{1}{\Omega_k^{n+1}}. \tag{9}$$

The first few coefficients are given below:

$$c_3 = -\frac{1}{45}b^2, \quad c_4 = -\frac{1}{36}ab, \quad c_5 = \frac{2}{945}b^3 - \frac{1}{112}a^2, \quad c_6 = \frac{1}{270}b^2 a, \quad \ldots \tag{10}$$

3. The dynamics of the zeros

The coefficients $c_n(a,b)$ defining the Laurent series of the function $u(z,a,b)$ around $z=0$ are polynomials in a and b, as is shown by (9). Actually, they are weighted homogeneous polynomials of total degree $n+1$, with weights 3 and 2 respectively. It means that each $c_n(a,b)$ satisfy the equation

$$c_n(\xi^3 a, \xi^2 b) = \xi^{n+1} c_n(a,b). \tag{11}$$

The function $u(z,a,b)$ is itself a homogeneous function of degree 1, i.e.

$$u(\xi^{-1}z, \xi^3 a, \xi^2 b) = \xi u(z,a,b). \tag{12}$$

From the Euler's theorem on homogeneous functions it follows that $u(z,a,b)$ solves the following PDE

$$\left(3a\frac{\partial}{\partial a} + 2b\frac{\partial}{\partial b} - z\frac{\partial}{\partial z} - 1\right) u(z,a,b) = 0. \tag{13}$$

There is however a second independent PDE solved by the function u. It reads

$$S^2(z,a,b)\left(a\frac{\partial}{\partial b} - \frac{\partial}{\partial z}\right) u(z,a,b) = 1. \tag{14}$$

The proof of this second equation is just by direct differentiation with respect to z. Indeed, Eq. (2) and the differential consequences of (5) imply that the derivative of the left hand side of Eq. (14) is zero. So the left hand side is a function of a and b only. This function is identified to be equal to 1 by the development of the left hand side itself around $z=0$. In our opinion, Eqs. (13) and (14) are very close to the equations for the derivatives of the Weierstrass elliptic function $\wp(z,g_2,g_3)$ with respect to its invariants (see e.g. [9]). Indeed in that case also one has two equations, one from the homogeneity properties of $\wp(z)$:

$$\left(4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3} - z\frac{\partial}{\partial z} - 2\right)\wp(z,g_2,g_3) = 0, \tag{15}$$

and the other involving the $\wp(z)$ and the $\zeta(z)$ functions (where $\zeta'(z) = -\wp$)

$$\left(12g_3\frac{\partial}{\partial g_2} + \frac{2}{3}g_2^2\frac{\partial}{\partial g_3} - 2\zeta(z,g_2,g_3)\frac{\partial}{\partial z} - 4\wp(z,g_2,g_3)\right)\wp(z,g_2,g_3) = \frac{2}{3}g_2 \tag{16}$$

In this case the elliptic invariants g_2 and g_3 are related to the poles ω_k of the function $\wp(z)$ by the relations

$$g_2 = 60\sum_{\substack{k=1\\ w_k\neq 0}} \frac{1}{w_k^4}, \quad g_3 = 140\sum_{\substack{k=1\\ w_k\neq 0}} \frac{1}{w_k^6}. \tag{17}$$

In the Airy's case, the "invariants" a and b are given by (see Eqs. (8) and (9))

$$b = -3\sum_{\substack{k=1\\ \Omega_k\neq 0}} \frac{1}{\Omega_k^2}, \quad a = -4\sum_{\substack{k=1\\ \Omega_k\neq 0}} \frac{1}{\Omega_k^3}. \tag{18}$$

The relation (13) and (14) allow to find the derivatives of the function u with respect to a and b in terms of u and S. But, also, allow to describe the variation of the zeros Ω_k in terms of the parameters a and b. Indeed $a\frac{\partial u}{\partial b} - \frac{\partial u}{\partial z}$ has double poles in $z = \Omega_k$, whereas S has double zeros in $z = \Omega_k$. By considering Ω_k functions of the parameters a and b one obtains the following equation for the derivative of Ω_k with respect to b:

$$\left(a\frac{\partial \Omega_k}{\partial b} + 1\right) S'(\Omega_k, a, b)^2 = 1 \tag{19}$$

The previous equation can be written more explicitly in terms of product of all other zeros thanks to the formula (3):

$$a\frac{\partial \Omega_k}{\partial b} + 1 = e^{-2} \prod_{\substack{n\neq k \\ \Omega_n \neq 0}} \left(1 - \frac{\Omega_k}{\Omega_n}\right)^{-2} e^{-2\frac{\Omega_k}{\Omega_n}}, \qquad \Omega_k \neq 0. \tag{20}$$

The derivative with respect to a can be deduced from Eq. (12), giving

$$3a\frac{\partial \Omega_k}{\partial a} + 2b\frac{\partial \Omega_k}{\partial b} + \Omega_k = 0 \tag{21}$$

Eqs. (20) and (21) define a system of an infinite number of differential equations describing the dynamics of the zeros of the solution $S(z, a, b)$ as the parameters a and b in Eq. (2) change. As explained in the introduction, to every solution of Eq. (1) it is associated to the particular solution $S(z, a, b)$ for suitable choices of a and b: more precisely to $S(z, a, b)$ it is associated to the solution of (1) having a zero at $z_0 = b/\eta^2$, where $\eta^3 = a$. The zeros z_k of the particular solutions (all equivalent up to a global scaling factor) of Eq. (1) having a zero at $z_0 = b/\eta^2$ are linearly related to the zeros of $\tau(z, a, b)$, i.e.

$$z_k = \eta\Omega_k + \frac{b}{\eta^2} \tag{22}$$

The zeros Ω_k can be determined, in principle, by the coefficients of the Laurent series c_n (see [14]): equations (20), (21) and (22) then describe the dynamics of the zeros of the solutions of Eq. (1) too.

Let us make some more remarks about formulas (21): when considering the evolution of the zeros by varying the values of b, the dynamics must comply with the constraints given by Eqs. (8) and (9), i.e. the sum of the inverse powers of the zeros are polynomials in a and b, such as

$$b = -3\sum_{\substack{k=1 \\ \Omega_k\neq 0}} \frac{1}{\Omega_k^2}, \quad a = -4\sum_{\substack{k=1 \\ \Omega_k\neq 0}} \frac{1}{\Omega_k^3}, \quad ab = -36\sum_{\substack{k=1 \\ \Omega_k\neq 0}} \frac{1}{\Omega_k^4}, \quad \ldots \tag{23}$$

Other constraints involving each zeros Ω_k can be obtained from Eq. (5) and the expansion (6). Indeed, as we will now show, the zeros must satisfy also the relations

$$\frac{2}{\Omega_k^2} + \sum_{n\neq k} \frac{1}{\Omega_n(\Omega_k - \Omega_n)} = 0, \qquad k \neq 0 \tag{24}$$

Eq. (24) can be proved in different ways. The simplest is to notice that the function $f(z) = u(z) - 1/(z - \Omega_k)$ must have a zero in Ω_k. Indeed, as shown in [14] (see Remark 2.2), the function $S(z, a, b)$ is such that

$$S(z, a, b) = S'(\Omega_k, a, b)S(z - \Omega_k, a, b + a\Omega_k), \tag{25}$$

where Ω_k is any of the zeros of $S(z, a, b)$ and $S'(\Omega_k, a, b)$ is the derivative of $S(z, a, b)$ with respect to z evaluated at $z = \Omega_k$. The converging Taylor series ($S(z, a, b)$ is entire) for $S(z, a, b)$ around zero is $S(z, a, b) = z + O(z^3)$, implying, from (25), that the Taylor series of $S(z, a, b)$ around Ω_k is $S(z, a, b) = z - \Omega_k + O((z - \Omega_k)^3)$. From the definition (4) of the function u then one has the following series around Ω_k

$$u(z, a, b) = \frac{1}{z - \Omega_k} + O((z - \Omega_k)) \tag{26}$$

The relation (26) is valid for any given Ω_k. By considering the expansion over the poles (6) and $f(z)|_{z=\Omega_k} = (u(z) - 1/(z - \Omega_k))|_{z=\Omega_k} = 0$, then one arrives at (24).

Relations among zeros like (10) or (24) are usually called Stieltjes–Calogero (see e.g. [1]) type relations when the zeros are those of orthogonal polynomials. The difference here is that the relations (24) are given for an infinite set of quantities. Eqs. (8) and (9), giving the equivalences (10) between the zeros and polynomials in a and b, can be used to obtain similar equivalences for the zeros of the classical Airy equation (1) (i.e. the equation without the parameters a and b). Indeed, by using the relationship (22) $z_k = \eta\Omega_k + b/\eta^2$ and remembering that the solution of Eq. (1) is fixed to have a zero at $z = z_0 = b/\eta^2$, one has $\Omega_k = (z_k - z_0)/\eta$. This is the value to be used in the right hand side of (9) for Ω_k. For the left hand side, we remember that the coefficients c_n are weighted homogeneous polynomials in a and b, satisfying (11). The value of a is equal to η^3 and b is given by $\eta^2 z_0$, so one has

$$c_n(\eta^3, \eta^2 z_0) = \eta^{n+1} c_n(1, z_0). \tag{27}$$

Putting all together one has the following result for the zeros of the Airy equation (1):

$$c_n(1, z_0) = -\sum_{\substack{k=1 \\ z_k \neq z_0}} \frac{1}{(z_k - z_0)^{n+1}}. \tag{28}$$

These equation appeared in [2] also, where they have been found by extending a result given in [7], relating the sum over the inverse powers of the zeros and the trace of a matrix depending on the coefficients of the Taylor series of the function considered. It is interesting to notice that here the coefficients $c_n(1, z_0)$ are given instead by a recursion. The relation (24) instead seems to be new. For the zeros z_k of the Airy equation (1) it gives

$$\frac{2}{(z_k - z_0)^2} + \sum_{n \neq k} \frac{1}{(z_k - z_0)(z_k - z_n)} = 0, \qquad z_k \neq z_0. \tag{29}$$

4. Numerics

The system of Eqs. (19)–(21) is in a certain sense solvable, since the dynamical quantities are given explicitly by the zeros of the solutions of the Airy equation. The description of the distribution of the zeros has been discussed elsewhere in the literature (see e.g. [8,11,14]). Also, the existence of Stieltjes–Calogero relations (10), (24) seems to be an analogy with the theory of integrable systems, since these relations can be considered constraints between the dynamical variable systems, just like the conserved quantities. There is however a difference with respect to classical integrable systems. Indeed, the solution of the differential equations (20) must be supplied with initial conditions. The initial conditions to be taken are just the set of the zeros $\Omega_k(a, b)$ for some fixed values of the parameters a and b. But we do not know what happens at the relations (10) or (24) if *arbitrary* initial conditions are taken, that is if the initial values $\Omega_k(a, b)$ are not those corresponding to the zeros of the solution for that particular choice of a and b. Actually it can be shown that this problem is related to the dynamics of the value distribution of the function $S(z, a, b)$, that is to the dynamics of the solution of $S(z, a, b) = c$, where c is a constant. A thorough analysis of this problem is not easy and deserves more research in the future to be completely addressed.

It is interesting to obtain some numerical considerations based on Eqs. (20). We solved numerically equation (20) by fixing the values of $a = 1$ and taking $b = -1$ as initial condition. Equations (20) then has been considered a set of ordinary differential equations with respect to the real independent variable b. We looked at the evolution of the two zeros closest to the origin of $S(z, 1, b)$. We used the initial conditions $\Omega_1(1, -1) = -2.16463140(5)$ and $\Omega_2(1, -1) = -3.74319657(7)$, corresponding to the values of the zeros at $b = -1$. The result is given in Fig. 1: as can be seen the zeros run on two lines almost parallel for positive values of b, i.e. their difference stabilizes on a given value. In Fig. 2 it is reported the plot of the evolution of $\Omega(a, b)$ as a function of b and for a fixed to be equal to 1, for values of b in the range $(-20, 20)$: now the initial values have been taken to be different to one of the zeros of $S(z, 1, b)$ for $b = -1$: a certain regularity in the solution was expected, since, as said, this should be the dynamics of the solution of the roots of the equation $S(z, 1, b) = c$ for suitable values of c, but indeed the bell shaped curve was unexpected.

5. Conclusions

In this work we described the dynamics of the zeros of the solution of the Airy equations (1) and (2): the zeros of the two equations are related by a linear relation, so we choose to analyze the characteristics of the solutions of Eq. (2) since they have other nice properties, like homogeneity with respect to the parameters (a, b) and the independent variable z (for further properties, like addition formulas generalizing trigonometric identities or a recursion based on Eqs. (8) and (9) the interested reader can look at [14]). There are different directions of

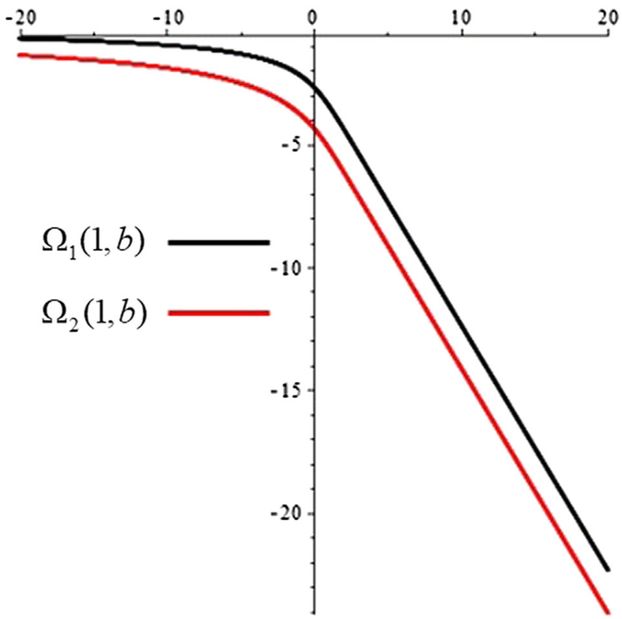

Fig. 1. The evolution of the first two zeros of $S(z, 1, b)$ for $b \in (-20, 20)$.

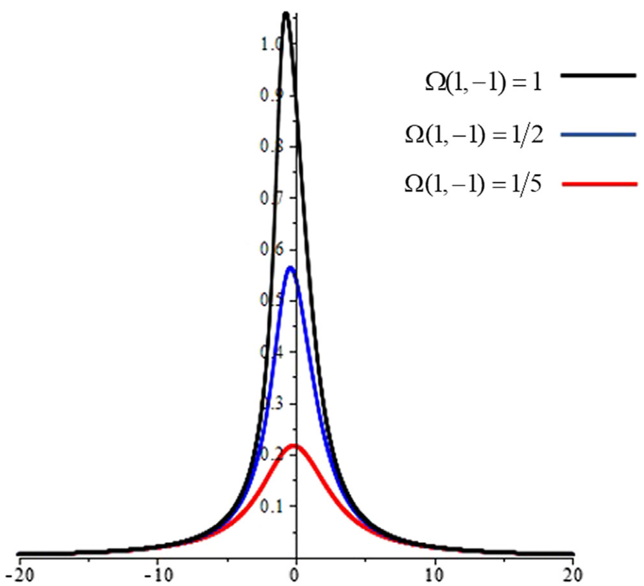

Fig. 2. The solution of Eq. (19) for $a = 1$ and the initial values $\Omega(1, -1) = 1$ (black), $\Omega(1, -1) = 1/2$ (blue) and $\Omega(1, -1) = 1/5$ (red).

research that may follow this work: for example it is known (see e.g. [3,4] and [6]) that the Airy's function Ai(z) is connected with special solutions of the Painlevè II equation. The results obtained here and in [14] may have interesting implications on the descriptions of the singularities of those exact solutions. Also, the Stieltjes relations obtained are known for the case of Eq. (1), apart equations (24) that seems to be new. By viewing at the parameter b as a time variable, the set of Eqs. (23) and (24) can be viewed as conserved quantities for the set of Eqs. (20). The analogy of these differential equations with the theory of integrable systems is intriguing and deserves more work to be analyzed. The possibility to describe, with the use of the differential equations (20), the dynamics of the value distributions of Airy functions, and the extension of this methodology to other transcendental functions, seems to be an interesting direction of research too and will be the subject of future investigations.

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