$t$-Intersection sets in $AG(r, q^2)$ and two-character multisets in $PG(3, q^2)$

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Abstract

In this article we construct new minimal intersection sets in $AG(r, q^2)$ with respect to hyperplanes, of size $q^{2r-1}$ and multiplicity $t$, where $t \in \{q^{2r-3} - q^{(3r-5)/2}, q^{2r-3} + q^{(3r-5)/2} - q^{(3r-3)/2}\}$, for $r$ odd or $t \in \{q^{2r-3} - q^{(3r-4)/2}, q^{2r-3} - q^{r-2}\}$, for $r$ even. As a byproduct, for any odd $q$ we get a new family of two-character multisets in $PG(3, q^2)$.

The essential idea is to investigate some point-sets in $AG(r, q^2)$ satisfying the opposite of the algebraic conditions required in [1] for quasi–Hermitian varieties.

Keywords: Hermitian variety, quadric, two-character set.

1 Introduction

All non–degenerate Hermitian varieties of $PG(r, q^2)$ are projectively equivalent; furthermore, they sport just two intersection numbers with hyperplanes, see [6]. Quasi–Hermitian varieties $\mathcal{V}$ of $PG(r, q^2)$ are combinatorial objects which have the same size and the same intersection numbers with hyperplanes as a (non–degenerate) Hermitian variety $\mathcal{H}$; see [1] for details and some constructions. In the present paper we shall consider varieties $\mathcal{V}$ arising by taking algebraic conditions opposite to those of [1] and show that these are in turn interesting geometric objects with 3 intersection numbers. The topic is also of interest for applications, as the projective system induced by $\mathcal{V}$ will determine linear codes with few weights; see [2] for a description of this correspondence.

Fix a projective frame in $PG(r, q^2)$ and assume the space to have homogeneous coordinates $(X_0, X_1, \ldots, X_r)$. Consider the affine plane $AG(r, q^2)$ whose infinite

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hyperplane $\Pi_\infty$ has equation $X_0 = 0$. Then, $\text{AG}(r, q^2)$ has affine coordinates $(x_1, x_2, \ldots, x_r)$ where $x_i = X_i / X_0$ for $i \in \{1, \ldots, r\}$.

Consider now the non–degenerate Hermitian variety $H$ with affine equation of the form

$$x_r^q - x_r = (b^q - b)(x_1^{q+1} + \ldots + x_{r-1}^{q+1}),$$

where $b \in GF(q^2) \setminus GF(q)$. The set of the points at infinity of $H$ is

$$F = \{(0, x_1, \ldots, x_r) | x_1^{q+1} + \ldots + x_{r-1}^{q+1} = 0\};$$

this can be regarded as a Hermitian cone of $PG(r - 1, q^2)$, projecting a Hermitian variety of $PG(r - 2, q^2)$ from the point $P_\infty := (0, \ldots, 0, 1)$. In particular, observe that the hyperplane $\Pi_\infty$ is tangent to $H$ at $P_\infty$.

For any $a \in GF(q^2)^*$ and $b \in GF(q^2) \setminus GF(q)$, let $B := B(a, b)$ be the affine algebraic equation of variation

$$x_r^q - x_r + a^q(x_1^{2q} + \ldots + x_{r-1}^{2q}) - a(x_1^2 + \ldots + x_{r-1}^2) = (b^q - b)(x_1^{q+1} + \ldots + x_{r-1}^{q+1}).$$

It is shown in [1] that $B(a, b)$, together with the points at infinity of $H$, as given by [2], is a quasi–Hermitian variety $V$ of $PG(r, q^2)$ provided that either of the following algebraic conditions are satisfied: for $q$ odd, $r$ is odd and $4a^{q+1} + (b^q - b)^2 \neq 0$, or $r$ is even and $4a^{q+1} + (b^q - b)^2$ is a non–square in $GF(q)$; for $q$ even, $r$ is odd, or $r$ is even and $\text{Tr} \left(a^{q+1}/(b^q + b^2)\right) = 0$.

In this paper, as stated before, we shall study the variety $B(a, b)$ when the opposite of the previous conditions holds. More precisely our main results are the following

**Proposition 1.1.** Suppose $q$ odd, $4a^{q+1} + (b^q - b)^2 = 0$ and $r$ odd. Then $B(a, b)$ is a set of $q^{2r-1}$ points of $\text{AG}(r, q^2)$ of characters:

- for $r \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$
  $$q^{2r-3} - q^{(3r-5)/2}, q^{2r-3}, q^{2r-3} - q^{(3r-5)/2} + q^{3(r-1)/2};$$

- for $r \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$
  $$q^{2r-3} + q^{(3r-5)/2} - q^{3(r-1)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-5)/2};$$

- for $r$ even,
  $$q^{2r-3} - q^{(3r-4)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}.$$

Furthermore $B(a, b)$ is always a minimal intersection set with respect to hyperplanes.
Theorem 1.2. Suppose $q$ odd and $4a^{q+1} + (b^q - b)^2 = 0$. In $PG(3, q^2)$ there exists a $2$–character multiset $\mathcal{B}(a, b)$ containing $\mathcal{B}(a, b)$ and characters either $q^3 - q^2$ and $2q^3 - q^2$ if $q \equiv 1 \pmod{4}$, or $q^2$, and $q^3 + q^2$ if $q \equiv 3 \pmod{4}$.

These results are proved respectively in Section 3 and in Section 4.

Finally, in Section 5 we prove that in the remaining cases we again get minimal intersection sets of the same size but multiplicity $q^{2r-3} - q^{r-2}$.

2 Preliminaries

2.1 Intersection sets with respect to hyperplanes

A set of points $\mathcal{B}$ in a projective or an affine space is a $t$–fold blocking set with respect to hyperplanes if every hyperplane contains at least $t$ points of $\mathcal{B}$. Such a set $\mathcal{B}$ is also known as a $t$-intersection set, or an intersection set with multiplicity $t$, or a multiple intersection set.

A point $P$ of a $t$-intersection set $\mathcal{B}$ is said to be essential if $\mathcal{B} \setminus \{P\}$ is not a $t$-intersection set. When all points of $\mathcal{B}$ are essential then $\mathcal{B}$ is minimal. If the size of the intersection of $\mathcal{B}$ with an arbitrary hyperplane takes $m$ values, say $v_1, \ldots, v_m$, then the non-negative integers $v_1, \ldots, v_m$ are called the characters of $\mathcal{B}$ and $\mathcal{B}$ is also an $m$-character set. We observe that if $\mathcal{B}$ is an $m$–character set consisting of $n$ points and spanning the projective space where it is contained, then the linear code having as columns of its generator matrix the coordinates of the points of $\mathcal{B}$ has exactly $m$ distinct nonzero weights and length $n$. The dimension $k$ of this code is the vector dimension of the subspace spanned by $\mathcal{B}$.

Quasi-Hermitian varieties are examples of 2-character sets of $PG(r, q^2)$. In [1] a new infinite family of quasi–Hermitian varieties have been constructed by modifying some point-hyperplane incidences in $PG(r, q^2)$. To this purpose, the authors kept the point set of $PG(r, q^2)$ but replaced the hyperplanes with their images under a suitable quadratic transformation, obtaining a non–standard model $\Pi$ of $PG(r, q^2)$. This model arises as follows.

Fix a non-zero element $a \in GF(q^2)$. For any choice $m = (m_1, \ldots, m_{r-1}) \in GF(q^2)^{r-1}$ and $d \in GF(q^2)$ let $Q_a(m, d)$ denote the quadric of equation

$$x_r = a(x_1^2 + \ldots + x_{r-1}^2) + m_1 x_1 + \ldots + m_{r-1} x_{r-1} + d. \quad (4)$$

Consider now the incidence structure $\Pi_a = (P, \Sigma)$ whose points are the points of $AG(r, q^2)$ and whose hyperplanes are the hyperplanes of $PG(r, q^2)$ through the infinite point $P_a(0, 0, \ldots, 0, 1)$ together with the quadrics $Q_a(m, d)$ as $m$ and $d$ range as indicated above.
Lemma 2.1. For every non-zero \( a \in \text{GF}(q^2) \), the incidence structure \( \Pi_a = (\mathcal{P}, \Sigma) \) is an affine space isomorphic to \( \text{AG}(r, q^2) \).

Completing \( \Pi_a \) with its points at infinity in the usual way gives a projective space isomorphic to \( \text{PG}(r, q^2) \). We shall make use of this non-standard model of \( \text{PG}(r, q^2) \) in our work.

2.2 Multisets

A multiset in a \( r \)-dimensional projective space \( \Pi \) is a mapping \( M : \Pi \to \mathbb{N} \) from points of \( \Pi \) into non-negative integers. The points of a multiset are the points \( P \) of \( \Pi \) with multiplicity \( M(P) > 0 \). Assume that the number of points of \( M \), each of them counted with its multiplicity, is \( n \). For any hyperplane \( \pi \) of \( \Pi \), the non-negative integer \( M(\pi) = \sum_{P \in \pi} M(P) \) is a character of the multiset \( M \), whereas \( n - M(\pi) \) is called a weight of \( M \). If the set \( \{ M(\pi) \}_{\pi \in \Pi} \) consists of two non-negative integers only, then \( M \) is a 2-character multiset.

Suppose the points of \( M \) span a projective space \( \text{PG}(r, q) \). Then, it is possible to regard the coordinates of the points of \( M \) as the columns of a generator matrix of a code \( C \) of length \( n \) and dimension \( r + 1 \). In this case it is straightforward to see that the weights of \( M \) are indeed exactly the weights of \( C \). We observe that points with multiplicity greater than one correspond to repeated components in \( C \).

3 Proof of Proposition 1.1

From now on, we shall always silently assume \( a \in \text{GF}(q^2)^* \), \( b \in \text{GF}(q^2) \setminus \text{GF}(q) \). Recall that for any quadric \( Q \), the radical \( \text{Rad}(Q) \) of \( Q \) is the subspace \( \text{Rad}(Q) := \{ x \in Q : \forall y \in Q, \langle x, y \rangle \subseteq Q \} \),

where, as usual, by \( \langle x, y \rangle \) we denote the line through \( x \) and \( y \). It is well known that \( \text{Rad}(Q) \) is a subspace of \( \text{PG}(r, q^2) \).

Assume \( B := B(a, b) \) to have Equation (3). It is straightforward to see that \( B(a, b) \) coincides with the affine part of the Hermitian variety \( \mathcal{H} \) of equation (1) in the space \( \Pi_a \); hence, any hyperplane \( \pi_{P_{\infty}} \) of \( \text{PG}(r, q^2) \) passing through \( P_{\infty} \) meets \( B \) in \( |\mathcal{H} \cap \pi_{P_{\infty}}| = q^{2r-3} \) points.

Now we are interested in the possible intersection sizes of \( B \) with a generic hyperplane \( \pi : x_r = m_1 x_1 + \cdots + m_{r-1} x_{r-1} + d \), of \( \text{AG}(r, q^2) \) with coefficients \( m_1, \ldots, m_r, d \in \text{GF}(q^2) \). This is the same as to study the intersection of \( \mathcal{H} \) with the quadrics \( Q_a(m, d) \). Choose \( \varepsilon \in \text{GF}(q^2) \setminus \text{GF}(q) \) such
that $\varepsilon^q = -\varepsilon$; for any $z \in \text{GF}(q^2)$ write $z = z^0 + \varepsilon z^1$ with $z^1, z^2 \in \text{GF}(q)$. The number $N$ of affine points which lie in $\mathcal{B} \cap \pi$ is the same as the number of points of the affine quadric $Q$ of $\text{AG}(2r - 2, q)$ of equation

$$
\sum_{i=1}^{r-1} ((b^1 + a^1)\varepsilon^2(x_i^1)^2 + 2a^0 x_i^0 x_i^1 + (a^1 - b^1)(x_i^1)^2) + \sum_{i=1}^{r-1} (m_i^0 x_i^1 + m_i^1 x_i^0) + d^1 = 0.
$$

(5)

Following the approach of [1], in order to compute $N$, we first count the number of points of the quadric at infinity $Q_\infty := Q \cap \Pi_\infty$ of $Q$ and then we determine $N = |Q| - |Q_\infty|$. Observe that the quadric $Q_\infty$ of $\text{PG}(2r - 3, q)$ has a matrix of the form

$$
A_\infty = \begin{pmatrix}
(a^1 - b^1)^2 & a^0 \\
(a^1 - b^1) a^0 & (b^1 + a^1)\varepsilon^2 \\
\vdots & \vdots \\
(a^1 - b^1) a^0 & (b^1 + a^1)\varepsilon^2 \\
\end{pmatrix}.
$$

Since $(a^0)^2 - \varepsilon^2[(a^1)^2 - (b^1)^2] = [a^{q+1} + (b^q - b)^2]/4 = 0$, we have $\det A_\infty = 0$. This is possible if, and only if,

$$
\det \begin{pmatrix}
(a^1 - b^1) & a^0 \\
(a^1 + b^1)\varepsilon^2 & a^0 \\
\end{pmatrix} = 0,
$$

that is, each of the $2 \times 2$ blocks on the main diagonal of $A_\infty$ has rank 1. Consequently, the rank of $A_\infty$ is exactly $r - 1$.

If $a^1 = b^1$, then $a^0 = 0$, the matrix $A_\infty$ is diagonal and the quadric $Q_\infty$ is projectively equivalent to

$$(x_1^1)^2 + (x_2^1)^2 + \cdots + (x_{r-1}^1)^2 = 0.$$

Otherwise, take

$$
M = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
-a^0/(a^1 - b^1) & 1 \\
\vdots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & -a^0/(a^1 - b^1) \\
\end{pmatrix};
$$

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a direct computation proves that

\[
M^T A \infty M = \begin{pmatrix}
  a^1 - b^1 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & a^1 - b^1
\end{pmatrix}.
\]

Hence, \( Q \infty \) is projectively equivalent to the quadric of rank \( r - 1 \) with equation

\[
(x_1^0)^2 + (x_2^0)^2 + \cdots + (x_{r-1}^0)^2 = 0.
\]

For \( r \) odd we see that in both cases \( Q \infty \) is either

- a cone with vertex \( \text{Rad}(Q \infty) \simeq \text{PG}(r - 2, q) \) and basis a hyperbolic quadric \( Q^+(r - 2, q) \) if \( q \equiv 1 \pmod{4} \) or \( r \equiv 1 \pmod{4} \), or

- a cone with vertex \( \text{Rad}(Q \infty) \simeq \text{PG}(r - 2, q) \) and basis an elliptic quadric \( Q^-(r - 2, q) \) if \( q \equiv 3 \pmod{4} \) and \( r \equiv 3 \pmod{4} \).

For \( r \) even, \( Q \infty \) is a cone with vertex \( \text{Rad}(Q \infty) \simeq \text{PG}(r - 2, q) \) and basis a parabolic quadric \( Q(r - 2, q) \).

We now move to investigate the quadric \( Q \). Clearly, its rank is either \( r - 1 \) or \( r \). Observe that

- \( Q \) has rank \( r - 1 \) if, and only if, there exist a linear function \( f : \text{GF}(q) \to \text{GF}(q) \) such that for all \( i = 1, \ldots, r - 1 \) we have \( m_i^1 = f(m_i^0) \); also, the value of \( d_1 \) turns out to be uniquely determined. Thus, the number of distinct possibilities for the parameters is exactly \( q^r \).

Write now \( \Pi \infty = \Sigma \oplus \text{Rad}(Q \infty) \). As \( \Sigma \) is disjoint from the radical of the quadratic form inducing \( Q \infty \), we have that \( \Sigma \cap Q \infty \) is a nondegenerate quadric (either hyperbolic, elliptic or parabolic according to the various conditions). Since \( Q \) has the same rank as \( Q \infty \), we have \( \dim \text{Rad}(Q) = \dim \text{Rad}(Q \infty) + 1 \). Observe that \( \text{Rad}(Q) \cap \Pi \infty \leq \text{Rad}(Q \infty) \). Thus, \( \text{Rad}(Q) \cap \Sigma = \{0\} \) and \( \Sigma \) is also a direct complement of \( \text{Rad}(Q) \). It follows that \( Q \) is cone of vertex a \( \text{PG}(r - 1, q) \) and basis a quadric of the same kind as the basis of \( Q \infty \).

- \( Q \) has rank \( r \) in the remaining \( q^{2r} - q^r \) possibilities. Here \( Q \) is a cone of vertex a \( \text{PG}(r - 2, q) \) and basis a parabolic quadric \( Q(r - 1, q) \) for \( r \) odd or \( Q \) is a cone of vertex a \( \text{PG}(r - 2, q) \) and basis a hyperbolic quadric \( Q^+(r - 1, q) \) or an elliptic quadric \( Q^-(r - 1, q) \) for \( r \) even.

We can now determine the complete list of sizes for \( r \) odd.
\[
|Q_{\infty}| = \frac{q^{2r-3} - 1}{q - 1} \pm q^{(3r-5)/2};
\]

- in case \( \text{rank}(Q) = r - 1 \), then
  \[
  |Q| = \frac{q^{2r-2} - 1}{q - 1} \pm q^{(r-1)/2};
  \]

- in case \( \text{rank}(Q) = r \),
  \[
  |Q| = \frac{q^{2r-2} - 1}{q - 1}
  \]

In particular, the possible values for \( |Q| - |Q_{\infty}| \) are
\[
q^{2r-3} + q^{3(r-1)/2} - q^{(3r-5)/2}, q^{2r-3} - q^{(3r-5)/2}
\]
for \( q \equiv 1 \pmod{4} \) or \( r \equiv 1 \pmod{4} \) and
\[
q^{2r-3} - q^{3(r-1)/2} + q^{(3r-5)/2}, q^{2r-3} + q^{(3r-5)/2}
\]
for \( q \equiv 3 \pmod{4} \) and \( r \equiv 3 \pmod{4} \).

When \( r \) is even we get:

- \[
  |Q_{\infty}| = \frac{q^{2r-3} - 1}{q - 1};
  \]

- in case \( \text{rank}(Q) = r - 1 \), then
  \[
  |Q| = \frac{q^{2r-2} - 1}{q - 1};
  \]

- in case \( \text{rank}(Q) = r \),
  \[
  |Q| = \frac{q^{2r-2} - 1}{q - 1} \pm q^{(3r-4)/2}.
  \]

Thus, the possible list of cardinalities for \( |Q| - |Q_{\infty}| \) is
\[
q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}, q^{2r-3} - q^{(3r-4)/2}.
\]

Now we are going to show that \( \mathcal{B}(a, b) \) is a minimal intersection set. First of all, we prove that for any \( P \in \mathcal{B}(a, b) \) there exists a subspace \( \Lambda_n(P) \cong \text{AG}(n, q^2) \),
\( 1 \leq n \leq r - 1 \) through \( P \) such that \( |\mathcal{B}(a, b) \cap \Lambda_n(P)| \leq q^{2n-1} - q^{n-1} \). The
argument is by induction on $n$. Assume $n = 1$. Then, for any $P \in \mathcal{B}$ there exists at least one line $\ell$ through $P$ such that $|\ell \cap \mathcal{B}| < q$, otherwise $\mathcal{B}$ would contain more than $q^{2r-1}$ points. Suppose now that the result holds for $n = 1, \ldots, r - 2$, take $P \in \mathcal{B}$ and suppose that any hyperplane $\pi$ through $P$ meets $\mathcal{B}$ in at least $q^{2r-3}$ points. By induction, there exists a subspace $\pi' := \Lambda_{r-2}(P) \simeq \text{AG}(r - 2, q^2)$ through $P$ meeting $\mathcal{B}$ in at most $q^{2r-5} - q^{r-3}$ points. By considering all hyperplanes containing $\pi'$ we get $|\mathcal{B}| \geq (q^2 + 1)(q^{2r-3} - q^{2r-5} + q^{r-3}) + q^{2r-5} - q^{r-3} > q^{2r-1}$, a contradiction. Thus, through any $N$ implies that $\mathcal{B}$ meets at least one line $\ell$. By induction, there exists a subspace $\sigma$ for which $\ell$ is a minimal intersection set.

**Corollary 3.1.** For $q$ odd and $4a^{q+1} + (b^q - b)^2 = 0$, the number of hyperplanes $N_j$ meeting $\mathcal{B}(a, b)$ in exactly $j$ points are as follows:

(a) for $r$ odd,

$$N_{q^{2r-3} + q^{3(r-5)/2}} = q^{2r} - q^r, \quad N_{q^{2r-3}} = \frac{q^{2r} - 1}{q^2 - 1} - 1,$$

$$N_{q^{2r-3} - q^{3(r-1)/2} + q^{3(r-5)/2}} = q^r.$$

(b) for $r$ even,

$$N_{q^{2r-3} - q^{3(r-4)/2}} = \frac{1}{2}(q^{2r} - q^r), \quad N_{q^{2r-3}} = q^r + \frac{q^{2r} - 1}{q^2 - 1} - 1,$$

$$N_{q^{2r-3} + q^{3(r-4)/2}} = \frac{1}{2}(q^{2r} - q^r).$$

**Proof.** Case (a) is a direct consequence of the arguments of Theorem 3.1. In Case (b), when $r$ is even, we need to count how often $\mathcal{Q}$ turns out to be elliptic rather than hyperbolic. For any choice of the parameters $m_1, \ldots, m_{r-1}, d$ there is exactly one quadric $\mathcal{Q}$ to consider. As $\mathcal{Q}_\infty$ is always a parabolic quadric, we can assume it to be fixed. Denote by $\sigma^0, \sigma^+, \sigma^-$ respectively the number of quadrics $\mathcal{Q}$ which are parabolic, elliptic or hyperbolic. Clearly $\sigma_0$ corresponds to the case in which $\text{rank}(\mathcal{Q}) = \text{rank}(\mathcal{Q}_\infty)$. We have

$$\sigma^+ + \sigma^0 + \sigma^- = q^{2r}, \quad \sigma^0 = q^r.$$

Each point of $\mathcal{B}(a, b)$ lies on $\frac{q^{2r-2}}{q^2 - 1}$ hyperplanes; of these $\frac{q^{2r-2} - 1}{q^2 - 1}$ pass through $P_\infty$ (and they must be discounted). Thus, we get

$$q^{2r-2}|\mathcal{B}| = q^{4r-3} = \sigma^0 q^{2r-3} + \sigma^+(q^{2r-3} + q^{3(r-3)/2}) + \sigma^-(q^{2r-3} - q^{3(r-4)/2}) = q^{2r-3}(\sigma^0 + \sigma^+ + \sigma^-) + q^{3(r-4)/2}(\sigma^+ - \sigma^-) = q^{4r-3} + (\sigma^+ - \sigma^-)q^{3(r-4)/2}.$$

Hence, $\sigma^+ = \sigma^- = \frac{1}{2}(q^{2r} - q^r)$. \qed
Remark 3.2. The quadric $Q_{a}(m, d)$ of Equation (4) shares its tangent hyperplane at $P_{\infty}$ with the Hermitian variety (1).

The problem of the intersection of the Hermitian variety $H$ with irreducible quadrics $Q$ having the same tangent plane at a common point $P \in Q \cap H$ has been considered for $r = 3$ in [3, 4].

4 A family of two-character multisets in $PG(3, q^2)$

In [2, Theorem 4.1] it is shown that for $r = 2$, $q$ odd and $4a^{q+1} + (b^q - b)^2 \neq 0$ or $r = 2$, $q$ even and $\text{Tr}(a^{q+1}/(b^q + b)^2) = 1$, the set $B(a, b)$ can be completed to a 2–character multiset $\overline{B}(a, b)$. An analogous result holds for $r = 3$. In this section we now prove Theorem 1.2.

Assume $q$ odd and $4a^{q+1} + (b^q - b)^2 = 0$. From the proof of Proposition 1.1 the quadric $Q_{\infty}$ is the union of two distinct planes for $q \equiv 1 \pmod{4}$ or just a line for $q \equiv 3 \pmod{4}$. Therefore, if $q \equiv 1 \pmod{4}$ then either

$$N = q^3 + q^2 + q + 1 - (2q^2 + q + 1) = q^3 - q^2$$

or

$$N = 2q^3 + q^2 + q + 1 - (2q^2 + q + 1) = 2q^3 - q^2,$$

according as $Q$ is either the join of a line to a conic or a pair of solids; hence, the list of intersection numbers of $B(a, b)$ with affine hyperplanes is $q^3 - q^2$, $q^3$ and $2q^3 - q^2$.

If $q \equiv 3 \pmod{4}$ we get either

$$N = q^3 + q^2 + q + 1 - q - 1 = q^3 + q^2,$$

or

$$N = q^2 + q + 1 - q - 1 = q^2,$$

according as $Q$ is either the join of a line to a conic or a plane; therefore, in this case, the intersection numbers are $q^2$, $q^3$ and $q^3 + q^2$.

Now consider the multiset $\overline{B}(a, b)$ in $PG(3, q^2)$ arising from $B(a, b)$ by assigning multiplicity bigger than 1 to just the point $P_{\infty}$.

More in detail the points of the 2–character multiset $\overline{B}(a, b)$ are exactly those of $B(a, b) \cup \{P_{\infty}\}$ where each affine point of $B(a, b)$ has multiplicity one, and $P_{\infty}$ has either multiplicity $q^3 - q^2$ for $q \equiv 1 \pmod{4}$, or multiplicity $q^2$ when $q \equiv 3 \pmod{4}$. Our theorem follows.

Remark 4.1. Let $C$ be the linear code associated to $\overline{B}(a, b)$. In the first case $C$ is a $[q^5 + q^3 - q^2, 4, q^5 - q^3]_{q^2}$ two-weight code, while in the second it has parameters $[q^5 + q^2, 4, q^5 - q^3]_{q^2}$. In either case the non–zero weights are $q^5$ and $q^3 - q^5$. 9
If \( A_i \) is the number of words in \( C \) of weight \( i \) then by Corollary 3.11 it follows that

\[
A_{q^i} = (q^6 - q^3 + 1)(q^2 - 1); \quad A_{q^i - q^i} = (q^4 + q^3 + q^2)(q^2 - 1).
\]

# 5 Intersection sets with multiplicity \( q^{2r-3} - q^{r-2} \)

We keep the notation of the previous sections and examine the remaining cases. Even though the results we obtain are a direct consequence of the construction of [1], we provide some further technical details so that this paper can be considered self-contained.

**Proposition 5.1.** Suppose \( r \) to be even and that either \( q \) is odd and \( 4a^{q+1} + (b^q - b)^2 \) is a non–zero square in \( GF(q) \) or \( q \) is even and \( \text{Tr}(a^{q+1}/(b^q + b)^2) = 1 \). Then, \( B(a, b) \) is a set of \( q^{2r-1} \) points of \( AG(r, q^2) \) with characters

\[
q^{2r-3} - q^{r-2}, q^{2r-3}, q^{2r-3} - q^{r-2} + q^{r-1}.
\]

This is also a minimal intersection set with respect to hyperplanes.

**Proof.** We first discuss the nature of \( Q_\infty \). Observe that, under our assumptions, for \( q \) odd \((-1)^{r-1} \text{det} A_\infty \) is always a square; hence, \( Q_\infty \) is a hyperbolic quadric.

For \( q \) even choose \( \epsilon \in GF(q^2) \setminus GF(q) \) such that \( \epsilon^2 + \epsilon + \nu = 0 \), for some \( \nu \in GF(q) \setminus \{1\} \) with \( \text{Tr}(\nu) = 1 \). Then, \( \epsilon^{q+\nu} + \nu^2 + \nu = 0 \). Therefore, \((\epsilon^q + \epsilon)^2 + (\epsilon^q + \epsilon) = 0\), whence \( \epsilon^q + \epsilon + 1 = 0 \). With this choice of \( \epsilon \), the system given by (3) and (4) reads as

\[
(a^1 + b^1)(x^0)^2 + [(a^0 + a^1) + \nu(a^1 + b^1)](x^1)^2 + b^1 x^0 x^1_1 + m^1_1 x^0_1 + (m^1_0 + m^1_1)x^1_1 \\
+ \ldots + (a^1 + b^1)(x^0_{r-1})^2 + [(a^0 + a^1) + \nu(a^1 + b^1)](x^1_{r-1})^2 + b^1 x^0_{r-1} x^1_{r-1} \\
+ m^1_{r-1} x^0_{r-1} + (m^0_{r-1} + m^1_{r-1})x^1_{r-1} + d^1 = 0.
\]

The discussion of the (possibly degenerate) quadric \( Q \) of Equation (6) may be carried out in close analogy to what has been done before.

Observe however that, as also pointed out in the remark before [5, Theorem 22.2.1], some caution is needed when quadrics and their classifications are studied in even characteristic. Indeed let \( A_\infty \) be the formal matrix associated to the quadric \( Q_\infty \) of equation

\[
(a^1 + b^1)(x^0)^2 + [(a^0 + a^1) + \nu(a^1 + b^1)](x^1)^2 + b^1 x^0 x^1 + \ldots \\
+ (a^1 + b^1)(x^0_{r-1})^2 + [(a^0 + a^1) + \nu(a^1 + b^1)](x^1_{r-1})^2 + b^1 x^0_{r-1} x^1_{r-1} = 0.
\]

Its determinant is equal to

\[
\text{det } A_\infty = [4(a^1 + b^1)(a^0 + a^1 + \nu(a^1 + b^1)) + (b^1)^2]^{r-1}.
\]
In order to encompass the case \( q \) even, \( \det A_\infty \) needs to be regarded as a formal function in the polynomial ring \( GF(q)[z_0, z_1, z_2, z_3] \) evaluated in \( (a^0, a^1, b^0, b^1) \). This gives \( \det A_\infty = b_1^2 \). Here \( b_1 \neq 0 \), by our assumption \( b^q \neq b \). From [5] Theorem 22.2.1 (i), the quadric \( Q_\infty \) must be non-degenerate. Furthermore, by [5] Theorem 22.2.1 (ii) and the successive Lemma 22.2.2 the nature of \( Q_\infty \) can be ascertained as follows. Let \( B \) the matrix obtained from \( A_\infty \) by omitting all the entries on its main diagonal, and define

\[
\alpha = \frac{\det B - (-1)^{r-1} \det A_\infty}{4 \det B}.
\]

A straightforward computation shows that

\[
\alpha = \frac{(b^1)^{2(r-1)} + (4(a^1 + b^1)(a^0 + a^1 + \nu(a^1 + b^1)) + (b^1)^2)^{r-1}}{4(b^1)^2(r-1)}.
\]

Regard \( \alpha \) also as a function in the polynomial ring \( GF(q)[z_0, z_1, z_2, z_3] \) evaluated in \( (a^0, a^1, b^0, b^1) \). Hence we get

\[
\alpha = \frac{(a^1 + b^1)(a^0 + a^1 + \nu(a^1 + b^1))}{(b^1)^2}.
\]

Arguing as in [1] p. 439, we see that \( \text{Tr}_{GF(q)|GF(2)}(\alpha) = 0 \) and, hence, \( Q_\infty \) is hyperbolic also for \( q \) even.

Now, in both cases \( q \) odd or \( q \) even we investigate the possible nature of \( Q \). Suppose \( Q \) to be non-singular; then

\[
N = \frac{(q^{r-1} + 1)(q^{r-1} - 1)}{q - 1} - \frac{(q^{r-1} + 1)(q^{r-2} - 1)}{q - 1} = q^{r-2}(q^{r-1} + 1).
\]

If \( Q \) is singular, then

\[
N = \frac{q(q^{r-1} + 1)(q^{r-2} - 1)}{q - 1} - \frac{(q^{r-1} + 1)(q^{r-2} - 1)}{q - 1} + 1 = q^{r-2}(q^{r-1} + 1) - q^{r-1}.
\]

This gives the possible intersection numbers.

Finally, in order to show that \( B(a, b) \) is a minimal \( (q^{2r-3} - q^{r-2}) \)-fold blocking set we can use the same techniques as those adopted to prove that \( B(a, b) \) is a minimal blocking set in Section 3 for \( q \) odd and \( 4a^q + 1 + (b^q - b)^2 = 0 \)

\[ \square \]

References


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