Down-linking \((K_v, \Gamma)\)-designs to \(P_3\)-designs

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Abstract

Let \(\Gamma'\) be a subgraph of a graph \(\Gamma\). We define a down-link from a \((K_v, \Gamma)\)-design \(B\) to a \((K_n, \Gamma')\)-design \(B'\) as a map \(f : B \to B'\) mapping any block of \(B\) into one of its subgraphs. This is a new concept, closely related with both the notion of metamorphosis and that of embedding. In the present paper we study down-links in general and prove that any \((K_v, \Gamma)\)-design might be down-linked to a \((K_n, \Gamma')\)-design, provided that \(n\) is admissible and large enough. We also show that if \(\Gamma' = P_3\), it is always possible to find a down-link to a design of order at most \(v + 3\). This bound is then improved for several classes of graphs \(\Gamma\), by providing explicit constructions.

Keywords: down-link; metamorphosis; embedding; \((K_v, \Gamma)\)-design.

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1 Introduction

Let \(K\) be a graph and \(\Gamma \leq K\). A \((K, \Gamma)\)-design, also called a \(\Gamma\)-decomposition of \(K\), is a set \(B\) of graphs all isomorphic to \(\Gamma\), called blocks, partitioning the edge-set of \(K\). Given a graph \(\Gamma\), the problem of determining the existence of \((K_v, \Gamma)\)-designs, also called \(\Gamma\)-designs of order \(v\), where \(K_v\) is the complete graph on \(v\) vertices, has been extensively studied; for surveys on this topic see, for instance, [3, 4].

We propose the following new definition.

**Definition 1.1.** Given a \((K, \Gamma)\)-design \(B\) and a \((K', \Gamma')\)-design \(B'\) with \(\Gamma' \leq \Gamma\), a down-link from \(B\) to \(B'\) is a function \(f : B \to B'\) such that \(f(B) \leq B\), for any \(B \in B\).

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By the definition of \((K, \Gamma)\)-design, a down-link is necessarily injective. When a function \(f\) as in Definition 1.1 exists, that is if each block of \(\mathcal{B}\) contains at least one element of \(\mathcal{B}'\) as a subgraph, it will be said that it is possible to down-link \(\mathcal{B}\) to \(\mathcal{B}'\).

In this paper we shall investigate the existence and some further properties of down-links between designs on complete graphs and outline their relationship with some previously known notions. More in detail, Section 2 is dedicated to the close interrelationship between down-links, metamorphoses and embeddings. In Section 3 we will introduce, in close analogy to embeddings, two problems on the spectra of down-links and determine bounds on their minima. In Section 4 down-links from any \((K_v, \Gamma)\)-design to a \(P_3\)-design of order \(n \leq v + 3\) are constructed; this will improve on the values determined in Section 3. In further Sections 5, 6, 7, 8 the existence of down-links to \(P_3\)-designs from, respectively, star-designs, kite-designs, cycle systems and path-designs are investigated by providing explicit constructions.

Throughout this paper the following standard notations will be used; see also [15]. For any graph \(\Gamma\), write \(V(\Gamma)\) for the set of its vertices and \(E(\Gamma)\) for the set of its edges. By \(t\Gamma\) we shall denote the disjoint union of \(t\) copies of graphs all isomorphic to \(\Gamma\). Given any set \(V\), the complete graph with vertex-set \(V\) is \(K_V\). As usual, \(K_{v_1, v_2, \ldots, v_m}\) is the complete \(m\)-partite graph with parts of size respectively \(v_1, \ldots, v_m\); when \(v = v_1 = v_2 = \cdots = v_m\) we shall simply write \(K_{mv}\). When we want to focus our attention on the actual parts \(V_1, V_2, \ldots, V_m\), the notation \(K_{V_1, V_2, \ldots, V_m}\) shall be used instead. The join \(\Gamma + \Gamma'\) of two graphs consists of the graph \(\Gamma \cup \Gamma'\) together with the edges connecting all the vertices of \(\Gamma\) with all the vertices of \(\Gamma'\); hence, \(\Gamma + \Gamma' = \Gamma \cup \Gamma' \cup K_{V(\Gamma), V(\Gamma')}\).

2 Down-links, metamorphoses, embeddings

As it will be shown, the concepts of down-link, metamorphosis and embedding are closely related.

Metamorphoses of designs have been first introduced by Lindner and Rosa in [18] in the case \(\Gamma = K_4\) and \(\Gamma' = K_3\). In recent years metamorphoses and their generalizations have been extensively studied; see for instance [9, 10, 17, 19, 21, 22]. We here recall the general notion of metamorphosis. Suppose \(\Gamma' \leq \Gamma\) and let \(\mathcal{B}\) be a \((\lambda K_v, \Gamma)\)-design. For each block \(B \in \mathcal{B}\) take a subgraph \(B' \leq B\) isomorphic to \(\Gamma'\) and put it into a set \(S\). If it is possible to reassemble all the remaining edges of \(\lambda K_v\) into a set \(R\) of copies of \(\Gamma'\), then \(S \cup R\) are the blocks of a \((\lambda K_v, \Gamma')\)-design, which is said to be a metamorphosis of \(\mathcal{B}\). Thus, if \(\mathcal{B}'\) is a metamorphosis of \(\mathcal{B}\) with \(\lambda = 1\), then there exists a down-link \(f : \mathcal{B} \to \mathcal{B}'\) given by \(f(B) = B'\). With a slight abuse
of notation we shall call metamorphoses all down-links from a \((K, \Gamma)\)-design to a \((K, \Gamma')\)-design.

There is also a generalization of metamorphosis, originally from [22], which turns out to be closely related to down-links. Suppose \(\Gamma' \leq \Gamma\) and let \(\mathcal{B}\) be a \(\lambda K_v, \Gamma)\)-design. Write \(n\) for the minimum integer \(n \geq v\) for which there exists a \(\lambda K_n, \Gamma')\)-design. Take \(X = V(K_v)\) and \(X \cup Y = V(K_n)\). For each block \(B \in \mathcal{B}\) extract a subgraph of \(B\) isomorphic to \(\Gamma'\) and put it into a set \(S\). Let also \(R\) be the set of all the remaining edges of \(\lambda K_v\). Let \(T\) be the set of edges of \(\lambda K_Y\) and of the \(\lambda\)-fold complete bipartite graph \(\lambda K_X,Y\). If it is possible to reassemble the edges of \(R \cup T\) into a set \(R'\) of copies of \(\Gamma'\), then \(S \cup R'\) are the blocks of a \((\lambda K_n, \Gamma')\)-design \(\mathcal{B}'\). In this case, one speaks of a metamorphosis of \(\mathcal{B}\) into a minimum \(\Gamma'\)-design. It is easy to see that for \(\lambda = 1\) these generalized metamorphoses also induce down-links.

Even if metamorphoses with \(\lambda = 1\) are all down-links, the converse is not true. For instance, all down-links from designs of order \(v\) to designs of order \(n < v\) are not metamorphoses. Example 2.1 shows that such down-links may exist.

Gluing of metamorphoses and down-links can be used to produce new classes of down-links from old, as shown by the following construction. Take \(\mathcal{B}\) as a \((K_v, \Gamma)\)-design with \(V(K_v) = X \cup \bigcup_{i=1}^{t} A_i\) and suppose \(X' \subseteq X\). Let \(\Gamma' \leq \Gamma\) and \(\mathcal{B}'\) be a \((K_{v-|X'|}, \Gamma')\)-design with \(V(K_{v-|X'|}) = V(K_v) \setminus X'\). Suppose that

\[
f_i : (K_{A_i}, \Gamma)\)-design \rightarrow (K_{A_i}, \Gamma')\)-design \quad \text{for any} \quad i = 1, \ldots, t,
\]

\[
h_{ij} : (K_{A_i}, A_j, \Gamma)\)-design \rightarrow (K_{A_i}, A_j, \Gamma')\)-design \quad \text{for} \quad 1 \leq i < j \leq t
\]

are metamorphoses and that

\[
g : (K_X, \Gamma)\)-design \rightarrow (K_{X \setminus X'}, \Gamma')\)-design,
\]

\[
g_i : (K_{X,A_i}, \Gamma)\)-design \rightarrow (K_{X \setminus X', A_i}, \Gamma')\)-design \quad \text{for any} \quad i = 1, \ldots, t
\]

are down-links. As

\[
K_v = \bigcup_{i=1}^{t} K_{A_i} \cup \bigcup_{1 \leq i < j \leq t} K_{A_i,A_j} \cup K_X \cup \bigcup_{i=1}^{t} K_{X,A_i}
\]

and

\[
K_{v-|X'|} = \bigcup_{i=1}^{t} K_{A_i} \cup \bigcup_{1 \leq i < j \leq t} K_{A_i,A_j} \cup K_{X \setminus X'} \cup \bigcup_{i=1}^{t} K_{X \setminus X', A_i},
\]

the function obtained by gluing together \(g\) and all of the \(f_i\)'s, \(h_{ij}\)'s and \(g_i\)'s provides a down-link from \(\mathcal{B}\) to \(\mathcal{B}'\).
Recall that an *embedding* of a design $\mathcal{B}'$ into a design $\mathcal{B}$ is a function $\psi : \mathcal{B}' \to \mathcal{B}$ such that $\Gamma \leq \psi(\Gamma)$, for any $\Gamma \in \mathcal{B}'$; see [24]. Existence of embeddings of designs has been widely investigated. In particular, a great deal of results are known on injective embeddings of path-designs; see, for instance, [12, 14, 23, 25, 26]. If $\psi : \mathcal{B}' \to \mathcal{B}$ is a bijective embedding, then $\psi^{-1}$ is a down-link from $\mathcal{B}$ to $\mathcal{B}'$. Clearly, a bijective embedding of $\mathcal{B}'$ into $\mathcal{B}$ might exist only if $\mathcal{B}$ and $\mathcal{B}'$ have the same number of blocks. This condition, while quite restrictive, does not necessarily lead to trivial embeddings, as shown in the following example.

**Example 2.1.** Consider the $\left(K_4, P_3\right)$-design

$$\mathcal{B}' = \{\Gamma'_1 = [1, 2, 3], \Gamma'_2 = [1, 3, 0], \Gamma'_3 = [2, 0, 1]\}$$

and the $\left(K_6, P_6\right)$-design

$$\mathcal{B} = \{\Gamma_1 = [4, 0, 5, 1, 2, 3], \Gamma_2 = [2, 5, 4, 1, 3, 0], \Gamma_3 = [5, 3, 4, 2, 0, 1]\}.$$

Define $\psi : \mathcal{B}' \to \mathcal{B}$ by $\psi(\Gamma'_i) = \Gamma_i$ for $i = 1, 2, 3$. Then, $\psi$ is a bijective embedding; consequently, $\psi^{-1}$ is a down-link from $\mathcal{B}$ to $\mathcal{B}'$.

### 3 Spectrum problems

Spectrum problems about the existence of embeddings of designs have been widely investigated; see [12, 13, 14, 23, 25, 26].

In close analogy, we pose the following questions about the existence of down-links:

(I) For each admissible $v$, determine the set $L_1\Gamma(v)$ of all integers $n$ such that there exists *some* $\Gamma$-design of order $v$ down-linked to a $\Gamma'$-design of order $n$.

(II) For each admissible $v$, determine the set $L_2\Gamma(v)$ of all integers $n$ such that *every* $\Gamma$-design of order $v$ can be down-linked to a $\Gamma'$-design of order $n$.

In general, write $\eta_i(v; \Gamma, \Gamma') = \inf L_i\Gamma(v)$. When the graphs $\Gamma$ and $\Gamma'$ are easily understood from the context, we shall simply use $\eta_i(v)$ instead of $\eta_i(v; \Gamma, \Gamma')$. The problem of the actual existence of down-links for given $\Gamma' \leq \Gamma$ is addressed in Proposition 3.2. We recall the following lemma on the existence of finite embeddings for partial decompositions, a straightforward consequence of an asymptotic result by R.M. Wilson [31, Lemma 6.1]; see also [6].
Lemma 3.1. Any partial \((K_v, \Gamma)\)-design can be embedded into a \((K_n, \Gamma)\)-design with \(n = O((v^2/2)v^2)\).

Proposition 3.2. For any \(v\) such that there exists a \((K_v, \Gamma)\)-design and any \(\Gamma' \leq \Gamma\), the sets \(L_1\Gamma(v)\) and \(L_2\Gamma(v)\) are non-empty.

Proof. Fix first a \((K_v, \Gamma)\)-design \(B\). Denote by \(K_v(\Gamma')\) the so called complete \((K_v, \Gamma')\)-design, that is the set of all subgraphs of \(K_v\) isomorphic to \(\Gamma'\), and let \(\zeta : B \rightarrow K_v(\Gamma')\) be any function such that \(\zeta(\Gamma) \leq \Gamma\) for all \(\Gamma \in B\). Clearly, the image of \(\zeta\) is a partial \((K_v, \Gamma')\)-design \(\mathcal{P}\); see [11]. By Lemma 3.1, there is an integer \(n\) such that \(\mathcal{P}\) is embedded into a \((K_n, \Gamma')\)-design \(B'\).

Let \(\psi : \mathcal{P} \rightarrow B'\) be such an embedding; then, \(\xi = \psi\zeta\) is, clearly, a down-link from \(B\) to a \(\Gamma'\)-design \(B'\) of order \(n\). Thus, we have shown that for any \(v\) such that a \(\Gamma'\)-design of order \(v\) exists, and for any \(\Gamma' \leq \Gamma\) the set \(L_1\Gamma(v)\) is non-empty.

To show that \(L_2\Gamma(v)\) is also non-empty, proceed as follows. Let \(\omega\) be the number of distinct \((K_v, \Gamma)\)-designs \(B_i\). For any \(i = 0, \ldots, \omega - 1\), write \(V(B_i) = \{0, \ldots, v-1\} + i \cdot v\). Consider now \(\Omega = \bigcup_{i=0}^{\omega-1} B_i\). Clearly, \(\Omega\) is a partial \(\Gamma\)-design of order \(\omega v\). As above, take \(K_{v\omega}(\Gamma')\) and construct a function \(\zeta : \Omega \rightarrow K_{v\omega}(\Gamma')\) associating to each \(\Gamma \in B_i\) a \(\zeta(\Gamma) \leq \Gamma\). The image \(\bigcup_i \zeta(B_i)\) is a partial \(\Gamma'\)-design \(\Omega'\). Using Lemma 3.1 once more, we determine an integer \(n\) and an embedding \(\psi\) of \(\Omega'\) into a \((K_n, \Gamma')\)-design \(B'\). For any \(i\), let \(\zeta_i\) be the restriction of \(\zeta\) to \(B_i\). It is straightforward to see that \(\psi\zeta_i : B_i \rightarrow B'\) is a down-link from \(B_i\) to a \((K_n, \Gamma')\)-design. It follows that \(n \in L_2\Gamma(v)\). \(\Box\)

Notice that the order of magnitude of \(n\) is \(v^2v^2\); yet, it will be shown that in several cases it is possible to construct down-links from \((K_v, \Gamma)\)-designs to \((K_n, \Gamma')\)-designs with \(n \approx v\).

Lower bounds on \(\eta(v; \Gamma, \Gamma')\) are usually hard to obtain and might not be strict; a easy one to prove is the following:

\[
(v-1)\sqrt{\frac{|E(\Gamma')|}{|E(\Gamma)|}} < \eta_1(v; \Gamma, \Gamma').
\]

4 Down-linking \(\Gamma\)-designs to \(P_3\)-designs

From this section onwards we shall fix \(\Gamma' = P_3\) and focus our attention on the existence of down-links to \((K_n, P_3)\)-designs. Recall that a \((K_n, P_3)\)-design exists if, and only if, \(n \equiv 0, 1(\text{mod}4)\); see [28]. We shall make extensive use of the following result from [30].

Theorem 4.1. Let \(\Gamma\) be a connected graph. Then, the edges of \(\Gamma\) can be partitioned into copies of \(P_3\) if and only if the number of edges is even.
When the number of edges is odd, $E(\Gamma)$ can be partitioned into a single edge together with copies of $P_3$.

Our main result for down-links from a general $(K_v, \Gamma)$-design is contained in the following theorem.

**Theorem 4.2.** For any $(K_v, \Gamma)$-design $\mathcal{B}$ with $P_3 \leq \Gamma$,

$$\eta_1(v) \leq \eta_2(v) \leq v + 3.$$  

**Proof.** For any block $B \in \mathcal{B}$, fix a $P_3 \leq B$ to be used for the down-link. Write $S$ for the set of all these $P_3$'s. Remove the edges covered by $S$ from $K_v$ and consider the remaining graph $R$. If each connected component of $R$ has an even number of edges, by Theorem 4.1, there is a decomposition $D$ of $R$ in $P_3$'s; $S \cup D$ is a decomposition of $K_v$; thus, $\eta_1(v) \leq \eta_2(v) \leq v$. If not, take $1 \leq w \leq 3$ such that $v + w \equiv 0, 1 \pmod{4}$. Then, the graph $R' = (K_v + K_w) \setminus S$ is connected and has an even number of edges. Thus, by Theorem 4.1, there is a decomposition $D$ of $R'$ into copies of $P_3$'s. It follows that $S \cup D$ is a $(K_{v+w}, P_3)$-design $\mathcal{B}'$. \hfill \Box

**Remark 4.3.** In Theorem 4.2, if $v \equiv 2 \pmod{4}$, then the order of the design $\mathcal{B}'$ is the smallest $m \geq v$ for which there exists a $(K_m, P_3)$-design. Thus, the down-links are actually metamorphoses to minimum $P_3$-designs. This is not the case for $v \equiv 0, 1 \pmod{4}$, as we cannot a priori guarantee that each connected component of $R$ has an even number of edges.

Theorem 4.2 might be improved under some further (mild) assumptions on $\Gamma$.

**Theorem 4.4.** Let $\mathcal{B}$ be a $(K_v, \Gamma)$-design.

a) If $v \equiv 1, 2 \pmod{4}$, $|V(\Gamma)| \geq 5$ and there are at least 3 vertices in $\Gamma$ with degree at least 4, then there exists a down-link from $\mathcal{B}$ to a $(K_{v-1}, P_3)$-design.

b) If $v \equiv 0, 3 \pmod{4}$, $|V(\Gamma)| \geq 7$ and there are at least 5 vertices in $\Gamma$ with degree at least 6, then there exists a down-link from $\mathcal{B}$ to a $(K_{v-3}, P_3)$-design.

**Proof.** a) Let $x, y \in V(K_v)$. Extract from any $B \in \mathcal{B}$ a $P_3 \leq B$ whose vertices are neither $x$ nor $y$ and use it for the down-link. This is always possible, since $|V(\Gamma)| \geq 5$ and there is at least one vertex in $\Gamma \setminus \{x, y\}$ of degree at least 2. Write now $S$ for the set of all of these $P_3$'s. Consider the graph $R = (K_{v-2} + \{\alpha\}) \setminus S$ where $K_{v-2} = K_v \setminus \{x, y\}$. This is a connected graph with an even number of edges; thus, by Theorem 4.1, there exists a decomposition $D$ of $R$ in $P_3$'s. Hence, $S \cup D$ provides the blocks of a $P_3$-design of order $v - 1$.
b) In this case consider 4 vertices \( \Lambda = \{x, y, z, t\} \) of \( V(K_v) \). By the assumptions, it is always possible to take a \( P_3 \) disjoint from \( \Lambda \) from each block of \( B \). We now argue as in the proof of part a).

The down-links constructed above are not, in general, to designs whose order is as small as possible; thus, theorems 4.2 and 4.4 do not provide the exact value of \( \eta_1(v) \), unless further assumptions are made.

Remark 4.5. In general, a \((K_n, P_3)\)-design can be trivially embedded into \( P_3 \)-designs of any admissible order \( m \geq n \). Thus, if \( n \in \mathcal{L}_1 \Gamma(v) \), then \( \{m \geq n \mid m \equiv 0, 1 \text{ (mod 4)}\} \subseteq \mathcal{L}_1 \Gamma(v) \). Hence,

\[
\mathcal{L}_1 \Gamma(v) = \{m \geq \eta_1(v) \mid m \equiv 0, 1 \text{ (mod 4)}\}.
\]

Thus, solving problems (I) and (II) turns out to be actually equivalent to determining exactly the values of \( \eta_1(v; \Gamma, P_3) \) and \( \eta_2(v; \Gamma, P_3) \).

For the remainder of this paper, we shall always silently apply Remark 4.5 in all the proofs.

5 Star-designs

In this section the existence of down-links from star-designs to \( P_3 \)-designs is investigated. We follow the notation introduced in Section 3, where \( \Gamma' = P_3 \) is understood. Recall that the star on \( k + 1 \) vertices \( S_k \) is the complete bipartite graph \( K_{1,k} \) with one part having a single vertex, say \( c \), called the center of the star, and the other part having \( k \) vertices, say \( x_i \) for \( i = 0, \ldots, k-1 \), called external vertices. In general, we shall write \( S_k = [c; x_0, x_1, \ldots, x_{k-1}] \).

In [29], Tarsi proved that a \((K_v, S_k)\)-design exists if, and only if, \( v \geq 2k \) and \( v(v-1) \equiv 0 \text{ (mod 2k)} \). When \( v \) satisfies these necessary conditions we shall determine the sets \( \mathcal{L}_1 S_k(v) \) and \( \mathcal{L}_2 S_k(v) \).

Proposition 5.1. For any admissible \( v \) and \( k > 3 \),

\[
\begin{align*}
\mathcal{L}_1 S_k(v) & \subseteq \{n \geq v - 1 \mid n \equiv 0, 1 \text{ (mod 4)}\}, \\
\mathcal{L}_2 S_k(2k) & \subseteq \{n \geq v - 1 \mid n \equiv 0, 1 \text{ (mod 4)}\}, \\
\mathcal{L}_2 S_k(v) & \subseteq \{n \geq v \mid n \equiv 0, 1 \text{ (mod 4)}\} \text{ for } v > 2k.
\end{align*}
\]

Proof. In a \((K_v, S_k)\)-design \( B \), the edge \([x_1, x_2]\) of \( K_v \) belongs either to a star of center \( x_1 \) or to a star of center \( x_2 \). Thus, there is possibly at most one vertex which is not the center of any star; (1) and (2) follow.

The condition (3) is obvious when any vertex of \( K_v \) is center of at least one star of \( B \). Suppose now that there exists a vertex, say \( x \), which is
not center of any star. Since \( v > 2k \), there exists also a vertex \( y \) which is center of at least two stars. Let \( S = [y; x, a_1, \ldots, a_{k-1}] \) and take, for any \( i = 1, \ldots, k-1 \), \( S^i \) as the star with center \( a_i \) and containing \( x \). Replace \( S \) in \( B \) with the star \( S' = [x; y, a_1, \ldots, a_{k-1}] \). Also, in each \( S^i \) substitute the edge \([a_i, x]\) with \([a_i, y]\). Thus, we have again a \((K_v, S_k)\)-design in which each vertex of \( K_v \) is the center of at least one star. This gives (3). \(\square\)

**Theorem 5.2.** Assume \( k > 3 \). For every \( v \geq 4k \) with \( v(v-1) \equiv 0 \pmod{2k} \),

\[
\mathcal{L}_1 S_k(v) = \{ n \geq v-1 \mid n \equiv 0, 1 \pmod{4} \}.
\]

**Proof.** By Proposition 5.1, it is enough to show \( \{ n \geq v-1 \mid n \equiv 0, 1 \pmod{4} \} \subseteq \mathcal{L}_1 S_k(v) \). We distinguish some cases:

a) \( v \equiv 0 \pmod{4} \). Since \( v \) is admissible and \( v \geq 4k \), by [8, Theorem 1] there always exists a \((K_v, S_k)\)-design \( B \) having exactly one vertex, say \( x \), which is not the center of any star. Select from each block of \( B \) a path \( P_3 \) whose vertices are different from \( x \). Use these \( P_3 \)'s for the down-link and remove their edges from \( K_v \). This yields a connected graph \( R \) having an even number of edges. So, by Theorem 4.1 \( R \) can be decomposed in \( P_3 \)'s; hence, there exists a down-link from \( B \) to a \((K_v, P_3)\)-design.

b) \( v \equiv 1, 2 \pmod{4} \). In this case there always exists a \((K_v, S_k)\)-design \( B \) having exactly one vertex, say \( x \), which is not center of any star and at least one vertex \( y \) which is center of exactly one star, say \( S \); see [8, Theorem 1]. Choose a \( P_3 \), say \( P = [x_1, y, x_2] \), in \( S \). Let now \( S' \) be the star containing the edge \([x_1, x_2]\) and pick a \( P_3 \) containing this edge. Select from each of the other blocks of \( B \) a \( P_3 \) whose vertices are different from \( x \) and \( y \). This is always possible since \( k > 3 \). Use all of these \( P_3 \)'s to construct a down-link. Remove from \( K_v \setminus \{x\} \) all of the edges of the \( P_3 \)'s, thus obtaining a graph \( R \) with an even number of edges. Observe that \( R \) is connected, as \( y \) is adjacent to all vertices of \( K_v \) different from \( x, x_1, x_2 \). Thus, by Theorem 4.1, \( R \) can be decomposed in \( P_3 \)'s. Hence, there exists a down-link from \( B \) to a \((K_v-1, P_3)\)-design.

c) \( v \equiv 3 \pmod{4} \). As neither \( n = v-1 \) nor \( n = v \) are admissible for \( P_3 \)-designs, the result follows arguing as in the proof of Theorem 4.2. \(\square\)

The condition \( v \geq 4k \) might be relaxed when \( k > 3 \) is a prime power, as shown by the following theorem.

**Theorem 5.3.** Let \( k > 3 \) be a prime power. For every \( 2k \leq v < 4k \) with \( v(v-1) \equiv 0 \pmod{2k} \),

\[
\mathcal{L}_1 S_k(v) = \{ n \geq v-1 \mid n \equiv 0, 1 \pmod{4} \}.
\]
Proof. Since \( k \) is a prime power, \( v \) can only assume the following values: \( 2k, 2k + 1, 3k, 3k + 1 \). For each of the allowed values of \( v \) there exists a \((K_v, S_k)\)-design with exactly one vertex which is not center of any star; see [8]. The result can be obtained arguing as in previous theorem.

Theorem 5.4. Let \( k > 3 \) and take \( v \) be such that \( v(v-1) \equiv 0 \pmod{2k} \). Then,

\[
\mathcal{L}_2S_k(2k) = \{n \geq v-1 \mid n \equiv 0, 1 \pmod{4}\};
\]
\[
\mathcal{L}_2S_k(v) = \{n \geq v \mid n \equiv 0, 1 \pmod{4}\} \text{ for } v > 2k.
\]

Proof. Let \( B \) be a \((K_{2k}, S_k)\)-design. Clearly, there is exactly one vertex of \( K_{2k} \) which is not the center of any star. By Proposition 5.1, it is enough to show that \( \{n \geq v-1 \mid n \equiv 0, 1 \pmod{4}\} \subseteq \mathcal{L}_2S_k(2k) \). The result can be obtained arguing as in step a) of Theorem 5.2 for \( k \) even and as in step b) of the same for \( k \) odd.

We now consider the case \( v > 2k \). As before, by Proposition 5.1, we just need to prove one of the inclusions. Suppose \( v \equiv 0, 1 \pmod{4} \). Let \( B \) be a \((K_v, S_k)\)-design. For \( k \) even, each star is a disjoint union of \( P_3 \)'s and the existence of a down-link to a \((K_v, P_3)\)-design is trivial. For \( k \) odd, observe that \( B \) contains an even number of stars. Hence, there is an even number of vertices \( x_0, x_1, x_2, \ldots, x_{2t-1} \) of \( K_v \) which are center of an odd number of stars. Consider the edges \([x_{2i}, x_{2i+1}]\) for \( i = 0, \ldots, t-1 \). From each star of \( B \), extract a \( P_3 \) which does not contain any of the aforementioned edges and use it for the down-link. If \( y \in K_v \) is the center of an even number of stars, then the union of all the remaining edges of stars with center \( y \) is a connected graph with an even number of edges; thus, it is possible to apply Theorem 4.1. If \( y \) is the center of an odd number of stars, then there is an edge \([x_{2i}, x_{2i+1}]\) containing \( y \). In this case the graph obtained by the union of all the remaining edges of the stars with centers \( x_{2i} \) and \( x_{2i+1} \) is connected and has an even number of edges. Thus, we can apply again Theorem 4.1. For \( v \equiv 2, 3 \pmod{4} \), the result follows as in Theorem 4.2.

6 Kite-designs

Denote by \( D = [a, b, c \bowtie d] \) the kite, a triangle with an attached edge, having vertices \( \{a, b, c, d\} \) and edges \([c, a], [c, b], [c, d], [a, b]\).

In [2], Bermond and Schönheim proved that a kite-design of order \( v \) exists if, and only if, \( v \equiv 0, 1 \pmod{8} \), \( v > 1 \). In this section we completely determine the sets \( \mathcal{L}_1D(v) \) and \( \mathcal{L}_2D(v) \) where \( \Gamma' = P_3 \) and \( v \), clearly, fulfills the aforementioned condition.

We need now to recall some preliminaries on difference families. For general definitions and in depth discussion, see [7]. Let \((G,+)\) be a group
and take $H \leq G$. A set $F$ of kites with vertices in $G$ is called a $(G, H, D, 1)$-

difference family (DF, for short), if the list $\Delta F$ of differences from $F$, namely
the list of all possible differences $x - y$, where $(x, y)$ is an ordered pair of adjacent
vertices of a kite in $F$, covers all the elements of $G \setminus H$ exactly once, while no element of $H$ appears in $\Delta F$.

**Proposition 6.1.** For every $v \equiv 0, 1 \pmod{8}$, $v > 1$,

\begin{align*}
L_1 D(v) & \subseteq \{ n \geq v - 1 \mid n \equiv 0, 1 \pmod{4} \}, \\
L_2 D(v) & \subseteq \{ n \geq v \mid n \equiv 0, 1 \pmod{4} \}.
\end{align*}

**Proof.** Let $\mathcal{B}$ and $\mathcal{B}'$ be respectively a $(K_v, D)$-design and a $(K_n, P_3)$-design. Suppose there are $x, y \in V(K_v) \setminus V(K_n)$ with $x \neq y$. Since there is at least one kite $D \in \mathcal{B}$ containing both $x$ and $y$, we see that it is not possible to extract any $P_3 \in \mathcal{B}'$ from $D$; thus $n \geq v - 1$. This proves (4).

As for (5), we distinguish two cases. For $v \equiv 0 \pmod{8}$, there does not exist a $P_3$-design of order $v - 1$. On the other hand, for any $v = 8t + 1$,

$$
\mathcal{F} = \{ [2i - 1, 3t + i, 0 \times 2i] \mid i = 1, \ldots, t \}
$$

is a $(\mathbb{Z}_{8t+1}, \{0\}, D, 1)$-DF. As a special case of a more general result proved in [7], the existence of such a difference family implies that of a cyclic $(K_{8t+1}, D)$-design $\mathcal{B}$. Thus, any $x \in V(K_{8t+1})$ has degree 3 in at least one block of $\mathcal{B}$. Hence, there is no down-link of $\mathcal{B}$ in a design of order less than $8t + 1$. \( \square \)

**Lemma 6.2.** For every integer $m = 2n + 1$ there exists a $(K_{mx}, D)$-design.

**Proof.** The set

$$
\mathcal{F} = \{ [(0, 0), (0, 2i), (2, i) \times (1, 0)], [(0, 0), (4, i), (1, -i) \times (6, i)] \mid i = 1, \ldots, n \}
$$

is a $(\mathbb{Z}_8 \times \mathbb{Z}_m, \mathbb{Z}_8 \times \{0\}, D, 1)$-DF. A special case of a result in [7] shows that any difference family with these parameters determines a $(K_{mx}, D)$-

**Proposition 6.3.** There exists a $(K_v, D)$-design with a vertex $x$ having
degree 2 in all the blocks in which it appears if and only if $v \equiv 1 \pmod{8}$, $v > 1$.

**Proof.** Clearly, $v \equiv 1 \pmod{8}$, $v > 1$, is a necessary condition for the existence of such a design. We will show that it is also sufficient. Assume $v = 8t + 1$, $t \geq 1$. Let $A_i = \{a_{i1}, a_{i2}, \ldots, a_{is}\}$, $i = 1, \ldots, t$ and write $V(K_v) = \{0\} \cup A_1 \cup A_2 \cup \cdots \cup A_t$. Clearly, $E(K_v)$ is the disjoint union of the sets of edges of $K_{0,A_i}$, $K_{A_i}$ and $K_{A_1,A_2,\ldots,A_t}$, for $i = 1, 2, \ldots, t$. 

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• Suppose \( t = 1 \), so that \( V(K_v) = \{0\} \cup A_1 \). An explicit kite decomposition of \( K_v = K_{0,A_1} \cup K_A \) where the degree of 0 is always 2 is given by

\[
\{[0, a_{11}, a_{12} \bowtie a_{16}], [0, a_{14}, a_{13} \bowtie a_{15}], [0, a_{16}, a_{15} \bowtie a_{17}], [0, a_{17}, a_{18} \bowtie a_{16}],
\]

\[
[a_{11}, a_{13}, a_{16} \bowtie a_{17}], [a_{11}, a_{17}, a_{14} \bowtie a_{16}], [a_{11}, a_{15}, a_{18} \bowtie a_{14}],
\]

\[
[a_{12}, a_{18}, a_{13} \bowtie a_{17}], [a_{14}, a_{15}, a_{12} \bowtie a_{17}].
\]

• Let now \( t = 2 \), so that \( V(K_v) = \{0\} \cup A_1 \cup A_2 \). There exists a kite decomposition of \( K_v \) where the degree of 0 is always 2. Such a decomposition results from the disjoint union of the previous kite decomposition of \( K_{0,A_1} \cup K_{A_1} \) and the kite decomposition of \( K_{0,A_2} \cup K_{A_2} \cup K_{0,A_1} \cup A_1 \cup A_2 \) here listed:

\[
\{[0, a_{22}, a_{21} \bowtie a_{18}], [0, a_{24}, a_{23} \bowtie a_{18}], [0, a_{26}, a_{25} \bowtie a_{18}], [0, a_{28}, a_{27} \bowtie a_{18}],
\]

\[
[a_{11}, a_{21}, a_{28} \bowtie a_{18}], [a_{11}, a_{27}, a_{22} \bowtie a_{18}], [a_{11}, a_{23}, a_{26} \bowtie a_{18}], [a_{11}, a_{25}, a_{24} \bowtie a_{18}],
\]

\[
[a_{12}, a_{27}, a_{21} \bowtie a_{17}], [a_{12}, a_{26}, a_{22} \bowtie a_{17}], [a_{12}, a_{25}, a_{23} \bowtie a_{17}], [a_{12}, a_{28}, a_{24} \bowtie a_{17}],
\]

\[
[a_{13}, a_{21}, a_{26} \bowtie a_{17}], [a_{13}, a_{22}, a_{25} \bowtie a_{17}], [a_{13}, a_{23}, a_{28} \bowtie a_{17}], [a_{13}, a_{24}, a_{27} \bowtie a_{17}],
\]

\[
[a_{14}, a_{25}, a_{21} \bowtie a_{16}], [a_{14}, a_{24}, a_{22} \bowtie a_{16}], [a_{14}, a_{23}, a_{27} \bowtie a_{16}], [a_{14}, a_{26}, a_{28} \bowtie a_{16}],
\]

\[
[a_{15}, a_{24}, a_{21} \bowtie a_{23}], [a_{15}, a_{23}, a_{22} \bowtie a_{28}], [a_{15}, a_{28}, a_{25} \bowtie a_{16}], [a_{15}, a_{26}, a_{27} \bowtie a_{25}],
\]

\[
[a_{24}, a_{26}, a_{16} \bowtie a_{23}].
\]

• Take \( v = 8t + 1, t \geq 3 \). For odd \( t \), the complete multipartite graph \( K_{t \times 8} \) always admits a kite decomposition; see Lemma 6.2. Thus, \( K_v \) has a kite decomposition which is the disjoint union of the kite decomposition of \( K_{0,A_i} \cup K_{A_i} \), for each \( i = 1, \ldots, t \) (compare this with the case \( t = 1 \)), and that of \( K_{A_1,A_2,\ldots,A_t} \). If \( t \) is even, write

\[
V(K_v) = \{0\} \cup A_1 \cup \cdots \cup A_{t-1} \cup A_t.
\]

As \( t - 1 \) is odd, the graph \( K_v \) has a kite decomposition which is the disjoint union of the kite decompositions of \( K_{0,A_i} \cup K_{A_i} \) (see the case \( t = 1 \)), \( K_{A_1,A_2,\ldots,A_{t-1}} \) and \( K_{0,A_i} \cup K_{A_i} \cup A_i \) (as in the case \( t = 2 \)), for \( i = 1, \ldots, t-1 \). In either case the degree of 0 is 2.

**Theorem 6.4.** For every \( v \equiv 0, 1 \pmod{8}, v > 1 \),

\[
L_1(D(v)) = \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{4}\}; \tag{6}
\]

\[
L_2(D(v)) = \{n \geq v \mid n \equiv 0, 1 \pmod{4}\}. \tag{7}
\]

**Proof.** Relation (6) follows from Proposition 6.1 and Proposition 6.3. Clearly, any \( D \)-design with a vertex \( x \) of degree 2 in all the blocks in which it appears can be down-linked to a \( P_3 \)-design of order \( v - 1 \).

In view of Proposition 6.1, to prove relation (7), it is sufficient to observe that each kite can be seen as the union of two \( P_3 \)'s. \qed
7 Cycle systems

Denote by $C_k$ the cycle on $k$ vertices, $k \geq 3$. It is well known that a $k$-cycle system of order $v$, that is a $(K_v, C_k)$-design, exists if, and only if, $k \leq v$, $v$ is odd and $v(v-1) \equiv 0 \pmod{2k}$. The if part of this theorem was solved by Alspach and Gavlas [1] for $k$ odd and by Šajna [27] for $k$ even.

In this section we shall provide some partial results on $\mathcal{L}_1 C_k(v)$ and $\mathcal{L}_2 C_k(v)$.

Theorem 7.1. For any admissible $v$,

$$\begin{align*}
\mathcal{L}_2 C_3(v) &= \mathcal{L}_1 C_3(v) = \{n \geq v \mid n \equiv 0, 1 \pmod{4}\}; \\
\mathcal{L}_2 C_4(v) &= \mathcal{L}_1 C_4(v) = \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{4}\}; \\
\mathcal{L}_2 C_5(v) &= \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{4}\} \subseteq \mathcal{L}_1 C_5(v);
\end{align*}$$

and for any $k \geq 6$

$$\left\{n \geq v - \left\lceil \frac{k-4}{3} \right\rceil \mid n \equiv 0, 1 \pmod{4}\right\} \subseteq \mathcal{L}_2 C_k(v) \subseteq \mathcal{L}_1 C_k(v).$$

Proof. Suppose $k = 3$. It is obvious that a down-link from a $(K_v, C_3)$-design $\mathcal{B}$ to a $P_3$-design of order less than $v$ cannot exists. When $v \equiv 1 \pmod{4}$, the triangles in $\mathcal{B}$ can be paired so that each pair share a vertex; see [16]. Let $T_1 = (1, 2, 3)$ and $T_2 = (1, 4, 5)$ be such a pair. Use the paths $[1, 2, 3]$ and $[1, 4, 5]$ for down-link and consider the path $[3, 1, 5]$. Observe that these three paths provide a decomposition of the edges of $T_1 \cup T_2$ in $P_3$’s. The proof is completed by repeating this procedure for all paired triangles. For $v \equiv 3 \pmod{4}$, proceed as in Theorem 4.2.

- Assume $k = 4$. Let $\mathcal{B}$ be a $(K_v, C_4)$-design. It is easy to see that, as in the case of the kites, the image of a $C \in \mathcal{B}$ in a $(K_n, P_3)$-design $\mathcal{B}'$ must necessarily leave out exactly one of the vertices of $C$. Obviously, any two vertices of $V(K_v)$ are contained together in at least one block of $\mathcal{B}$; thus, $\mathcal{L}_2 C_4(v) \subseteq \mathcal{L}_1 C_4(v) \subseteq \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{4}\}$. We now prove the reverse inclusion: take $x \in V(K_v)$ and delete from each $C \in \mathcal{B}$ with $x \in V(C)$ the edges passing through $x$, thus obtaining a $P_3$, say $P$. Let the image of $C$ under the down-link be $P$. Observe now that the blocks not containing $x$ can still be decomposed into two $P_3$’s. Thus, it is possible to construct a down-link from $\mathcal{B}$ to a $P_3$-design of order $v - 1$.

- Take $k = 5$. Note that a $(K_v, C_5)$-design $\mathcal{B}$ necessarily satisfies either of the following:

1) there exist $x, y \in V(K_v)$ such that $x$ and $y$ appear in exactly one block $B$, wherein they are adjacent;
2) every pair of vertices of $K_v$ appear in exactly 2 blocks. In other words, $\mathcal{B}$ is a Steiner pentagon system; see [20].

We will show that it is always possible to down-link $\mathcal{B}$ to a $P_3$-design of order $n \geq v - 1$ admissible. Suppose $v \equiv 1 \mod 4$. If $\mathcal{B}$ satisfies 1), then select from each block a $P_3$ whose vertices are different from $x$ and $y$. Use these $P_3$’s to construct the down-link. Observe that $K_{v-1} = K_v \setminus \{x\}$ minus the edges used for the down-link is a connected graph; thus the assertion follows from Theorem 4.1. If $\mathcal{B}$ satisfies 2), select from each block a $P_3$ whose vertices are different from $x$ and $y$. Note that this is always possible, unless the cycle is $C = (x, a, b, y, c)$. In this case, select from $C$ the path $P = [a, b, y]$. Note that none of the selected paths contains $x$; thus, their union is a subgraph $S$ of $K_{v-1} = K_v \setminus \{x\}$. It is easy to see that each vertex $v \neq b$ of $K_{v-1} \setminus S$ is adjacent to $y$. Thus, either $K_{v-1} \setminus S$ is connected or it consists of the isolated vertex $b$ and a connected component. In both cases it is possible to apply Theorem 4.1 to obtain a $(K_{v-1}, P_3)$-design. When $v \equiv 3 \mod 4$, argue as in Theorem 4.2.

- Let $k \geq 6$ and denote by $\mathcal{B}$ a $(K_v, C_k)$-design. Write $t = \left\lceil \frac{k-4}{3} \right\rceil$. Take $t + 1$ distinct vertices $x_1, x_2, \ldots, x_t, y \in V(K_v)$. Observe that it is always possible to extract from each block $C \in \mathcal{B}$ a $P_3$ whose vertices are different from $x_1, \ldots, x_t, y$, as we are forbidding at most $2\left\lceil \frac{k-4}{3} \right\rceil + 2$ edges from any $k$-cycle; consequently, the remaining edges cannot be pairwise disjoint. Use these $P_3$’s for the down-link. Write $S$ for the image of the down-link, regarded as a subgraph of $K_{v-t} = K_v \setminus \{x_1, x_2, \ldots, x_t\}$. Observe that the edges of $K_{v-t}$ not contained in $S$ form a connected graph $R$. When $R$ has an even number of edges, namely $v - \left\lceil \frac{k-4}{3} \right\rceil \equiv 0, 1 \mod 4$, the result is a direct consequence of Theorem 4.1 and we are done. Otherwise add $u = 1$ or $u = 2$ vertices to $K_{v-t}$ and then apply Theorem 4.1 to $R' = (K_{v-t} + K_u) \setminus S$. \hfill $\square$

**Remark 7.2.** It is not possible to down-link a $(K_v, C_5)$-design with Property 2) to $P_3$-designs of order smaller than $v-1$. On the other hand if a $(K_v, C_5)$-design enjoys Property 1), then it might be possible to obtain a down-link to a $P_3$-design of order smaller than $v-1$, as shown by the following example.

**Example 7.3.** Consider the cyclic $(K_{11}, C_5)$-design $\mathcal{B}$ presented in [5]:

$$\mathcal{B} = \{ [0, 8, 7, 3, 5], [1, 9, 8, 4, 6], [2, 10, 9, 5, 7], [3, 0, 10, 6, 8], [4, 1, 0, 7, 9], [5, 2, 1, 8, 10], [6, 3, 2, 9, 0], [7, 4, 3, 10, 1], [8, 5, 4, 0, 2], [9, 6, 5, 1, 3], [10, 7, 6, 2, 4] \}.$$

Note that 0 and 1 appear together in exactly one block. It is possible to down-link $\mathcal{B}$ to the following $P_3$-design of order 9:

$$\mathcal{B}' = \{ [8, 7, 3], [8, 4, 6], [9, 5, 7], [10, 6, 8], [7, 9, 4], [8, 10, 5], [6, 3, 2], [4, 3, 10], [8, 5, 4], [3, 9, 6], [4, 10, 7] \} \cup \{ [3, 5, 2], [3, 8, 9], [7, 2, 10], [10, 9, 2], [6, 7, 4], [4, 2, 8], [2, 6, 5] \}.$$
8 Path-designs

In [28], Tarsi proved that the necessary conditions for the existence of a \((K_v, P_k)\)-design, namely \(v \geq k\) and \(v(v - 1) \equiv 0 \pmod{2(k - 1)}\), are also sufficient. In this section we investigate down-links from path-designs to \(P_3\)-designs and provide partial results for \(L_1 P_k(v)\) and \(L_2 P_k(v)\).

**Theorem 8.1.** For any admissible \(v > 1\),

\[
L_1 P_4(v) = \{ n \geq v - 1 \mid n \equiv 0,1 \pmod{4} \}; \quad (8)
\]

\[
L_2 P_4(v) \subseteq \{ n \geq v \mid n \equiv 0,1 \pmod{4} \}. \quad (9)
\]

**Proof.** Let \(\mathcal{B}\) and \(\mathcal{B}'\) be respectively a \((K_v, P_4)\)-design and a \((K_n, P_3)\)-design. Suppose there exists a down-link \(f : \mathcal{B} \rightarrow \mathcal{B}'\). Clearly, \(n > v - 2\). Hence, \(L_2 P_4(v) \subseteq L_1 P_4(v) \subseteq \{ n \geq v - 1 \mid n \equiv 0,1 \pmod{4} \}\).

To show the reverse inclusion in (8) we prove the actual existence of designs providing down-links attaining the minimum. For the case \(v \equiv 1, 2 \pmod{4}\) we refer to Subsection 8.1. For \(v \equiv 3 \pmod{4}\), it is possible to argue as in Theorem 4.2. For \(v \equiv 0 \pmod{4}\), observe that a \((K_v, P_4)\)-design exists if, and only if \(v \equiv 0, 4 \pmod{12}\). In particular, for \(v = 4\), the existence of a down-link from a \((K_4, P_4)\)-design to a \((K_4, P_3)\)-design is trivial. For \(v > 4\), arguing as in Subsection 8.1, we can obtain a \((K_v, P_4)\)-design \(\mathcal{B}\) with a vertex \(0 \in V(K_v)\) having degree 1 in each block wherein it appears. Hence, it is possible to choose for the down-link a \(P_3\) not containing 0 from any block of \(\mathcal{B}\). Denote by \(S\) the set of all of these \(P_3\)'s and consider the complete graph \(K_{v-1} = K_v \setminus \{0\}\). Let now \(R = (K_{v-1} + \{\alpha\}) \setminus S\). Clearly, \(R\) is a connected graph with an even number of edges. Hence, by Theorem 4.1, \(\eta_1(v) = v\).

In order to prove (9), it is sufficient to show that for any admissible \(v\) there exists a \((K_v, P_4)\)-design \(\mathcal{B}\) wherein no vertices can be deleted. In particular, this is the case if each vertex of \(K_v\) has degree 2 in at least one block of \(\mathcal{B}\). First of all note that in a \((K_v, P_4)\)-design there is at most one vertex with degree 1 in each block where it appears. Suppose that there actually exists a \((K_v, P_4)\)-design \(\overline{B}\) with a vertex \(x\) as above. It is not hard to see that there is in \(\overline{B}\) at least one block \(P^1 = [x, a, b, c]\) such that the vertices \(a\) and \(b\) have degree 2 in at least another block. Let \(P^2 = [x, c, d, e]\). By reassembling the edges of \(P^1 \cup P^2\) it is possible to replace in \(\overline{B}\) these two paths with \(P^3 = [b, a, x, c], P^4 = [b, c, d, e]\) if \(b \neq e\) or \(P^5 = [a, x, c, b], P^6 = [c, d, b, a]\) if \(b = e\). Thus, we have again a \((K_v, P_4)\)-design. By the assumptions on \(a\) and \(b\) all the vertices of this new design have degree 2 in at least one block. \(\square\)

Arguing exactly as in the proof of Theorem 7.1 it is possible to prove the following result.
Theorem 8.2. Let \( k > 4 \). For any admissible \( v > 1 \),

\[
\left\{ n \geq v - \left\lceil \frac{k - 6}{3} \right\rceil \mid n \equiv 0, 1 \pmod{4} \right\} \subseteq \mathcal{L}_2 P_k(v) \subseteq \mathcal{L}_1 P_k(v).
\]

8.1 A construction

The aim of the current subsection is to provide for any admissible \( v \equiv 1, 2 \pmod{4} \) a \((K_v, P_4)\)-design \( \mathcal{B} \) with a vertex having degree 1 in every block in which it appears. It will be then possible to provide a down-link from \( \mathcal{B} \) into a \((K_{v-1}, P_3)\)-design \( \mathcal{B}' \), as needed in Theorem 8.1. Recall that if \((v-1)(v-2) \not\equiv 0 \pmod{4} \), no \((K_{v-1}, P_3)\)-design exists. Thus this condition is necessary for the existence of a down-link with the required property. We shall prove its sufficiency by providing explicit constructions for all \( v \equiv 1, 6, 9, 10 \pmod{12} \). The approach outlined in Section 2 shall be extensively used, by constructing a partition of the vertices of the graph \( K_v \) in such a way that all the induced complete and complete bipartite graphs can be down-linked to decompositions in \( P_3 \)'s of suitable subgraphs of \( K_{v-1} \); these, in turn, shall yield a decomposition of \( \mathcal{B}' \) with an associated down-link.

Write \( V(K_v) = X_\ell \cup A_1 \cup \cdots \cup A_t \), where \( X_\ell = \{0\} \cup \{1, \ldots, \ell - 1\} \) for \( \ell = 6, 9, 10, 13 \) and \( |A_i| = 12 \) for all \( i = 1, \ldots, t \). We first construct a \((K_{X_\ell}, P_3)\)-design \( \mathcal{B} \) which can be down-linked to a \((K_{X_\ell \setminus \{0\}}, P_3)\)-design \( \mathcal{B}' \). The possible cases are as follows.

- \( \ell = 6 \):

\[
\mathcal{B} = \{[0, 1, 2, 4], [0, 2, 3, 5], [0, 3, 4, 1], [0, 4, 5, 2], [0, 5, 1, 3]\}
\]

\[
\mathcal{B}' = \{[1, 2, 4], [2, 3, 5], [3, 4, 1], [4, 5, 2], [5, 1, 3]\}
\]

- \( \ell = 9 \):

\[
\mathcal{B} = \{[0, 1, 2, 4], [0, 2, 3, 5], [0, 3, 4, 6], [0, 4, 5, 7], [0, 5, 6, 8], [0, 6, 7, 1], [0, 7, 8, 2], [0, 8, 1, 3], [5, 8, 4, 1], [2, 5, 1, 6], [3, 6, 2, 7], [4, 7, 3, 8]\}
\]

\[
\mathcal{B}' = \{[1, 2, 4], [2, 3, 5], [3, 4, 6], [4, 5, 7], [5, 6, 8], [6, 7, 1], [7, 8, 2], [8, 1, 3], [8, 4, 1], [5, 1, 6], [3, 6, 2], [7, 3, 8]\} \cup \{[2, 5, 8], [2, 7, 4]\}
\]

- \( \ell = 10 \):

\[
\mathcal{B} = \{[0, 1, 2, 4], [0, 2, 3, 5], [0, 3, 4, 6], [0, 4, 5, 7], [0, 5, 6, 8], [0, 6, 7, 9], [0, 7, 8, 1], [0, 8, 9, 2], [0, 9, 1, 3], [1, 4, 8, 2], [2, 6, 9, 4], [4, 7, 2, 5], [5, 9, 3, 7], [7, 1, 5, 8], [8, 3, 6, 1]\}
\]

\[
\mathcal{B}' = \{[1, 2, 4], [2, 3, 5], [3, 4, 6], [4, 5, 7], [5, 6, 8], [6, 7, 9], [7, 8, 1], [8, 9, 2], [9, 1, 3], [1, 4, 8], [6, 9, 4], [4, 7, 2], [9, 3, 7], [7, 1, 5], [3, 6, 1]\} \cup \{[8, 2, 6], [2, 5, 9], [5, 8, 3]\}
\]
• \( \ell = 13: \)

\[
\mathcal{B} = \{[0, 1, 2, 4], [0, 2, 3, 5], [0, 3, 4, 6], [0, 4, 5, 7], [0, 5, 6, 8], [0, 6, 7, 9], [0, 7, 8, 10], [0, 8, 9, 11], [0, 9, 10, 12], [0, 10, 11, 1], [0, 11, 12, 2], [0, 12, 1, 3], [1, 4, 9, 5], [2, 5, 10, 6], [3, 6, 11, 7], [4, 7, 12, 8], [5, 8, 1, 9], [5, 9, 10, 11], [5, 10, 7, 15], [5, 12, 9, 3], [7, 3, 2, 11], [6, 12, 4, 11], [11, 8, 2, 6], [6, 1, 10, 4], [4, 8, 3, 12]\}.
\]

\[
\mathcal{B}' = \{[1, 2, 4], [2, 3, 5], [3, 4, 6], [4, 5, 7], [5, 6, 8], [6, 7, 9], [7, 8, 10], [8, 9, 11], [9, 10, 12], [10, 11, 1], [11, 12, 2], [12, 1, 3], [1, 4, 9, 5], [5, 10, 6], [3, 6, 11, 7], [7, 12, 8], [5, 8, 1, 9], [9, 2, 10], [5, 11, 3], [7, 1, 5], [5, 12, 9], [7, 2, 11], [6, 12, 4], [8, 2, 6], [6, 1, 10], [8, 3, 12] \cup \{(9, 5, 2), [11, 7, 4], [1, 9, 6], [3, 10, 7], [9, 3, 7], [4, 11, 8], [10, 4, 8]\}.
\]

We now consider down-links between designs on complete bipartite graphs. For \( X = \{0, 1, 2\} \) and \( Y =\{a, b, c, d, e, f\} \), there is a metamorphosis of the \((K_{X,Y}, P_4)\)-design

\[
\mathcal{B} = \{[0, a, 1, d], [0, b, 1, e], [0, c, 1, f], [0, d, 2, a], [0, e, 2, b], [0, f, 2, c]\}
\]
to the \((K_{X,Y}, P_3)\)-design

\[
\mathcal{B}' = \{[a, 1, d], [b, 1, e], [c, 1, f], [d, 2, a], [e, 2, b], [f, 2, c], [a, 0, b], [c, 0, d], [e, 0, f]\}.
\]

Note that if we remove the paths \([a, 0, b], [c, 0, d], [e, 0, f]\) from \( \mathcal{B}' \), we obtain a bijective down-link from \( \mathcal{B} \) to the \((K_{X\setminus \{0\}, Y}, P_3)\)-design

\[
\mathcal{B}'' = \{[a, 1, d], [b, 1, e], [c, 1, f], [d, 2, a], [e, 2, b], [f, 2, c]\}.
\]

Thus, we have actually obtained a metamorphosis \( \mu : (K_{3,6}, P_4)\)-design \(\rightarrow (K_{3,6}, P_3)\) and a down-link \(\delta : (K_{3,6}, P_4)\)-design \(\rightarrow (K_{2,6}, P_3)\)-design. By gluing together copies of \(\mu\) we get metamorphoses of \(P_3\)-decompositions into \(P_3\)-decompositions of \(K_{6,6}, K_{6,12}, K_{9,12}, K_{12,12}\). Likewise, using \(\delta\) we also determine down-links from \(P_4\)-decompositions of \(K_{6,6}, K_{6,12}\) and \(K_{9,12}\) to \(P_3\)-decompositions of respectively \(K_{5,6}, K_{5,12}\) and \(K_{8,12}\). For our construction, it will also be necessary to provide a metamorphosis of a \((K_{12}, P_4)\)-design \(\mathcal{B}\) into a \((K_{12}, P_3)\)-design \(\mathcal{B}'\). This is given by

\[
\mathcal{B} = \{[1, 2, 3, 5], [1, 3, 4, 6], [1, 4, 5, 7], [1, 5, 6, 8], [1, 6, 7, 9], [1, 7, 8, 10], [1, 8, 9, 11], [1, 9, 10, 12], [1, 10, 11, 2], [1, 11, 12, 3], [1, 12, 2, 4], [2, 5, 10, 6], [3, 6, 11, 7], [4, 7, 12, 8], [5, 8, 2, 9], [6, 9, 3, 10], [7, 10, 4, 11], [8, 11, 5, 12], [9, 12, 6, 2], [10, 2, 7, 3], [11, 3, 8, 4], [12, 4, 9, 5]\};
\]

\[
\mathcal{B}' = \{[2, 3, 5], [3, 4, 6], [4, 5, 7], [5, 6, 8], [6, 7, 9], [7, 8, 10], [8, 9, 11], [9, 10, 12], [10, 11, 2], [11, 12, 3], [12, 2, 4], [5, 10, 6], [6, 11, 7], [7, 12, 8], [8, 2, 9], [9, 3, 10], [10, 4, 11], [11, 5, 12], [12, 6, 2], [2, 7, 3], [3, 8, 4], [4, 9, 5]\} \cup \{(1, 2, 5), [1, 3, 6], [1, 4, 7], [1, 5, 8], [1, 6, 9], [1, 7, 10], [1, 8, 11], [1, 9, 12], [1, 10, 2], [1, 11, 3], [1, 12, 4]\}.
\]

Consider now a \((K_v, P_4)\)-design with \( v = \ell + 2t \) where \( \ell = 1, 6, 9, 10 \) and \( t > 1 \).
• For \( v = 1 + 12t \), write

\[
K_{1+12t} = (tK_{1,12} \cup tK_{12}) \cup \left( \frac{t}{2} \right) K_{12,12} = tK_{13} \cup \left( \frac{t}{2} \right) K_{12,12}.
\]

The down-link here is obtained by gluing down-links from \( P_4 \)-decompositions of \( K_{13} \) to \( P_3 \)-decompositions of \( K_{12} \) with metamorphoses of \( P_4 \)-decompositions of \( K_{12,12} \) into \( P_3 \)-decompositions.

• For \( v = 6 + 12t \), consider

\[
K_{6+12t} = K_6 \cup tK_{12} \cup tK_{12,12} \cup \left( \frac{t}{2} \right) K_{12,12}.
\]

Down-link the \( P_4 \)-decompositions of \( K_6 \) and \( K_{6,12} \) to respectively \( P_3 \)-decompositions of \( K_5 \) and \( K_{5,12} \) and consider metamorphoses of the \( P_4 \)-decompositions of \( K_{12} \) and \( K_{12,12} \) into \( P_3 \)-decompositions.

• For \( v = 9 + 12t \), let

\[
K_{9+12t} = K_9 \cup tK_{12} \cup tK_{9,12} \cup \left( \frac{t}{2} \right) K_{12,12}.
\]

We know how to down-link the \( P_4 \)-decompositions of \( K_9 \) and \( K_{9,12} \) to \( P_3 \)-decompositions of respectively \( K_8 \) and \( K_{8,12} \). As before, there are metamorphoses of the \( P_4 \)-decompositions of both \( K_{12} \) and \( K_{12,12} \) into \( P_3 \)-decompositions.

• For \( v = 10 + 12t \), observe that

\[
K_{10+12t} = K_{10} \cup tK_{12} \cup tK_{10,12} \cup \left( \frac{t}{2} \right) K_{12,12} =
K_{10} \cup tK_{12} \cup tK_{1,12} \cup tK_{9,12} \cup \left( \frac{t}{2} \right) K_{12,12} = K_{10} \cup tK_{13} \cup tK_{9,12} \cup \left( \frac{t}{2} \right) K_{12,12}.
\]

We know how to down-link \( P_4 \)-decompositions of \( K_{10} \), \( K_{13} \) and \( K_{9,12} \) to \( P_3 \)-decompositions of respectively \( K_9 \), \( K_{12} \) and \( K_{8,12} \). As for the \( K_{12,12} \) we argue as in the preceding cases.

References


[26] Quattrocchi, G., Embedding handcuffed designs in \(D\)-designs, where \(D\) is the triangle with attached edge, Discrete Math. 261 (2003), 413–434.


