

# Minimum distance of Symplectic Grassmann codes

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## Abstract

In this paper we introduce symplectic Grassmann codes, in analogy to ordinary Grassmann codes and orthogonal Grassmann codes, as projective codes defined by symplectic Grassmannians. Lagrangian–Grassmannian codes are a special class of symplectic Grassmann codes. We describe all the parameters of line symplectic Grassmann codes and we provide the full weight enumerator for the Lagrangian–Grassmannian codes of rank 2 and 3.

*Keywords:* Symplectic Grassmannian, Dual Polar Space, Error Correcting Code, Lagrangian Grassmannian Code

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## 1. Introduction

Grassmann codes have been introduced in [20, 21] as generalizations of first order Reed–Muller codes and they have been extensively investigated ever since. These are projective codes arising from the Plücker embedding of a  $k$ –Grassmannian. Their parameters, as well as some of their higher weights have been fully determined in [17]. A further point of interest is that the weight distribution provides some interesting insight on the geometry of the embedding itself.

Codes arising from the Plücker embedding of the  $k$ –Grassmannian of an orthogonal polar space have been introduced in a recent series of papers [4, 5, 6]. In [4], we computed the minimum distance for the codes arising from orthogonal dual polar spaces of rank 2 and 3 and provided a general bound on their minimum distance. More recently, in [6], for  $q$  odd the minimum distance for all line polar Grassmann codes of orthogonal type has been determined. In [5] an encoding scheme, as well as strategies for decoding and error correction, has been proposed for line polar Grassmann codes.

The aim of the present paper is to provide results analogous to those of [4, 6] for codes arising from the Plücker embedding of  $k$ –Grassmannians of symplectic type.

More in detail, we shall denote by  $\mathcal{W}(n, k)$ , the projective code defined by the image under the Plücker embedding of the  $k$ –symplectic Grassmannian  $\Lambda_{n,k}$  defined by a non–degenerate alternating bilinear form  $\sigma$  on a vector space  $V := V(2n, q)$  of dimension  $2n$  over a finite field  $\mathbb{F}_q$ . This will be referred as a *symplectic Grassmann code*.

The paper is organized as follows: in Section 2 some basic notions about projective codes and symplectic Grassmannians are recalled; Section 3 is dedicated to line symplectic Grassmann codes and contains our main results for  $k = 2$ ; Section 4 is devoted to the case of rank  $k = 3$ . Overall, in these sections we prove the following.

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**Main Theorem.** *The code  $\mathcal{W}(n, k)$  has parameters*

$$N = \prod_{i=0}^{k-1} (q^{2n-2i} - 1) / (q^{i+1} - 1), \quad K = \binom{2n}{k} - \binom{2n}{k-2}.$$

Furthermore,

- For  $k = 2$ , its minimum distance is  $q^{4n-5} - q^{2n-3}$ ;
- For  $n = k = 3$ , its minimum distance is  $q^6 - q^4$ .

Finally, in Section 5 we discuss, in the general case of symplectic Grassmann codes, some further bounds for the minimum distance arising from higher weights of ordinary Grassmann codes.

We point out that the code  $\mathcal{W}(n, n)$  where  $k = n$ , corresponding to the so called *dual polar space*, has already been introduced under the name of Lagrangian-Grassmannian code of rank  $n$  in [7], where also some bounds on the parameters have been obtained.

## 2. Preliminaries

A  $[N, K, d_{\min}]$  projective system  $\Omega \subseteq \text{PG}(K-1, q)$  is just a set of  $N$  distinct points in  $\text{PG}(K-1, q)$  whose span is  $\text{PG}(K-1, q)$  and such that there is a hyperplane  $\Sigma$  of  $\text{PG}(K-1, q)$  with  $\#(\Omega \setminus \Sigma) = d_{\min}$  and for any hyperplane  $\Sigma'$  of  $\text{PG}(K-1, q)$ ,

$$\#(\Omega \setminus \Sigma') \geq d_{\min}.$$

It is well known that existence of a  $[N, K, d_{\min}]$  projective system is equivalent to that of a projective linear code  $\mathcal{C}$  with the same parameters; in particular

$$d_{\min}(\mathcal{C}) = \#\Omega - \max\{\#(\Omega \cap \Sigma) : \Sigma \text{ hyperplane of } \text{PG}(K-1, q)\}. \quad (1)$$

Indeed, by taking as generator matrix  $G$  the matrix whose columns are the coordinates of the points of  $\Omega$  normalized in some way, several codes can be obtained. As the order of the points, the choice of coordinates as well as the normalization adopted change, we obtain potentially different codes arising from  $\Omega$ , but all of these turn out to be equivalent. As such, in the following discussion, they will be silently identified and we shall write  $\mathcal{C} = \mathcal{C}(\Omega)$ . The spectrum of the intersections of  $\Omega$  with the hyperplanes of  $\text{PG}(K-1, q)$  provides the list of the weights of  $\mathcal{C}$ ; we refer to [23] for further details.

Let now and throughout the paper  $V := V(2n, q)$  be a  $2n$ -dimensional vector space equipped with a non-degenerate bilinear alternating form  $\sigma$ . Denote by  $\mathcal{G}_{2n, k}$  the  $k$ -Grassmannian of the projective space  $\text{PG}(V)$ , that is the point-line geometry whose points are the  $k$ -dimensional subspaces of  $V$  and whose lines are the sets

$$\ell_{W, T} := \{X : W \leq X \leq T, \dim X = k\}$$

with  $\dim W = k-1$  and  $\dim T = k+1$ . A projective embedding of  $\mathcal{G}_{2n, k}$  is a function  $e : \mathcal{G}_{2n, k} \rightarrow \text{PG}(U)$  such that  $\langle e(\mathcal{G}_{2n, k}) \rangle = \text{PG}(U)$  and each line of  $\mathcal{G}_{2n, k}$  is mapped onto a line of  $\text{PG}(U)$ . The dimension of  $U$  is called *dimension of the embedding*. It is well known that the geometry  $\mathcal{G}_{2n, k}$  affords a projective embedding  $e_k^{gr} : \mathcal{G}_{2n, k} \rightarrow \text{PG}(\bigwedge^k V)$  by means of Plücker coordinates. In particular,  $e_k^{gr}$  maps an arbitrary  $k$ -dimensional subspace  $\langle v_1, v_2, \dots, v_k \rangle$  of  $V$  to the point  $\langle v_1 \wedge v_2 \wedge \dots \wedge v_k \rangle$  of  $\text{PG}(\bigwedge^k V)$ . The image  $e_k^{gr}(\mathcal{G}_{2n, k})$  is a projective variety

of  $\text{PG}(\bigwedge^k V)$ , usually denoted by the symbol  $\mathbb{G}(2n-1, k-1)$ , see [14, Lecture 6], called the *Grassmann variety*.

The symplectic Grassmannian  $\Lambda_{n,k}$  induced by  $\sigma$ , is defined for  $k = 1, \dots, n$  as the subgeometry of  $\mathcal{G}_{2n,k}$  having as points the totally  $\sigma$ -isotropic subspaces of  $V$  of dimension  $k$  and as lines

- for  $k < n$ , the sets of the form

$$\ell_{W,T} := \{X : W \leq X \leq T, \dim X = k\}$$

with  $T$  totally isotropic and  $\dim W = k-1$ ,  $\dim T = k+1$ .

- for  $k = n$ , the sets of the form

$$\ell_W := \{X : W \leq X, \dim X = n\}$$

with  $\dim W = n-1$ ,  $W$  totally isotropic.

For  $k = n$ ,  $\Lambda_{n,n}$  is usually called *dual polar space of rank  $n$*  or *Lagrangian Grassmannian*.

The image of  $\Lambda_{n,k}$  under the Plücker embedding  $e_k^{gr}$  is a subvariety  $\mathbb{L}(n-1, k-1)$  of the Grassmann variety  $\mathbb{G}(2n-1, k-1)$ .

Let  $\Sigma = \langle \mathbb{L}(n-1, k-1) \rangle < \text{PG}(\bigwedge^k V)$ . It is well known, see [9, 19], that

$$\dim \Sigma = \binom{2n}{k} - \binom{2n}{k-2};$$

indeed, the variety  $\mathbb{L}(n-1, k-1)$  is the full intersection of  $\mathbb{G}(2n-1, k-1)$  with a suitable subspace of  $\bigwedge^k V$  of codimension  $\binom{2n}{k-2}$ .

The following formula provides the length of  $\mathcal{W}(n, k)$ :

$$\#\mathbb{L}(n-1, k-1) = \#\Lambda_{n,k} = \prod_{i=0}^{k-1} (q^{2n-2i} - 1)/(q^{i+1} - 1). \quad (2)$$

As pointed out before, the pointset of  $\mathbb{L}(n-1, k-1)$  is a projective system of  $\text{PG}(\Sigma)$ ; this determines a projective code which we shall denote by  $\mathcal{W}(n, k)$  and call it a *symplectic Grassmann code*. A straightforward consequence of the remarks presented above is the following lemma.

**Lemma 2.1.** *The code  $\mathcal{W}(n, k)$  has length  $N = \#\mathbb{L}(n-1, k-1)$  and dimension  $K = \dim(\Sigma)$ .*

### 3. Line Symplectic Grassmann Codes

Throughout this section  $\mathcal{S} := W(2n-1, q)$  denotes the non-degenerate symplectic polar space of rank  $n$  defined by a fixed non-degenerate alternating bilinear form  $\sigma$  on  $V$ . By  $\theta$  we shall denote a different (possibly degenerate) alternating bilinear form on  $V$ . We shall also write  $\perp_\sigma$  and  $\perp_\theta$  for the orthogonality relations induced by  $\sigma$  and  $\theta$  respectively. Recall that the *radical* of  $\theta$  is the set

$$\text{Rad } \theta := \{x \in \mathcal{S} : x^\perp_\theta = \text{PG}(V)\} = \{x \in \mathcal{S} : \forall y \in \mathcal{S}, \theta(x, y) = 0\}.$$

For  $k = 2$ , Expression (2) and  $K = \dim(\Sigma)$  become

$$N := \#\Lambda_{n,2} = \frac{(q^{2n} - 1)(q^{2n-2} - 1)}{(q-1)(q^2 - 1)}; \quad K := 2n^2 - n - 1.$$

It is well known that any bilinear alternating form  $\theta$  determines a hyperplane of  $\bigwedge^2 V$  and conversely; hence, the minimum distance of  $\mathcal{W}(n, 2)$  can be deduced from the maximum number of lines which are simultaneously totally isotropic for both the forms  $\theta$  and  $\sigma$ , under the assumption  $\theta \neq \sigma$ . In order to determine this number we follow an approach similar to that of [6, Lemma 3.2].

**Lemma 3.1.** *Suppose  $M$  is the matrix representing  $\sigma$  and  $S$  the matrix representing  $\theta$  with respect to a given reference system. For any  $p \in \mathcal{S}$  we have  $p^{\perp\sigma} \subseteq p^{\perp\theta}$  if, and only if,  $p$  is an eigenvector of the matrix  $M^{-1}S$ .*

*Proof.* Since  $M$  is non-singular, if  $M^{-1}Sp = 0$ , then  $Sp = 0$ , that is to say  $p \in \text{Rad}\theta$ , i.e.  $p^{\perp\theta} = \text{PG}(V)$ . In this case, obviously,  $p^{\perp\sigma} \subseteq p^{\perp\theta}$ .

When  $p \notin \text{Rad}\theta$ , we have  $\dim p^{\perp\sigma} = \dim p^{\perp\theta}$ . Thus,  $p^{\perp\sigma} \subseteq p^{\perp\theta}$  if, and only if,  $p^{\perp\sigma} = p^{\perp\theta}$ , that is to say the systems of equations  $x^T M p = 0$  and  $x^T S p = 0$  are equivalent. This yields  $Sp = \lambda M p$  for some  $\lambda \neq 0$ , whence  $p$  is an eigenvector of eigenvalue  $\lambda$  for  $M^{-1}S$ .  $\square$

Write now

$$N_0 := \#\{p \in \mathcal{S} : p^{\perp\sigma} \not\subseteq p^{\perp\theta}\}, \quad N_1 := \#\{p \in \mathcal{S} : p^{\perp\sigma} \subseteq p^{\perp\theta}\}.$$

Clearly,  $N_0 = \frac{q^{2n}-1}{q-1} - N_1$ .

For any  $p \in \mathcal{S}$ , a line  $\ell$  through  $p$  is both totally  $\sigma$ -isotropic and  $\theta$ -isotropic if, and only if,  $\ell \in p^{\perp\sigma} \cap p^{\perp\theta}$ . In particular,

- if  $p^{\perp\sigma} \subseteq p^{\perp\theta}$ , then  $\frac{q^{2n-2}-1}{q-1}$  lines through  $p$  are both  $\sigma$ - and  $\theta$ -isotropic;
- if  $p^{\perp\sigma} \not\subseteq p^{\perp\theta}$ , then  $p^{\perp\sigma} \cap p^{\perp\theta}$  is a subspace of codimension 2 in  $\text{PG}(V)$  and the number of lines which are both  $\sigma$ - and  $\theta$ -isotropic is  $\frac{q^{2n-3}-1}{q-1}$ .

Denote now by  $\eta$  the number of lines of  $\mathcal{S}$  which are simultaneously totally  $\sigma$ - and  $\theta$ -isotropic. As each line contains  $(q+1)$  points, we have

$$(q+1)\eta = N_0 \frac{q^{2n-3}-1}{q-1} + N_1 \frac{q^{2n-2}-1}{q-1} = q^{2n-3}N_1 + \frac{(q^{2n}-1)(q^{2n-3}-1)}{(q-1)^2}. \quad (3)$$

Clearly,  $\eta$  is maximum when  $N_1$  is maximum. In the remainder of this section we shall determine exactly how large  $N_1$  can be.

**Lemma 3.2.** *Let  $M$  and  $S$  be as in Lemma 3.1. If the matrix  $M^{-1}S$  has just two eigenspaces, one of dimension  $2n-2$ , the other of dimension 2, then the number of eigenvectors of  $M^{-1}S$  is maximum.*

*Proof.* It is straightforward to see that, in order for the number of eigenvectors of  $M^{-1}S$  to be maximum, we need  $M^{-1}S$  to be diagonalizable.

Observe that since  $S$  is antisymmetric, its rank is necessarily even; in particular, the rank of  $M^{-1}S$  is also even. Suppose first that  $M^{-1}S$  has just two eigenvalues,  $\lambda \neq 0$  is one of them and the corresponding eigenspace is maximum and has dimension  $g$ . The number of simultaneously  $\sigma$ - and  $\theta$ -isotropic lines  $\ell = \langle v_1, v_2 \rangle$  is the same as the number of lines which are simultaneously  $\sigma$ - and  $(\theta - \lambda\sigma)$ -isotropic, as  $\sigma(v_1, v_2) = 0$  and  $\theta(v_1, v_2) = 0$  yields  $(\theta - \lambda\sigma)(v_1, v_2) = \theta(v_1, v_2) - \lambda\sigma(v_1, v_2) = 0$ . The latter alternating form, say  $\theta' = \theta - \lambda\sigma$  is represented by the matrix  $S' = S - \lambda M$ . In particular, we can replace  $S$  with  $S'$  and we get

$$M^{-1}S' = M^{-1}(S - \lambda M) = M^{-1}S - \lambda I.$$

For this new matrix, 0 is an eigenvalue with eigenspace of dimension  $g$ . Thus,  $g$  must be even. We conclude that the maximum number of eigenvectors occurs when  $g = 2n - 2$  and there is a further eigenspace of dimension 2, that is

$$\#V_\lambda + \#V_\mu - 2 = q^{2n-2} + q^2 - 2.$$

Suppose now that there are at least 3 distinct eigenspaces for  $M^{-1}S$ , say  $V_\alpha, V_\beta$  and  $V_\gamma$  of dimensions respectively  $a, b, c$  with  $a \leq b \leq c$ . Then, the number  $\zeta$  of eigenvectors is

$$\#V_\alpha + \#V_\beta + \#V_\gamma - 3 \leq \zeta \leq q^a + q^b + q^{2n-a-b} - 3.$$

Clearly  $a, b \geq 1$  and  $2n - a - b \geq c \geq a, b$ . In particular, the upper bound is maximum for  $a = b = 1$ , in which case we get

$$\zeta \leq 2q + q^{2n-2} - 3 < q^2 + q^{2n-2} - 2.$$

Thus, the maximum number of eigenvectors which can be obtained with just 2 eigenspaces, say  $V_\alpha$  and  $V_\beta$  is larger than what is possible with at least 3 distinct eigenspaces. This completes the proof.  $\square$

The previous lemma gives

$$N_1 \leq \frac{q^{2n-2} - 1}{q - 1} + \frac{q^2 - 1}{q - 1}.$$

Plugging in this value in (3) we obtain

$$\eta \leq \frac{q^{4n-3} + q^{4n-4} - q^{4n-5} - q^{2n-1} - 2q^{2n-2} + q^{2n-3} + 1}{(q-1)(q^2-1)}. \quad (4)$$

whence we get the following lemma.

**Lemma 3.3.**

$$d_{\min}(\mathcal{W}(n, 2)) \geq q^{4n-5} - q^{2n-3}.$$

*Proof.* By Equation (1), we have  $d_{\min} \geq N - \eta$ . The estimate follows from (4).  $\square$

We are now ready to prove our main theorem for line symplectic Grassmann codes.

**Theorem 3.4.** *The minimum distance of the code  $\mathcal{W}(n, 2)$  is  $d_{\min}(\mathcal{W}(n, 2)) = q^{4n-5} - q^{2n-3}$ .*

*Proof.* We shall show that, given a non-degenerate alternating form  $\sigma$  represented by a matrix  $M$ , it is always possible to define an alternating form  $\theta$  represented by a matrix  $S$  such that  $M^{-1}S$  has only two eigenspaces, one of dimension  $2n - 2$  and the other of dimension 2.

In order to prove this, let  $\ell = \langle v_1, v_2 \rangle$  be a line of  $\text{PG}(2n - 1, q)$  which is not  $\sigma$ -isotropic and define an alternating form  $\theta$  such that  $\theta(v_1, v_2) = \sigma(v_1, v_2)$  and  $\text{Rad } \theta = \ell^\perp$ .

Take  $B = B_1 \cup B_2$  to be an ordered basis of  $V$  where  $B_1 = (v_1, v_2)$  and  $B_2$  is an ordered basis of  $\ell^\perp$ .

Let  $M$  be the matrix representing  $\sigma$  with respect to  $B$ . We can suppose  $\theta(v_1, v_2) = \sigma(v_1, v_2) = 1$ ; thus  $M = \text{diag}(M_{11}, M_{22})$  is a block diagonal matrix where

$$M_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } M_{22} = \begin{pmatrix} O_{n-1} & I_{n-1} \\ -I_{n-1} & O_{n-1} \end{pmatrix}$$

with  $O_{n-1}$  the null  $(n-1) \times (n-1)$ -matrix and  $I_{n-1}$  the  $(n-1) \times (n-1)$ -identity matrix. By construction, the matrix  $S$  representing  $\theta$  with respect to  $B$  is also block diagonal  $S = \text{diag}(S_{11}, O_{2n-2})$  with  $S_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $O_{2n-2}$  the null  $(2n-2) \times (2n-2)$ -matrix.

Hence  $M^{-1}S$  is the block diagonal matrix  $M^{-1}S = \text{diag}(I_2, O_{2n-2})$ . Clearly,  $M^{-1}S$  has only two eigenspaces, one of dimension  $2n-2$  and the other of dimension 2. The thesis now follows from Lemma 3.2 and Lemma 3.3.  $\square$

The proof of Theorem 3.4 holds also for the code  $\mathcal{W}(2, 2)$  arising from the dual polar space  $\Lambda_{2,2}$ . However, in this case we can easily provide the full weight enumerator.

**Proposition 3.5.** *The code  $\mathcal{W}(2, 2)$  has exactly 3 nonzero weights, namely  $q^3 + q$ ,  $q^3 - q$  and  $q^3$  and the following weight enumerator*

Weight	# Codewords
$q^3 - q$	$q^2(q^2 + 1)(q - 1)/2$
$q^3$	$q^4 - 1$
$q^3 + q$	$q^2(q^2 - 1)(q - 1)/2$

*Proof.* The Lagrangian-Grassmannian  $\mathbb{L}(1, 1)$  is a non-singular hyperplane section of the ordinary line-Grassmannian  $\mathbb{G}(3, 1)$  of  $\text{PG}(3, q)$ . In particular  $\mathbb{L}(1, 1) = \mathbb{G}(3, 1) \cap \Sigma = Q(4, q)$ , where  $\Sigma$  is a suitable hyperplane of  $\text{PG}(4, q)$ , depending only on  $\sigma$  and  $Q(4, q)$  is a non-singular parabolic quadric. Thus, the code  $\mathcal{W}(2, 2)$  is the same as the code determined by the projective system of  $Q(4, q)$  in a  $\text{PG}(4, q)$ . The 3 weights of this code correspond to hyperplanes which meet  $Q(4, q)$  in either a quadratic cone, an elliptic quadric  $Q^-(3, q)$  or an hyperbolic quadric  $Q^+(3, q)$ . In particular, the hyperplanes of  $\text{PG}(4, q)$  lie in 3 orbits under the action of the orthogonal group  $O(4, q)$ . The tangent hyperplanes to  $Q(4, q)$  are  $q^3 + q^2 + q + 1$ , and each of them determines a cone consisting of  $q^2 + q + 1$  points; thus the associated weight is  $q^3$ . As for the remaining two orbits, the one determining elliptic quadrics, (having an elliptic quadric cardinality  $q^2 + 1$ ), consists of  $q^2(q^2 - 1)/2$  elements, while that inducing hyperbolic quadrics, (having a hyperbolic quadric cardinality  $(q + 1)^2$ ), has size  $q^2(q^2 + 1)/2$ ; see [15, Chapter 22]. The corresponding weights are  $q^3 + q$  and  $q^3 - q$ . As each hyperplane corresponds to  $(q - 1)$  words, this provides the complete enumerator.  $\square$

#### 4. Lagrangian-Grassmannian codes of rank 3

In this section we shall provide the full weight enumerator for the Lagrangian-Grassmannian code  $\mathcal{W}(3, 3)$ , and discuss some codes arising from different embeddings of the symplectic Grassmannian  $\Lambda_{3,3}$ .

**Theorem 4.1.** *For  $k = n = 3$ , the minimum distance of the code  $\mathcal{W}(3, 3)$  is  $q^6 - q^4$ . The enumerator is as follows*

Weight	# Codewords
$q^6 - q^4$	$\frac{1}{2}q^2(q^2 + 1)(q^2 + q + 1)(q^3 + 1)(q - 1)$
$q^6$	$(q + 1)^2(q^2 - q + 1)(q^2 + 1)(q^6 - q^3 + 1)(q - 1)$
$q^6 + q^3$	$q^9(q^4 - 1)(q - 1)$
$q^6 + q^4$	$\frac{1}{2}q^2(q + 1)(q^6 - 1)(q - 1)$

Furthermore, all codewords of minimum weight lie in the same orbit.

*Proof.* The theorem is a direct consequence of the classification of the classical (geometric) hyperplanes of the dual polar space  $\Lambda_{3,3}$  of rank 3 arising from the Grassmann embedding, as provided in [8, 10]. We refer, in particular, to [8, Tables 1,2,3] for the exact number of hyperplanes of each type and the cardinalities of their intersection.  $\square$

**Remark 4.2.** We point out that for  $q = 2^h$ , the Lagrangian-Grassmannian  $\Lambda_{n,n}$  always affords the spin embedding in  $\text{PG}(2^n - 1, q)$ ; see [2] for a description. In particular,  $\Lambda_{3,3}$  can also be embedded in  $\text{PG}(7, q)$ . Such an embedding gives rise to a projective system with parameters  $N = q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1$  and  $K = 8$ . The corresponding code has just two weights, see [10], namely

Weight	# Codewords
$q^6$	$(q^2 - 1)(q^2 + 1)(q^3 + 1)$
$q^6 + q^3$	$(q^7 - q^3)(q - 1)$ .

We observe that for  $q = 2$ , this determines a  $[135, 8, 64]$  code, and the best known code with length  $N = 135$  and dimension  $K = 8$  has minimum distance  $d = 65$  (see [13]).

Likewise,  $\Lambda_{4,4}$  can be embedded in  $\text{PG}(15, q)$  and there it also determines a 2-weight code of parameters  $[N, K] = [(q^4 + 1)(q^3 + 1)(q^2 + 1)(q + 1), 16]$  and weights  $q^{10}$  and  $q^{10} + q^7$ ; see [3].

**Remark 4.3.** For  $q = 2$ , the universal embedding of  $\Lambda_{3,3}$  is different from the Grassmann embedding and it spans a  $\text{PG}(14, 2)$ ; see [1, 16]. As such it determines a code of length  $N = 135$  and dimension  $K = 15$  with weight enumerator as follows, see [10, 18]:

Weight	# Codewords	Weight	# Codewords
30	36	70	10368
48	630	72	7680
54	1120	78	1080
62	3780	80	378.
64	7695		

In particular, the minimum distance in this case is 30.

## 5. Further bounds on the minimum distance

As  $\mathbb{L}(n - 1, k - 1)$  is a section of  $\mathbb{G}(2n - 1, k - 1)$  with a subspace of codimension  $\binom{2n}{k-2}$ , it is possible to provide a bound on the minimum distance of  $\mathcal{W}(n, k)$  in terms of higher weights of the projective Grassmann code induced by the projective system  $\mathbb{G}(n - 1, k - 1)$ . Recall that the  $r$ -th higher weight of a code  $\mathcal{C}$  induced by a projective system  $\Omega$  consisting of  $N$  points is

$$d_r := N - \max\{\#(\Omega \cap \Pi) : \Pi \text{ projective subspace of codimension } r \text{ in } \langle \Omega \rangle\};$$

see [24] for the definition and some properties, as well as [22] for its geometric interpretation; in the case of Grassmann codes they have been extensively studied in [11, 12, 17]. As  $\mathcal{W}(n, k)$  can be regarded as the intersection of the Grassmannian  $\mathbb{G}(2n - 1, k - 1)$  with a suitable subspace  $\Sigma$  of codimension  $\binom{2n}{k-2}$ , we have

$$\begin{aligned} d_{\min}(\mathcal{W}(n, k)) &= \#\mathcal{W}(n, k) - \max\{\#(\mathbb{G}(2n - 1, k - 1) \cap \Pi) : \Pi \leq \Sigma, \dim(\Sigma/\Pi) = 1\} \geq \\ &\#\mathcal{W}(n, k) - \max\{\#(\mathbb{G}(2n - 1, k - 1) \cap \Pi) : \text{codim } \wedge^k V(\Pi) = \binom{2n}{k-2} + 1\} = \\ &\#\mathcal{W}(n, k) - \#\mathbb{G}(2n - 1, k - 1) + d_s, \end{aligned}$$

where  $s = \binom{2n}{k-2} + 1$  and  $d_s$  is the  $s$ -th higher weight of the Grassmann code arising from  $\mathcal{G}_{2n,k}$ . In general, this bound is not sharp. This can be seen directly by considering the case of the code  $\mathcal{W}(n, 2)$ . Indeed, using the the second highest weight of the Grassmann code, see [17], we see that

$$\begin{aligned} d_{\min}(\mathcal{W}(n, 2)) &\geq \frac{(q^{2n} - 1)(q^{2n-2} - 1)}{(q - 1)(q^2 - 1)} - \begin{bmatrix} 2n \\ 2 \end{bmatrix}_q + q^{2(2n-2)-1}(q + 1) = \\ &\frac{(q^{2n} - 1)(q^{2n-2} - q^{2n-1})}{(q - 1)(q^2 - 1)} + q^{2(2n-2)-1}(q + 1) = \frac{q^{4n-2} - 2q^{4n-3} + q^{4n-5} + q^{2n-1} - q^{2n-2}}{(q - 1)(q^2 - 1)} \approx \\ & q^{4n-5} - q^{4n-6}. \end{aligned}$$

This, however, is quite far away from the correct value for line symplectic Grassmann codes, namely  $d_{\min}(\mathcal{W}(n, 2)) = q^{4n-5} - q^{2n-3}$ , as we have determined in Section 3.

We point out that in [7, Proposition 5], an upper bound on the minimum distance for Lagrangian-Grassmannian codes is given in terms of the dimension of the Lagrangian-Grassmannian variety, that is

$$d_{\min}(\mathcal{W}(n, n)) \leq q^{n(n+1)/2}.$$

By Section 4, we see that this bound is not sharp for  $n = 2$  and  $n = 3$ .

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