# Down-linking $(K_v, \Gamma)$-designs to $P_3$-designs

A. Benini, L. Giuzzi and A. Pasotti*

**Abstract**

Let $\Gamma'$ be a subgraph of a graph $\Gamma$. We define a *down-link* from a $(K_v, \Gamma)$-design $\mathcal{B}$ to a $(K_n, \Gamma')$-design $\mathcal{B}'$ as a map $f : \mathcal{B} \to \mathcal{B}'$ mapping any block of $\mathcal{B}$ into one of its subgraphs. This is a new concept, closely related with both the notion of *metamorphosis* and that of *embedding*. In the present paper we study down-links in general and prove that any $(K_v, \Gamma)$-design might be down-linked to a $(K_n, \Gamma')$-design, provided that $n$ is admissible and large enough. We also show that if $\Gamma' = P_3$, it is always possible to find a down-link to a design of order at most $v + 3$. This bound is then improved for several classes of graphs $\Gamma$, by providing explicit constructions.

**Keywords:** down-link; metamorphosis; embedding; $(K_v, \Gamma)$-design.
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## 1 Introduction

Let $K$ be a graph and $\Gamma \leq K$. A $(K, \Gamma)$-*design*, also called a $\Gamma$-*decomposition* of $K$, is a set $\mathcal{B}$ of graphs all isomorphic to $\Gamma$, called *blocks*, partitioning the edge-set of $K$. Given a graph $\Gamma$, the problem of determining the existence of $(K_v, \Gamma)$-designs, also called $\Gamma$-*designs of order* $v$, where $K_v$ is the complete graph on $v$ vertices, has been extensively studied; for surveys on this topic see, for instance, [3, 4].

We propose the following new definition.

**Definition 1.1.** *Given a* $(K, \Gamma)$*-design* $\mathcal{B}$ *and a* $(K', \Gamma')$*-design* $\mathcal{B}'$ *with* $\Gamma' \leq \Gamma$*, a* down-link *from* $\mathcal{B}$ *to* $\mathcal{B}'$ *is a function* $f : \mathcal{B} \to \mathcal{B}'$ *such that* $f(B) \leq B$*, for any* $B \in \mathcal{B}$*.*

*anna.benini@ing.unibs.it, luca.giuzzi@ing.unibs.it, anita.pasotti@ing.unibs.it, Dipartimento di Matematica, Facoltà di Ingegneria, Università degli Studi di Brescia, Via Valotti 9, I-25133 Brescia (IT).

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By the definition of $(K,\Gamma)$-design, a down-link is necessarily injective. When a function $f$ as in Definition 1.1 exists, that is if each block of $\mathcal{B}$ contains at least one element of $\mathcal{B}'$ as a subgraph, it will be said that it is possible to *down-link* $\mathcal{B}$ to $\mathcal{B}'$.

In this paper we shall investigate the existence and some further properties of down-links between designs on complete graphs and outline their relationship with some previously known notions. More in detail, Section 2 is dedicated to the close interrelationship between down-links, metamorphoses and embeddings. In Section 3 we will introduce, in close analogy to embeddings, two problems on the spectra of down-links and determine bounds on their minima. In Section 4 down-links from any $(K_v,\Gamma)$-design to a $P_3$-design of order $n \leq v+3$ are constructed; this will improve on the values determined in Section 3. In further Sections 5, 6, 7, 8 the existence of down-links to $P_3$-designs from, respectively, star-designs, kite-designs, cycle systems and path-designs are investigated by providing explicit constructions.

Throughout this paper the following standard notations will be used; see also [15]. For any graph $\Gamma$, write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. By $t\Gamma$ we shall denote the disjoint union of $t$ copies of graphs all isomorphic to $\Gamma$. Given any set $V$, the complete graph with vertex-set $V$ is $K_V$. As usual, $K_{v_1,v_2,\dots,v_m}$ is the complete $m$–partite graph with parts of size respectively $v_1,\dots,v_m$; when $v = v_1 = v_2 = \cdots = v_m$ we shall simply write $K_{m\times v}$. When we want to focus our attention on the actual parts $V_1, V_2,\dots,V_m$, the notation $K_{V_1,V_2,\dots,V_m}$ shall be used instead. The join $\Gamma + \Gamma'$ of two graphs consists of the graph $\Gamma \cup \Gamma'$ together with the edges connecting all the vertices of $\Gamma$ with all the vertices of $\Gamma'$; hence, $\Gamma + \Gamma' = \Gamma \cup \Gamma' \cup K_{V(\Gamma),V(\Gamma')}$.

## 2 Down-links, metamorphoses, embeddings

As it will be shown, the concepts of down-link, metamorphosis and embedding are closely related.

Metamorphoses of designs have been first introduced by Lindner and Rosa in [18] in the case $\Gamma = K_4$ and $\Gamma' = K_3$. In recent years metamorphoses and their generalizations have been extensively studied; see for instance [9, 10, 17, 19, 21, 22]. We here recall the general notion of metamorphosis. Suppose $\Gamma' \leq \Gamma$ and let $\mathcal{B}$ be a $({}^{\lambda}K_v,\Gamma)$-design. For each block $B \in \mathcal{B}$ take a subgraph $B' \leq B$ isomorphic to $\Gamma'$ and put it into a set $S$. If it is possible to reassemble all the remaining edges of ${}^{\lambda}K_v$ into a set $R$ of copies of $\Gamma'$, then $S \cup R$ are the blocks of a $({}^{\lambda}K_v,\Gamma')$-design, which is said to be a metamorphosis of $\mathcal{B}$. Thus, if $\mathcal{B}'$ is a metamorphosis of $\mathcal{B}$ with $\lambda = 1$, then there exists a down-link $f : \mathcal{B} \to \mathcal{B}'$ given by $f(B) = B'$. With a slight abuse of notation we shall call *metamorphoses* all down-links from a $(K,\Gamma)$-design to a $(K,\Gamma')$-design.

There is also a generalization of metamorphosis, originally from [22], which turns out to be closely related to down-links. Suppose $\Gamma' \le \Gamma$ and let $\mathcal{B}$ be a $({}^{\lambda}K_v,\Gamma)$-design. Write $n$ for the minimum integer $n \ge v$ for which there exists a $({}^{\lambda}K_n,\Gamma')$-design. Take $X = V(K_v)$ and $X \,\dot\cup\, Y = V(K_n)$. For each block $B \in \mathcal{B}$ extract a subgraph of $B$ isomorphic to $\Gamma'$ and put it into a set $S$. Let also $R$ be the set of all the remaining edges of ${}^{\lambda}K_v$. Let $T$ be the set of edges of ${}^{\lambda}K_Y$ and of the $\lambda$-fold complete bipartite graph ${}^{\lambda}K_{X,Y}$. If it is possible to reassemble the edges of $R \cup T$ into a set $R'$ of copies of $\Gamma'$, then $S \cup R'$ are the blocks of a $({}^{\lambda}K_n,\Gamma')$-design $\mathcal{B}'$. In this case, one speaks of a metamorphosis of $\mathcal{B}$ into a minimum $\Gamma'$-design. It is easy to see that for $\lambda = 1$ these generalized metamorphoses also induce down-links.

Even if metamorphoses with $\lambda = 1$ are all down-links, the converse is not true. For instance, all down-links from designs of order $v$ to designs of order $n < v$ are not metamorphoses. Example 2.1 shows that such down-links may exist.

Gluing of metamorphoses and down-links can be used to produce new classes of down-links from old, as shown by the following construction. Take $\mathcal{B}$ as a $(K_v,\Gamma)$-design with $V(K_v) = X \,\dot\cup\, \dot\bigcup_{i=1}^{t} A_i$ and suppose $X' \subseteq X$. Let $\Gamma' \le \Gamma$ and $\mathcal{B}'$ be a $(K_{v-|X'|},\Gamma')$-design with $V(K_{v-|X'|}) = V(K_v) \setminus X'$. Suppose that

$$f_i : (K_{A_i},\Gamma)\text{-design} \longrightarrow (K_{A_i},\Gamma')\text{-design} \quad \text{for any } i = 1,\dots,t,$$

$$h_{ij} : (K_{A_i,A_j},\Gamma)\text{-design} \longrightarrow (K_{A_i,A_j},\Gamma')\text{-design} \quad \text{for } 1 \le i < j \le t$$

are metamorphoses and that

$$g : (K_X,\Gamma)\text{-design} \longrightarrow (K_{X\setminus X'},\Gamma')\text{-design},$$

$$g_i : (K_{X,A_i},\Gamma)\text{-design} \longrightarrow (K_{X\setminus X',A_i},\Gamma')\text{-design} \quad \text{for any } i = 1,\dots,t$$

are down-links. As

$$K_v = \dot\bigcup_{i=1}^{t} K_{A_i} \,\dot\cup \dot\bigcup_{1\le i<j\le t} K_{A_i,A_j} \,\dot\cup\, K_X \,\dot\cup \dot\bigcup_{i=1}^{t} K_{X,A_i}$$

and

$$K_{v-|X'|} = \dot\bigcup_{i=1}^{t} K_{A_i} \,\dot\cup \dot\bigcup_{1\le i<j\le t} K_{A_i,A_j} \,\dot\cup\, K_{X\setminus X'} \,\dot\cup \dot\bigcup_{i=1}^{t} K_{X\setminus X',A_i},$$

the function obtained by gluing together $g$ and all of the $f_i$'s, $h_{ij}$'s and $g_i$'s provides a down-link from $\mathcal{B}$ to $\mathcal{B}'$.

Recall that an *embedding* of a design $\mathcal{B}'$ into a design $\mathcal{B}$ is a function $\psi : \mathcal{B}' \to \mathcal{B}$ such that $\Gamma \leq \psi(\Gamma)$, for any $\Gamma \in \mathcal{B}'$; see [24]. Existence of embeddings of designs has been widely investigated. In particular, a great deal of results are known on injective embeddings of path-designs; see, for instance, [12, 14, 23, 25, 26]. If $\psi : \mathcal{B}' \to \mathcal{B}$ is a *bijective* embedding, then $\psi^{-1}$ is a down-link from $\mathcal{B}$ to $\mathcal{B}'$. Clearly, a bijective embedding of $\mathcal{B}'$ into $\mathcal{B}$ might exist only if $\mathcal{B}$ and $\mathcal{B}'$ have the same number of blocks. This condition, while quite restrictive, does not necessarily lead to trivial embeddings, as shown in the following example.

**Example 2.1.** *Consider the* $(K_4, P_3)$*-design*

$$\mathcal{B}' = \{\Gamma'_1 = [1,2,3], \Gamma'_2 = [1,3,0], \Gamma'_3 = [2,0,1]\}$$

*and the* $(K_6, P_6)$*-design*

$$\mathcal{B} = \{\Gamma_1 = [4,0,5,1,2,3], \Gamma_2 = [2,5,4,1,3,0], \Gamma_3 = [5,3,4,2,0,1]\}.$$

*Define* $\psi : \mathcal{B}' \to \mathcal{B}$ *by* $\psi(\Gamma'_i) = \Gamma_i$ *for* $i = 1,2,3$*. Then,* $\psi$ *is a bijective embedding; consequently,* $\psi^{-1}$ *is a down-link from* $\mathcal{B}$ *to* $\mathcal{B}'$*.*

# 3 Spectrum problems

Spectrum problems about the existence of embeddings of designs have been widely investigated; see [12, 13, 14, 23, 25, 26].

In close analogy, we pose the following questions about the existence of down-links:

(I) For each admissible $v$, determine the set $\mathcal{L}_1\Gamma(v)$ of all integers $n$ such that there exists *some* $\Gamma$-design of order $v$ down-linked to a $\Gamma'$-design of order $n$.

(II) For each admissible $v$, determine the set $\mathcal{L}_2\Gamma(v)$ of all integers $n$ such that *every* $\Gamma$-design of order $v$ can be down-linked to a $\Gamma'$-design of order $n$.

In general, write $\eta_i(v;\Gamma,\Gamma') = \inf \mathcal{L}_i\Gamma(v)$. When the graphs $\Gamma$ and $\Gamma'$ are easily understood from the context, we shall simply use $\eta_i(v)$ instead of $\eta_i(v;\Gamma,\Gamma')$.

The problem of the actual existence of down-links for given $\Gamma' \leq \Gamma$ is addressed in Proposition 3.2. We recall the following lemma on the existence of finite embeddings for partial decompositions, a straightforward consequence of an asymptotic result by R.M. Wilson [31, Lemma 6.1]; see also [6].

**Lemma 3.1.** *Any partial* $(K_v, \Gamma)$*-design can be embedded into a* $(K_n, \Gamma)$*-design with* $n = O((v^2/2)^{v^2})$.

**Proposition 3.2.** *For any* $v$ *such that there exists a* $(K_v, \Gamma)$*-design and any* $\Gamma' \le \Gamma$*, the sets* $\mathcal{L}_1\Gamma(v)$ *and* $\mathcal{L}_2\Gamma(v)$ *are non-empty.*

*Proof.* Fix first a $(K_v, \Gamma)$-design $\mathcal{B}$. Denote by $K_v(\Gamma')$ the so called *complete* $(K_v, \Gamma')$*-design*, that is the set of all subgraphs of $K_v$ isomorphic to $\Gamma'$, and let $\zeta : \mathcal{B} \to K_v(\Gamma')$ be any function such that $\zeta(\Gamma) \le \Gamma$ for all $\Gamma \in \mathcal{B}$. Clearly, the image of $\zeta$ is a partial $(K_v, \Gamma')$-design $\mathcal{P}$; see [11]. By Lemma 3.1, there is an integer $n$ such that $\mathcal{P}$ is embedded into a $(K_n, \Gamma')$-design $\mathcal{B}'$. Let $\psi : \mathcal{P} \to \mathcal{B}'$ be such an embedding; then, $\xi = \psi\zeta$ is, clearly, a down-link from $\mathcal{B}$ to a $\Gamma'$-design $\mathcal{B}'$ of order $n$. Thus, we have shown that for any $v$ such that a $\Gamma$-design of order $v$ exists, and for any $\Gamma' \le \Gamma$ the set $\mathcal{L}_1\Gamma(v)$ is non-empty.

To show that $\mathcal{L}_2\Gamma(v)$ is also non-empty, proceed as follows. Let $\omega$ be the number of distinct $(K_v, \Gamma)$-designs $\mathcal{B}_i$. For any $i = 0, \dots, \omega - 1$, write $V(\mathcal{B}_i) = \{0, \dots, v-1\} + i \cdot v$. Consider now $\Omega = \bigcup_{i=0}^{\omega-1} \mathcal{B}_i$. Clearly, $\Omega$ is a partial $\Gamma$-design of order $v\omega$. As above, take $K_{v\omega}(\Gamma')$ and construct a function $\zeta : \Omega \to K_{v\omega}(\Gamma')$ associating to each $\Gamma \in \mathcal{B}_i$ a $\zeta(\Gamma) \le \Gamma$. The image $\bigcup_i \zeta(\mathcal{B}_i)$ is a partial $\Gamma'$-design $\Omega'$. Using Lemma 3.1 once more, we determine an integer $n$ and an embedding $\psi$ of $\Omega'$ into a $(K_n, \Gamma')$-design $\mathcal{B}'$. For any $i$, let $\zeta_i$ be the restriction of $\zeta$ to $\mathcal{B}_i$. It is straightforward to see that $\psi\zeta_i : \mathcal{B}_i \to \mathcal{B}'$ is a down-link from $\mathcal{B}_i$ to a $(K_n, \Gamma')$-design. It follows that $n \in \mathcal{L}_2\Gamma(v)$. □

Notice that the order of magnitude of $n$ is $v^{2v^2}$; yet, it will be shown that in several cases it is possible to construct down-links from $(K_v, \Gamma)$-designs to $(K_n, \Gamma')$-designs with $n \approx v$.

Lower bounds on $\eta(v; \Gamma, \Gamma')$ are usually hard to obtain and might not be strict; a easy one to prove is the following:

$$(v - 1)\sqrt{\frac{|E(\Gamma')|}{|E(\Gamma)|}} < \eta_1(v; \Gamma, \Gamma').$$

# 4 Down-linking $\Gamma$-designs to $P_3$-designs

From this section onwards we shall fix $\Gamma' = P_3$ and focus our attention on the existence of down-links to $(K_n, P_3)$-designs. Recall that a $(K_n, P_3)$-design exists if, and only if, $n \equiv 0, 1 \pmod 4$; see [28]. We shall make extensive use of the following result from [30].

**Theorem 4.1.** *Let* $\Gamma$ *be a connected graph. Then, the edges of* $\Gamma$ *can be partitioned into copies of* $P_3$ *if and only if the number of edges is even.*

*When the number of edges is odd, $E(\Gamma)$ can be partitioned into a single edge together with copies of $P_3$.*

Our main result for down-links from a general $(K_v, \Gamma)$-design is contained in the following theorem.

**Theorem 4.2.** *For any $(K_v, \Gamma)$-design $\mathcal{B}$ with $P_3 \le \Gamma$,*

$$\eta_1(v) \le \eta_2(v) \le v + 3.$$

*Proof.* For any block $B \in \mathcal{B}$, fix a $P_3 \le B$ to be used for the down-link. Write $S$ for the set of all these $P_3$'s. Remove the edges covered by $S$ from $K_v$ and consider the remaining graph $R$. If each connected component of $R$ has an even number of edges, by Theorem 4.1, there is a decomposition $D$ of $R$ in $P_3$'s; $S \cup D$ is a decomposition of $K_v$; thus, $\eta_1(v) \le \eta_2(v) \le v$. If not, take $1 \le w \le 3$ such that $v + w \equiv 0, 1 \pmod 4$. Then, the graph $R' = (K_v + K_w) \setminus S$ is connected and has an even number of edges. Thus, by Theorem 4.1, there is a decomposition $D$ of $R'$ into copies of $P_3$'s. It follows that $S \cup D$ is a $(K_{v+w}, P_3)$-design $\mathcal{B}'$. □

**Remark 4.3.** *In Theorem 4.2, if $v \equiv 2, 3 \pmod 4$, then the order of the design $\mathcal{B}'$ is the smallest $m \ge v$ for which there exists a $(K_m, P_3)$-design. Thus, the down-links are actually metamorphoses to minimum $P_3$-designs. This is not the case for $v \equiv 0, 1 \pmod 4$, as we cannot* a priori *guarantee that each connected component of $R$ has an even number of edges.*

Theorem 4.2 might be improved under some further (mild) assumptions on $\Gamma$.

**Theorem 4.4.** *Let $\mathcal{B}$ be a $(K_v, \Gamma)$-design.*

*a) If $v \equiv 1, 2 \pmod 4$, $|V(\Gamma)| \ge 5$ and there are at least 3 vertices in $\Gamma$ with degree at least 4, then there exists a down-link from $\mathcal{B}$ to a $(K_{v-1}, P_3)$-design.*

*b) If $v \equiv 0, 3 \pmod 4$, $|V(\Gamma)| \ge 7$ and there are at least 5 vertices in $\Gamma$ with degree at least 6, then there exists a down-link from $\mathcal{B}$ to a $(K_{v-3}, P_3)$-design.*

*Proof.* a) Let $x, y \in V(K_v)$. Extract from any $B \in \mathcal{B}$ a $P_3 \le B$ whose vertices are neither $x$ nor $y$ and use it for the down-link. This is always possible, since $|V(\Gamma)| \ge 5$ and there is at least one vertex in $\Gamma \setminus \{x, y\}$ of degree at least 2. Write now $S$ for the set of all of these $P_3$'s. Consider the graph $R = (K_{v-2} + \{\alpha\}) \setminus S$ where $K_{v-2} = K_v \setminus \{x, y\}$. This is a connected graph with an even number of edges; thus, by Theorem 4.1, there exists a decomposition $D$ of $R$ in $P_3$'s. Hence, $S \cup D$ provides the blocks of a $P_3$-design of order $v - 1$.

b) In this case consider 4 vertices $\Lambda = \{x, y, z, t\}$ of $V(K_v)$. By the assumptions, it is always possible to take a $P_3$ disjoint from $\Lambda$ from each block of $\mathcal{B}$. We now argue as in the proof of part a). □

The down-links constructed above are not, in general, to designs whose order is as small as possible; thus, theorems 4.2 and 4.4 do not provide the exact value of $\eta_1(v)$, unless further assumptions are made.

**Remark 4.5.** *In general, a $(K_n, P_3)$-design can be trivially embedded into $P_3$-designs of any admissible order $m \geq n$. Thus, if $n \in \mathcal{L}_i\Gamma(v)$, then $\{m \geq n \mid m \equiv 0, 1 (\mathrm{mod}\, 4)\} \subseteq \mathcal{L}_i\Gamma(v)$. Hence,*

$$\mathcal{L}_i\Gamma(v) = \{m \geq \eta_i(v) \mid m \equiv 0, 1 (\mathrm{mod}\, 4)\}.$$

*Thus, solving problems* (I) *and* (II) *turns out to be actually equivalent to determining exactly the values of $\eta_1(v; \Gamma, P_3)$ and $\eta_2(v; \Gamma, P_3)$.*

For the remainder of this paper, we shall always silently apply Remark 4.5 in all the proofs.

# 5 Star-designs

In this section the existence of down-links from star-designs to $P_3$-designs is investigated. We follow the notation introduced in Section 3, where $\Gamma' = P_3$ is understood. Recall that the *star* on $k+1$ vertices $S_k$ is the complete bipartite graph $K_{1,k}$ with one part having a single vertex, say $c$, called the *center* of the star, and the other part having $k$ vertices, say $x_i$ for $i = 0, \ldots, k-1$, called *external vertices*. In general, we shall write $S_k = [c; x_0, x_1, \ldots, x_{k-1}]$.

In [29], Tarsi proved that a $(K_v, S_k)$-design exists if, and only if, $v \geq 2k$ and $v(v-1) \equiv 0 \pmod{2k}$. When $v$ satisfies these necessary conditions we shall determine the sets $\mathcal{L}_1 S_k(v)$ and $\mathcal{L}_2 S_k(v)$.

**Proposition 5.1.** *For any admissible $v$ and $k > 3$,*

$$\begin{aligned} \mathcal{L}_1 S_k(v) &\subseteq \{n \geq v-1 \,|\, n \equiv 0, 1 \pmod 4\}, &(1)\\ \mathcal{L}_2 S_k(2k) &\subseteq \{n \geq v-1 \,|\, n \equiv 0, 1 \pmod 4\}, &(2)\\ \mathcal{L}_2 S_k(v) &\subseteq \{n \geq v \,|\, n \equiv 0, 1 \pmod 4\} \textit{ for } v > 2k. &(3)\end{aligned}$$

*Proof.* In a $(K_v, S_k)$-design $\mathcal{B}$, the edge $[x_1, x_2]$ of $K_v$ belongs either to a star of center $x_1$ or to a star of center $x_2$. Thus, there is possibly at most one vertex which is not the center of any star; (1) and (2) follow.

The condition (3) is obvious when any vertex of $K_v$ is center of at least one star of $\mathcal{B}$. Suppose now that there exists a vertex, say $x$, which is not center of any star. Since $v > 2k$, there exists also a vertex $y$ which is center of at least two stars. Let $S = [y; x, a_1, \dots, a_{k-1}]$ and take, for any $i = 1, \dots, k-1$, $S^i$ as the star with center $a_i$ and containing $x$. Replace $S$ in $\mathcal{B}$ with the star $S' = [x; y, a_1, \dots, a_{k-1}]$. Also, in each $S^i$ substitute the edge $[a_i, x]$ with $[a_i, y]$. Thus, we have again a $(K_v, S_k)$-design in which each vertex of $K_v$ is the center of at least one star. This gives (3). □

**Theorem 5.2.** *Assume $k > 3$. For every $v \geq 4k$ with $v(v-1) \equiv 0 (\mathrm{mod}\, 2k)$,*

$$\mathcal{L}_1 S_k(v) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\}.$$

*Proof.* By Proposition 5.1, it is enough to show $\{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\} \subseteq \mathcal{L}_1 S_k(v)$. We distinguish some cases:

a) $v \equiv 0 \pmod 4$. Since $v$ is admissible and $v \geq 4k$, by [8, Theorem 1] there always exists a $(K_v, S_k)$-design $\mathcal{B}$ having exactly one vertex, say $x$, which is not the center of any star. Select from each block of $\mathcal{B}$ a path $P_3$ whose vertices are different from $x$. Use these $P_3$'s for the down-link and remove their edges from $K_v$. This yields a connected graph $R$ having an even number of edges. So, by Theorem 4.1 $R$ can be decomposed in $P_3$'s; hence, there exists a down-link from $\mathcal{B}$ to a $(K_v, P_3)$-design.

b) $v \equiv 1, 2 \pmod 4$. In this case there always exists a $(K_v, S_k)$-design $\mathcal{B}$ having exactly one vertex, say $x$, which is not center of any star and at least one vertex $y$ which is center of exactly one star, say $S$; see [8, Theorem 1]. Choose a $P_3$, say $P = [x_1, y, x_2]$, in $S$. Let now $S'$ be the star containing the edge $[x_1, x_2]$ and pick a $P_3$ containing this edge. Select from each of the other blocks of $\mathcal{B}$ a $P_3$ whose vertices are different from $x$ and $y$. This is always possible since $k > 3$. Use all of these $P_3$'s to construct a down-link. Remove from $K_v \setminus \{x\}$ all of the edges of the $P_3$'s, thus obtaining a graph $R$ with an even number of edges. Observe that $R$ is connected, as $y$ is adjacent to all vertices of $K_v$ different from $x, x_1, x_2$. Thus, by Theorem 4.1, $R$ can be decomposed in $P_3$'s. Hence, there exists a down-link from $\mathcal{B}$ to a $(K_{v-1}, P_3)$-design.

c) $v \equiv 3 \pmod 4$. As neither $n = v-1$ nor $n = v$ are admissible for $P_3$-designs, the result follows arguing as in the proof of Theorem 4.2. □

The condition $v \geq 4k$ might be relaxed when $k > 3$ is a prime power, as shown by the following theorem.

**Theorem 5.3.** *Let $k > 3$ be a prime power. For every $2k \leq v < 4k$ with $v(v-1) \equiv 0 \pmod{2k}$,*

$$\mathcal{L}_1 S_k(v) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\}.$$

*Proof.* Since $k$ is a prime power, $v$ can only assume the following values: $2k, 2k+1, 3k, 3k+1$. For each of the allowed values of $v$ there exists a $(K_v, S_k)$-design with exactly one vertex which is not center of any star; see [8]. The result can be obtained arguing as in previous theorem. $\square$

**Theorem 5.4.** *Let $k > 3$ and take $v$ be such that $v(v-1) \equiv 0 \pmod{2k}$. Then,*

$$\mathcal{L}_2 S_k(2k) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\};$$

$$\mathcal{L}_2 S_k(v) = \{n \geq v \mid n \equiv 0, 1 \pmod 4\} \textit{ for } v > 2k.$$

*Proof.* Let $\mathcal{B}$ be a $(K_{2k}, S_k)$-design. Clearly, there is exactly one vertex of $K_{2k}$ which is not the center of any star. By Proposition 5.1, it is enough to show that $\{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\} \subseteq \mathcal{L}_2 S_k(2k)$. The result can be obtained arguing as in step a) of Theorem 5.2 for $k$ even and as in step b) of the same for $k$ odd.

We now consider the case $v > 2k$. As before, by Proposition 5.1, we just need to prove one of the inclusions. Suppose $v \equiv 0, 1 \pmod 4$. Let $\mathcal{B}$ be a $(K_v, S_k)$-design. For $k$ even, each star is a disjoint union of $P_3$'s and the existence of a down-link to a $(K_v, P_3)$-design is trivial. For $k$ odd, observe that $\mathcal{B}$ contains an even number of stars. Hence, there is an even number of vertices $x_0, x_1, x_2, \ldots, x_{2t-1}$ of $K_v$ which are center of an odd number of stars. Consider the edges $[x_{2i}, x_{2i+1}]$ for $i = 0, \ldots, t-1$. From each star of $\mathcal{B}$, extract a $P_3$ which does not contain any of the aforementioned edges and use it for the down-link. If $y \in K_v$ is the center of an even number of stars, then the union of all the remaining edges of stars with center $y$ is a connected graph with an even number of edges; thus, it is possible to apply Theorem 4.1. If $y$ is the center of an odd number of stars, then there is an edge $[x_{2i}, x_{2i+1}]$ containing $y$. In this case the graph obtained by the union of all the remaining edges of the stars with centers $x_{2i}$ and $x_{2i+1}$ is connected and has an even number of edges. Thus, we can apply again Theorem 4.1. For $v \equiv 2, 3 \pmod 4$, the result follows as in Theorem 4.2. $\square$

# 6 Kite-designs

Denote by $D = [a, b, c \bowtie d]$ the *kite*, a triangle with an attached edge, having vertices $\{a, b, c, d\}$ and edges $[c, a], [c, b], [c, d]$, $[a, b]$.

In [2], Bermond and Schönheim proved that a kite-design of order $v$ exists if, and only if, $v \equiv 0, 1 \pmod 8$, $v > 1$. In this section we completely determine the sets $\mathcal{L}_1 D(v)$ and $\mathcal{L}_2 D(v)$ where $\Gamma' = P_3$ and $v$, clearly, fulfills the aforementioned condition.

We need now to recall some preliminaries on difference families. For general definitions and in depth discussion, see [7]. Let $(G, +)$ be a group and take $H \le G$. A set $\mathcal{F}$ of kites with vertices in $G$ is called a $(G,H,D,1)$-*difference family* (DF, for short), if the list $\Delta\mathcal{F}$ of differences from $\mathcal{F}$, namely the list of all possible differences $x-y$, where $(x,y)$ is an ordered pair of adjacent vertices of a kite in $\mathcal{F}$, covers all the elements of $G \smallsetminus H$ exactly once, while no element of $H$ appears in $\Delta\mathcal{F}$.

**Proposition 6.1.** *For every $v \equiv 0,1 \pmod 8$, $v > 1$,*

$$\mathcal{L}_1 D(v) \subseteq \{n \ge v-1 \mid n \equiv 0,1 \pmod 4\}, \quad (4)$$
$$\mathcal{L}_2 D(v) \subseteq \{n \ge v \mid n \equiv 0,1 \pmod 4\}. \quad (5)$$

*Proof.* Let $\mathcal{B}$ and $\mathcal{B}'$ be respectively a $(K_v,D)$-design and a $(K_n,P_3)$-design. Suppose there are $x,y \in V(K_v) \smallsetminus V(K_n)$ with $x \ne y$. Since there is at least one kite $D \in \mathcal{B}$ containing both $x$ and $y$, we see that it is not possible to extract any $P_3 \in \mathcal{B}'$ from $D$; thus $n \ge v-1$. This proves (4).

As for (5), we distinguish two cases. For $v \equiv 0 \pmod 8$, there does not exist a $P_3$-design of order $v-1$. On the other hand, for any $v = 8t+1$,

$$\mathcal{F} = \{[2i-1, 3t+i, 0 \bowtie 2i] \mid i = 1,\dots,t\}$$

is a $(\mathbb{Z}_{8t+1},\{0\},D,1)$-DF. As a special case of a more general result proved in [7], the existence of such a difference family implies that of a cyclic $(K_{8t+1},D)$-design $\mathcal{B}$. Thus, any $x \in V(K_{8t+1})$ has degree 3 in at least one block of $\mathcal{B}$. Hence, there is no down-link of $\mathcal{B}$ in a design of order less than $8t+1$. □

**Lemma 6.2.** *For every integer $m = 2n+1$ there exists a $(K_{m\times 8},D)$-design.*

*Proof.* The set

$$\mathcal{F} = \{[(0,0),(0,2i),(2,i) \bowtie (1,0)],[(0,0),(4,i),(1,-i) \bowtie (6,i)] \mid i = 1,\dots,n\}$$

is a $(\mathbb{Z}_8 \times \mathbb{Z}_m, \mathbb{Z}_8 \times \{0\}, D, 1)$-DF. A special case of a result in [7] shows that any difference family with these parameters determines a $(K_{m\times 8},D)$-design. □

**Proposition 6.3.** *There exists a $(K_v,D)$-design with a vertex $x$ having degree 2 in all the blocks in which it appears if and only if $v \equiv 1 \pmod 8$, $v > 1$.*

*Proof.* Clearly, $v \equiv 1 \pmod 8$, $v > 1$, is a necessary condition for the existence of such a design. We will show that it is also sufficient. Assume $v = 8t+1$, $t \ge 1$. Let $A_i = \{a_{i1}, a_{i2}, \dots, a_{i8}\}$, $i = 1,\dots,t$ and write $V(K_v) = \{0\} \cup A_1 \cup A_2 \cup \dots \cup A_t$. Clearly, $E(K_v)$ is the disjoint union of the sets of edges of $K_{0,A_i}$, $K_{A_i}$ and $K_{A_1,A_2,\dots,A_t}$, for $i = 1,2,\dots,t$.

- Suppose $t = 1$, so that $V(K_v) = \{0\} \cup A_1$. An explicit kite decomposition of $K_v = K_{0,A_1} \cup K_{A_1}$ where the degree of 0 is always 2 is given by

$$\{[0, a_{11}, a_{12} \bowtie a_{16}], [0, a_{14}, a_{13} \bowtie a_{15}], [0, a_{16}, a_{15} \bowtie a_{17}], [0, a_{17}, a_{18} \bowtie a_{16}],$$
$$[a_{11}, a_{13}, a_{16} \bowtie a_{17}], [a_{11}, a_{17}, a_{14} \bowtie a_{16}], [a_{11}, a_{15}, a_{18} \bowtie a_{14}],$$
$$[a_{12}, a_{18}, a_{13} \bowtie a_{17}], [a_{14}, a_{15}, a_{12} \bowtie a_{17}]\}.$$

- Let now $t = 2$, so that $V(K_v) = \{0\} \cup A_1 \cup A_2$. There exists a kite decomposition of $K_v$ where the degree of 0 is always 2. Such a decomposition results from the disjoint union of the previous kite decomposition of $K_{0,A_1} \cup K_{A_1}$ and the kite decomposition of $K_{0,A_2} \cup K_{A_2} \cup K_{A_1,A_2}$ here listed:

$$\{[0, a_{22}, a_{21} \bowtie a_{18}], [0, a_{24}, a_{23} \bowtie a_{18}], [0, a_{26}, a_{25} \bowtie a_{18}], [0, a_{28}, a_{27} \bowtie a_{18}],$$
$$[a_{11}, a_{21}, a_{28} \bowtie a_{18}], [a_{11}, a_{27}, a_{22} \bowtie a_{18}], [a_{11}, a_{23}, a_{26} \bowtie a_{18}], [a_{11}, a_{25}, a_{24} \bowtie a_{18}],$$
$$[a_{12}, a_{27}, a_{21} \bowtie a_{17}], [a_{12}, a_{26}, a_{22} \bowtie a_{17}], [a_{12}, a_{25}, a_{23} \bowtie a_{17}], [a_{12}, a_{28}, a_{24} \bowtie a_{17}],$$
$$[a_{13}, a_{21}, a_{26} \bowtie a_{17}], [a_{13}, a_{22}, a_{25} \bowtie a_{17}], [a_{13}, a_{23}, a_{28} \bowtie a_{17}], [a_{13}, a_{24}, a_{27} \bowtie a_{17}],$$
$$[a_{14}, a_{25}, a_{21} \bowtie a_{16}], [a_{14}, a_{24}, a_{22} \bowtie a_{16}], [a_{14}, a_{23}, a_{27} \bowtie a_{16}], [a_{14}, a_{26}, a_{28} \bowtie a_{16}],$$
$$[a_{15}, a_{24}, a_{21} \bowtie a_{23}], [a_{15}, a_{23}, a_{22} \bowtie a_{28}], [a_{15}, a_{28}, a_{25} \bowtie a_{16}], [a_{15}, a_{26}, a_{27} \bowtie a_{25}],$$
$$[a_{24}, a_{26}, a_{16} \bowtie a_{23}]\}.$$

- Take $v = 8t + 1$, $t \geq 3$. For odd $t$, the complete multipartite graph $K_{t\times 8}$ always admits a kite decomposition; see Lemma 6.2. Thus, $K_v$ has a kite decomposition which is the disjoint union of the kite decomposition of $K_{0,A_i} \cup K_{A_i}$, for each $i = 1, \dots, t$ (compare this with the case $t = 1$), and that of $K_{A_1,A_2,\dots,A_t}$. If $t$ is even, write

$$V(K_v) = \{0\} \cup A_1 \cup \cdots \cup A_{t-1} \cup A_t.$$

As $t - 1$ is odd, the graph $K_v$ has a kite decomposition which is the disjoint union of the kite decompositions of $K_{0,A_t} \cup K_{A_t}$ (see the case $t = 1$), $K_{A_1,A_2,\dots,A_{t-1}}$ and $K_{0,A_i} \cup K_{A_i} \cup K_{A_i,A_t}$ (as in the case $t = 2$), for $i = 1, \dots, t-1$. In either case the degree of 0 is 2. □

**Theorem 6.4.** *For every $v \equiv 0, 1 \pmod 8$, $v > 1$,*

$$\mathcal{L}_1 D(v) = \{n \geq v - 1 \mid n \equiv 0, 1 \pmod 4\}; \quad (6)$$
$$\mathcal{L}_2 D(v) = \{n \geq v \mid n \equiv 0, 1 \pmod 4\}. \quad (7)$$

*Proof.* Relation (6) follows from Proposition 6.1 and Proposition 6.3. Clearly, any $D$-design with a vertex $x$ of degree 2 in all the blocks in which it appears can be down-linked to a $P_3$-design of order $v - 1$.

In view of Proposition 6.1, to prove relation (7), it is sufficient to observe that each kite can be seen as the union of two $P_3$'s. □

# 7 Cycle systems

Denote by $C_k$ the cycle on $k$ vertices, $k \geq 3$. It is well known that a $k$-cycle system of order $v$, that is a $(K_v, C_k)$-design, exists if, and only if, $k \leq v$, $v$ is odd and $v(v-1) \equiv 0 \pmod{2k}$. The *if part* of this theorem was solved by Alspach and Gavlas [1] for $k$ odd and by Šajna [27] for $k$ even.

In this section we shall provide some partial results on $\mathcal{L}_1 C_k(v)$ and $\mathcal{L}_2 C_k(v)$.

**Theorem 7.1.** *For any admissible $v$,*

$$\mathcal{L}_2 C_3(v) = \mathcal{L}_1 C_3(v) = \{n \geq v \mid n \equiv 0, 1 \pmod 4\};$$
$$\mathcal{L}_2 C_4(v) = \mathcal{L}_1 C_4(v) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\};$$
$$\mathcal{L}_2 C_5(v) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\} \subseteq \mathcal{L}_1 C_5(v);$$

*and for any $k \geq 6$*

$$\left\{ n \geq v - \left\lfloor \frac{k-4}{3} \right\rfloor \,\middle|\, n \equiv 0, 1 \pmod 4 \right\} \subseteq \mathcal{L}_2 C_k(v) \subseteq \mathcal{L}_1 C_k(v).$$

*Proof.* • Suppose $k = 3$. It is obvious that a down-link from a $(K_v, C_3)$-design $\mathcal{B}$ to a $P_3$-design of order less than $v$ cannot exists. When $v \equiv 1 \pmod 4$, the triangles in $\mathcal{B}$ can be paired so that each pair share a vertex; see [16]. Let $T_1 = (1, 2, 3)$ and $T_2 = (1, 4, 5)$ be such a pair. Use the paths $[1, 2, 3]$ and $[1, 4, 5]$ for down-link and consider the path $[3, 1, 5]$. Observe that these three paths provide a decomposition of the edges of $T_1 \cup T_2$ in $P_3$'s. The proof is completed by repeating this procedure for all paired triangles. For $v \equiv 3 \pmod 4$, proceed as in Theorem 4.2.

• Assume $k = 4$. Let $\mathcal{B}$ be a $(K_v, C_4)$-design. It is easy to see that, as in the case of the kites, the image of a $C \in \mathcal{B}$ in a $(K_n, P_3)$-design $\mathcal{B}'$ must necessarily leave out exactly one of the vertices of $C$. Obviously, any two vertices of $V(K_v)$ are contained together in at least one block of $\mathcal{B}$; thus, $\mathcal{L}_2 C_4(v) \subseteq \mathcal{L}_1 C_4(v) \subseteq \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\}$. We now prove the reverse inclusion: take $x \in V(K_v)$ and delete from each $C \in \mathcal{B}$ with $x \in V(C)$ the edges passing through $x$, thus obtaining a $P_3$, say $P$. Let the image of $C$ under the down-link be $P$. Observe now that the blocks not containing $x$ can still be decomposed into two $P_3$'s. Thus, it is possible to construct a down-link from $\mathcal{B}$ to a $P_3$-design of order $v-1$.

• Take $k = 5$. Note that a $(K_v, C_5)$-design $\mathcal{B}$ necessarily satisfies either of the following:

1) there exist $x, y \in V(K_v)$ such that $x$ and $y$ appear in exactly one block $B$, wherein they are adjacent;

2) every pair of vertices of $K_v$ appear in exactly 2 blocks. In other words, $\mathcal{B}$ is a Steiner pentagon system; see [20].

We will show that it is always possible to down-link $\mathcal{B}$ to a $P_3$-design of order $n \geq v-1$ admissible. Suppose $v \equiv 1 \pmod 4$. If $\mathcal{B}$ satisfies 1), then select from each block a $P_3$ whose vertices are different from $x$ and $y$. Use these $P_3$'s to construct the down-link. Observe that $K_{v-1} = K_v \setminus \{x\}$ minus the edges used for the down-link is a connected graph; thus the assertion follows from Theorem 4.1. If $\mathcal{B}$ satisfies 2), select from each block a $P_3$ whose vertices are different from $x$ and $y$. Note that this is always possible, unless the cycle is $C = (x, a, b, y, c)$. In this case, select from $C$ the path $P = [a, b, y]$. Note that none of the selected paths contains $x$; thus, their union is a subgraph $S$ of $K_{v-1} = K_v \setminus \{x\}$. It is easy to see that each vertex $v \neq b$ of $K_{v-1} \setminus S$ is adjacent to $y$. Thus, either $K_{v-1} \setminus S$ is connected or it consists of the isolated vertex $b$ and a connected component. In both cases it is possible to apply Theorem 4.1 to obtain a $(K_{v-1}, P_3)$-design. When $v \equiv 3 \pmod 4$, argue as in Theorem 4.2.

- Let $k \geq 6$ and denote by $\mathcal{B}$ a $(K_v, C_k)$-design. Write $t = \lfloor \frac{k-4}{3} \rfloor$. Take $t+1$ distinct vertices $x_1, x_2, \dots, x_t, y \in V(K_v)$. Observe that it is always possible to extract from each block $C \in \mathcal{B}$ a $P_3$ whose vertices are different from $x_1, \dots, x_t, y$, as we are forbidding at most $2\lfloor \frac{k-4}{3} \rfloor + 2$ edges from any $k$-cycle; consequently, the remaining edges cannot be pairwise disjoint. Use these $P_3$'s for the down-link. Write $S$ for the image of the down-link, regarded as a subgraph of $K_{v-t} = K_v \setminus \{x_1, x_2, \dots, x_t\}$. Observe that the edges of $K_{v-t}$ not contained in $S$ form a connected graph $R$. When $R$ has an even number of edges, namely $v - \lfloor \frac{k-4}{3} \rfloor \equiv 0, 1 \pmod 4$, the result is a direct consequence of Theorem 4.1 and we are done. Otherwise add $u = 1$ or $u = 2$ vertices to $K_{v-t}$ and then apply Theorem 4.1 to $R' = (K_{v-t} + K_u) \setminus S$. □

**Remark 7.2.** *It is not possible to down-link a $(K_v, C_5)$-design with Property 2) to $P_3$-designs of order smaller than $v-1$. On the other hand if a $(K_v, C_5)$-design enjoys Property 1), then it might be possible to obtain a down-link to a $P_3$-design of order smaller than $v-1$, as shown by the following example.*

**Example 7.3.** *Consider the cyclic $(K_{11}, C_5)$-design $\mathcal{B}$ presented in [5]:*

$$\mathcal{B} = \{ [0,8,7,3,5],[1,9,8,4,6],[2,10,9,5,7],[3,0,10,6,8],[4,1,0,7,9],[5,2,1,8,10], [6,3,2,9,0],[7,4,3,10,1],[8,5,4,0,2],[9,6,5,1,3],[10,7,6,2,4] \}.$$

*Note that* 0 *and* 1 *appear together in exactly one block. It is possible to down-link $\mathcal{B}$ to the following $P_3$-design of order* 9*:*

$$\mathcal{B}' = \{ [8,7,3],[8,4,6],[9,5,7],[10,6,8],[7,9,4],[8,10,5],[6,3,2],[4,3,10],[8,5,4], [3,9,6],[4,10,7] \} \cup \{ [3,5,2],[3,8,9],[7,2,10],[10,9,2],[6,7,4],[4,2,8],[2,6,5] \}.$$

# 8 Path-designs

In [28], Tarsi proved that the necessary conditions for the existence of a $(K_v, P_k)$-design, namely $v \geq k$ and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient. In this section we investigate down-links from path-designs to $P_3$-designs and provide partial results for $\mathcal{L}_1 P_k(v)$ and $\mathcal{L}_2 P_k(v)$.

**Theorem 8.1.** *For any admissible $v > 1$,*

$$\mathcal{L}_1 P_4(v) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\}; \tag{8}$$

$$\mathcal{L}_2 P_4(v) \subseteq \{n \geq v \mid n \equiv 0, 1 \pmod 4\}. \tag{9}$$

*Proof.* Let $\mathcal{B}$ and $\mathcal{B}'$ be respectively a $(K_v, P_4)$-design and a $(K_n, P_3)$-design. Suppose there exists a down-link $f : \mathcal{B} \to \mathcal{B}'$. Clearly, $n > v-2$. Hence, $\mathcal{L}_2 P_4(v) \subseteq \mathcal{L}_1 P_4(v) \subseteq \{n \geq v-1 \mid n \equiv 0, 1 \pmod 4\}$.

To show the reverse inclusion in (8) we prove the actual existence of designs providing down-links attaining the minimum. For the case $v \equiv 1, 2 \pmod 4$ we refer to Subsection 8.1. For $v \equiv 3 \pmod 4$, it is possible to argue as in Theorem 4.2. For $v \equiv 0 \pmod 4$, observe that a $(K_v, P_4)$-design exists if, and only if $v \equiv 0, 4 \pmod{12}$. In particular, for $v = 4$, the existence of a down-link from a $(K_4, P_4)$-design to a $(K_4, P_3)$-design is trivial. For $v > 4$, arguing as in Subsection 8.1, we can obtain a $(K_v, P_4)$-design $\mathcal{B}$ with a vertex $0 \in V(K_v)$ having degree 1 in each block wherein it appears. Hence, it is possible to choose for the down-link a $P_3$ not containing 0 from any block of $\mathcal{B}$. Denote by $S$ the set of all of these $P_3$'s and consider the complete graph $K_{v-1} = K_v \smallsetminus \{0\}$. Let now $R = (K_{v-1} + \{\alpha\}) \smallsetminus S$. Clearly, $R$ is a connected graph with an even number of edges. Hence, by Theorem 4.1, $\eta_1(v) = v$.

In order to prove (9), it is sufficient to show that for any admissible $v$ there exists a $(K_v, P_4)$-design $\mathcal{B}$ wherein no vertices can be deleted. In particular, this is the case if each vertex of $K_v$ has degree 2 in at least one block of $\mathcal{B}$. First of all note that in a $(K_v, P_4)$-design there is at most one vertex with degree 1 in each block where it appears. Suppose that there actually exists a $(K_v, P_4)$-design $\overline{\mathcal{B}}$ with a vertex $x$ as above. It is not hard to see that there is in $\overline{\mathcal{B}}$ at least one block $P^1 = [x, a, b, c]$ such that the vertices $a$ and $b$ have degree 2 in at least another block. Let $P^2 = [x, c, d, e]$. By reassembling the edges of $P^1 \cup P^2$ it is possible to replace in $\overline{\mathcal{B}}$ these two paths with $P^3 = [b, a, x, c]$, $P^4 = [b, c, d, e]$ if $b \neq e$ or $P^5 = [a, x, c, b]$, $P^6 = [c, d, b, a]$ if $b = e$. Thus, we have again a $(K_v, P_4)$-design. By the assumptions on $a$ and $b$ all the vertices of this new design have degree 2 in at least one block. □

Arguing exactly as in the proof of Theorem 7.1 it is possible to prove the following result.

**Theorem 8.2.** *Let $k > 4$. For any admissible $v > 1$,*

$$\left\{n \geq v - \left\lfloor \frac{k-6}{3} \right\rfloor \mid n \equiv 0, 1 \pmod 4\right\} \subseteq \mathcal{L}_2 P_k(v) \subseteq \mathcal{L}_1 P_k(v).$$

## 8.1 A construction

The aim of the current subsection is to provide for any admissible $v \equiv 1, 2 \pmod 4$ a $(K_v, P_4)$-design $\mathcal{B}$ with a vertex having degree 1 in every block in which it appears. It will be then possible to provide a down-link from $\mathcal{B}$ into a $(K_{v-1}, P_3)$-design $\mathcal{B}'$, as needed in Theorem 8.1. Recall that if $(v-1)(v-2) \not\equiv 0 \pmod 4$, no $(K_{v-1}, P_3)$-design exists. Thus this condition is necessary for the existence of a down-link with the required property. We shall prove its sufficiency by providing explicit constructions for all $v \equiv 1, 6, 9, 10 \pmod{12}$. The approach outlined in Section 2 shall be extensively used, by constructing a partition of the vertices of the graph $K_v$ in such a way that all the induced complete and complete bipartite graphs can be down-linked to decompositions in $P_3$'s of suitable subgraphs of $K_{v-1}$; these, in turn, shall yield a decomposition of $\mathcal{B}'$ with an associated down-link.

Write $V(K_v) = X_\ell \cup A_1 \cup \cdots \cup A_t$, where $X_\ell = \{0\} \cup \{1, \ldots, \ell - 1\}$ for $\ell = 6, 9, 10, 13$ and $|A_i| = 12$ for all $i = 1, \ldots, t$. We first construct a $(K_{X_\ell}, P_4)$-design $\mathcal{B}$ which can be down-linked to a $(K_{X_\ell \setminus \{0\}}, P_3)$-design $\mathcal{B}'$. The possible cases are as follows.

- $\ell = 6$:

$$\mathcal{B} = \{[0,1,2,4], [0,2,3,5], [0,3,4,1], [0,4,5,2], [0,5,1,3]\}$$

$$\mathcal{B}' = \{[1,2,4], [2,3,5], [3,4,1], [4,5,2], [5,1,3]\}$$

- $\ell = 9$:

$$\mathcal{B} = \{[0,1,2,4], [0,2,3,5], [0,3,4,6], [0,4,5,7], [0,5,6,8], [0,6,7,1], [0,7,8,2], [0,8,1,3], [5,8,4,1], [2,5,1,6], [3,6,2,7], [4,7,3,8]\}$$

$$\mathcal{B}' = \{[1,2,4], [2,3,5], [3,4,6], [4,5,7], [5,6,8], [6,7,1], [7,8,2], [8,1,3], [8,4,1], [5,1,6], [3,6,2], [7,3,8]\} \cup \{[2,5,8], [2,7,4]\}$$

- $\ell = 10$:

$$\mathcal{B} = \{[0,1,2,4], [0,2,3,5], [0,3,4,6], [0,4,5,7], [0,5,6,8], [0,6,7,9], [0,7,8,1], [0,8,9,2], [0,9,1,3], [1,4,8,2], [2,6,9,4], [4,7,2,5], [5,9,3,7], [7,1,5,8], [8,3,6,1]\}$$

$$\mathcal{B}' = \{[1,2,4], [2,3,5], [3,4,6], [4,5,7], [5,6,8], [6,7,9], [7,8,1], [8,9,2], [9,1,3], [1,4,8], [6,9,4], [4,7,2], [9,3,7], [7,1,5], [3,6,1]\} \cup \{[8,2,6], [2,5,9], [5,8,3]\}$$

- $\ell = 13$:

$$\mathcal{B} = \{[0,1,2,4],[0,2,3,5],[0,3,4,6],[0,4,5,7],[0,5,6,8],[0,6,7,9],[0,7,8,10],[0,8,9,11],[0,9,10,12],[0,10,11,1],[0,11,12,2],[0,12,1,3],[1,4,9,5],[2,5,10,6],[3,6,11,7],[4,7,12,8],[5,8,1,9],[6,9,2,10],[5,11,3,10],[10,7,1,5],[5,12,9,3],[3,7,2,11],[6,12,4,11],[11,8,2,6],[6,1,10,4],[4,8,3,12]\}.$$

$$\mathcal{B}' = \{[1,2,4],[2,3,5],[3,4,6],[4,5,7],[5,6,8],[6,7,9],[7,8,10],[8,9,11],[9,10,12],[10,11,1],[11,12,2],[12,1,3],[1,4,9],[5,10,6],[3,6,11],[7,12,8],[5,8,1],[9,2,10],[5,11,3],[7,1,5],[5,12,9],[7,2,11],[6,12,4],[8,2,6],[6,1,10],[8,3,12]\} \cup \{[9,5,2],[11,7,4],[1,9,6],[3,10,7],[9,3,7],[4,11,8],[10,4,8]\}.$$

We now consider down-links between designs on complete bipartite graphs. For $X = \{0,1,2\}$ and $Y = \{a,b,c,d,e,f\}$, there is a metamorphosis of the $(K_{X,Y}, P_4)$-design

$$\mathcal{B} = \{[0,a,1,d],[0,b,1,e],[0,c,1,f],[0,d,2,a],[0,e,2,b],[0,f,2,c]\}$$

to the $(K_{X,Y}, P_3)$-design

$$\mathcal{B}' = \{[a,1,d],[b,1,e],[c,1,f],[d,2,a],[e,2,b],[f,2,c],[a,0,b],[c,0,d],[e,0,f]\}.$$

Note that if we remove the paths $[a,0,b]$,$[c,0,d]$,$[e,0,f]$ from $\mathcal{B}'$, we obtain a bijective down-link from $\mathcal{B}$ to the $(K_{X\setminus\{0\},Y}, P_3)$-design

$$\mathcal{B}'' = \{[a,1,d],[b,1,e],[c,1,f],[d,2,a],[e,2,b],[f,2,c]\}.$$

Thus, we have actually obtained a metamorphosis $\mu : (K_{3,6}, P_4)$-design $\to$ $(K_{3,6}, P_3)$-design and a down-link $\delta : (K_{3,6}, P_4)$-design $\to$ $(K_{2,6}, P_3)$-design. By gluing together copies of $\mu$ we get metamorphoses of $P_4$-decompositions into $P_3$-decompositions of $K_{6,6}$, $K_{9,6}$, $K_{6,12}$, $K_{9,12}$, $K_{12,12}$. Likewise, using $\delta$ we also determine down-links from $P_4$-decompositions of $K_{6,6}$, $K_{6,12}$ and $K_{9,12}$ to $P_3$-decompositions of respectively $K_{5,6}$, $K_{5,12}$ and $K_{8,12}$. For our construction, it will also be necessary to provide a metamorphosis of a $(K_{12}, P_4)$-design $\mathcal{B}$ into a $(K_{12}, P_3)$-design $\mathcal{B}'$. This is given by

$$\mathcal{B} = \{[1,2,3,5],[1,3,4,6],[1,4,5,7],[1,5,6,8],[1,6,7,9],[1,7,8,10],[1,8,9,11],[1,9,10,12],[1,10,11,2],[1,11,12,3],[1,12,2,4],[2,5,10,6],[3,6,11,7],[4,7,12,8],[5,8,2,9],[6,9,3,10],[7,10,4,11],[8,11,5,12],[9,12,6,2],[10,2,7,3],[11,3,8,4],[12,4,9,5]\};$$

$$\mathcal{B}' = \{[2,3,5],[3,4,6],[4,5,7],[5,6,8],[6,7,9],[7,8,10],[8,9,11],[9,10,12],[10,11,2],[11,12,3],[12,2,4],[5,10,6],[6,11,7],[7,12,8],[8,2,9],[9,3,10],[10,4,11],[11,5,12],[12,6,2],[2,7,3],[3,8,4],[4,9,5]\} \cup \{[1,2,5],[1,3,6],[1,4,7],[1,5,8],[1,6,9],[1,7,10],[1,8,11],[1,9,12],[1,10,2],[1,11,3],[1,12,4]\}.$$

Consider now a $(K_v, P_4)$-design with $v = \ell + 12t$ where $\ell = 1,6,9,10$ and $t > 1$.

- For $v = 1 + 12t$, write

$$K_{1+12t} = (tK_{1,12} \cup tK_{12}) \cup \binom{t}{2} K_{12,12} = tK_{13} \cup \binom{t}{2} K_{12,12}.$$

The down-link here is obtained by gluing down-links from $P_4$-decompositions of $K_{13}$ to $P_3$-decompositions of $K_{12}$ with metamorphoses of $P_4$-decompositions of $K_{12,12}$ into $P_3$-decompositions.

- For $v = 6 + 12t$, consider

$$K_{6+12t} = K_6 \cup tK_{12} \cup tK_{6,12} \cup \binom{t}{2} K_{12,12}.$$

Down-link the $P_4$-decompositions of $K_6$ and $K_{6,12}$ to respectively $P_3$-decompositions of $K_5$ and $K_{5,12}$ and consider metamorphoses of the $P_4$-decompositions of $K_{12}$ and $K_{12,12}$ into $P_3$-decompositions.

- For $v = 9 + 12t$, let

$$K_{9+12t} = K_9 \cup tK_{12} \cup tK_{9,12} \cup \binom{t}{2} K_{12,12}.$$

We know how to down-link the $P_4$-decompositions of $K_9$ and $K_{9,12}$ to $P_3$-decompositions of respectively $K_8$ and $K_{8,12}$. As before, there are metamorphoses of the $P_4$-decompositions of both $K_{12}$ and $K_{12,12}$ into $P_3$-decompositions.

- For $v = 10 + 12t$, observe that

$$K_{10+12t} = K_{10} \cup tK_{12} \cup tK_{10,12} \cup \binom{t}{2} K_{12,12} =$$

$$K_{10} \cup tK_{12} \cup tK_{1,12} \cup tK_{9,12} \cup \binom{t}{2} K_{12,12} = K_{10} \cup tK_{13} \cup tK_{9,12} \cup \binom{t}{2} K_{12,12}.$$

We know how to down-link $P_4$-decompositions of $K_{10}$, $K_{13}$ and $K_{9,12}$ to $P_3$-decompositions of respectively $K_9$, $K_{12}$ and $K_{8,12}$. As for the $K_{12,12}$ we argue as in the preceding cases.

# References


[1] Alspach, B., Gavlas, H., *Cycle decompositions of $K_n$ and $K_n - I$*, J. Combin. Theory Ser. B **81** (2001), 77–99.

[2] Bermond, J.C., Schönheim, J., *G-decomposition of $K_n$, where G has four vertices or less*, Discrete Math. **19** (1977), 113–120.

[3] Bosák, J., "Decompositions of graphs", Mathematics and its Applications (1990), *Kluwer Academic Publishers Group.*

[4] Bryant, D., El-Zanati, S., *Graph Decompositions*, CRC Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz, CRC Press (2006), 477–486.

[5] Buratti, M., Del Fra, A., *Existence of cyclic $k$-cycle systems of the complete graph*, Discrete Math. **261** (2003), 113–125.

[6] Buratti, M., Pasotti, A., *Combinatorial designs and the theorem of Weil on multiplicative character sums*, Finite Fields Appl. **15** (2009), 332–344.

[7] Buratti, M., Pasotti, A., *Graph decompositions with the use of difference matrices*, Bull. Inst. Combin. Appl. **47** (2006), 23–32.

[8] Cain, P., *Decomposition of complete graphs into stars*, Bull. Austral. Math. Soc. **10** (1974), 23–30.

[9] Chang, Y., Lo Faro, G., Tripodi, A., *Determining the spectrum of* Meta$(K_4 + e > K_4, \lambda)$ *for any* $\lambda$, Discrete Math. **308** (2008), 439–456.

[10] Chang, Y., Lo Faro, G., Tripodi, A., *The spectrum of* Meta$(K_4 - e > K_3 + e, \lambda)$ *with any* $\lambda$, Util. Math. **72** (2007), 3–22.

[11] Colbourn, C.J., Hamm, R.C., Lindner, C.C., Rodger, A., *Embedding partial graph designs, block designs, and triple systems with* $\lambda > 1$, Canad. Math. Bull. **29** (1986), 385–391.

[12] Colbourn, C.J., Ling, A.C.H., Quattrocchi, G., *Embedding path designs into kite systems*, Discrete Math. **297** (2005), 38–48.

[13] Doyen, J., Wilson, R.M., *Embeddings of Steiner triple systems*, Discrete Math. **5** (1973), 229–239.

[14] Gionfriddo, M., Quattrocchi, G., *Embedding balanced $P_3$-designs into (balanced) $P_4$-designs*, Discrete Math. **308** (2008), 155–160.

[15] Harary, F., "Graph Theory" (1969), Addison-Wesley.

[16] Horák, P., Rosa, A., *Decomposing Steiner triple systems into small configurations*, Ars Combin. **26** (1988), 91–105.

[17] Küçükçifçi, S., Lindner, C.C., *The metamorphosis of $\lambda$-fold block designs with block size four into $\lambda$-fold kite systems*, J. Combin. Math. Combin. Comput. **40** (2002), 241–252.

[18] Lindner, C.C., Rosa, A., *The metamorphosis of $\lambda$-fold block designs with block size four into $\lambda$-fold triple systems*, J. Stat. Plann. Inference **106** (2002), 69–76.

[19] Lindner, C.C., Rosa, A., *The metamorphosis of block designs with block size four into $(K_4 - e)$-designs*, Util. Math. **6** (2002), 33–46.

[20] Lindner, C.C., Stinson, D.R., *Steiner pentagon systems*, Discrete Math. **52** (1984), 67–74.

[21] Lindner, C.C., Street, A.P., *The metamorphosis of $\lambda$-fold block designs with block size four into $\lambda$-fold 4-cycle systems*, Bull. Inst. Combin. Appl. **28** (2000), 7–18, Bull. Inst. Combin. Appl. **29** (2000), 88 (Corrigendum).

[22] Ling, A.C.H., Milici, S., Quattrocchi, G., *Two generalizations of the metamorphosis definition*, Bull. Inst. Combin. Appl. **43** (2005), 58–66.

[23] Milici, S., Quattrocchi, G., *Embedding handcuffed designs with block size* 2 *or* 3 *in* 4*-cycle systems*, Combinatorics (Assisi, 1996). Discrete Math. **208/209** (1999), 443–449

[24] Quattrocchi, G., *Embedding $G_1$-designs into $G_2$-designs, a short survey*, Rend. Sem. Mat. Messina Ser. II **8** (2001), 129–143.

[25] Quattrocchi, G., *Embedding path designs in* 4*-cycle systems*, Combinatorics (Palermo, 1998). Discrete Math. **255** (2002), 349–356.

[26] Quattrocchi, G., *Embedding handcuffed designs in D-designs, where D is the triangle with attached edge*, Discrete Math. **261** (2003), 413–434.

[27] S̃ajna, M., *Cycle decompositions III: complete graphs and fixed length cycles*, J. Combin. Designs **10** (2002), 27–78.

[28] Tarsi, M., *Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs*, J. Combin. Theory Ser. A **34** (1983), 60–70.

[29] Tarsi, M., *Decomposition of complete multigraphs into stars*, Discrete Math. **26** (1979), 273–278.

[30] West, D.B., "Introduction to graph theory", 2nd edition (2006), Dorling Kindersley India.

[31] Wilson, R.M., *Construction and uses of pairwise balanced designs*, Math. Centre Tracts **55** (1974), 18–41.