1 - Introduction

In [5] we defined and studied the algebraic structure called weakly divisible nearring (wd-nearring). In [1, 2] a special class of finite wd-nearrings on $\mathbb{Z}_{p^n}$, $p$ prime, was constructed: on the group $(\mathbb{Z}_{p^n}, +)$ of the residue classes (mod $p^n$) a multiplication “$*$” can be defined in such a way that $(\mathbb{Z}_{p^n}, +, *)$ becomes a wd-nearring. Afterwards, in [3, 4] Partially Balanced Incomplete Block Designs (PBIBDs) and codes were obtained starting from the wd-nearrings of [1, 2] and formulae for computing their parameters could be derived just making use of the combinatorial properties of the constructed algebraic structure.

In [9] the construction of [1, 2] was generalized to any wd-nearring. Applying Prop. 1 of [9], in Example 2.1 of this paper a wd-nearring $N = (\mathbb{Z}_2^7, +, *)$ is constructed on the elementary abelian group $(\mathbb{Z}_2^7, +)$ and a PBIBD is obtained from $N$. Using the algebraic properties of $N = (\mathbb{Z}_2^7, +, *)$, all the parameters of the PBIBD are computed.

Since it seems reasonable to think the construction and the method to compute all the parameters in [3] could be extended to some additional classes of wd-nearrings, the aim of this paper is to study in more depth the algebraic structure of any finite wd-nearring, especially with regard to determining the size of the elements of significant structures in $N$, as partitions, normal chains and products. In the next paragraph, the main definitions and properties of a finite wd-nearring are recalled (Remark 2.1) and the most significant results presented in this paper are summarized (Remark 2.2).
In particular we obtain $B$ Sylow subgroup of both $U$. Finite weakly divisible nearrings

In the sequel the subset $\{1, 2, \ldots, m\} \subseteq N$ could be denoted by $I_m$ and $a/b$ will be sometimes used as $a$ divides $b$.

**Definition 2.1** A nearring $N$ is called weakly divisible (wd-nearring) if the following condition is satisfied:

$$\forall a, b \in N \; \exists x \in N \; | \; a \ast x = b \; \text{or} \; b \ast x = a$$

**Remark 2.1** In [5] it is proved that a finite wd-nearring $N$ is the disjoint union of $Q$, the set of all the nilpotent elements, and $C$, the set of all the left cancellable elements, that is $N = C \cup Q$ and $C \cap Q = \emptyset$. In the finite case, from Theorem 8 of [5] we know that:

(a) The set $C$ of the left cancellable elements is the disjoint union of $m$ isomorphic groups. We will call them “the $B_{e_i}$’s”, $e_i$ being the identity of $B_{e_i}$ and a left identity of $N$, for $i \in I_m$. The map $\pi : B_{e_i} \rightarrow B_{e_i}$, defined by $\pi(x) = x \ast e_i$ for $x \in B_{e_i}$, is a (multiplicative) group isomorphism, for $i, h \in I_m$.

(b) The set $Q$ of the nilpotent elements is the prime radical of $N$, it coincides with the Jacobson radicals and contains every right invariant subset, that is any subset $H$ of $N$ such that $HN \subseteq H$. Obviously, any zero divisor belongs to $Q$.

**Remark 2.2** In Paragraph 3 we find that, for a finite wd-nearring $N$, there are integers $t$ and $r$ such that $|N| = t^r$ and $|Q| = t^{-r}$, so $|C| = (t - 1)t^{r-1}$. Moreover, for $j \in I_{r-1}$, we are able to find partitions for the right annihilators of $q^j$, $q$ being any nilpotent such that $q \ast N = Q$ and $Ann(q^j) = \{ y \in N \; | \; q^j \ast y = 0 \}$. More precisely, we have $Ann(q) = q^{-1} \ast C \cup \{0\}$ and $Ann(q^j) = q^{-r-j} \ast C \cup Ann(q^{-r-j})$. So, since $Q$ can be seen as the right annihilator of $q^{-r-1}$, it results in $Q = q \ast C \cup q^2 \ast C \cup \ldots \cup q^{-r-1} \ast C \cup \{0\}$. Also $|Ann(q^j)| = t^j$ and $|q^j \ast C| = (t - 1)t^{r-j-1}$.

In Paragraph 4 we study the algebraic structure of one of the $B_{e_i}$’s, say $B_e$. We know that $|B_e| = hk$, where $h|(t - 1)$ and $k|t^{r-1}$. We prove that $B_e$ contains two normal chains of subgroups: $F_e(q) \subseteq F_e(q^2) \subseteq \cdots \subseteq F_e(q^{-r-1})$ and $U_e(q) \subseteq U_e(q^2) \subseteq \cdots \subseteq U_e(q^{-r-1})$, so we investigate the orders of their elements. In particular we obtain $|F_e(q^{-r-1})| = h_{r-1}k$, where $h_{r-1}|h$, and $|U_e(q^{-r-1})| = k$, thus $B_e$ results in the semidirect product between $U_e(q^{-r-1})$ and a suitable complement of order $h$.

In Paragraph 5, in addition, $t$ is a prime and, consequently, $|B_e| = ht^o$, $|U_e(q^{-r-1})| = t^o$ and $|F_e(q^{-r-1})| = h_{r-1}t^o$. Hence $U_e(q^{-r-1})$ results in the $t$-Sylow subgroup of both $B_e$ and $F_e(q^{-r-1})$ and $\frac{|U_e(q^{-r-1})|}{|F_e(q^{-r-1})|} \in \{1, t\}$.

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3In the following, $X \cup Y$ will denote the disjoint union of $X$ and $Y$. 2
2.1 - Example

**First step - Construction of a wd-nearring**

Here is what we need:

- the elementary abelian group \( (\mathbb{Z}_7^2, +) \),
- an automorphism group of \( (\mathbb{Z}_7^2, +) \), \( \Phi := \{ \text{id}, \gamma : (x, y) \rightarrow (x, -y) \} \),
- a nilpotent endomorphism of \( (\mathbb{Z}_7^2, +) \), \( \psi : (x, y) \rightarrow (y, 0) \).

We begin by choosing the representatives of the \( \Phi \)-orbits:

<table>
<thead>
<tr>
<th>( \Phi )-orbits, ( x \in \mathbb{Z}_7 )</th>
<th>representatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { (x, 1), (x, 6) } )</td>
<td>( (x, 1) )</td>
</tr>
<tr>
<td>( { (x, 2), (x, 5) } )</td>
<td>( (x, 2) )</td>
</tr>
<tr>
<td>( { (x, 3), (x, 4) } )</td>
<td>( (x, 3) )</td>
</tr>
<tr>
<td>( { (x, 0) } )</td>
<td>( (x, 0) )</td>
</tr>
</tbody>
</table>

Let \( E \) denote the set of the chosen representatives for \( \Phi \) on \( \mathbb{Z}_7^2 \setminus \text{Im} \psi \). We can verify that all the conditions required in [9] Prop.1 are satisfied, in particular:

- for all \( n \in \mathbb{Z}_7^2 \), there are \( i = 0, 1, \) \( \varphi \in \Phi \) and \( e \in E \) such that \( n = \psi^i \varphi(e) \)

so that a multiplication “\(*\)” on \( (\mathbb{Z}_7^2, +) \) can be defined in the following way:

\[
n * m = \psi^i \varphi(e) * m = \psi^i \varphi(m)
\]

Now \( N = (\mathbb{Z}_7^2, +, *) \) results in a wd-nearring with:

- the set of the nilpotent elements \( Q = \ker \psi = \text{Im} \psi = \{(y, 0) \in \mathbb{Z}_7^2 | y \in \mathbb{Z}_7 \} \);
- the set of the cancellable elements \( C = N \setminus Q = \bigcup_{e \in E} \Phi(e) \).

\( \Phi \) acts fixed point free on \( C \) and \( C \) is partitioned into \( \Phi \)-orbits, each of them results in a multiplicative group with the representative as identity.

**Second step - Construction of a tactical configuration on \( N \)**

We will proceed with adapting the method of Hall (see [8]). The raw materials needed are a finite non empty set \( X \), a transitive permutation group \( G \) on \( X \) with an intransitive subgroup \( S \). Now:

- \( X = \mathbb{Z}_7^2 \);
- \( G = (\mathbb{Z}_7^2, +) \rtimes \Phi \), the natural semidirect sum \( ((n, \phi_1) + (m, \phi_2) = (n + \phi_1(m), \phi_1 \phi_2)) \);
- \( S = \{(0, \phi) \in G \rtimes \Phi, \phi \in \Phi \} \).

We choose an element in \( E \), say \( e = (0, 1) \), and we consider the set \( N * (0, 1) = \{(0, 1), (0, 6), (1, 0), (6, 0), (0, 0)\} \). It is easy to see that \( N * (0, 1) \) is a union of orbits of \( \Phi \), being \( N * (0, 1) = \Phi((0, 1)) \cup \Phi((1, 0)) \cup \Phi((6, 0)) \cup \Phi((0, 0)) \).

A direct computation shows that \( S \) results in the stabilizer of \( N * (0, 1) \) in \( G \), hence distinct elements of \( (\mathbb{Z}_7^2, +) \) determine distinct cosets of \( S \) in \( G \). Thereby, when \( (x, y) \in \mathbb{Z}_7^2 \), the sets \( N * (0, 1) + (x, y) \) are the distinct blocks of a tactical configuration whose parameters are \( (v, b, r, k) = (49, 49, 5, 5) \).

**Third step - Construction of an association scheme on \( N \)**

We will continue to apply the method of Hall. The raw materials needed are
the stabilizer \( G_n \) of any element \( n \in N \), the \( G_n \)-orbits partitioning \( N \) and the sets \( U = \Delta \cup (-\Delta) \) obtained by forming the union of any orbit \( \Delta \) and the orbit \(-\Delta \). Now:

- \( n = (0,0) \) and \( G_n = \Phi \);
- \( U_1 = \{(0,1),(0,6)\} = \Delta_1 = -\Delta_1 \) self paired
- \( U_2 = \{(0,2),(0,5)\} = \Delta_2 = -\Delta_2 \) "
- \( U_3 = \{(0,3),(0,4)\} = \Delta_3 = -\Delta_3 \) "
- \( U_4 = \{(1,1),(1,6)\} \cup \{(6,6),(6,1)\} = \Delta_4 \cup (-\Delta_4) \) paired
- \( U_5 = \{(1,2),(1,5)\} \cup \{(6,5),(6,2)\} = \Delta_5 \cup (-\Delta_5) \) "
- \( U_6 = \{(1,3),(1,4)\} \cup \{(6,4),(6,3)\} = \Delta_6 \cup (-\Delta_6) \) "
- \( U_7 = \{(2,1),(2,6)\} \cup \{(5,6),(5,1)\} = \Delta_7 \cup (-\Delta_7) \) "
- \( U_8 = \{(2,2),(2,5)\} \cup \{(5,5),(5,2)\} = \Delta_8 \cup (-\Delta_8) \) "
- \( U_9 = \{(2,3),(2,4)\} \cup \{(5,4),(5,3)\} = \Delta_9 \cup (-\Delta_9) \) "
- \( U_{10} = \{(3,1),(3,6)\} \cup \{(4,6),(4,1)\} = \Delta_{10} \cup (-\Delta_{10}) \) "
- \( U_{11} = \{(3,2),(3,5)\} \cup \{(4,5),(4,2)\} = \Delta_{11} \cup (-\Delta_{11}) \) "
- \( U_{12} = \{(3,3),(3,4)\} \cup \{(4,4),(4,3)\} = \Delta_{12} \cup (-\Delta_{12}) \) "
- \( U_{13} = \{(1,0)\} \cup \{(6,0)\} = \Delta_{13} \cup (-\Delta_{13}) \) "
- \( U_{14} = \{(2,0)\} \cup \{(5,0)\} = \Delta_{14} \cup (-\Delta_{14}) \) "
- \( U_{15} = \{(3,0)\} \cup \{(4,0)\} = \Delta_{15} \cup (-\Delta_{15}) \) "

Two elements will be called ith-associates if their difference belongs to \( U_i \), for \( i = 1, \ldots, 15 \). Hence, we obtain 15 relations which constitute an Association Scheme whose parameters are:

- the numbers \( n_i \) of the ith-associates of any element
  \[ n_1 = n_2 = n_3 = n_{13} = n_{14} = n_{15} = 2, \quad n_4 = \cdots = n_{12} = 4 \]
- the numbers \( n^k_{ij} \) of the elements which are ith-associates of \((a,b)\) and jth-associates of \((c,d)\) when \((a,b)\) and \((c,d)\) are kth-associates.

These parameters are organized into 15 symmetric square matrices of order 15, denoted by \( P^k = (p^k_{ij}) \) with \( k = 1, \ldots, 15 \). The values of the \( p^k_{ij} \) were calculated directly, using the algebraic properties of \( (\mathbb{Z}_2^3,+,*) \). Below you can find a way to obtain \( P^k \) for any \( k = 1, \ldots, 15 \).

Let \( O \) and \( I \) denote the zero matrix and the identity matrix of order 3 respectively. Let \( A^t \) denote the transpose of \( A \). Moreover, let:

\[
\begin{align*}
\bar{A} &= \begin{pmatrix} A & O & O & O & O \\ O & 2A & O & O & 2A_1 \\ O & O & 2A & O & 2A_2 \\ O & O & O & 2A & 2A_3 \\ O & 2A_1^t & 2A_2^t & 2A_3^t & O \end{pmatrix} \\
B_1 &= \begin{pmatrix} O & A & O & O & A_1 \\ A & O & A & O & A_2 \\ O & A & O & A & A_1 + A_3 \\ O & O & A & A & A_2 + A_3 \\ A_1^t & A_2^t & A_1^t + A_3^t & A_2^t + A_3^t & O \end{pmatrix} \\
B_2 &= \begin{pmatrix} O & A & O & A & A_1 + A_3 \\ O & O & A & A & A_2 \\ O & A & O & A & A_1 + A_2 \\ O & A & O & A & A_1 + A_2 \\ A_1^t & A_2^t + A_3^t & A_1^t + A_3^t & A_2^t + A_3^t & O \end{pmatrix} \\
C_1 &= \begin{pmatrix} O & 2I & O & O & 0 \\ 2I & O & 2I & O & 0 \\ O & 2I & O & 2I & 0 \\ O & 2I & O & 2I & 0 \\ O & 0 & O & 0 & A \end{pmatrix} \\
C_2 &= \begin{pmatrix} O & 0 & O & O & 0 \\ 0 & 0 & O & 2I & 0 \\ 0 & 0 & O & 2I & 0 \\ 2I & O & 2I & O & 0 \\ O & 0 & O & 0 & A \end{pmatrix}
\end{align*}
\]
Then:
\[
P^1 = A, \quad P^4 = B_1, \quad P^7 = B_2, \quad P^{10} = B_3, \quad P^{13} = C_1 \quad \text{with}
\]
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
\[
P^2 = A, \quad P^5 = B_1, \quad P^8 = B_2, \quad P^{11} = B_3, \quad P^{14} = C_2 \quad \text{with}
\]
\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
\[
P^3 = A, \quad P^6 = B_1, \quad P^9 = B_2, \quad P^{12} = B_3, \quad P^{15} = C_3 \quad \text{with}
\]
\[
A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Forth step - The partial balance**

Finally, from [8] the tactical configuration results in partially balanced with respect to the association scheme, that is any two \( i \)-th associate elements belong exactly to \( \lambda_i \) blocks. We can compute easily the parameters of the partial balance: \( \lambda_1 = \lambda_4 = \lambda_{13} = 2, \quad \lambda_2 = \lambda_{14} = 1, \quad \lambda_i = 0 \) otherwise.

Notice that our ability to compute the parameters of the PBIBD depends both on the small size and the algebraic properties of the nearring \( N \). The more the size of \( N \) increases, the more the knowledge of the algebraic structure becomes essential. This is why we want to know more about the algebraic structure of any finite \( \mathbb{W} \)-nearring.

### 3 - The set \( Q \) of the nilpotent elements

Hereinafter \( N \) denotes a \( \mathbb{W} \)-nearring, \( Q \) is the set of its nilpotent elements and \( C \) is the set of its cancellable ones. We will always assume \( Q \neq \{0\} \).

The smallest positive integer \( n \) such that \( x^n = 0 \) (\( H^n = \{0\} \)) will be denoted by \( \nu(x) (\nu(H)) \). From [5] we learn that \( Q \) is monogenic, that is there exists \( q \in Q \) such that \( Q = q + N \). Such an element will be called generator of \( Q \). Obviously, if \( \nu(q) = n \), for any \( p \in Q \) we have \( p = q^t \ast c \), for some \( t \in \mathbb{I}_n \) and \( c \in C \).

Following propositions state a lot of useful properties of the generators of \( Q \).

**Proposition 3.1** Let \( q \) be a generator of \( Q \) and let \( \nu(q) = r \), then \( \nu(Q) = r \).

**Proof.** Consider \( \prod_{i=1}^r x_i \) where \( x_i \in Q \) for all \( i \in I_r \). We have \( \prod_{i=1}^r x_i = \prod_{i=1}^r q^t \ast n_i \), where \( n_i \in N \) for all \( i \in I_r \). As \( q^t = 0 \) and the I.F.P. holds in a nearring \( N \) has the I.F.P. if for every \( a, b, n \in N \), \( a \ast b = 0 \) implies \( a \ast n \ast b = 0 \).
finite wd-nearring, we obtain $\prod_{i=1}^{r} x_i = 0$. \hfill\diamond

**Proposition 3.2** Let $q$ be a generator of $Q$ and let $\nu(q) = r$, then the following statements are equivalent.

(a) $p$ is an element of $Q$ and $\nu(p) = r$;
(b) $p$ is of the form $q \ast c$, where $c \in C$;
(c) $p$ is a generator of $Q$.

**Proof.** (a) $\Rightarrow$ (b) Let $p$ be an element of $Q$. Then $p = q^t \ast c$, for some $t \in I_r$ and $c \in C$. Applying Proposition 3.1 we obtain $p^{r-t+1} = (p^{r-t} \ast q^t) \ast c = 0 \ast c = 0$. Hence it must be $r - t + 1 \geq r$, and this implies $t = 1$.

(b) $\Rightarrow$ (c) Obvious, because $(q \ast c) \ast N = q \ast (c \ast N) = q \ast N = Q$, for $c \in C$.

(c) $\Rightarrow$ (a) Let $p$ be a generator of $Q$ and $\nu(p) = s$. Applying Proposition 3.1 we have $\nu(Q) = s$. But we know that $\nu(Q) = r$, so $s = r$. \hfill\diamond

**Proposition 3.3** Let $p$ and $q$ be generators of $Q$ and $\nu(q) = \nu(p) = r$, then $p^t \ast C = q^t \ast C$, for all $j \in I_r$.

**Proof.** For every $c \in C$, $p^t \ast c$ belongs to $Q$, so $p^t \ast c = q^t \ast c'$, for some $t \in I_r$ and $c' \in C$. If $j < t$, $p^{r-t+j} \ast c = (p^{r-t} \ast q^t) \ast c' = 0 \ast c' = 0$, from previous Proposition 3.1. As a zero divisor of $N$ must belong to $Q$ and $c \not\in Q$, it must be $p^{r-t+j} = 0$, but this is impossible. Analogously, if $j > t$, $(q^{r-j} \ast p^t) \ast c = (q^{r-j} \ast q^t) \ast c'$ implies $0 \ast c = 0 = q^{r-(j-t)} \ast c'$ and again $q^{r-(j-t)} = 0$ is impossible. So $j = t$. \hfill\diamond

**Proposition 3.4** Let $q$ be a generator of $Q$ and $\nu(q) = r$. Then, for $k, j \in I_{r-1}$ with $k \neq j$,

(a) $C \ast q^j \subseteq q^j \ast C$;
(b) $q^j \ast C \cap q^k \ast C = \emptyset$;
(c) $\text{Ann}(q^j) = q^{r-j} \ast N = q^{r-j} \ast C \cup q^{r-j+1} \ast C \cup \ldots \cup q^{r-1} \ast C \cup \{0\}$;
(d) $Q = q \ast C \cup q^2 \ast C \cup \ldots \cup q^{r-1} \ast C \cup \{0\}$.

**Proof.** (a) We know that for any $c \in C$ the element $c \ast q^j$ belongs to $Q$, so it is of the form $q^i \ast c'$ for some $k \in I_r$ and $c' \in C$. As in previous Proposition 3.3, from $c \ast q^j = q^i \ast c'$ we obtain $j = k$. Hence $C \ast q^j \subseteq q^i \ast C$.

(b) If $x \in q^j \ast C \cap q^k \ast C$, we have $q^j \ast c = x = q^k \ast c'$ for some $c, c' \in C$ and $j = k$ follows as before, but now we have $k \neq j$.

(c) We start showing that $\text{Ann}(q^j) = q^{r-j} \ast N$. Obviously $q^j \ast (q^{r-j} \ast N) = 0 \ast N = \{0\}$, thus $q^{r-j} \ast N \subseteq \text{Ann}(q^j)$. Vice versa, if $x \in \text{Ann}(q^j)$ then $x \ast N$ must belong to $Q$, so $x = q^i \ast c$ for some $i \in I_r$ and $c \in C$. From $q^i \ast x = q^i \ast c = 0$ we have $q^i \ast q^{r-j} = 0$, and this forces $j + i \geq r$. Hence $x = q^{r-j} \ast q^{i-r-j} \ast c \in q^{r-j} \ast N$, and this implies $q^{r-j} \ast N \supseteq \text{Ann}(q^j)$. Moreover, $q^{r-j} \ast N = q^{r-j} \ast (C \cup Q) = q^{r-j} \ast C \cup q^{r-j} \ast Q = q^{r-j} \ast C \cup q^{r-j} \ast N = \ldots = q^{r-j} \ast C \cup q^{r-j+1} \ast C \cup \ldots \cup q^{r-1} \ast C \cup \{0\}$.

(d) Obvious, as $Q = \text{Ann}(q^{r-1})$ and we can apply previous point (c). \hfill\diamond

\textit{Ann}(x) = \{y \in N | x \ast y = 0\} is called the right annihilator of $x$ (here it is an ideal of $N$).
Lemma 3.1 Let $N$ be a finite $wd$-nearring with $|N| = n$ and $|Q| = m$. Let $q$ be any generator of $Q$ and $r = \nu(q)$. Then, for $j \in I_{t-1}$,
(a) $|\text{Ann}(q^j)||\text{Ann}(q^{t-j})| = n$;
(b) $|q^j * C||\text{Ann}(q^j)| = n = m$;
(c) $|\text{Ann}(q^j)| = (n/m)^j$.

Proof. (a) From Proposition 3.4 (c), we know that $|\text{Ann}(q^j)| = |q^{t-j} * N|$. If $q^{t-j} * n_1 = q^{t-j} * n_2$, then $q^{t-j} * (n_1 - n_2) = 0$ implies $n_1 \in n_2 + \text{Ann}(q^{t-j})$ and vice versa. So $|q^{t-j} * N| = |N|/|\text{Ann}(q^{t-j})|$, that is $|\text{Ann}(q^j)| = n$.
(b) Let $c_1, c_2 \in C$. If $q^j * c_1 = q^j * c_2$, then $q^j * (c_1 - c_2) = 0$ implies $c_1 \in c_2 + \text{Ann}(q^j)$ and vice versa. Since $c + \text{Ann}(q^j) \subseteq C$ for all $c \in C$, we obtain $|q^j * C| = |C|/|\text{Ann}(q^j)| = (n - m)/|\text{Ann}(q^j)|$.
(c) From Proposition 3.4 (c), we have $\text{Ann}(q^j) = q^{t-j} * C \cup \text{Ann}(q^{t-j})$, so $|\text{Ann}(q^j)| = |q^{t-j} * C| + |\text{Ann}(q^{t-j})|$. Applying previous points (a) and (b), we obtain $|\text{Ann}(q)| = n/m$, as $\text{Ann}(q^{t-j}) = Q$, and $|\text{Ann}(q^j)| = (n/m)|\text{Ann}(q^{t-j})|$. So $|\text{Ann}(q^j)| = (n/m)^j$.

Theorem 3.1 Let $N$ be a finite $wd$-nearring with $|N| = n$, $|Q| = m$ and $|N : Q| = n/m = t$. Let $q$ be any generator of $Q$ and $r = \nu(q)$, then
(a) $|N| = t^r$ and $|Q| = t^{r-1}$;
(b) $|\text{Ann}(q^j)| = t^j$ and $|q^j * C| = (t - 1)t^{j-1}$, for $j \in I_{t-1}$.

Proof. (a) Since $Q = \text{Ann}(q^{t-1})$, applying previous Lemma 3.1 (c), we obtain $|Q| = |\text{Ann}(q^{t-1})| = t^{r-1}$ and $|N| = |Q| = n/m = t$.
(b) From previous Lemma 3.1 (b) we know that $|\text{Ann}(q^j)| = (n/m)^j = t^j$. Moreover, $|q^j * C| = (n - m)/t^j$ with $n = t^r$ and $m = t^{r-1}$, so $|q^j * C| = (t - 1)t^{j-1}$, $\forall j \in I_{t-1}$.

Notice that, generally, the set $E = \{e_1, \ldots, e_m\}$ results in the set of the left identities of $N$ and also, from Definition 2.1, every element of $N$ has at least a right identity. Thus, both the set of the left identities of any element of $N$ and the set of the right ones are certainly non-empty.

Remark 3.1 In the $\mathbb{Z}_{pq}$ case (see [1, 2]), if $e$ is an idempotent right identity of any generator of $Q$, say $q$, in $B_e$ the sets of the left and right identities of $q$ coincide and $e$ is the only left (and right) identity of $q$ in $B_e$ if and only if the order of $B_e$ is a divisor of $t - 1$. From previous Example 2.1 we can see that it is not always true.

Return to the Example 2.1 - Now, we have $|Q| = 7$ and $t - 1 = 6$. Each non zero element of $Q$ results in a generator of $Q$ itself. So, fixing $q = (1, 0)$ as a generator and without loss of generality, we have

$$B_{(0,1)} = \Phi((0,1)) = \{(0,1), (0,6)\}$$
$$B_{(0,1)} * (1,0) = \{(1,0)\} \subset \{(1,0), (6,0)\} = (1,0) * B_{(0,1)}$$

Thus, all the elements of $B_{(0,1)}$ are left identities of $(1,0)$ but the only right identity of $(1,0)$ in $B_{(0,1)}$ is $(0,1)$. Moreover, $B_{(0,1)}$ has order 2, but even if 2 is a divisor of $t - 1 = 6$, $(0,1)$ has more then one left identity.
So, in the following paragraphs 4.1 and 4.2 we are just dealing with the sets of the left or right identities of \( q^j \), \( j = 1, \ldots, r - 1 \), where \( q \) is a generator of \( Q \) and \( r = \nu(q) \).

4. - The set \( C \) of the left cancellable elements

In what follows we will always assume \( |N| = t^r \) for some integer \( t > 1 \). Here we recall again (see Remark 2.1) that \( C \) is a multiplicative semigroup, disjoint union of \( m \) isomorphic groups, the \( B_{e_i} \), \( e_i \) being the identity of \( B_{e_i} \).

**Remark 4.1** From previous Theorem 3.1 we learn that \( |C| = (1 - 1)t^{-1} \), thus \( |B_{e_i}| = hk \), where \( h \) divides \( t - 1 \) and \( k \) divides \( t^{-1} \), for \( i \in I_m \).

4.1 - Left identities of \( q^j \)

**Definition 4.1** Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). The set of all the left identities of \( q^j \) will be denoted by \( F(q^j) \), that is
\[
F(q^j) = \{ x \in N | x \cdot q^j = q^j \}, \quad \text{for } j \in I_{r-1}
\]

**Proposition 4.1** Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). Then \( F(q) \subseteq F(q^2) \subseteq \cdots \subseteq F(q^{r-1}) \subseteq C \) is a chain of multiplicative semigroups.

**Proof.** Obviously, \( x, y \in F(q^j) \) implies \( x \cdot y \in F(q^j) \). Moreover \( F(q^{j}+1) \subseteq F(q^j) \), as \( x \cdot q^j = q^j \) implies \( x \cdot q^{j+1} = q^{j+1}, \forall j \in I_{r-1} \). Finally, let \( x \in F(q^{j-1}) \). If \( x \in Q \), then \( x = q^s \cdot c \), for some \( s \in I_{r-1} \) and \( c \in C \). Hence, \( q^{j-1} = x \cdot q^{j-1} = q^s \cdot c \cdot q^{j-1} = 0 \), because the I.F.P. holds now. But \( q^{j-1} = 0 \) is clearly impossible, so \( x \in C \).

**Definition 4.2** Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). The set of all the left identities of \( q^j \) belonging to \( B_{e_i} \) will be denoted by \( F_{e_i}(q^j) \), that is
\[
F_{e_i}(q^j) = F(q^j) \cap B_{e_i} = \{ x \in B_{e_i} | x \cdot q^j = q^j \}, \quad \text{for } j \in I_{r-1} \text{ and } i \in I_m
\]

**Remark 4.2** \( F_{e_i}(q^j) \) is non empty, because \( e_i \in F_{e_i}(q^j) \), \( \forall j \in I_{r-1}, i \in I_m \).

**Proposition 4.2** Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). Then, for \( i, h \in I_m \) and \( j \in I_{r-1} \),

(a) \( F_{e_i}(q) \subseteq F_{e_i}(q^2) \subseteq \cdots \subseteq F_{e_i}(q^{r-1}) \subseteq B_{e_i} \) is a normal chain of multiplicative subgroups of \( B_{e_i} \);

(b) \( F_{e_i}(q^j) \) and \( F_{e_h}(q^j) \) are isomorphic groups.

**Proof.** (a) Previous Proposition 4.1 implies that \( F_{e_i}(q) \subseteq \cdots \subseteq F_{e_i}(q^{r-1}) \) is a chain of multiplicative subsemigroups of \( B_{e_i} \), \( \forall i \in I_m \). Now, let \( x \in F_{e_i}(q^j) \) and \( x^{-1} \) be the inverse of \( x \) in \( B_{e_i} \). From \( q^j = x \cdot q^j \) we have \( x^{-1} \cdot q^j = x^{-1} \cdot x \cdot q^j = e_i \cdot q^j = q^j \), so \( x^{-1} \in F_{e_i}(q^j) \). Hence \( F_{e_i}(q^j) \) results in a subgroup of \( B_{e_i} \) with \( e_i \) as identity. Moreover, from Proposition 3.4(a) we learn that for all \( c \in C \) there exists \( c' \in C \) such that \( c \cdot q^j = q^j \cdot c' \). Thus \( \forall x \in B_{e_i}, \forall y \in F_{e_i}(q^j) \) we have \( x^{-1} \cdot y \cdot x \cdot q^j = x^{-1} \cdot y \cdot q^j \cdot x' = x^{-1} \cdot q^j \cdot x' = x^{-1} \cdot x \cdot q^j = e_i \cdot q^j = q^j \)
Hence $F_{e_i}(q^j)$ is a normal subgroup of $B_{e_i}$.

(b) It can be easily verified that $\pi(F_{e_i}(q^j)) = F_{e_i}(q^j)$, where $\pi$ is the isomorphism defined as in Remark 2.1.

\[ \text{Proposition 4.3} \]
Let $q$ be a generator of $Q$, $\nu(q) = r$, $|B_{e_i}| = hk$ and $|F_{e_i}(q^j)| = h_j k_j$, where $h_1 | \ldots | h_{r-1} | h | (t-1)$ and $k_1 | \ldots | k_{r-1} | k | tr^{-1}$. Then $\frac{k}{k_j}$ divides $t^{r-(j+1)}$ for $j \in I_{r-1}$ and, in particular, $k_{r-1} = k$.

\[ \text{Proof.} \]
Firstly, we observe that $C \ast q^j = C \ast e_i \ast q^j = B_{e_i} \ast q^j$ and, from Proposition 3.4 (a), $B_{e_i} \ast q^j \subseteq q^j \ast C$, for $i \in I_m$. Secondly, for any fixed $c \in C$, we can show that the right translation $\delta_c : B_{e_i} \ast q^j \rightarrow B_{e_i} \ast q^j \ast c$ is a bijection for all $j \in I_{r-1}$. In fact, let $b_1 \ast q^j \ast c = b_2 \ast q^j \ast c$, for $b_1, b_2 \in B_{e_i}$. Obviously $c \in B_{e_i}$ for some $s \in I_m$. Let $c^{-1}$ be the inverse of $c$ in $B_{e_i}$. Then $b_1 \ast q^j \ast c^{-1} = b_2 \ast q^j \ast c \ast c^{-1}$ implies $b_1 \ast q^j \ast c = b_2 \ast q^j \ast c$. Multiplying on the right by an idempotent right identity of $q^j$ and keeping in mind that $e_s$ is a left identity of $N$, we obtain $b_1 \ast q^j = b_2 \ast q^j$, hence $\delta_c$ results in injective and hence bijective. Moreover, for any fixed $c \in C$, either $B_{e_i} \ast q^j \cap B_{e_i} \ast q^j \ast c = \emptyset$ or $B_{e_i} \ast q^j = B_{e_i} \ast q^j \ast c$. In fact, if $y$ belongs to $B_{e_i} \ast q^j \cap B_{e_i} \ast q^j \ast c$ we have $y = b_1 \ast q^j = b_2 \ast q^j \ast c$, for some $b_1, b_2 \in B_{e_i}$. Hence $q^j = b_1^{-1} \ast b_2 \ast q^j \ast c$ implies $b \ast q^j = b \ast b_1^{-1} \ast b_2 \ast q^j \ast c \in B_{e_i} \ast q^j \ast c$, for any $b \in B_{e_i}$. So $B_{e_i} \ast q^j \subseteq B_{e_i} \ast q^j \ast c$ implies $B_{e_i} \ast q^j = B_{e_i} \ast q^j \ast c$. We can deduce that the elements of $q^j$ are equally shared in each $B_{e_i} \ast q^j$, $\forall i \in I_m$, so $|B_{e_i} \ast q^j| = |B_{e_i} : F_{e_i}(q^j)| = \frac{k}{k_j}$ must divide $|q^j \ast C| = (t-1)r^{-j+1}$ (see Proposition 3.1). In particular, $\frac{k}{k_j}$ divides 1, so $k_{r-1} = k$.

\[ \text{Remark 4.3} \]
Since $F(q^j) = \bigcup_{i=1}^m F_{e_i}(q^j)$, we have that $F(q^j) \subseteq \cdots \subseteq F(q^{r-1}) \subseteq C$ is a chain of multiplicative subsemigroups of $N$, each of them results in a disjoint union of $m$ isomorphic groups.

4.2 - Right identities of $q^j$

From Definition 2.1 we know that every element of $N$ has at least a right identity, so the set of all the right identities of any element $x$ of $N$ is certainly non empty.

Now we are dealing with the set of all the right identities of $q^j$, $j = 1, \ldots, r-1$, where $q$ is a generator of $Q$ and $\nu(q) = r$.

\[ \text{Definition 4.3} \]
Let $q$ be a generator of $Q$ and $\nu(q) = r$. The set of all the right identities of $q^j$ will be denoted by $U(q^j)$, that is $U(q^j) = \{ x \in N | q^j \ast x = q^j \}$, for $j \in I_{r-1}$.

\[ \text{Proposition 4.4} \]
Let $q$ be a generator of $Q$ and $\nu(q) = r$. Then, for $j \in I_{r-1}$,

(a) $U(q) \subseteq U(q^2) \subseteq \cdots \subseteq U(q^{r-1}) \subseteq C$ is a chain of subsemigroups of $C$;

(b) $U(q^j) = u + \text{Ann}(q^j)$, $u$ being any right identity of $q^j$;

(c) $|U(q^j)| = \nu^j$. 


Proof. (a) Obvious, as in Proposition 4.1.
(b) Let $u$ be any right identity of $q^i$. Then $q^i * x = q^i = q^i * u$, thus $q^i * (x - u) = 0$ implies $x - u \in Ann(q^i)$. Conversely, let $y$ be any element of $Ann(q^i)$, then $q^i * (u + y) = q^i * u + q^i * y = q^i$.
(c) From previous point (b) and Theorem 3.1, $|U(q^i)| = |Ann(q^i)| = t^i$.

Definition 4.4 Let $q$ be a generator of $Q$ and $\nu(q) = r$. The set of all the right identities of $q^i$ belonging to $B_{e_i}$ will be denoted by $U_{e_i}(q^i)$, that is $U_{e_i}(q^i) = U(q^i) \cap B_{e_i} = \{ x \in B_{e_i} \mid q^i * x = q^i \}$, for $j \in I_{r-1}$ and $i \in I_m$.

Remark 4.4 For all $j \in I_{r-1}$, $U_{e_i}(q^i)$ is non empty if and only if $q^i * e_i = q^i$, that is if and only if $e_i \in U_{e_i}(q^i)$.

Remark 4.5 If $q^i * e_i \neq q^i$ then $e_i \in U_{e_i}(p^i)$, where $p = q * e_i$ results in a generator of $Q$.

Proposition 4.5 Let $q$ be a generator of $Q$ and $\nu(q) = r$. Then, for $i, h \in I_m$ and $j \in I_{r-2}$,
(a) if $q^i * e_i = q^i$, then $U_{e_i}(q^i) \subseteq U_{e_i}(q^{i+1}) \subseteq \cdots \subseteq U_{e_i}(q^{r-1})$ is a normal chain of multiplicative subgroups of $B_{e_i}$;
(b) $U_{e_i}((q * e_i)^j)$ and $U_{e_i}((q * e_h)^j)$ are isomorphic groups;
(c) if $q^i * B_{e_i} \cap q^j * B_{e_h}$ is non empty, then $q^i * B_{e_i} = q^j * B_{e_h}$.

Proof. (a) If $q^i * e_i = q^i$, from previous Proposition 4.4(a) we know that $U_{e_i}(q^i)$ is a non empty subsemigroup of $U_{e_i}(q^{i+1})$ with $e_i$ as identity (see Remark 4.4), $\forall j \in I_{r-2}$. Let now $x \in U_{e_i}(q^i)$. The inverse of $x$ in $B_{e_i}$ belongs to $U_{e_i}(q^i)$ because it is an integer power of $x$ (see [5], Th.8), so $U_{e_i}(q^i)$ results in a subgroup of $B_{e_i}$. In order to show that $U_{e_i}(q^i)$ is normal in $B_{e_i}$, firstly we prove that an element $x$ of $B_{e_i}$ belongs to $U_{e_i}(q^i)$ if and only if $x * (c + Ann(q^i)) = c + Ann(q^i)$, for all $c \in C$. In fact $x \in U_{e_i}(q^i)$ implies $q^i * x = q^i$. Thus $q^i * x * e = q^i * e$ and this implies $x * e \in C + Ann(q^i)$, $\forall e \in C$. Hence, keeping in mind that $x * Ann(q^i) = Ann(q^i)$ for all $x \in C$ and $j \in I_{r-1}$, $x * (c + Ann(q^i)) = x * (c + Ann(q^j)) = c + Ann(q^i)$, for all $c \in C$. Conversely, if $x \in B_{e_i}$ and $x * (c + Ann(q^j)) = c + Ann(q^i)$ for all $c \in C$, choosing $c = e_i$ we have $x * (e_i + Ann(q^i)) = e_i + Ann(q^i)$. Hence $q^i * (x * (e_i + Ann(q^i))) = q^i * (e_i + Ann(q^i))$ and this implies $q^i * x = q^i$, that is $x \in U_{e_i}(q^i)$.

Applying previous characterization, $\forall y \in B_{e_i}, \forall x \in U_{e_i}(q^i)$ and $\forall e \in C$ we have $y^{-1} * x * y * (c + Ann(q^i)) = y^{-1} * x * (y * (c + Ann(q^i))) = y^{-1} * x * (y * c + Ann(q^i)) = y^{-1} * (y * c + Ann(q^i)) = y^{-1} * y * c + y^{-1} * Ann(q^i) = c + Ann(q^i)$. Thus $y^{-1} * x * y \in U_{e_i}(q^i)$ and this implies $U_{e_i}(q^i)$ is normal.

(b) It can be easily verified that $\pi(U_{e_i}(q^i)) = U_{e_h}(q^i)$, where $\pi$ is the isomorphism defined as in Remark 2.1 (a).

(c) $q^i * B_{e_i} \cap q^j * B_{e_h} \neq \emptyset$ implies $q^i * x = q^j * y$ for some $x \in B_{e_i}$ and $y \in B_{e_h}$. Multiplying by $e_i$ on the right we obtain $q^i * x = q^i * y * e_i$, so $q^i * y * (e_i - e_h) = 0$. Let $y^{-1}$ be the inverse of $y$ in $B_{e_h}$. Applying the I.F.P., we obtain $q^i * y * y^{-1} * n * (e_i - e_h) = 0$, so $q^i * n * e_i = q^i * n * e_h$ for all $n \in N$. Hence $q^i * B_{e_i} = q^j * B_{e_i} * e_h = q^j * B_{e_h}$.
Remark 4.6 From what precedes we can say that \( U(q^i) \) is a semigroup containing exactly \( m_j \) idempotent right identities of \( q^i \), say \( e_{i_1}, e_{i_2}, \ldots, e_{i_{m_j}} \). Then \( U(q^i) \) results in the disjoint union of \( m_j \) isomorphic groups, the \( U_{e_{i_\lambda}}(q^i) \)'s, for \( \lambda \in I_{m_j} \), that is \( U(q^i) = \bigcup_{\lambda=1}^{m_j} U_{e_{i_\lambda}}(q^i) \) and \( m_j = \frac{|U(q^i)|}{|U_{e_{i_\lambda}}(q^i)|} \), for \( j \in I_{r-1} \).

Now we are able to state a theorem about the algebraic structure of the \( B_e \)'s. Since the \( B_e \)'s are isomorphic groups, we will confine our attention to one of them, say \( B_e \), \( e \) being its identity. Since each non zero idempotent is a right identity of some generator of \( Q \), let \( q \) be a generator of \( Q \) such that \( q * e = q \).

Actually, the following Theorem 4.1 could be inferred from Prop. 4 of \([9]\), changing the contest appropriately. Anyway, here we give a short direct proof.

**Theorem 4.1** Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). Let \( e \) be an idempotent right identity of \( q \) and \( |B_e| = hk \), where \( h \mid (t-1) \) and \( k \mid t^{r-1} \). Then
(a) \( U_e(q^{-1}) \) is a normal subgroup of \( B_e \) of order \( k \);
(b) \( B_e \) results in the semidirect product of \( U_e(q^{-1}) \) and a complement \( U' \) of order \( h \).

**Proof.** (a) From Proposition 4.5 (a) we know that \( U_e(q^{-1}) \) is a normal subgroup of \( B_e \). Moreover, since the elements of \( U(q^{-1}) \) are shared into disjoint subgroups isomorphic to \( U_e(q^{-1}) \) (see Proposition 4.5 (b)), the order of \( U_e(q^{-1}) \) divides \( |U(q^{-1})| = t^{r-1} \) (see Proposition 4.4(c)). In addition, the index of \( U_e(q^{-1}) \) in \( B_e \) equals the cardinality of \( q^{-1} * B_e \) and \( q^{-1} * B_e \) must divide \( |q^{-1} * C| \) divides \( t-1 \) (see Proposition 4.5 (c) and Theorem 3.1). Thus, \( \nu(B_e : U_e(q^{-1})) \) divides \( t-1 \) and \( |U_e(q^{-1})| = k \).

(b) \( U_e(q^{-1}) \) is a normal subgroup of \( B_e \) whose order and index are coprime, so \( B_e \) results in the semidirect product between \( U_e(q^{-1}) \) and its Schur-Zassenhaus complement (see \([7]\)).

**Remark 4.1** Let \( q \) be a generator of \( Q \), \( \nu(q) = r \) and \( |B_e| = hk \) where \( h \) divides \( t-1 \) and \( k \) divides \( t^{r-1} \). The following statements are equivalent.
(a) \( |F_e(q^{-1})| = 1 \);
(b) \( |B_e| = h \) and \( |F_e(q^i)| = |U_e(q^i)| = 1 \), \( \forall j \in I_{r-1} \);
(c) \( |B_e| = h \) and \( B_e * q^i = q^i * B_e \), \( \forall j \in I_{r-1} \).

**Proof.** (a) \( \Rightarrow (b) \) Obviously \( |F_e(q^{-1})| = 1 \) implies \( |F_e(q^i)| = 1 \), \( \forall j \in I_{r-1} \).
Moreover, from Proposition 4.3 we know that \( |F_e(q^{-1})| = h_{r-1}k \), where \( h_{r-1} \) divides \( t-1 \), so \( k = h_{r-1} = 1 \). Hence \( |U_e(q^{-1})| = 1 \) implies both \( |B_e| = h \) and \( |U_e(q^i)| = 1 \), \( \forall j \in I_{r-1} \).

(b) \( \Rightarrow (c) \) \( |F_e(q^i)| = |U_e(q^i)| = 1 \), \( \forall j \in I_{r-1} \) implies \( |B_e * q^i| = |B_e| = |q^i * B_e| \).
Since \( B_e * q^i \subseteq q^i * B_e \) (see Proposition 3.4 (a)), then \( B_e * q^i = q^i * B_e \).

(c) \( \Rightarrow (a) \) \( B_e * q^{-1} = q^{-1} * B_e \) implies \( |F_e(q^{-1})| = |U_e(q^{-1})| = |B_e| = h \) implies \( k = 1 \), so \( |U_e(q^{-1})| = 1 = |F_e(q^{-1})| \).
Proposition 4.6 Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). If \( |B_e| = h \), where \( h \) divides \( t - 1 \), then \( |U_e(q^j)| = 1 \), \( |F_e(q^j)| = h_j \) and \( h_j | B_e \ast q^j | = |q^j \ast B_e| = h, \forall j \in I_{r-1} \).

Proof. Since we know that \( |B_e| = hk \), where \( h \) divides \( t - 1 \) and \( k \) divides \( t^{r-1} \), our hypothesis forces \( k = 1 \), hence \( |U_e(q^j)| = 1 \) (see Proposition 4.5 and Theorem 4.1) and \( |F_e(q^j)| = h_j \), where \( h_j \) divides \( h \), \( \forall j \in I_{r-1} \) (see Proposition 4.3). So, \( |q^j \ast B_e| = |B_e : U_e(q^j)| = h \) and \( |B_e \ast q^j| = |B_e : F_e(q^j)| = \frac{h}{h_j} \).

5 Let \( t \) be a prime number

If \( t \) is a prime number, the orders of \( N \) and \( Q \) are prime powers, so \( N \) and \( Q \) are (additive) \( t \)-groups. We also know that \( |B_e| = hk \), where \( h \) divides \( t - 1 \) and \( k \) divides \( t^{r-1} \), so \( k = t^\alpha \), with \( 0 \leq \alpha \leq r - 1 \).

Theorem 5.1 Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). Let \( e \) be any idempotent right identity of \( q \) and \( |B_e| = h t^\alpha \), where \( t \) is a prime, \( h \) divides \( t - 1 \) and \( 0 \leq \alpha \leq r - 1 \). Then \( |U_e(q^{j-1})| = t^\alpha \) and \( U_e(q^{j-1}) \subseteq F_e(q^{j-1}) \).

Proof. From previous Theorem 4.1 we know that \( |U_e(q^{j-1})| = t^\alpha \). Moreover, as \( |F_e(q^{j-1})| = h_{r-1} t^\alpha \), with \( h_{r-1} \) and \( t^\alpha \) relatively prime, we know that \( F_e(q^{j-1}) \) contains a subgroup of order \( t^\alpha \), say \( F_e \). Obviously \( F_e \) is a \( t \)-Sylow subgroup of \( B_e \). As \( U_e(q^{j-1}) \) is normal in \( B_e \), it is the only \( t \)-Sylow subgroup of \( B_e \). Hence, \( U_e(q^{j-1}) = F_e \).

Remark 5.1 If \( t \) is a prime number, from Propositions 4.4, 4.5 and previous Theorem 5.1 we know that

(a) \( |U_e(q^{j-1})| = t^\alpha \);
(b) \( |U_e(q)| = 1 \) or \( t \);
(c) if \( |U_e(q)| = t^\alpha \) then \( |U_e(q^{j+1})| = t^\alpha \) for \( j \in I_{r-2} \).

Thus, we can easily deduce the following

Proposition 5.1 Let \( q \) be a generator of \( Q \) and \( \nu(q) = r \). Let \( e \) be any idempotent right identity of \( q \) and \( |B_e| = h t^\alpha \), where \( t \) is a prime, \( h \) divides \( t - 1 \) and \( 0 \leq \alpha \leq r - 1 \). If \( |U_e(q^{\alpha + j})| = t^\alpha \) then

\[
\begin{align*}
\text{for } j \leq \alpha & : |U_e(q^j)| = t^\alpha \quad \text{and} \quad |q^j \ast B_e| = h t^{\alpha - j} \\
\text{for } j \geq \alpha & : |U_e(q^j)| = t^\alpha \quad \text{and} \quad |q^j \ast B_e| = h \\
\end{align*}
\]

If \( |U_e(q^{\alpha - j})| = 1 \) then

\[
\begin{align*}
\text{for } j \leq r - \alpha - 1 & : |U_e(q^j)| = 1 \quad \text{and} \quad |q^j \ast B_e| = h t^{\alpha} \\
\text{for } j \geq r - \alpha - 1 & : |U_e(q^j)| = t^{r-\alpha+1} \quad \text{and} \quad |q^j \ast B_e| = h t^{r-j-1} \\
\end{align*}
\]

\( \diamond \)
References


Abstract

In [5] the algebraic structure called weakly divisible nearring (wd-nearring) was defined and studied. In [1, 2] a special class of wd-nearrings was constructed and its combinatorial properties was investigated. In [3] PBIBDs were derived from a class of wd-nearrings and their parameters were calculated thanks to the knowledge of the algebraic structure. In [9] a generalization of the construction of [3] to more general cases, this paper is devoted to a more in depth study of the algebraic structure of any finite wd-nearring $N$, especially with regard to determining the size of the elements of significant structures in $N$, as partitions, normal chains and products.

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