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A constructive solution to the Oberwolfach problem with a large cycle

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ABSTRACT

For every 2-regular graph F of order v, the Oberwolfach problem $OP(F)$ asks whether there is a 2-factorization of K_v (v odd) or K_v minus a 1-factor (v even) into copies of F. Posed by Ringel in 1967 and extensively studied ever since, this problem is still open.
In this paper we construct solutions to $OP(F)$ whenever F contains a cycle of length greater than an explicit lower bound. Our constructions combine the amalgamation-detachment technique with methods aimed at building 2-factorizations with an automorphism group having a nearly-regular action on the vertex-set.

1. Introduction

Let K_v^* denote the complete graph K_v if v is odd, or K_v minus the edges of a 1-factor I if v is even. Given any 2-regular graph F of order v, the *Oberwolfach problem* $OP(F)$ asks for a decomposition (i.e., a partition of the edge set) of K_v^* into copies of F. It was originally posed by Ringel in 1967 when the order v is odd, and then extended to even orders by Huang, Kotzig, and Rosa [20] in 1979. We notice that the even variant can be seen as the *maximum packing version* of the original problem posed by Ringel. Although it has received much attention over the past 55 years, OP remains open. Complete and constructive solutions are known when the cycles of F have the same length (see [2,19]), when F is bipartite (see [6]), when F has exactly two cycles (see [24]), or for orders belonging to an infinite set of primes (see [7]) or twice a prime (see [1]).

We refer to [14, Section VI.12] for further results, up to 2006, concerning the solvability of infinite instances of OP, which however settle only a small fraction of the general problem. We notice, in particular, that no complete solution is known as soon as F has 3 or more cycles. A recent survey on constructive resolution methods that proved successful for solving a large portion of the Oberwolfach problem can be found in [13]. We notice that, opposite to the even case, which concerns 2-factorizations of $K_{2n} - I$, the variant to OP that deals with 2-factorizations of K_{2n} with additional copies of a given one factor I has been considered only recently in [3,21,22,27].

Letting $L = \{^{\mu_1}\ell_1, \dots, {}^{\mu_u}\ell_u\}$ be a multiset of integers $\ell_1, \dots, \ell_u \geq 3$, with multiplicity $\mu_1, \dots, \mu_u$, respectively, we write $F \simeq [L]$ or $F \simeq [^{\mu_1}\ell_1, \dots, {}^{\mu_u}\ell_u]$, whenever F is a 2-regular graph whose list of cycle-lengths is L. In this case, we may use the notation $OP(^{\mu_1}\ell_1, \dots, {}^{\mu_u}\ell_u)$ in place of $OP(F)$. We point out that OP(F) has a solution whenever $|V(F)| \leq 60$ [15,23], except when $F \in \{[^2 3], [^4 3], [4, 5], [^2 3, 5]\}$.

Constructive solutions to $OP(F)$ are given in [12] whenever

$$F \simeq [x, 2m_1, \dots, 2m_t, {}^2\ell_1, \dots, {}^2\ell_u], \tag{1.1}$$

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and x is greater than an explicit lower bound. Condition (1.1) places a constraint on the *cycle structure* of F which cannot contain an odd number of ℓ-cycles for every odd $\ell \neq x$. It is worth mentioning that [12] provides a similar result for the *minimum covering* and the 2-*fold variants* of OP, but in the second case (the 2-fold variant) there is no restriction on the cycle structure of F.

The aim of this paper is to generalize the main result of [12] by dropping restriction (1.1). More precisely, we prove Theorem 1.1 which employs the following notation: given a list L of positive integers (not necessarily distinct), we write L_0 and L_1 to represent the multiset of even and odd elements of L, respectively. Note that by $|L|$ we mean the size of L as a multiset and let $\max(\varnothing) = 0$.

Theorem 1.1. *$OP(y, \ell_1, \ell_2, \ldots, \ell_u)$ has an explicit solution whenever*

$$y \geq 3b + 24b_0 + 28b_1 + 119,$$

where $b = \sum_{i=1}^{u} \ell_i$, $b_0 = 2|L_0|\,(\max(L_0) + 3)$, $b_1 = 7^{|L_1|-1}(2\max(L_1) + 1)$ and $L = \{\ell_1, \ell_2, \ldots, \ell_u\}$.

In other words, Theorem 1.1 (proved in Section 3) constructs solutions to $OP(F)$ for every arbitrary 2-regular graph F with a large cycle of length greater than an explicit lower bound, thus taking us one step closer to a complete constructive solution of the Oberwolfach Problem.

The emphasis placed by Theorem 1.1 on providing 'explicit' solutions and an 'explicit' lower bound aims to point out the constructive approach used in this paper, which is antithetical to the purely existential ones, such as those based on probabilistic methods that recently allowed in [16] to obtain a non-constructive asymptotic solution to the Oberwolfach problem for orders greater than a lower bound that is however unquantified.

Theorem 1.1 exploits two results, Corollary 2.2 and Theorem 2.6, obtained via completely different methods, described in Section 2. Corollary 2.2, proven in [18], extends $(1,2)$-decompositions (see Section 2.1) of K_m^* to 2-factorizations of K_n^* $(m < n)$ by making use of the very powerful *amalgamation-detachment technique* introduced by Hilton [17]. Theorem 2.6 provides solutions to $OP(x, {}^2\ell_1, \ldots, {}^2\ell_u)$ satisfying the *matching property* (see Section 2.2). Here, the method used is based on constructing solutions to OP with a *pyramidal automorphism group*.

2. Preliminaries

In this section we introduce the basic notions and the preliminary results we need to prove Theorem 1.1.

Throughout the paper, all graphs are simple and finite. Given a subgraph F of G (briefly, $F \subset G$), we denote by $G \setminus F$ the graph G minus the edges of F. We refer to the number ℓ of edges of a *path* or a *cycle* as their length, and speak of an ℓ-path ($\ell \geq 1$) or an ℓ-cycle ($\ell \geq 3$), respectively. In particular, we denote by $P = \langle p_1, \ldots, p_\ell \rangle$ the ℓ-path whose edges are $\{p_i, p_{i+1}\}$, for $1 \leq i \leq \ell - 1$ and write $(p_1, \ldots, p_\ell)$ to denote the ℓ-cycle obtained from P by adding the edge $\{p_1, p_\ell\}$. A *linear forest* (resp. 2-*regular* graph) is the vertex-disjoint union of paths (resp. cycles) with at least one vertex of degree 2. We are hence preventing a linear forest to be a *matching*, that is, the vertex disjoint union of 1-paths.

A graph F, which is not a matching, whose vertices have degree 1 or 2, will be called a $(1,2)$-*graph*. Hence, F can be either a linear forest, a 2-regular graph, or the vertex-disjoint union of two such graphs. The list of the cycle-lengths of F is referred to as the *cycle structure* of F and denoted by $cs(F)$. Therefore, $cs(F) = \{{}^{\mu_1}\ell_1, \ldots, {}^{\mu_u}\ell_u\}$ means that F is the vertex-disjoint union of a (possibly empty) linear forest and u distinct 2-regular graphs, each containing exactly $\mu_i \geq 1$ cycles of length $\ell_i \geq 3$, for $1 \leq i \leq u$. An arbitrary 2-regular graph with cycle structure $L = \{\ell_1, \ldots, \ell_u\}$ will be denoted by $[L]$ or $[\ell_1, \ldots, \ell_u]$. If F is isomorphic to $[L]$, we write $F \simeq [L]$.

A *factor* of a simple graph G is a subgraph F of G such that $V(F) = V(G)$. When F is a matching, 2-regular, or a $(1,2)$-graph, we speak of a 1-factor, 2-factor, or $(1,2)$-factor of G, respectively. We recall that K_v^* is either K_v when v is odd, or $K_v \setminus I$ when v is even, where I is a 1-factor of K_v.

A *decomposition* of a simple graph G is a set $\mathcal{G}$ of graphs whose edge-sets partition $E(G)$. We speak of a 2-decomposition or a $(1,2)$-decomposition if each graph in $\mathcal{G}$ is 2-regular or a $(1,2)$-graph, respectively. Furthermore, if all graphs in $\mathcal{G}$ are also factors of G, then we speak of a *factorization*, 2-factorization or $(1,2)$-factorization of G, respectively.

The proof of our main result, Theorem 1.1, is based on Corollary 2.2 and Theorem 2.6, obtained through completely different methods which we describe in the following.

2.1. Extending $(1,2)$-decompositions to 2-factorizations

Corollary 2.2 is based on the *amalgamation-detachment technique* introduced by Hilton [17] to extend a path decomposition of K_m to a Hamiltonian cycle decomposition of K_n $(m < n)$. This constructive method was then used in [18] to solve $OP(x, \ell, \ldots, \ell)$ for every sufficiently large x.

We start by recalling the crucial result in [18], that is, Theorem 2.1, and then provide a reduced version of it, Corollary 2.2, stated by using the terminology of graph decompositions. Theorem 2.1 requires the basic notions on edge-colored graphs, which we recall in the following.

An edge-coloring of a simple graph G with t colors is a function γ mapping $E(G)$ onto a set $C = \{c_1, \dots, c_t\}$ of colors. In this case, we say that G is t-edge-colored and for each i we denote by $G(c_i)$ the subgraph induced by the edges colored c_i. It is not difficult to see that $\mathcal{G} = \{G(c_1), \dots, G(c_t)\}$ is a decomposition of G. Conversely, any decomposition of G naturally induces an edge coloring of G, by assigning distinct colors to the graphs of the decomposition. Given a graph G' containing G, an edge-coloring γ' of G' is called an extension of γ, if γ' coincides with γ over the edges of G.

Finally, we denote by $\delta_{max}(G)$ the maximum degree of the vertices of G, and recall that a composition of n is a sequence $(s_1, \dots, s_t)$ of positive integers that sums to n.

Theorem 2.1. *[18, Theorem 6] Let m and n be integers, $1 \le m \le n$, and let $(s_1, \dots, s_t)$ be a composition of $n-1$, where each $s_i \in \{1, 2\}$. Let K_m be edge-colored with t colors $c_1, \dots, c_t$, and let f_i denote the number of edges colored c_i. This coloring can be extended to an edge-coloring of K_n where each $K_n(c_i)$ is an s_i-factor, and if $s_i = 2$ then $K_n(c_i)$ contains exactly one more cycle than $K_m(c_i)$, if and only if the following conditions hold for $1 \le i \le t$:*

(1) $f_i \ge s_i \left(m - \frac{n}{2}\right)$,
(2) $s_i n$ *is even,*
(3) $\delta_{max}(K_m(c_i)) \le s_i$.

The following result represents a reduced version of Theorem 2.1, restated using the terminology of graph decompositions.

Corollary 2.2. *Let $\mathcal{F} = \{F_1, \dots, F_b\}$ be a $(1,2)$-decomposition of $K^*_{2a+\epsilon}$, where $\epsilon \in \{1, 2\}$. If the following condition holds*

$$b \ge 2a - \min_i \frac{|E(F_i)| - \epsilon}{2}, \tag{2.1}$$

then there exists a 2-factorization $\mathcal{F}^+ = \{F^+_1, \dots, F^+_b\}$ of $K^*_{2b+\epsilon}$ such that

$$F_i \subset F^+_i \quad \text{and} \quad |cs(F^+_i)| = |cs(F_i)| + 1.$$

Proof. Set $m = 2a + \epsilon$, $n = 2b + \epsilon$, $t = b + \epsilon - 1$ and let $K^*_m = K_m \setminus I$ when m is even, that is, $\epsilon = 2$. We make use of the $(1,2)$-decomposition $\mathcal{F}$ and edge-color K_m with t colors $c_1, \dots, c_t$ so that

$$K_m(c_i) = \begin{cases} F_i & \text{if } 1 \le i \le b, \\ I & \text{if } i = b+1 \text{ and } \epsilon = 2. \end{cases}$$

Each F_i has maximum degree 2, and we set $s_i = \delta_{max}(K_m(c_i)) = 2$ for $1 \le i \le b$, while $s_{b+1} = \delta_{max}(K_m(c_{b+1})) = 1$ when $\epsilon = 2$. Note that $(s_1, \dots, s_t)$ is a composition of n. Therefore, conditions 2 and 3 of Theorem 2.1 are satisfied. Finally, let $f_i = |E(F_i)|$, for $1 \le i \le b$, while $f_{b+1} = |I| = m/2$ when m is even. Considering that $|E(F_1)| \le m = 2a + \epsilon$, by the inequality (2.1) we get $a \le b$ which implies, when m is even, that $f_{b+1} = \frac{m}{2} \ge s_{b+1}(m - \frac{n}{2})$ (condition 1 of Theorem 2.1 for $i = b+1$). Furthermore, (2.1) is equivalent to saying that $n \ge 2m - \min_i f_i$ which is in turn equivalent to condition 1 of Theorem 2.1 for $1 \le i \le b$. Therefore, there is an edge-coloring of K_n with the same t colors $c_1, \dots, c_t$ such that

(1) $K_n(c_i)$ is an s_i-factor of K_n containing $K_m(c_i)$, for $1 \le i \le t$;
(2) $K_n(c_i)$ contains exactly one more cycle than $K_n(c_i)$, for $1 \le i \le b$.

Letting $F^+_i = K_n(c_i)$ and considering that F^+_i is a 1-factor of K_n if and only if $i = b+1$ and $\epsilon = 2$, the set $\mathcal{F}^+ = \{F^+_1, \dots, F^+_b\}$ provides the desired 2-factorization of K^*_n. □

2.2. Solutions to OP satisfying the matching property

As mentioned above, the proof of Theorem 1.1 is also based on Theorem 2.6 which constructs solutions to suitable instances of OP satisfying the matching property $(\mathcal{M})$ defined below. Theorem 2.6 is partly proven in [10,12] where the methods used are aimed at building *pyramidal* 2*-factorizations*. More precisely, a solution to $OP(F)$ is called pyramidal if it has an automorphism group Γ fixing 1 or 2 vertices (according to the parity of $|V(F)|$) and acting sharply transitively on the remaining. Pyramidal solutions can be equivalently described as follows: if $|V(F)| = 2k + \epsilon$ with $\epsilon \in \{1, 2\}$, a solution $\mathcal{F}$ to $OP(F)$ is called ϵ-pyramidal (or just pyramidal) over an additive group Γ (not necessarily abelian) of order $2k$, if we can label the vertices of F over $\Gamma \cup \{\infty_1, \infty_\epsilon\}$ so that $\mathcal{F} = \{F + \gamma \mid \gamma \in \Gamma\}$, where $F + \gamma$ (the right translate of F by γ) is the graph obtained from F by replacing each vertex $x \notin \{\infty_1, \infty_\epsilon\}$ with $x + \gamma$. The group of right translations induced by Γ over $V(F) = \Gamma \cup \{\infty_1, \infty_\epsilon\}$ represents an automorphism group of $\mathcal{F}$ fixing ∞_1 and ∞_ϵ and acting sharply transitively on the remaining vertices. We notice that one usually uses the term 1-rotational in place of 1-pyramidal. A more general description of pyramidal 2-factorizations is given in [5,11], while some recent results showing the effectiveness of the pyramidal approach can be found in [4,9].

Theorem 2.3, proven in [8] in a more general setting, shows how to construct a 1-rotational solution to $OP(\ell_0, \ell_1, \dots, \ell_u)$. Letting ℓ_0 denote the length of the cycles through ∞_1, this solution can be easily extended (see [11,20]) to a 2-pyramidal solution of $OP(\ell_0+1, \ell_1, \dots, \ell_u)$. Before stating Theorem 2.3, we recall that given a graph G with $V(G) \subset \Gamma \cup \{\infty_1, \infty_2\}$, where Γ is any additive group not necessarily abelian, the *list of differences* of G, is the multiset $\Delta G = \{x - y \mid (x, y) \in \Gamma \times \Gamma, \{x, y\} \in E(G)\}$ of all differences of adjacent vertices of G distinct from ∞_1 and ∞_2.

Theorem 2.3 *([8]). Let $F \simeq [\ell_0, \ell_1, \dots, \ell_u]$ with $V(F) = \mathbb{Z}_{2a} \cup \{\infty_1\}$. If $F + a = F$ and $\Delta F \supset \mathbb{Z}_{2a} \setminus \{0\}$, then the set $\mathcal{F} = \{F + i \mid i \in \mathbb{Z}_{2a}\}$ is a 1-rotational solution of $OP(F)$.*

Theorem 2.6 makes use of a general doubling construction described in [10]. This construction, when applied to a graceful labeling (defined below) of suitable (1, 2)-graphs, allows us to construct 2-regular graphs satisfying the assumptions of Theorem 2.3, hence pyramidal solutions to OP. This first part of Theorem 2.6 (under assumption 1) is proven in [10, Theorem 6.4]. However, we recall here its proof since we need to further show that the pyramidal solutions obtained satisfy the matching property $(\mathcal{M})$ which will be crucial to prove the main result.

From now on, an arbitrary (1, 2)-graph with cycle structure $L = \{\ell_1, \dots, \ell_u\}$ and exactly one path-component of length k will be denoted by $[k \mid L]$ or $[k \mid \ell_1, \dots, \ell_u]$. If the graph T is isomorphic to $[k \mid L]$, we write $T \simeq [k \mid L]$. Such a graph has been called a *zillion graph* in [12].

We recall that a *graceful labeling* of $[k \mid \ell_1, \dots, \ell_u]$ is a graph $T \simeq [k \mid \ell_1, \dots, \ell_u]$ with vertices in $\mathbb{Z}$, such that $V(T) = \{0, \dots, a\}$ and $\Delta T = \{\pm 1, \dots, \pm a\}$, where $a = k + \sum_{i=1}^{u} \ell_i$. Graceful labelings of $[k \mid L]$ are built in [12] whenever $k > B(L)$, where the lower bound $B(L)$ depends on the cycle structure L and it is defined as follows:

$$B(L) = 6b_0 + 7b_1 + 29 \text{ where,}$$
$$b_0 = 2|L_0|(\max(L_0) + 3), \text{ and } b_1 = 7^{|L_1|-1}(2\max(L_1) + 1).$$

Theorem 2.4 *([12]). $[k \mid L]$ has a graceful labeling whenever $k \geq B(L)$.*

We call a *halving* of a 2-regular graph $G \simeq [x, {}^2\ell_1, \dots, {}^2\ell_u]$ any subgraph $h(G)$ such that

$$cs(h(G)) = cs(G \setminus h(G)) = \{\ell_1, \dots, \ell_u\}. \tag{2.2}$$

Clearly, $h(G)$ can always be obtained by choosing u cycles of lengths $\ell_1, \dots, \ell_u$, respectively, and then adding an edge of the x-cycle of G. Note that both $h(G)$ and $G \setminus h(G)$ are certainly (1, 2)-graphs when $u \geq 1$.

We say that a solution $\mathcal{G}$ to $OP(x, {}^2\ell_1, \dots, {}^2\ell_u)$ satisfies the matching property $(\mathcal{M})$ if

$(\mathcal{M})$ there is a matching M such that
(a) $cs(G \setminus M) = \{\ell_1, \dots, \ell_u\} = cs(h(G') \cup M)$, and
(b) $h(G') \cup M$ is a (1, 2)-graph,
for some distinct graphs $G, G' \in \mathcal{G}$ and some halving $h(G')$ of G'.

Example 2.5. Here, we show a solution to $OP(x, {}^23, {}^24)$, with $x \in \{3, 4\}$, that satisfies the matching property.

Set $V(K_{17}) = \mathbb{Z}_{16} \cup \{\infty_1\}$ and consider the 2-factor G of K_{17} defined as the vertex-disjoint union of the following cycles

$$(\infty_1, 2, 10), (3, 6, 4), (11, 14, 12), (0, 5, 1, 7), (8, 13, 9, 15).$$

One can check that $\mathcal{G} = \{G + i \mid 1 \leq i \leq 8\}$ is a pyramidal solution to $OP({}^33, {}^24)$. Note that $G = G + 8 \in \mathcal{G}$; also, setting $G' = G + 1 \in \mathcal{G}$, we have that,

$$G' = (\infty, 3, 11) \cup (4, 7, 5) \cup (12, 15, 13) \cup (1, 6, 2, 8) \cup (9, 14, 10, 0)$$

and $h(G') = \langle \infty, 3 \rangle \cup (12, 15, 13) \cup (9, 14, 10, 0)$ is a halving of G'. Taking the matching $M = \{\{\infty, 2\}, \{1, 5\}, \{4, 6\}\}$, one can check that

(1) $cs(G \setminus M) = \{3, 4\} = cs(h(G') \cup M)$, and
(2) $h(G') \cup M$ is a (1, 2)-graph.

Therefore, $\mathcal{G}$ satisfies the matching property.

To obtain a solution to $OP({}^23, {}^34)$ satisfying the matching property we proceed in a similar way. Set $V(K_{18}) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ and consider the 2-factor G of K_{18} defined below:

$$G = (\infty_1, 2, \infty_2, 10) \cup (3, 6, 4) \cup (11, 14, 12) \cup (0, 5, 1, 7) \cup (8, 13, 9, 15).$$

One can check that $\mathcal{G} = \{G + i \mid 1 \leq i \leq 8\}$ is a pyramidal solution to $OP(^2 3, {}^3 4)$. As before $G = G + 8 \in \mathcal{G}$, and letting $G' = G + 1 \in \mathcal{G}$, we have that

$$G' = (\infty, 3, \infty_2, 11) \cup (4, 7, 5) \cup (12, 15, 13) \cup (1, 6, 2, 8) \cup (9, 14, 10, 0)$$

and $h(G') = \langle \infty, 3 \rangle \cup (12, 15, 13) \cup (9, 14, 10, 0)$ is a halving of G'. One can check that $\mathcal{G}$ satisfies the matching property with respect to M, G and $h(G')$.

We end this section building solutions to $OP(x,\ ^2\ell_1, \ldots,\ ^2\ell_u)$ that satisfies the matching property.

Theorem 2.6. *There exists a pyramidal solution to $OP(x,\ ^2\ell_1, \ldots,\ ^2\ell_u)$, with $u \geq 1$, that satisfies the matching property when either*

(1) *there exists a graceful labeling of $\left[\lfloor \frac{x-3}{2} \rfloor \mid \ell_1, \ldots, \ell_u\right]$, or*
(2) *$x \geq 2B(L) + 3$ with $L = \{\ell_1, \ldots, \ell_u\}$.*

Proof. Let $x = 2k + \epsilon$, with $\epsilon \in \{1, 2\}$, and set $a = k + \sum_{i=1}^{u} \ell_i$. Note that $a \geq 2$ since $u \geq 1$, and $\lfloor \frac{x-3}{2} \rfloor = k - 1$. Since Theorem 2.4 guarantees the existence of a graceful labeling of $[k - 1 \mid \ell_1, \ldots, \ell_u]$ whenever $k - 1 \geq B(L)$, that is, $x \geq 2B(L) + 3$, it is enough to prove the assertion under assumption 1. Therefore, let $T = P \cup R$ be a graceful labeling of $[k - 1 \mid \ell_1, \ldots, \ell_u]$, where P is a $(k - 1)$-path disjoint from the 2-regular graph $R \simeq [\ell_1, \ldots, \ell_u]$. Hence,

$$V(T) = \{0, \ldots, a - 1\} \quad \text{and} \quad \Delta T = \{\pm 1, \ldots, \pm(a - 1)\}. \tag{2.3}$$

Let p_0 and p_1 denote the end-vertices of P, let $\mathcal{E}$ be the graph with $E(\mathcal{E}) = \{\{\infty_1, p_0\}, \{\infty_1, p_0 + a\}, \{p_1, p_1 + a\}\}$, and set $\mathcal{C} = P \cup (P + a) \cup \mathcal{E}$. Finally, set

$$G = T \cup (T + a) \cup \mathcal{E} = R \cup (R + a) \cup \mathcal{C}$$

By (2.3), $V(G) = V(T) \cup V(T + a) \cup \{\infty_1\} = \{0, \ldots, 2a - 1\} \cup \{\infty_1\}$. Also, recalling that P and R are vertex-disjoint, it follows that $P, R, P + a, R + a$ are vertex-disjoint, as well. Hence, $\mathcal{C}$ is a $(2k + 1)$-cycle, and $G \simeq [2k + 1,\ ^2\ell_1, \ldots,\ ^2\ell_u]$.

From now on, we consider the vertices of G and its subgraphs modulo $2a$, hence $V(G) = \mathbb{Z}_{2a} \cup \{\infty_1\}$. By (2.3) and considering that $a \in \Delta\mathcal{E}$, we have that $\Delta G \supset \Delta T \cup \Delta\mathcal{E} \supset \mathbb{Z}_{2a} \setminus \{0\}$. Also, by construction, $G + a = G$. Therefore, Theorem 2.3 guarantees that $\mathcal{G} = \{G + i \mid i \in \mathbb{Z}_{2a}\}$ is a 1-rotational solution to $OP(2k + 1,\ ^2\ell_1, \ldots,\ ^2\ell_u)$.

We now show that $\mathcal{G}$ satisfies the matching property. Let $H = Q \cup R$ be a halving of G obtained by choosing the 1-path Q in $\mathcal{C}$ so that

$$V(Q) = \begin{cases} \{\infty_1, p_0\} & \text{if } 1 \leq p_0 \leq a - 1, \\ \{\infty_1, p_0 + a\} & \text{if } p_0 = 0. \end{cases}$$

Also, let $M = Q \cup N$ be a matching of H, where N consists of u edges belonging to the u distinct cycles of R. Recalling that $V(R) \subset \{0, \ldots, a - 1\}$, it is not difficult to see that the matching N of R can be chosen so that $0 \notin V(N)$, that is

$$V(N) \subset \{1, \ldots, a - 1\}.$$

Clearly, $cs(G \setminus M) = \{\ell_1, \ldots, \ell_u\}$. Also, $H' = H + (a + 1)$ is a halving of $G' = G + (a + 1) = G + 1$ which is a graph of $\mathcal{G}$ distinct from G, since $a > 1$. Note that H' is the vertex-disjoint union of the 1-path $Q' = Q + (a + 1)$ and the 2-regular graph $R' = R + (a + 1)$, where

$$V(R') \subset \{0\} \cup \{a + 1, \ldots, 2a - 1\}.$$

Considering the vertex-sets of Q, N and R', we conclude that M and R' are vertex-disjoint. Also, $M \cup Q'$ is either a matching or a linear forest. Hence $M \cup H'$ is a $(1, 2)$-graph with $cs(M \cup H') = cs(R') = \{\ell_1, \ldots, \ell_u\}$. This proves that $\mathcal{G}$ satisfies the matching property with respect to M, the two distinct graphs G and G', and the halving H' of G', thus showing the assertion for $\epsilon = 1$.

Consider the matching I containing the edges $e_\infty = \{\infty_1, \infty_2\}$ and $e_i = \{p_1 + i, p_1 + a + i\}$, for $0 \leq i < a$. Clearly, I is a 1-factor of K_{2a+2}, with $V(K_{2a+2}) = \mathbb{Z}_{2a} \cup \{\infty_1, \infty_2\}$. Now let G^* be the graph obtained from G by removing the edge e_0 and then joining its end-vertices to ∞_2, and set

$$\mathcal{G}^* = \{G^* + i \mid 0 \leq i < a\}.$$

Note that $G^* \simeq [2k + 2,\ ^2\ell_1, \ldots,\ ^2\ell_u]$. Since each $G^* + i$ can be obtained from $G + i$ by (performing the same operation of) inserting ∞_2 along the edge $e_i \in E(G + i)$, it follows that $\mathcal{G}^*$ is a 2-factorization of $K_{2a+2} \setminus I$ into copies of G^*. Hence, $\mathcal{G}^*$ is a 2-pyramidal solution of $OP(2k + 2,\ ^2\ell_1, \ldots,\ ^2\ell_u)$. Since the operations performed to obtain $\mathcal{G}^*$ do not involve edges of M, then $\mathcal{G}^*$ satisfies the matching property, with respect to M, the two distinct graphs G^* and $(G')^*$, and the halving H' of $(G')^*$, thus showing the assertion for $\epsilon = 2$. $\square$

3. The proof of Theorem 1.1

The idea behind the proof of the main result (Theorem 1.1) turns out to be similar to the one that in [18] allows the authors to solve $OP(x, {}^{\mu}\ell)$ whenever x is sufficiently large. We start with a solution of $OP(x, {}^{2}\ell_1, \dots, {}^{2}\ell_u)$ of order m (Theorem 2.6) and we decompose each of its factors into two $(1,2)$-graphs (halvings) whose cycle structure is $\{\ell_1, \dots, \ell_u\}$. We have thus obtained a decomposition $\mathcal{G}$ of K_m^* into $(1,2)$-graphs having the same cycle structure, which by means of Corollary 2.2 extends to a solution of $OP(y, \ell_1, \dots, \ell_u)$ (for a suitable y) whose order is $\equiv 1$ or $2 \pmod 4$ according to the parity of x (Theorem 3.1.(1)). To deal with the remaining classes of orders we need to suitably break one graph of $\mathcal{G}$ and redistribute its pieces between the remaining graphs of $\mathcal{G}$ without altering their cycle structure (Theorem 3.1.(2)). This can be done whenever the initial solution to $OP(x, {}^{2}\ell_1, \dots, {}^{2}\ell_u)$ satisfies the matching property (Theorem 2.6).

Theorem 3.1. *Let $\mathcal{G}$ be a solution to $OP(x, {}^{2}\ell_1, \dots, {}^{2}\ell_u)$ $(u \geq 1)$ and let $\epsilon \equiv x \pmod 2$ with $\epsilon \in \{1,2\}$. Then $OP(y, \ell_1, \dots, \ell_u)$ is solvable whenever the following conditions hold:*

(1) $y = 2x + 3\sum_{\beta=1}^{u} \ell_\beta - \epsilon$, *or*
(2) $y = 2x + 3\sum_{\beta=1}^{u} \ell_\beta - \epsilon - 2$, *provided that $\mathcal{G}$ satisfies $(\mathcal{M})$.*

Proof. Set $x = 2k + \epsilon$ $(k \geq 1)$ and $a = k + \sum_{\beta=1}^{u} \ell_\beta$ $(\ell_1, \dots, \ell_u \geq 3)$. Also, let $\mathcal{G} = \{G_1, \dots, G_a\}$ be a solution to $OP(2k + \epsilon, {}^{2}\ell_1, \dots, {}^{2}\ell_u)$, that is, a 2-factorization of $K_{2a+\epsilon}^*$ where $cs(G_\alpha) = \{2k+\epsilon, {}^{2}\ell_1, \dots, {}^{2}\ell_u\}$ for $1 \leq \alpha \leq a$.

For each α, let $h(G_\alpha)$ be a halving of G_α, and set

$$F_{2\alpha-1} = h(G_\alpha) \quad \text{and} \quad F_{2\alpha} = G_\alpha \setminus h(G_\alpha).$$

Clearly, $F_{2\alpha-1}$ and $F_{2\alpha}$ decompose G_α, hence $\mathcal{F} = \{F_i \mid 1 \leq i \leq 2a\}$ is a $(1,2)$-decomposition of $K_{2a+\epsilon}^*$, and by (2.2), each F_i is a $(1,2)$-graph such that $cs(F_i) = \{\ell_1, \dots, \ell_u\}$. Since $|\mathcal{F}| = 2a$, condition (2.1) holds, hence Corollary 2.2 guarantees the existence of a 2-factorization $\mathcal{F}^+ = \{F_i^+ \mid 1 \leq i \leq 2a\}$ of $K_{4a+\epsilon}$ where

$$F_i \subset F_i^+, \text{and } |cs(F_i^+)| = |cs(F_i)| + 1,$$

which imply that $cs(F_i^+) = cs(F_i) \cup \{y_i\} = \{\ell_1, \dots, \ell_u, y\}$, and

$$y_i = 4a + \epsilon - \sum_{\beta} \ell_\beta = 4k + 3\sum_{\beta} \ell_\beta + \epsilon = 2x + 3\sum_{\beta} \ell_\beta - \epsilon = y,$$

for $1 \leq i \leq 2a$. In other words, $\mathcal{F}^+$ is a solution to $OP(y, \ell_1, \dots, \ell_u)$, and this proves the first part of the theorem.

Now assume that $\mathcal{G}$ satisfies the matching property $(\mathcal{M})$: without loss of generality, we can assume that there is a matching M of G_1 such that

$$\begin{aligned} cs(G_1 \setminus M) = \{\ell_1, \dots, \ell_u\} = cs(F_3 \cup M), \text{ and} \\ F_3 \cup M \text{ is a } (1,2)\text{-graph.} \end{aligned} \tag{3.1}$$

Set $\overline{F}_2 = G_1 \setminus M$, $\overline{F}_3 = F_3 \cup M$, and $\overline{F}_i = F_i$ for $4 \leq i \leq 2a$. By (3.2), and recalling that F_1 and F_2 decompose G_1, and $M \subset G_1$, it follows that

$$\overline{\mathcal{F}} = \left\{\overline{F}_i \mid 2 \leq i \leq 2a\right\} = (\mathcal{F} \setminus \{F_1, F_2, F_3\}) \cup \left\{\overline{F}_2, \overline{F}_3\right\}$$

is a $(1,2)$-decomposition of $K_{2a+\epsilon}$ into $b = 2a-1$ graphs. Considering that each $\overline{F}_i$ contains at least one path-component of length ≥ 1 and a cycle-component of length ≥ 3, it follows that $|E(\overline{F}_i)| \geq 4$ for $2 \leq i \leq 2a$. Therefore, $\overline{\mathcal{F}}$ satisfies condition (2.1), hence Corollary 2.2 guarantees the existence of a 2-factorization $\overline{\mathcal{F}}^+ = \left\{\overline{F}_i^+ \mid 2 \leq i \leq 2a\right\}$ of $K_{2b+\epsilon} = K_{4a+\epsilon-2}$ such that

$$\overline{F_i} \subset \overline{F}_i^+, \quad \text{and} \quad |cs(\overline{F}_i^+)| = |cs(\overline{F}_i)| + 1,$$

for $2 \leq i \leq 2a$. Reasoning as before, we conclude that $\overline{\mathcal{F}}^+$ is a solution to $OP(y, \ell_1, \dots, \ell_u)$ where

$$y = 4a + \epsilon - 2 - \sum_{\beta=1}^{u} \ell_\beta = 4k - 3\sum_{\beta=1}^{u} \ell_\beta + \epsilon - 2 = 2x - 3\sum_{\beta=1}^{u} \ell_\beta - \epsilon - 2,$$

and this completes the proof. □

Example 3.2. Here, we follow the proof of Theorem 3.1, and by starting with a solution to $OP(x, {}^2\ell_1, {}^2\ell_2)$, with $x \in \{3, 4\}$ and $(\ell_1, \ell_2) = (3, 4)$, we construct a solution to $OP(y, 3, 4)$, for $y \in \{24, 25, 26, 27\}$.

Let $\epsilon \in \{1, 2\}$ and take the solution $\mathcal{G} = \{G_\alpha \mid 1 \le \alpha \le a = 8\}$ of $\mathrm{OP}(2+\epsilon, {}^2 3, {}^2 4)$ considered in Example 2.5, where $G_\alpha = G + \alpha$, and G is the 2-regular graph defined below, according to the values of ϵ: if $\epsilon = 1$, then

$$G = (\infty_1, 2, 10) \cup (3, 6, 4) \cup (11, 14, 12) \cup (0, 5, 1, 7) \cup (8, 13, 9, 15),$$

otherwise,

$$G = (\infty_1, 2, \infty_2, 10) \cup (3, 6, 4) \cup (11, 14, 12) \cup (0, 5, 1, 7) \cup (8, 13, 9, 15).$$

Note that $G = G_8 \in \mathcal{G}$. For $1 \le \alpha \le 8$, consider the halving

$$h(G_\alpha) = \langle \infty, 2+\alpha \rangle \cup (11+\alpha, 14+\alpha, 12+\alpha) \cup (8+\alpha, 13+\alpha, 9+\alpha, 15+\alpha)$$

of G_α, and set $F_{2\alpha-1} = h(G_\alpha)$ and $F_{2\alpha} = G_\alpha \setminus h(G_\alpha)$. Clearly,

$$\mathcal{F} = \{F_i \mid 1 \le i \le 2a = 16\}$$

is a $(1,2)$-decomposition of $K^*_{16+\epsilon}$. Since $b = |\mathcal{F}| = 16$, condition (2.1) holds, hence Corollary 2.2 guarantees the existence of a 2-factorization $\mathcal{F}^+ = \{F_i^+ \mid 1 \le i \le 16\}$ of $K_{4a+\epsilon} = K_{32+\epsilon}$ where each F_i^+ is a 2-regular graph of order $32+\epsilon$ containing F_i and exactly one more cycle than F_i, of length say y_i. Since $cs(F_i) = \{3, 4\}$, we have that $y_i = 25+\epsilon$ and $\mathcal{F}^+$ is a solution to $OP(25+\epsilon, 3, 4)$.

It is left to build a solution to $OP(24, 3, 4)$ and $OP(25, 3, 4)$. In Example 2.5, we showed that $\mathcal{G}$ satisfies the matching property $(\mathcal{M})$: more precisely, by taking the matching $M = \{\{\infty, 2\}, \{1, 5\}, \{4, 6\}\}$ of $G_8 = G$ and recalling that each G_α decomposes into $F_{2\alpha-1}$ and $F_{2\alpha}$ (both halvings of G_α), one can check that

$$\begin{aligned} &cs(G_8 \setminus M) = \{3, 4\} = cs(F_1 \cup M), \text{ and} \\ &\qquad F_1 \cup M \text{ is a } (1,2)\text{-graph.} \end{aligned} \tag{3.2}$$

Recall that $a = 8$ and let $\overline{\mathcal{F}}$ be the $(1,2)$-decomposition of $K_{2a+\epsilon}$ obtained from $\mathcal{F}$ by replacing F_{15} and F_{16} with $G_8 \setminus M$, and then replacing F_1 with $F_1 \cup M$. By (3.2), we have that $cs(\overline{F}) = \{3, 4\}$, for every $\overline{F} \in \overline{\mathcal{F}}$. Also, note that $b = |\overline{\mathcal{F}}| = 2a - 1 = 15$, and considering that $|E(F_i)| \ge 7$, it follows that $\overline{\mathcal{F}}$ satisfies condition (2.1). Hence, Corollary 2.2 guarantees the existence of a 2-factorization $\overline{\mathcal{F}}^+ = \left\{\overline{F}_i^+ \mid 1 \le i \le 2a-1 = 15\right\}$ of $K_{2b+\epsilon} = K_{30+\epsilon}$ where each $\overline{F_i}^+$ is a 2-regular graph of order $30+\epsilon$ containing $\overline{F}_i$ and exactly one more cycle than $\overline{F}_i$, of length say y_i. Since $cs(F_i) = \{3, 4\}$, we have that $y_i = 23+\epsilon$ and $\mathcal{F}^+$ is a solution to $OP(23+\epsilon, 3, 4)$.

We are now ready to prove the main result of this paper, which we restate below.

Theorem 1.1. *$OP(y, \ell_1, \ell_2, \dots, \ell_u)$ has an explicit solution whenever*

$$y \ge 3b + 24b_0 + 28b_1 + 119,$$

where $b = \sum_{i=1}^u \ell_i$, $b_0 = 2|L_0|\,(\max(L_0) + 3)$, $b_1 = 7^{|L_1|-1}(2\max(L_1) + 1)$ *and* $L = \{\ell_1, \ell_2, \dots, \ell_u\}$.

Proof. Since $OP(y)$ and $OP(y, \ell)$ are completely solved (see, [14, Section VI.12] and [12]), we can assume $u \ge 2$. Consider the maps $\epsilon : \mathbb{N} \to \{1, 2\}$ and $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$\epsilon(y) \equiv y + b \pmod 2, \quad f(y) = \frac{y + \epsilon(y) - 3b}{2} + \begin{cases} 0 & \text{if } \epsilon(y) \equiv y + b \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Now set $y_0 = 3b + 24b_0 + 28b_1 + 119$. Considering that $y_0 + b \equiv 3 \pmod 4$, then $\epsilon(y_0) = 1$, and it is not difficult to check that

$$\min_{y_0 \le y} f(y) = f(y_0) = 12b_0 + 14b_1 + 61.$$

Hence, for every $\overline{y} \ge y_0$, Theorem 2.6 constructs a pyramidal solution to $OP(f(\overline{y}), {}^2\ell_1, \dots, {}^2\ell_u)$ that satisfies the matching property. Finally, Theorem 3.1 constructs a solution to $OP(\overline{y}, \ell_1, \dots, \ell_u)$. □

4. Conclusions

In this paper we construct solutions to the Oberwolfach problem $OP(F)$ for every 2-regular graph F with a cycle whose length is greater than an explicit lower bound: Theorem 1.1. This result makes use of two results, Corollary 2.2 and Theorem 2.6 obtained via completely different methods. Corollary 2.2, proven in [18], extends $(1,2)$-decompositions of K_m^* to 2-factorizations of K_n^* $(m < n)$ by making use of the very powerful amalgamation-detachment technique introduced by Hilton [17]. Theorem 2.6 constructs solutions to $OP(x, {}^2\ell_1, \ldots, {}^2\ell_u)$ satisfying the matching property. Here, the method used is based on constructing 2-factorizations with a pyramidal automorphism group.

The main idea behind the proof of Theorem 1.1 can be easily generalized as follows. We start with a solution $\mathcal{F} = \{F_1, \ldots, F_a\}$ to $OP(x, {}^\mu\ell_1, \ldots, {}^\mu\ell_u)$ of order v. Then, we decompose each factor F_i of $\mathcal{F}$ into μ $(1,2)$-graphs $F_{i,1}, \ldots, F_{i,\mu}$ with the same cycle structure: $cs(F_{i,j}) = \{\ell_1, \ldots, \ell_u\}$. In other words, we separate out μ sets of cycles of length $\ell_1, \ldots, \ell_u$ and then add to each set a portion of the x-cycle of F. We have obtained a $(1,2)$-decomposition $\mathcal{G}$ of K_v^*, and by applying Corollary 2.2, we construct a 2-factorization that solves $OP(y, \ell_1, \ldots, \ell_u)$ for a specific value of y. In other words, we have proven the following.

Theorem 4.1. *If $OP(x, {}^\mu\ell_1, \ldots, {}^\mu\ell_u)$ of order $2w+\epsilon$, with $\epsilon \in \{1,2\}$, has a solution, then there is a solution to $OP(y, \ell_1, \ldots, \ell_u)$ with $y = 2w\mu + \epsilon - \sum \ell_i$.*

Note that the order of $OP(y, \ell_1, \ldots, \ell_u)$ is $2w\mu + \epsilon \equiv \epsilon \pmod{2\mu}$. To deal with the remaining classes of orders $\pmod{2\mu}$ it is enough to manipulate the intermediate $(1,2)$-decomposition $\mathcal{G}$, by decomposing i graphs in $\mathcal{G}$ $(1 \le i < \mu)$ into suitable linear forests that can then be added to the remaining graphs in $\mathcal{G}$ to form larger $(1,2)$-graphs but with the same initial cycle structure. This way we end up producing a solution to

$$OP(y, \ell_1, \ldots, \ell_u) \text{ with } y = 2w\mu + \epsilon - 2i - \sum \ell_i. \tag{4.1}$$

Succeeding to solve (4.1), for every $1 \le i < \mu$, would then lead to solve $OP(y, \ell_1, \ldots, \ell_u)$ for every $y > f(x_0, \mu)$, provided that we can solve $OP(x, {}^\mu\ell_1, \ldots, {}^\mu\ell_u)$ for every $x > x_0$. Note that f would be an increasing function of both x_0 and μ. An explicit value for the lower bound x_0 is given in [12] and it grows as μ increases. Therefore, the best possible lower bound on y, based on the results of [12], can be achieved when $\mu = 2$. This is the reason why all constructions in this paper make use of solutions to $OP(x, {}^2\ell_1, \ldots, {}^2\ell_u)$.

We conclude with two tables showing the smallest value $\overline{y}$, given by Theorem 1.1, that guarantees the solvability of $OP(y, \ell_1, \ell_2)$ for every $y \ge \overline{y}$, when $3 \le \ell_1 < \ell_2 \le 8$. We exclude the cases where $\ell_1 = \ell_2$ since for them a better lower bound, that is $\overline{y} = 5$, is given in [24]. Further partial results can be found in [26].

$\overline{y}$	672	2299	774	3089	876	3879	790	1017	908	1215	1026
ℓ_1	3	3	3	3	3	3	4	4	4	4	4
ℓ_2	4	5	6	7	8	9	5	6	7	8	9

$\overline{y}$	892	3095	994	3885	1010	1221	1128	1112	3891	1230
ℓ_1	5	5	5	5	6	6	6	7	7	8
ℓ_2	6	7	8	9	7	8	9	8	9	9

An improvement to Theorem 1.1 containing a lower bound on y that is linear in the remaining u cycle lengths will be given in a paper in preparation [25].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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