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GRADIENT THERMOVISCOELASTICITY BASED ON IRREVERSIBLE THERMODYNAMICS

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To my beloved wife, Rahil,
and to my sweet angel, Sana.

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Abstract

Nowadays resonators constitute an important part of advanced technologies such as Micro-electromechanical and Nano-electromechanical systems (MEMS, NEMS). They are employed for instance as high precision actuators or micro-sensors, as well as in atomic force microscopy and semiconductors. Micro- and nanoresonators exhibit loss of sensitivity as a result of internal energy dissipation. The responsible mechanism for energy loss is known to be of thermoelastic damping where thermal and mechanical fields are coupled.

In this type of damping, mechanical deformation causes temperature gradients happen from compressed areas toward stretched regions of microbeam. System which is in non-equilibrium state seeks to regain its equilibrium though exertion of heat. However it is an irreversible process which accompanies production of entropy and ultimately leads to dissipation of energy.

As can be seen the process of thermoelastic damping is a complex phenomenon which involves the interplay of mechanics and irreversible thermodynamics. This complexity is escalated when we notice that significant discrepancies have been recorded between experimental observations and theoretical/computational predictions. A major inhibitor for achieving higher accuracies is known to be incapability of classical continuum theories in describing mechanical fields at tiny scales such as micro- and nanometers. More recent theories, however, such as gradient elasticity have been emerged to fulfill such a need to more resolved description of advanced materials behaviors.

This study aims to provide a more accurate description and understanding toward the phenomenon of thermoelastic damping. The complexity of this phenomenon at small scales is resolved through incorporation of gradient theories (stress and strain gradients in particular) into the classical thermoviscoelasticity model on the ground of irreversible thermodynamics.

Thereby in this thesis theories of irreversible thermodynamics, gradient elasticity alongside viscoelasticity and thermoelastic damping have been studied through chapters 1 to 4. Then in chapter 5 the theory of gradient thermoviscoelasticity has been developed based on the principles explained in previous chapters. In this chapter first the general thermodynamic framework has been established, then constitutive models for both stress gradient and strain gradient materials are developed. Different energy functionals for each of the models have been postulated, each of which are functions of stress, strain and gradients of stress and strain. Different mathematical techniques and theorems such as Stieltjes biconvolutive integrals and Onsager's reciprocal theory have been employed throughout the development process. At the end constitutive relations for dissipative parties in entropy production have been formulated in a matrix-vector form and a number of illustrative examples presented. Finally in chapter 6 of this thesis conclusions along with suggestions for future works have been presented.

Sommario

Oggigiorno i risonatori costituiscono una parte importante delle tecnologie avanzate come i sistemi microelettromeccanici e nanoelettromeccanici (MEMS, NEMS). Sono impiegati ad esempio come attuatori o microsensori di alta precisione, così come nella microscopia a forza atomica e nei semiconduttori. I micro e nanorisonatori mostrano una perdita di sensibilità a causa della dissipazione interna dell'energia. È noto che il meccanismo responsabile della perdita di energia è lo smorzamento termoelastico in cui i campi termici e meccanici sono accoppiati.

In questo tipo di smorzamento, la deformazione meccanica fa sì che i gradienti di temperatura si verifichino dalle aree compresse verso le regioni tese del microbeam. Un sistema che si trova in uno stato di non equilibrio cerca di ritrovare il suo equilibrio attraverso la produzione di calore. Tuttavia è un processo irreversibile che accompagna la produzione di entropia e alla fine porta alla dissipazione di energia.

Come si può vedere il processo di smorzamento termoelastico è un fenomeno complesso che coinvolge l'interazione di meccanica e termodinamica irreversibili. Questa complessità aumenta quando notiamo che sono state registrate discrepanze significative tra le osservazioni sperimentali e le previsioni teoriche/computazionali. È noto che un importante inibitore per il raggiungimento di precisioni più elevate è l'incapacità delle teorie del continuo classico nel descrivere i campi meccanici su piccola scala come micro e nanometri. Teorie più recenti, tuttavia, come l'elasticità a gradiente sono emerse per soddisfare l'esigenza di descrizione più accurata dei comportamenti dei materiali avanzati.

Questo studio mira a fornire una descrizione e una comprensione più accurate del fenomeno dello smorzamento termoelastico. La complessità di questo fenomeno su piccola scala è risolta attraverso l'incorporazione delle teorie a gradiente (gradienti di sforzo e deformazione in particolare) nel modello classico di termoviscoelasticità sulla base della termodinamica irreversibile.

Pertanto, in questa tesi, le teorie della termodinamica irreversibile, dell'elasticità a gradiente, della viscoelasticità e dello smorzamento termoelastico sono state studiate nei capitoli da 1 a 4. Quindi nel capitolo 5 è stata sviluppata la teoria della termoviscoelasticità a gradiente sulla base dei principi spiegati nei capitoli precedenti. In questo capitolo viene dapprima stabilito il quadro termodinamico generale, quindi vengono sviluppati i modelli costitutivi sia per il gradiente di sforzo che per il gradiente di deformazione. Sono stati ricavati diversi funzionali energetici per ciascuno dei modelli, ciascuno dei quali è funzione di sforzo, deformazione e gradienti di sforzo e deformazione. Diverse tecniche e teoremi matematici come gli integrali biconvolutivi di Stieltjes e la teoria reciproca di Onsager sono stati impiegati. Alla fine le relazioni costitutive per le parti dissipative nella produzione di entropia sono state formulate in forma matriciale. Infine, nel capitolo 6 di questa tesi sono state presentate le conclusioni insieme a suggerimenti per lavori futuri.

Contents

Nomenclature

$'$	Admissible term
$*$	Convolution of Two Function
$\bar{\theta}$	Prescribed temperature
$\bar{\mathbf{q}}$	Prescribed heat flux
$\bar{\mathbf{u}}$	Prescribed displacement
$\mathbf{b}$	Body forces
$\mathbf{e}$	Conjugate Strain Tensor
$\mathbf{n}$	Unit outward normal vector
$\mathbf{q}$	Heat Flux
$\mathbf{t}$	Surface traction vector
$\mathbf{u}$	Displacement Field
$\boldsymbol{\eta}$	Conjugate Strain Tensor
$\boldsymbol{\Phi}$	Dissipated Power Density
$\boldsymbol{\sigma}$	Stress Tensor
$\boldsymbol{\varepsilon}$	Strain Tensor
$\circ$	Convolution product over half time interval
$\dot{s}^{pr}$	Rate of Entropy Production
$\dot{}$	Derivative with respect to time
ℓ	Characteristic Length
η	Relaxation Time
$\forall$	For all
Λ	Dissipation Function

$\langle,\rangle$ Standard bilinear form with time integral in the sense of Stieltjes

$\langle,\rangle_c$ Bilinear form with time integral in the sense of Stieltjes but standard with respect to space

$\mathcal{F}^G$ Gurtin's functional

$\mathcal{G}$ Gibbs Energy Function

$\mathcal{L}$ Navier operator

μ Coefficient of Viscosity

Ω Region occupied by a solid body

$\partial\Omega$ Boundary of the solid body

$\partial\Omega_\theta$ Constrained region of the boundary for heat conduction

$\partial\Omega_p$ Loaded region of the boundary for heat conduction

$\partial\Omega_u$ Constrained region of the boundary for elasticity

ψ Helmholtz free Energy Function

ρ Density

$\square$ Pseudo-convolution

τ Relaxation Time

θ Excess Temperature

$_1$ Variable defined over the first interval

$_2$ Variable defined over the second interval

$_,$ Partial spatial derivative with respect to a component/variable

c Specific Heat

$E(t)$ Relaxation Function

E_{ijhk} Relaxation tensor

I Identity operator

$J(t)$ Creep Compliance

J_i Thermodynamic Flux

J_{ijhk} Creep Tensor

k Thermal Conductivity

K_{ij} Memory Function

p_i	Surface traction vector
q_i	Heat flux vector component
R	Nonlocality Residual
s	Entropy
T	Absolute Temperature
t	Time
u	Internal Energy
W	Strain Energy Density Function
X_i	Thermodynamic Driving Force
$y(t)$	Function for the biconvolutive bilinear form
$\mathbf{C}$	Elastic Stiffness Tensor

Chapter 1

Irreversible Thermodynamics

1.1 Introduction

Generally by investigating a phenomenon in nature, we are more interested in the processes rather than the states.This is while in equilibrium thermodynamics, or better to name it thermostatics, processes are approximated only based on initial and final states of the system, or they are viewed as a succession of equilibrium states. Unfortunately neither of these approaches is capable to accurately capture and trace the process, as a time evolving phenomenon. In other words, they do not consider or include the rate at which a process is happening. An extension of thermodynamics that addresses rate of physical processes in a non-equilibrium states is called the irreversible thermodynamics [1, 2].

Fundamentals of irreversible thermodynamics are based on pioneering works of Onsager (1931)[3, 4] and Prigogine (1968)[5]. Onsager Reciprocity Theorem in fact provides us a basis to locally define intensive and extensive parameters as we need them in describing irreversible processes. Theory of irreversible thermodynamics was further developed through works of Meixner[6], de Groot and Mazur [7], and Casimir [8].

In non-equilibrium thermodynamics macroscopic quantities, through which a non-equilibrium state is described, are in fact extrapolations of variables that are used to describe the system which thermodynamically is in equilibrium.

Usually formulating a physical problem based on irreversible thermodynamics requires generalization of equilibrium thermodynamics. In that sense non-equilibrium thermodynamics offers three areas of application. First addressing and generalizing the classical transport problems such as heat, mass and momentum equations. Secondly it can be employed in coupling thermal, mechanical, chemical and electromagnetic effects. And third line of application would include stationary non-dissipative states in which although their properties are time invariant but they exhibit non-homogeneous distribution of variables along with non-vanishing values of flux [9].

Non-equilibrium thermodynamics originates from the fact that for example living biological systems should live in a state far from equilibrium. Thermodynamically this could be possible through flow of energy and information. Real world phenomena are irreversible through which increase in entropy is inevitable. The application of irreversible thermodynamics to continuous media requires the calculation of the entropy balance equations and in particular derivation of the entropy source strength in the balance equation [10].

1.2   Equilibrium Thermodynamics

Equilibrium thermodynamics investigates properties of systems in equilibrium state at macro level. Equilibrium state is a time-independent state meaning that the system is not disturbed by any flux of matter, energy, charge, momentum and so on. It could be equipped to study reversible processes through considering a reversible process as a continuum succession of equilibrium states.

It is worthy of mentioning that the term thermodynamics implicitly conveys the concept of time; since the formalism does not encompass time it might be more proper to use the term thermostatics for such a branch of science, but as the term thermodynamics is being used widely regardless of time dependence or time independence we stick to the term thermodynamics here as well [1].

As a bit of background, thermodynamics in fact stems from a broader area of science namely Mechanics. What distinguish thermodynamics as a relatively independent branch of science would be the notions of heat and temperature that are not present in mechanics. First contribution to establishment of such a branch was made notably by efforts of Fourier, Carnot, Kelvin, Clausius and Gibbs. A milestone work was done by James P. Joule who recognized heat as a form of energy which can be transferred to work and vice versa. His accomplishments led to construction of bases of the first law of thermodynamics. William Thomson (also known as Lord Kelvin) viewed the mechanical action of heat by appealing two laws that later on were named as the first and second laws of thermodynamics. Important notions of internal energy and entropy were introduced by Clausius. Ludwig Boltzmann developed a mechanistic approach for description of heat transport through relating the behavior of particles at microscopic level to macroscopic properties of the system. Another prominent scientist, namely Josiah Williard Gibbs, noticed the missing concept of time in thermodynamics and among his works he developed the theory of stability employing concepts of convex or concave functions and also expanding the notions of chemical potential as well as statistical ensembles. There are other leading scientists with undisputed role in development of thermodynamics whose mentioning their contribution is far from being exhaustive here [3, 11].

1.3   Fundamental Concepts from Equilibrium Thermodynamics

Roots of equilibrium thermodynamics extend to macroscopic physics whose main objective is to explain transformation of energy in all its forms. As stated, equilibrium thermodynamics is in fact the generalization of mechanics which introduces three main notions including state, internal energy and entropy.

State could be viewed as the ensemble of quantities called state variables. Definiteness of these state variables lead to identification of properties of the system. To be fully functional, these state variables need to be independent of each other and also need to be easily accessible through experiments.

The concept of internal energy is devised as a complement to the kinetic energy with a pure mechanical origin where notions of heat and temperature are missing. It is however not feasible to be measured, rather considering a system (in macroscopic scale) as a composition of individual particles, enables us to define internal energy as the mean value

of the kinetic and interacting energies of the particles. This way it can be presumed as a function of measurable properties of a system such as mass, volume and temperature.

Similar to internal energy, entropy can not be measured yet it is used to characterize a system. We are however able to measure the changes in entropy. Entropy is a nonconserved and an extensive property of a system. It is a state function which any change in state is associated to change in entropy.

As mentioned before, the state of a system is specified by ensemble of quantities called state variables. State variables in fact characterize the state of the system under study. Assuming that a system undergoes a transition between two equilibrium states A and B, it should be noted that state variables do not depend on the specific path the system takes to traverse between A and B. Proper selection of state variables is usually a matter of experiments and theory. Equilibrium by its own is a state which is time independent and generally homogeneous. Equilibrium state is free of any effecting disturbance such as fluxes of matter or energy. However in a non-equilibrium state state variables are functions of time and space as we have exchange of matter and energy between the system and environment. It should be noted that definition of equilibrium would not be complete without taking into account the notion of entropy. For example in the case of an isolated system which does not have any exchanges with its surroundings, the equilibrium is characterized by a maximum of entropy. Also definition of equilibrium is subjective as there exist fluctuations inherent to each equilibrium state. Plus to the fact that it is also a function of available data as well as desired accuracy of the observations.

There exist two types of state variables namely extensive variables and intensive variables. Extensive variables such as mass, volume and energy are functions of size/mass or extent of the system. In other words, considering a composite system, value of an extensive variable is equal to the sum of the values of each compartment of the system. On the other hand intensive variables, such as temperature, pressure or electrochemical potential, share the same value all over a system. Intuitively, for example the temperature of a portion of a system is the same as temperature of the whole system.

Classical thermodynamics goes beyond the restrictions associated to equilibrium states through considering a class of precesses which can be explained as continuum sequence of neighboring equilibrium states. Such processes are called quasi-static processes and are constructed upon slight modification in state variables of an equilibrium state at each stage. A quasi-static process can be either reversible or irreversible. A reversible process from $1 \rightarrow 2 \rightarrow 3$ is modeled as a continuum sequence of equilibrium states in an infinitesimally slow manner. In a reversing transformation i.e. from equilibrium state 3 to equilibrium state 1, state variables take the same values as in the direct transformation. It should be noted that in the reverse direction exchange of the matter and energy with environment would be of opposite sign. On the contrary, an irreversible process, which is not reversible, occurs at a finite velocity and is modeled as a discrete series of equilibrium states. In this case and during a reverse transformation, at each stage input of external energy from surroundings needs to be reset to its initial state. As a couple of examples of irreversible processes friction, shocks, flow of viscous fluids and chemical reactions can be named.

1.4 Laws of Thermodynamics

Naming the laws of thermodynamics has not happened chronologically or based on the time of recognition. For example the zeroth law was formulated latest and the first law was formulated second after the second law.

1.4.1 The Zeroth Law

The zeroth law states that if systems 1 and 2 are separately in thermal equilibrium with system 3, then they are in a thermal equilibrium with each other. This law of thermodynamics in fact defines the concept of temperature.

1.4.2 The First Law or Balance of Energy

Notions of energy and heat are introduced into thermodynamics through the first law. Considering a system with adiabatic walls which has no interaction with outside other than mechanical work W. This mechanical work can be exemplified through expansion of its volume. According to the famous result of Joule's experiments, the work can be measured by the decrease in potential energy of a slowly falling weight which is $W = mgh$ where h is displacement, g acceleration of gravity and m would be the weight of the piston. The work produced through transition from state A to state B is shown to get determined only based on initial and final states and independent of the taken path. Accordingly W can be defined as the difference between internal energy at initial and final states of A and B as $\Delta U = U(B) - U(A)$ where U is a state variable which is called internal energy. Thereby we can write

$$W = \Delta U \tag{1.4.1}$$

The above relation can be used to calculate the internal energy of a system regardless of its nature.

Now assume that the adiabatic wall is removed and the system can have exchange with the outer world. This time generally work would not be equal to the change in internal energy. The difference between the two is called Q which denotes the heat exchange between the system and environment. Accordingly one can write

$$\Delta U - W = Q \tag{1.4.2}$$

The above expression in fact represents the first law of thermodynamics which in a more familiar form can be written as

$$\Delta U = Q + W \tag{1.4.3}$$

stating that the change in internal energy would be equal to sum of the exchanged heat and work done. In differential form it is written as

$$dU = đQ + đW \tag{1.4.4}$$

where the stroke through symbol d stands for inexact differentials, meaning that they depend on the path and not only on the initial and final states.

It is worthy of mentioning that the first law is valid for both reversible and irreversible processes, also regardless of the status of the states A and B. It works either in equilibrium or in non-equilibrium conditions.The outcome of the first law is that the energy is conserved in all thermodynamic processes.

1.4.3 The Second Law

The first law by its own still lacks a clear establishment of the direction of the processes within a system. For example it does not consider the impossibility of the transfer of heat from a cold body to a warm body, nor on impossibility of a complete conversion of heat into the work. The second law in fact imposes further thermodynamical constraints or restrictions to create a framework within which the spontaneous evolution of natural processes in an specific direction is explained. The first formulations of the first law were developed by Clausius and Kelvin and were stated in form of impossibility of some processes to happen. Consequently there are two statements attributed to Clausius and Kelvin. According to Clausius it is not possible to create a system that transfers heat from a cold body to a warm body without exertion of any work done on the system by the environment. Kelvin stated that: it is not possible to create a machine working a full cycle which can take heat from a single reservoir and convert it entirely to work. Second law benefits from introduction of a new fundamental and universal (in the case of equilibrium systems) notion in thermodynamics state variables, namely entropy, and its behavior throughout a process which will be discussed in the following sections.

1.4.4 The Third Law

This law relates to the study of thermodynamic quantities where the absolute temperature approaches zero. This law was formulated by Nernst in studying chemical equilibrium. Through his studies he concluded that the entropy change in any isothermal process approaches zero as the temperature tends to zero,

$$(\Delta S)_{T \to 0} \to 0 \tag{1.4.5}$$

Though this form of the third law is sufficient for most of thermodynamical situations, sometimes stronger Planck's statement i.e. $(S \to 0$ as $T \to 0)$ is preferred.

1.5 Entropy Production and Entropy Flow

Considering the system shown in figure 1.1 in which two reservoirs at temperatures T_2 and T_1 are connected together through a thermally insulated rod, heat flows between the reservoirs along the rod. It is assumed that the temperatures of the reservoirs remain constant during the process. Due to heat conduction between reservoirs system reaches a state in which the rate of the heat flow, i.e. dQ/dt, entering the rod would be equal to rate of the heat leaving the rod, i.e. $-dQ/dt$. In this case temperature along the rod varies spatially while it remains constant in time.

Therefore the temperature of the rod would be a function of location but independent of time. System in overall would be in a stationary or steady state situation but not in equilibrium as it demands the temperature to be uniform throughout the system. Now assuming that heat flows into the left-hand side of the bar as a result of an infinitesimal difference in temperature (meaning that the process is reversible) the reservoir 2 loses entropy at the rate

$$\frac{dS_2}{dt} = -\frac{1}{T_2} \frac{dQ}{dt} \tag{1.5.1}$$
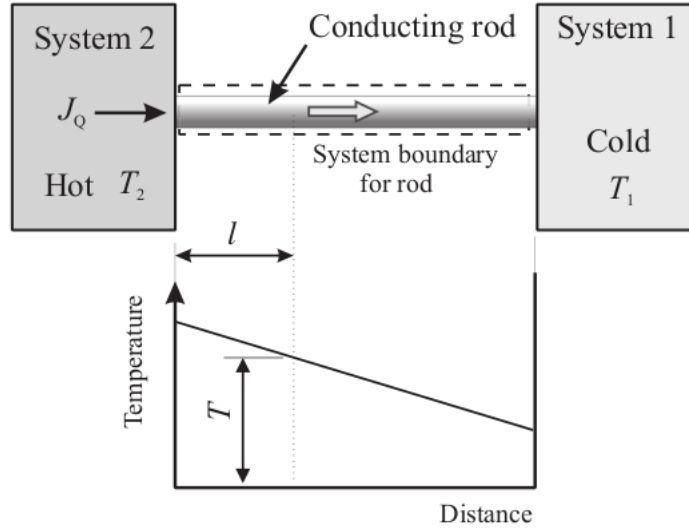
Figure 1.1: Heat Conduction along a bar in steady-state [12]

following the same line of reasoning the reservoir at the right hand side obtains entropy at the rate

$$\frac{dS_1}{dt} = \frac{1}{T_1}\frac{dQ}{dt} \qquad (1.5.2)$$

The total changes of entropy for the system thus reads as

$$\frac{dS}{dt} = \frac{dS_1}{dt} + \frac{dS_2}{dt} = \frac{dQ}{dt}\Big[\frac{1}{T_1} - \frac{1}{T_2}\Big] = \frac{dQ}{dt}\Big[\frac{T_2 - T_1}{T_2 T_1}\Big] \qquad (1.5.3)$$

For the system at hand $T_2 > T_1$ hence the rate of change of entropy would be positive $\frac{dS}{t} > 0$.

Considering a point at the bar with distance l from the left hand side, the value that a thermometer shows is T. As mentioned before this reading is time independent and is a result of reading the value at an equilibrium state of the system. So the system is in a steady state situation and entropy of each point of the bar is invariant with time. But a net transfer of entropy from left to right of the system is observed. This means that the entropy is in fact flowing along the rod. The entropy of the whole system is increasing with time which is phenomenon known as the entropy production [13].

One fundamental difference between reversible and irreversible processes is that irreversible processes are always accompanied with increase in entropy. Due to local equilibrium hypothesis, entropy is considered to be a state function in an non-equilibrium process as well [14–16]. As a result of that entropy of the system, as well as the entropy of the environment, increases. To put previous formulation in a new notation where e denotes the environment and i stands for internal interactions of a system, one can write:

$$dS = dS_e + dS_i \qquad (1.5.4)$$

Herein dS will always be positive $dS > 0$ since the process we are dealing with is an irreversible process. However dS_e could be either positive or negative depending on the

type of the boundary between the system and the environment as well as the type of the irreversible process. For example if the boundary is diathermic, due to the irreversible process the temperature of the system decreases and the system draws some heat (denoted by dQ) from the environment (which is at the constant temperature T), then:

$$dS_e = -\frac{dQ}{T}$$ (1.5.5)

and

$$dS_i = \frac{dQ}{T} + dS_e^i$$ (1.5.6)

In the above equation $\frac{dQ}{T}$ implies the increase in the entropy of the system as a result of the heat flow from the environment which is equal to $-dS_e$. The second term on the other hand implies the increase of the entropy of the system as a result of internal changes such as temperature or number of particles. Therefore we can write:

$$dS = dS_e + dS_i = dS_e^i$$ (1.5.7)

Noting that dS is positive, dS_e^i must also positive. If we assume that system expels the heat to the environment

To summarize the points around entropy it should be stated that entropy is an extensive property therefore the entropy of a system composed of several parts would be the sum of entropies of each part. It is a function of state and would be a measure of order or disorder of a system [17, 18].

Consider a continuum body of material (which can also be seen as a system) containing a total mass m and and volume V and surface represented by Γ. The rate of variation of such an extensive quantity, i.e. dS, can be written as the sum of the rate of exchange with environment d^eS/dt and the rate of entropy produced internally d^iS/dt:

$$\frac{dS}{dt} = \frac{d^eS}{dt} + \frac{d^iS}{dt}$$ (1.5.8)

The entropy crossing surface of the boundary per unit area and unit time is called entropy flux and is denoted by $\boldsymbol{J}^s$. The entropy produced per unit volume and unit time within the system is called rate of entropy production and is indicated by σ^s. Then we can write

$$\frac{d^eS}{dt} = -\int_\Gamma \boldsymbol{J}^s \cdot \mathbf{n} d\Gamma$$ (1.5.9)

$$\frac{d^iS}{dt} = \int_V \sigma^s dV$$ (1.5.10)

where $\boldsymbol{n}$ is the unit normal pointing outward.

Once specific entropy s is defined as $S = \int \rho s dV$ and using the above relationships balance of entropy reads as

$$\int \rho \frac{ds}{dt} dV = -\int_\Gamma \boldsymbol{J}^s \cdot \boldsymbol{n} d\Gamma + \int_V \sigma^s dV$$ (1.5.11)

then through Gauss and Reynolds theorems the above relation is reshaped into the following as

$$\int \rho \frac{ds}{dt} dV = -\int_V \nabla \cdot \boldsymbol{J}^s dV + \int_V \sigma^s dV \qquad (1.5.12)$$

Assuming that above expression is valid for any volume V and the integrands are continuous in terms of space then the above global equation can be converted to a local version in form of a local balance relation as

$$\rho \frac{ds}{dt} = -\nabla \cdot \boldsymbol{J}^s + \sigma^s \qquad (1.5.13)$$

and by virtue of second law we have

$$\sigma^s \geq 0 \qquad (1.5.14)$$

where the quality sign denotes the reversible processes. This is an important quantity as the product $T\sigma^s$ would be a measure of dissipation of energy in a thermodynamical system that when minimized depicts the enhanced efficiency.

1.6 Fluxes and Affinities

To establish the relationship in irreversible processes the approach would be to build a macroscopic description of irreversible processes and to determine the balance equations of local variables in the framework of fundamental laws of thermodynamics. Again as mentioned before irreversible processes are accompanied by the generation and flow of entropy that need to be considered. Additionally we are facing the flow of energy, matter or both. Thereby in irreversible processes flow of some physical quantity would be inevitable. This physical quantity is called flux or current and usually denoted by J_i. Subscript i here distinguishes different types of fluxes may exist in an irreversible process. The cause or source of the flux is defined as affinity, potential or driving force and is termed by X_i. Getting back to the heat flow example, the heat flow would be the flux and temperature gradient would be the corresponding affinity [18–20].

In some irreversible processes, especially coupled ones, it happens that two or more sources or affinities contribute to an individual flux. As an example, moving charged particles may be generated by both temperature gradient and gradient of electric potential. Fluxes that are function of many affinities are called coupled fluxes and such a dependency can be shown as

$$J_i = J_i(X_1, X_2, X_3, ...) \qquad (1.6.1)$$

1.7 Linear Irreversible Processes

At equilibrium state of a thermodynamical system there is no change in entropy or free energy. On the other hand an irreversible process is accompanied with energy dissipation and at the same time entropy is generated continuously as described before. The rate

at which energy is dissipated is the product of the temperature and the rate of entropy production (i.e. $T\sigma$) with:

$$T\sigma = JX \tag{1.7.1}$$

where J denotes a generalized flux, and X a generalized force.

The above relationship between force and flux, upon existence, suggests that flux must naturally depend on the force. Then it may be written as as a function of the force X. In irreversible processes that take place in systems that are not far from the equilibrium, affinities or forces are not too large. Therefore the fluxes J_i can be expanded using the Taylor's expansion. Omitting higher order and nonlinear terms, and keeping only the first order or linear terms, one can write

$$JX = \sum_0^\infty a_n X^n = \sum_0^\infty \frac{f^{(0)}0}{n!} X^n = J0 + J'0\frac{X}{1!} \tag{1.7.2}$$

noting that $J0 = 0$ represents the equilibrium. By neglecting high order terms we can see the linear relationship between flux and force as

$$J\alpha X \tag{1.7.3}$$

Assuming existence of multiple irreversible processes and consequently multiple fluxes, one can proceed and write

$$J_i(X_1, X_2, X_3, ...) = \left(\frac{\partial J_i}{\partial X_1}\right)_0 X_1 + \left(\frac{\partial J_i}{\partial X_2}\right)_0 X_2 + \left(\frac{\partial J_i}{\partial X_3}\right)_0 X_3.... \tag{1.7.4}$$

Or

$$J_i(X_1, X_2, X_3, ...) = \sum_{j=1}^n \left(\frac{\partial J_i}{\partial X_j}\right)_0 X_j \tag{1.7.5}$$

where $\left(\frac{\partial J_i}{\partial X_j}\right)_0$ would be the rate of change of J_i with X_j when the system tends to equilibrium state.

Concisely the above equation can be written as

$$J_i = \sum_{j=1}^n L_{ij} X_j \tag{1.7.6}$$

where

$$L_{ij} = \left(\frac{\partial J_i}{\partial X_j}\right)_0 \tag{1.7.7}$$

Relations of above type are usually called linear phenomenological relations, stemming from an approximation through which the system is assumed to be not far from equilibrium, so that only the nonlinear terms in Taylor's expansion can be neglected and in other

words only the linear terms are influencing the thermodynamics of a system. With those considerations the above relation can be a representative of an irreversible process [21].

Since the nonlinear terms are omitted in these equations, irreversible processes that can be explained through these equations are usually called linear irreversible processes. However, if the process is far from the equilibrium state, then the nonlinear terms in the related Taylor's expansion could not be neglected anymore. This originates from the fact that the affinities and fluxes are large so that all terms in Taylor's expansion are influencing. These processes are called nonlinear irreversible processes. Onsager developed the theory for studying the linear irreversible processes which is outlined in following sections.

1.8 Onsager's Reciprocal Relations

Thermodynamics of irreversible processes formulated throughout works of Onsager, Prigogine, Mexiner, de Groot, Mazur and others investigates small deviations from equilibrium state in open system. Although the term thermodynamics is used for this branch of science, it is also suggested to consider it as a branch of kinetics as the concept of rate is introduced in this theory through which the rate of irreversible processes are explained.

In his pioneering works [3, 4] Onsager, based on principle of microscopic reversibility, develops so called Reciprocal Relations which enable describing the situation of coupled phenomena in which two or more irreversible transport processes such as heat conduction, electric conduction or mass diffusion take place simultaneously. The challenge here is that in such a situation fluxes interfere with each other and a specific flux would affect all other components of the system. Before Onsager such a problem was addressed by W. Thompson through a thermoelectric phenomenon. Consider J_1 as the electric current and J_2 as the heat flow. The driving force for the current would be electromotive force which is shown by X_1. Correspondingly the driving force for the heat flow would be:

$$X_2 = -\frac{1}{T} grad T$$

where T shows the temperature. So assuming that the fluxes of heat and current are completely independent, then one can write:

$$X_1 = R_1 J_1$$
$$X_2 = R_2 J_2$$

where R_1 denotes the electrical resistance and R_2 stands for the heat resistance. However this assumption is not correct as these two phenomena are coupled and interfere with each other. In other words they are affecting each other, as a result of which each force incorporates contribution from both fluxes. Thereby the related phenomenological relations read as:

$$X_1 = R_{11} J_1 + R_{12} J_2$$
$$X_2 = R_{21} J_1 + R_{22} J_2$$

Onsager devised a reciprocal relationship more rigorously with the assumption of microscopic reversibility along with certain theorems from fluctuation theory. He considers

a situation in which energy dissipation is not only the result of difference in temperature on side of a medium, but also electrical conductivity plays a role in energy dissipation.

Onsager used linear phenomenological relations to formulate coupled irreversible processes encompassing n-fluxes and n-affinities based on phenomenological coefficients L_{ij} as,

$$J_1 = L_{11}X_1 + L_{12}X_2 + L_{13}X_3 + ...L_{1n}X_n$$
$$J_2 = L_{21}X_1 + L_{22}X_2 + L_{23}X_3 + ...L_{2n}X_n$$
$$J_3 = L_{31}X_1 + L_{32}X_2 + L_{33}X_3 + ...L_{3n}X_n$$
$$..$$
$$..$$
$$J_n = L_{n1}X_1 + L_{n2}X_2 + L_{n3}X_3 + ...L_{nn}X_n$$

(1.8.1)

In the above algebraic relations phenomenological coefficients L_{ij} represent properties such as generalized mobility or conductance.

On the other hand forces or affinities X_i which drive the fluxes can be written in terms of fluxes as

$$X_i = \sum_{j=1}^{n} R_{ij}J_j, i = 1, 2, 3, ...n$$

(1.8.2)

In contrary to phenomenological coefficients, coefficients of type R_{ij} represent the properties of generalized resistance or friction. It should be pointed out that coefficients with repeating indexes such as $L_{11}, L_{22}, L_{33}...L_{nn}$ also $R_{11}, R_{22}, R_{33}...R_{nn}$ are related to phenomena of the same type of fluxes and affinities, so they are usually called straight coefficients. On the other hand coefficients with different indexes such as L_{ij} and $R_{ij}; i \neq j$ represent interactions between fluxes and affinities of different type, hence they are called cross coefficients. Characterizing these coefficients leads to formulate the complete description for fluxes and their driving forces within a thermodynamic systems with irreversible processes.

Knowledge of either L_{ij} or R_{ij} would provide sufficient insight for characterizing an irreversible process. Thereby they are related to each other. Considering a simple case of two fluxes and two affinities as:

$$J_1 = L_{11}X_1 + L_{12}X_2 \quad \text{and} \quad X_1 = R_{11}J_1 + R_12J_2$$ (1.8.3)
$$J_2 = L_{21}X_1 + L_{22}X_2 \quad \text{and} \quad X_2 = R_{21}J_1 + R_{22}J_2$$ (1.8.4)

and solving above relation we obtain

$$L_{11} = R_{22}[|L_{11}L_{22} - L_{12}L_{21}|];$$
$$L_{22} = R_{11}[|L_{11}L_{22} - L_{12}L_{21}|]$$

(1.8.5)

and

$$L_{12} = -R_{12}[|L_{11}L_{22} - L_{12}L_{21}|];$$
$$L_{21} = -R_{21}[|L_{11}L_{22} - L_{12}L_{21}|]$$

(1.8.6)

It can be seen that in the case of simple cases such as when there exist only two fluxes, it will be resulting in four linear coefficients that need to be experimentally determined. However in general cases thermodynamic phenomena involve large number of fluxes and affinities, as a result of which number of coefficients that need to be determined increases significantly.

Onsager developed a formulation through which the number of linear coefficients were reduced significantly. He however imposed some restrictions to construct such a framework. According to his theorem, upon careful and proper selection of fluxes and affinities, the cross coefficients show to be symmetric, meaning

$$L_{21} = L_{12} \quad \text{and} \quad R_{21} = R_{12} \tag{1.8.7}$$

or in a more general representation

$$L_{ij} = L_{ji} \quad \text{and} \quad R_{ij} = Rji \tag{1.8.8}$$

The above relations demonstrate Onsager's reciprocal relations of linear coefficients.

The proper selection of currents and affinities that obey the Onsager's relations are identified to satisfy the following relation in that the rate of generation of internal entropy can be explained as a linear combination of fluxes and forces as:

$$\sigma = \frac{(dS)^{int}_{sys}}{dt} = \sum_{i=1}^{n} J_i X_i \tag{1.8.9}$$

Rate of generation of the internal energy follows the Gibb's relation as:

$$Tds = du + pdv - \sum_i \mu_i c_i \tag{1.8.10}$$

where μ_i and c_i respectively denote chemical potential and concentration of the i component. The above Gibb's relation is valid for any system which is in equilibrium. However Onsager employed that in the case of linear irreversible processes. Through the perturbation theory it was illustrated that Gibb's relation is still applicable to systems that are not far away from equilibrium. It was also later shown that the criteria for being displaced from the equilibrium and keeping the Gibb's validity is more relaxed compared to a limit within which the linearity of Taylor's expansion is satisfied for irreversible processes. As a result of these investigations, employment of Gibb's relation for linear irreversible processes was fully justified [9, 22].

After a careful selection of currents (fluxes) and driving forces (affinities), it is necessary to experimentally determine the phenomenological coefficients, i.e. L_{ij} or R_{ij}. More often at the initial steps of the process these coefficients vary in the course of time as the flux evolves. Thereby measurements are not possible at this stage. However once the system has reached steady state, net flux at a particular point tends to zero, meaning that as much flux enters the point as leaves it. This state is invariant with respect to time. It should be noted that the time invariant or steady state situation differs from equilibrium state as in the equilibrium energy is minimum and entropy is maximum and it does not change with time. Also as mentioned before there is no flux and affinities that disturb the system in equilibrium conditions. On the contrary in time invariant steady state entropy at a constant rate is generated which leads to energy dissipation. In steady state, entropy of the system can be either minimum or maximum. In fact steady state is a generalization of the equilibrium state [23–25].

1.9 Matrix Representation of Coupled Linear Relations

Following the tradition of linear algebra in representing a system of equations in matrix form, one can write force and flux relations in a compact form as:

$$\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad (1.9.1)$$

and

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \qquad (1.9.2)$$

In cases of number of fluxes and forces exceeding two, matrices and vectors can be expanded as following

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ . \\ . \\ . \\ J_n \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & ... & L_{1n} \\ L_{11} & L_{12} & L_{13} & ... & L_{1n} \\ L_{11} & L_{12} & L_{13} & ... & L_{1n} \\ L_{11} & L_{12} & L_{13} & ... & L_{1n} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ L_{n1} & L_{n2} & L_{n3} & ... & L_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ . \\ . \\ . \\ X_n \end{bmatrix} \qquad (1.9.3)$$

correspondingly equations can be written as

$$J_1 = L_{11}X_1 + L_{12}X_2 + L_{13}X_3 + ...L_{1n}X_n$$
$$J_2 = L_{21}X_1 + L_{22}X_2 + L_{23}X_3 + ...L_{2n}X_n$$
$$J_3 = L_{31}X_1 + L_{32}X_2 + L_{33}X_3 + ...L_{3n}X_n$$
$$..$$
$$..$$
$$J_n = L_{n1}X_1 + L_{n2}X_2 + L_{n3}X_3 + ...L_{nn}X_n$$

$$(1.9.4)$$

1.10 Application of Onsager's Method to Thermoelectricity

Thermoelectricity is applied to a configuration in which two metallic wires with two different temperatures produce the electric potential as a result of such a difference in temperature. This phenomenon was discovered at 1821 by John Seebeck where he recorded the coupling between two potentials (affinities) i.e. electrochemical potential and temperature gradient. Thereafter, Jean Peltier observed that fluxes of heat and electric current also could be coupled. It was Lars Onsager that in 1931 formulated these coupled thermodynamic forces and fluxes based on principles of linear irreversible thermodynamics. Thermoelectricity is known as a typical example for illustrating application of Onsager's reciprocal theory in coupled phenomena [23].

Movement of electrons within a piece of metal can be mimicked as a perfect gas but containing charged fermions namely electrons. Based on this analogy, electrochemical

potential μ_e can be viewed as partial pressure. Accordingly electrochemical potential can be given as

$$\mu_e = qV + \mu_c \tag{1.10.1}$$

where q denotes the charge carried by particle (herein electron), V would be the electric potential and as mentioned μ_c would be electrochemical potential. Electrochemical potential is defined as the energy needed to embed a carrier particle (electron) into a thermodynamic system.
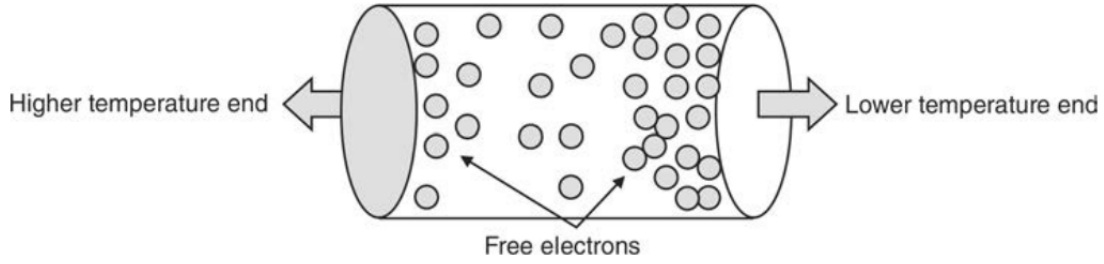
Figure 1.2: Distribution of free electrons within a conductor with
temperature gradient [26]

As can be seen in Figure 1.2, density of electrons at the side with lower temperature is more than the other side with higher temperature. Accumulation of electrons leads to the increase in electric potential difference i.e. voltage at the sides of the conductor. Clearly lower temperature end exhibits higher potential due to the presence of a larger number of electrons compared to the other side. Generated electric field makes electrons move from lower temperature end to the higher temperature. On the other hand, considering the effect of temperature, electrons tend to move from the higher temperature end to the lower temperature end as a result of excess energy they have. Such a movement creates a flux of electrons on opposite direction. In steady state situations flux of electrons from right to left as a result of electric potential is balanced by the flux due to temperature difference. It can be observed here that temperature difference also contributes into creating electric potential alongside the conductor. Thereby both electron flux and heat flux, as well as both temperature difference and electrochemical potential, are coupled together.This situation where both fluxes as well as their corresponding affinities are coupled together provides an excellent example for observing the effect of coupled phenomena in linear irreversible thermodynamics and to demonstrate the application of Onsager's theorem on describing such a phenomenon mathematically. Thermodynamical state variables in this example include temperature T and electrochemical potential μ_c. The system is non-equilibrium in steady state, however as explained before in a justification made by Onsager and Prigogine the Gibb's relation can be applied to such a system in steady state. Therefore one can write

$$G = U - TS - N\mu_e \tag{1.10.2}$$

or

$$dG = dU - TdS - SdT - Nd\mu_e - \mu_e dN \tag{1.10.3}$$

Considering δQ and δW respectively as changes in heat and the work done by the system, then we have

$$\delta Q = TdS \text{ and } \delta W = \mu_e dN \qquad (1.10.4)$$

where U denotes the internal energy and N stands for the number of free electrons (carriers) in the thermodynamical system. Changes in internal energy can be represented as

$$dU = \delta Q + \delta W = TdS + \mu_e dN \qquad (1.10.5)$$

through simple substitution we obtain

$$dG = -SdT - Nd\mu_e \qquad (1.10.6)$$

also we have

$$S = -\left(\frac{\partial G}{\partial T}\right)_{\mu_e} \qquad (1.10.7)$$

$$N = -\left(\frac{\partial G}{\partial \mu_e}\right)_T \qquad (1.10.8)$$

According to equation (1.10.4) and defining $\mathbf{J}_N, \mathbf{J}_s, \mathbf{J}_Q$ and $\mathbf{J}_e$ respectively as particle flux, entropy flux, heat flux and electrical flux (also known as electric current density) then on can write

$$\mathbf{J}_Q = T\mathbf{J}_s \quad \text{and} \quad \mathbf{J}_e = q\mathbf{J}_n \qquad (1.10.9)$$

where q denotes the electric charge of charge carrying particles which within a metal with a charge carrier as electrons it would be $-e$.

Driving forces or affinities for aforementioned fluxes respectively include electrochemical potential $\Delta\mu_e$, temperature difference ΔT. By considering $\mathbf{E}$ as the intensity of electric field produced as a result of gradient of electrochemical potential, the force experienced by carrier particles of charge q would be

$$\mathbf{F}_e = q\mathbf{E} \qquad (1.10.10)$$

Accordingly the force or affinity resulted from gradient of temperature may be expressed as

$$\mathbf{F}_T = -\nabla\mathbf{T} \qquad (1.10.11)$$

Now employing Onsager results one can express the coupled relations for particle flux and entropy flux as

$$\begin{bmatrix} J_N \\ J_S \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} F_e \\ F_T \end{bmatrix} \qquad (1.10.12)$$

or in another form we have

$$\begin{bmatrix} \frac{J_e}{q} \\ \frac{J_Q}{T} \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} q\mathbf{E} \\ -\nabla\mathbf{T} \end{bmatrix} \qquad (1.10.13)$$

which can be read as

$$J_e = q^2 L_{11}\mathbf{E} + qL_{12}(-\nabla\mathbf{T}) \tag{1.10.14}$$

and

$$J_Q = qTL_{21}\mathbf{E} + TL_{22}(-\nabla\mathbf{T}) \tag{1.10.15}$$

In steady state particle or electric flux will be zero, thus

$$q^2 L_{11}\mathbf{E} = qL_{12}(\nabla\mathbf{T}) \tag{1.10.16}$$

In steady state, i.e. when there is no particle or electric flux, the ratio of electric field to the gradient of temperature $\frac{E}{\nabla T}$ is called Seebeck coefficient and is shown by α. Then one car write

$$\alpha = (\frac{E}{\nabla T})_{noparticle/electricflux} = \frac{L_{12}}{qL_{11}} \tag{1.10.17}$$

Terms such as $qL_{12}(\nabla\mathbf{T})$ and $TL_{22}(-\nabla\mathbf{T})$ are generated as a result of externally imposed temperature differences. In absence of externally imposed temperature gradient, if an external electrostatic potential difference or electric field is applied at the ends of the conductor, an electric flux i.e. J_e will be generated which flows through the conductor. Under these conditions temperature difference between the ends of the conductor will be produced which is known as Peltier effect. The ratio of the electric field to the heat flux under the condition of $(-\nabla\mathbf{T}) = 0$ is called Peltier coefficient which is represented by Π.

Therefore

$$\Pi = (\frac{J_Q}{J_e})_{(\nabla T=0)} = T\frac{L_{12}}{qL_{11}}, \tag{1.10.18}$$

and

$$\Pi = T\frac{L_{12}}{qL_{11}} = T\alpha. \tag{1.10.19}$$

Experimental observations can demonstrate the reliability of the formalism of linear irreversible processes developed by Onsager.

1.11   Extended Irreversible Thermodynamics (EIT)

Classical Irreversible Thermodynamics (CIT) is unable to describe phenomena and situations such as materials with memory, non-Newtonian fluids, or the processes that take place far from the equilibrium or at high frequencies and short wavelengths. These are restrictions arising inherently in the frame of local equilibrium thermodynamics. CIT postulates that local relations between thermal and mechanical properties are the same as those applicable to a uniform system at equilibrium. However there are lots of other processes that happen far away from equilibrium state. As a result of that CIT and its postulations in linear regime would not be valid any more. To cope with such situations and to go beyond the limitations associated with CIT, the field of irreversible thermodynamics has been developed further into a new branch called Extended Irreversible Thermodynamics (EIT) [27, 28] which can be employed in phenomena which can not accurately be described through linear irreversible thermodynamics.

1.12 Zwanzig's Generalization

In irreversible thermodynamics there exists the following relation as

$$d\alpha_j/dt = \sum_k L_{jk} F_k(\alpha_1, \alpha_2, ..., \alpha_n) \qquad (1.12.1)$$

between deviation of jth state variable and thermodynamic force. Thermodynamic force is defined as derivative of entropy with respect to α.

In Onsager's theory it is assumed that the system responses to the applied force immediately.This assumption works well in variety of situations such as Navier-Stokes equation in fluids or Fick's law of diffusion. However there exist situations in which a rather accurate description of the system requires causality to be taken into account.As an example memory effects in viscoelastic materials is famous.

In experiments, such as measuring the relaxation of a stretched polymer, incorporation of all influencing phenomena is not possible. For example a rather full description of the state includes a vast majority of molecular variables. Thereby in such practical situations one goes for measuring only a few of these and proceeds to conclusion ignoring others as if they did not exist. This demonstrates the significance of causal responses existing in various engineering applications.

As stated, Onsager's derivation relies on the assumption of instantaneous responses. To cope with such a limitation in Onsager relations, Zwanzig developed a causal theory in the following form:

$$d\alpha_j(t)/dt = \sum_k \int_0^t ds K_{jk}(s) F_k(\alpha_1(t-s), ..., \alpha_n(t-s)) \qquad (1.12.2)$$

Transport coefficients are related to memory functions $K_{jk}(s)$ as

$$L_{jk} = \int_{-0}^{\tau} ds K_{jk}(s) \qquad (1.12.3)$$

Zwanzig's theory has been implemented in the current study as a generalization to Onsager's theory.

1.12.1 Outline of the method

Macroscopic state variables are translated into molecular variables through phase functions which represent a large number of molecules. There is a characteristic associated to these phase functions as they vary slowly in time relatively to the rates of individual molecular processes.

It is assumed that system under consideration is initially in thermal equilibrium. Also it is assumed that the system is free of external time-dependent forces or boundary conditions.

Firstly we need to decide on a set of variables which are supposed to describe the macroscopic state of the system. These variables would be in fact the phase functions $A_1(x), A_2(x), A_n(x)$ with x denoting the position of the system in phase space.

We consider certain macroscopic characteristics for our phase functions. They are parametrized by position and velocities of large number of molecules with small time derivatives relative to individual molecular quantities. Next the system is set up in a way that is always in thermal equilibrium. This brings certain constraints so that the phase functions $A(x)$ keep maintaining certain values. At the next step constraints are stripped allowing the system to relax. There after the probability distribution of the $A(x)$ is computed as a function of time.

Molecular Picture

Molecularly the state of the system is resolved through the position x. Point wisely the system moves from its initial position x_0 to x_t at time t. Correspondingly the phase functions change form $A(x_0)$ to $A(x_t)$.

The set containing $a_1, a_2, ...a_n$ is denoted by a. A surface in phase space is defined with a set of equations as $A(x) = a$, where x represents a point on this surface. Any point x on the surface $S(a)$ yields to the same set of numbers a. Initial values are defined as $a_0 = A(x_0)$.

Full statistical resolution of the outcomes of repeated experiments is addressed through the probability distribution $g(a; t)$ in a space. a space is a space of n dimensions . Each point in this space is labeled through $a_1, a_2, ..., a_n$. Accordingly the probability at time t reads as

$$g(a; t)da = g(a_1, a_2, ..., a_n; t)da_1 da_2 ... da_n \tag{1.12.4}$$

The initial value of the a-space distribution reads as

$$g(a; 0) = \delta(a - a_0) = \prod_{j=1}^{n} \delta(a_j - a_{j0}) \tag{1.12.5}$$

where a_0 denotes the set of initial values as assigned. In other words, $g(a; t)$ represents the probability of transition from initial surface $S(a_0)$ to some arbitrary surface $S(a)$ in time t.

1.12.2 Derivation of kinetic equation

Phase space is a space in which all possible states of a system are resolved where each state is related to one specific point in the phase space.

Equations describing molecular motion are written in the form of Liouville's equation. It is known that the probability distribution function in phase space $f(x; t)$ satisfies the equation

$$i\partial f(x; t)/\partial t = \mathcal{L}f(x; t) \tag{1.12.6}$$

where $\mathcal{L}$ would be the Liouville operator defined in terms of Hamiltonian function $H(x)$ and the Poisson bracket $,_{P,B.}$ as

$$\mathcal{L}f = iH, f_{P.B} \tag{1.12.7}$$

The relation for the probability distribution in a space and in phase space would be

$$g(a;t) = \int dx \delta(A(x) - a) f(x;t) \tag{1.12.8}$$

where replacing the $f(x;t)$ with Liouville operator yields the following relation for distribution function as

$$g(a;t) = \int dx \delta(A(x) - a) e^{-itL} \delta(A(x) - a_0)/W(a_0) \tag{1.12.9}$$

where $W(a)$ is the structure function of the surface and is defined as

$$W(a) = \int dx \delta(A(x) - a) \tag{1.12.10}$$

Projection operators

The subspace of functions $A(x)$ is described through a projection operator P. The projection operator is defined as

$$G_1(x) = PG(x) = \frac{\int dx' \delta(A(x') - A(x)) G(x')}{2} \tag{1.12.11}$$

Kinetic equation for f_1

Time dependency of the projected part of a phase space distribution function is derived by separating the distribution function into relevant and irrelevant parts through the projection operator. The resulting equation represents a memory effect or causality which reads as

$$i\frac{\partial f_1(t)}{\partial t} = PLf_1(t) - i\int_0^t ds PLe^{-is(1-P)L}(1 - P)Lf_1(t - s) + PL^{-it(1-P)L}f_2(0) \tag{1.12.12}$$

1.12.3 An exact form for $g(a;t)$

The exact relation for $g(a;t)$ includes

$$\frac{\partial g(a;t)}{\partial t} + \sum_{j=1}^{n} \frac{\partial}{\partial a_j} v_j(a) g(a;t) = \int_0^t ds \int d' \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial a_j} W(a) K_{jk}(a, a'; s) \times \frac{\partial}{\partial a'} \frac{g(a'; t - s)}{W(a')} \tag{1.12.13}$$

where

$$v_j(a) = \langle dA_j/dt; a \rangle \tag{1.12.14}$$

and

$$K_{jk}(a, a'; s) = \langle \frac{dA_j}{dt} e^{-is(1-P)L}(1 - P)\frac{dA_k}{dt} \delta(A(x) - a'); a \rangle \tag{1.12.15}$$

The variable $v_j(a)$ is the average rate of change of A_j on the surface $S(a)$. Subsequently $K_{jk}(a, a'; s)$ would be the memory function which would be a correlation function in time and in a space.

1.12.4 An approximation to Kinetic equation

The exact equation for $g(a;t)$ as introduced before is too complicated to be employed in practical situations. However it is possible to derive an approximation for such a function based on macroscopical properties of phase functions in which they vary very slowly with time. One can keep dA/dt terms in the exact form only up to the first and second powers and ignoring the rest. Noting that the term $v_j(a)$ if of the first order in dA/dt and exhibit no higher order contributions, it will be kept as is in the exact form. The term $K_{jk}(a, a'; s)$ is of the second order though exhibits higher order contributions in dA/dt in implicit manner. Hence those contributions should be disposed.

Such a treatment yields a memory function which is diagonal to terms of the second order in a space as

$$K_{jk}(a, a'; s) = \delta(a - a')K_{jk}(a; s) + O(\dot{A}^3) \tag{1.12.16}$$

The coefficient $K_{jk}(a, s)$ is a time-correlation function in deviation of time derivatives from equilibrium

$$K_{jk}(a; s) = \langle [\dot{A}_j(s) - v_j(a)][\dot{A}_k(0) - v_k(a)]; a \rangle \tag{1.12.17}$$

where

$$\dot{A}_j(s) = e^{isL}\dot{A}_j(0) \tag{1.12.18}$$

would be defined as the value of dA/dt at time s where the operator $exp[-is(1 - P)L]$ is no further required.
This leads to the following approximate relation

$$\frac{\partial g(a;t)}{\partial t} + \sum \frac{\partial}{\partial a_j}v_j(a)g(a;t)$$

$$= \int_0^t ds \sum_{j=1} \sum_{k=1} \frac{\partial}{\partial a_j}\left\{ K_{jk}(a; s)W(a)\frac{\partial}{\partial a_k}\frac{g(a;t-s)}{W(a)} \right\} + O(\dot{A}^3 g)$$

1.12.5 Derivation of transport equation

The above obtained relation for $g(a;t)$ is too detailed and might not be useful in case of most practical situations. Transport equations in statistical sense describe the average of observed behavior in experiments. Observations at macroscopic level need to be reproducible meaning that no fluctuations in average behavior should be observed.
The quantity $\alpha_j(t)$ defines the ensemble average of the phase function at time t as

$$\alpha_j(t) = \int da a_j g(a;t) \tag{1.12.19}$$

The time derivative of α_j can be calculated from the above equation. Assuming a sharp macroscopic characteristic for the a-space distribution, ensemble average of functions can be replaced by function of ensemble averages. Thereby with the following definition for thermodynamic force as

$$F_k(a) = \frac{\partial}{\partial a_k}logW(a) \tag{1.12.20}$$

the transport equation would read as

$$\frac{d\alpha_j(t)}{dt} = v_j(\alpha(t)) + \int_0^t ds \sum_l \left\{ K_{jl}(\alpha(t-s);s)F_l(\alpha(t-s)) + \frac{\partial}{\partial\alpha_l}K_{jl}(\alpha(t-s);s) \right\}$$

(1.12.21)

To obtain simple relations comparable to Onsager relations it is assumed that phase functions $A(x)$ are chosen so that $v_j(a)$ is vanished and memory functions only depend on those coefficients that are constants of the motion. As a result of these assumption the following final form could be obtained as

$$\frac{d\alpha_j(t)}{dt} = \int_0^t ds \sum_k K_{jk}(s)F_k(\alpha(t-s))$$

(1.12.22)

with memory functions obeying the reciprocal relations as

$$K_{jk}(a;s) = K_{kj}(a;s)$$

(1.12.23)

A   A summary of principles of classical irreversible thermodynamics

Due to their rare clarity, the present exposition is taken from those of Biot [18] and Eringen [21]. Suppose we have a thermodynamic system with variable temperature T (absolute temperature) which, in the initial state of equilibrium, is at the uniform temperature T_0 of the surrounding environment. Often, from an engineering point of view, the surrounding environment at a fixed temperature is the atmosphere. We consider the whole of our system plus the surrounding environment as a single isolated system. Suppose the surrounding environment is large enough to maintain own temperature to the constant value T_0. Suppose that n external forces act on the system, which can be of various nature, p_i, $i = 1, ..., n$, where $p_1 = T_0$. Suppose the system has n degrees of freedom defined by n state variables Q_i, $i = 1, ..., n$ where we assume $Q_1 \equiv S$ being S the entropy of the system in question. Suppose that the state variables have been chosen so that $Q_i = 0$ in correspondence to the equilibrium state and are such that

$$\delta W = p_i \mathrm{d}Q_i \tag{A.1}$$

where δW is the work done on the system by the external forces p_i. Therefore $p_1 Q_1 = T_0 S$. The total entropy variation $\mathrm{d}\Sigma$ is the sum of the entropy variation $\mathrm{d}S$ of the system and the surrounding environment $\mathrm{d}S_0$:

$$\mathrm{d}\Sigma = \mathrm{d}S_0 + \mathrm{d}S \tag{A.2}$$

By the second law of thermodynamics and taking into account that the system is isolated we have

$$\mathrm{d}\Sigma \geq 0 \tag{A.3}$$

The first law of thermodynamics tells us that

$$T_0 \mathrm{d}S_0 = -\delta Q = -(\mathrm{d}U - \delta W) \tag{A.4}$$

where δQ is the heat variation of the system. Consider the Gibbs equation of thermostatics

$$T \mathrm{d}S = \mathrm{d}U - p_i^R \mathrm{d}Q_i \qquad i = 2, ..., n \tag{A.5}$$

where p_i^R are the reversible parts of the forces p_i

$$p_i = p_i^R + X_i \qquad i = 2, ..., n \tag{A.6}$$

and X_i are the irreversible parts, also called Onsager forces. Prigogine [5] using the methods of statistical mechanics has shown that the Gibbs equation is valid also for systems not far from equilibrium. Taking $p_1^R = T$ and $X_1 = T_0 - T$ we have

$$\mathrm{d}U = p_i^R \mathrm{d}Q_i \qquad i = 1, ..., n \tag{A.7}$$

$$T_0 \mathrm{d}\Sigma = X_i \mathrm{d}Q_i \qquad i = 1, ..., n \tag{A.8}$$

where $U = U(Q_1, ..., Q_n)$ is an equation of state. From (2.7) and (2.8) we also obtain

$$p_i^R = \frac{\partial U}{\partial Q_i} \tag{A.9}$$

$$X_i = T_0 \frac{\partial \Sigma}{\partial Q_i} \tag{A.10}$$

expressions defining the purely reversible forces p_i^R and the Onsager forces X_i. Turns out

$$T_0 \frac{\mathrm{d}\Sigma}{\mathrm{d}t} = X_i \dot{Q}_i = X_i J_i \tag{A.11}$$

where $J_i \equiv \dot{Q}_i$ are called flows, while $\frac{\mathrm{d}\Sigma}{\mathrm{d}t}$ is called entropy production. The Onsager principle of reciprocity states that the Onsager forces are linear functions of the flows with the matrix of symmetric coefficients

$$X_i = b_{ij} J_j \tag{A.12}$$
$$b_{ij} = b_{ji}, \quad b_{ij} \geq 0 \tag{A.13}$$

It is usual to use the thermodynamics densities which are defined as the thermodynamic variables per unit mass of the system under examination. For linear systems it is also possible to define thermodynamic variables per unit of volume:

$$\chi = \frac{\Sigma}{V} \tag{A.14}$$

$$s = \frac{S}{V} \tag{A.15}$$

$$u = \frac{U}{V} \tag{A.16}$$

$$q_i = \frac{Q_i}{V} \tag{A.17}$$

where V is the volume of the system. The principles of thermodynamics with reference to the closed system formed by the system in question and by the external environment can be summarized as follows

$$\frac{\delta Q}{M} = \mathrm{d}u - p_i \mathrm{d}q_i \tag{A.18}$$

$$\dot{\chi} \geq 0 \tag{A.19}$$

$$\mathrm{d}u = p_i^R \mathrm{d}q_i \tag{A.20}$$

$$X_i = T_0 \frac{\partial \chi}{\partial q_i} = b_{ij} \dot{q}_j = p_i - p_i^R \tag{A.21}$$

$$b_{ij} = b_{ji}, \quad b_{ij} \geq 0 \tag{A.22}$$

which can be summarized as

$$T_0 \dot{\sigma} = X_i \dot{q}_i = b_{ij} \dot{q}_i \dot{q}_j = b_{ij}^{-1} X_i X_j. \tag{A.23}$$

Chapter 2

Gradient Elasticity

2.1 Introduction

Gradient theories may be viewed as a subcategory of generalized continuum mechanics, developed in response to some experimental observations which can not be described through conventional mathematical, numerical and computational tools available in the context of classical continuum theories. With technological advancements in structural studies of materials at micro and nano levels, phenomena are observed that look to be highly influenced by size effects stemming from the presence of elements, such as inhomogeneities, with length-scales considerably shorter than the matrix they reside in. Incorporating such effects with desired accuracy is not feasible through conventional methods in mechanics.

For example, it is observed that stiffness of a specimen depends on the size of the sample prepared for the testing [29]. In other words, material response exhibits dependence on a new intrinsic length scale that can not be captured through Cauchy continuum theory. Or in the case of nano-sized high performance structures such as nano-wires, nano-films and nano-tubes, where the surface to volume ratio is not negligible anymore, behavior and effects of the surface should be taken into account [30]. Such a phenomenon is called capillary effect and according to Mindlin [31], a second gradient of strain is needed to fully characterize its kinematics.

So the main goal of gradient theories is to capture the gradient effects and incorporate them into the material model [32]. They have been successfully used to address micro-/nano-structural inhomogeneities and they encompass the usage of tensors with ranks higher than two.

Gradient methods could be divided into two theories, namely strain gradient theories and stress gradient theories; in both of which, strain energy function depends on the strain. They lead to boundary value problems and could be used for numerical implementations [33] which are discussed in the following sections.

2.2 Gradient Models

Incorporating length scale effects in more recent models in elasticity or plasticity, for example, occurs through the introduction of higher gradients of strain (or constitutive

variables in general).

Historically, however, the origin of gradient theories goes back to works by Cauchy [34, 35] in which the idea of application of higher order spatial derivatives was suggested for enhanced estimation of the response of discrete lattice models where the size of the elementary volume appeared as an additional constitutive parameter. Thereafter, in 1887, molecular rotations, alongside molecular displacement, were introduced by Voigt [36, 37]. The remarkable advancement in the generalized continuum theories happened through the work of Cosserat brothers [38]in which kinematics of particles were enriched by incorporating both translational and rotational degrees of freedom.

In 1960s, and due to increased attention in the community, a second major revival towards development of gradient theories occurred and a number of landmark works were accomplished, among which studies done by Aero and Kuvshinski [39], Pal'mov [40], Kunin [41] as well as Toupin [42, 43], Mindlin and Tiersten [44], Mindlin [31, 45, 46], Mindlin and Eshel [47], Kroner [48, 49] and Green and Rivlin [50, 51] can be named. In particular the work done by Germain [52] should be pointed out where method of virtual powers was employed to address the general equilibrium equation of first strain gradient for materials with a microstructure equipped with inherent extra degrees of freedom. The field was further developed in modern times through the prominent work of Eringen [53] in which he reformulated his original integral-type constitutive relations into a differential type one.

However, and as a drawback, these models were very complex as they were including many parameters. Aifantis and co-workers [54–56] contributed to the field by developing models in which the number of higher-order terms were significantly reduced compared to its preceding similar works. This reduction led to reduced number of constants in constitutive relations.

2.3 Strain Gradient Theories

There exists various formats of gradient theories encompassing microstructural and size-effects. For example they could be viewed as mono or multi-scale supporting formulations; or based on the origin of extra variables in the model (due to the presence of micro/nano structures) other theories such as Cosserat-type theories, couple stress methods or micropolar theories can be considered. However herein we focus on linear elasticity gradient models and mention landmark works in this domain.

2.3.1 Mindlin's 1964 Theory

In Mindlin's theory [45], micro and macro scale effects kinematically are separated and addressed. Consequently, kinetic energy density, as well as the deformation energy density are formulated based on parameters originating from both scales. For the kinetic energy $\mathcal{K}$ one can write:

$$\mathcal{K} = \frac{1}{2}\rho \dot{u}_i \dot{u}_i + \frac{1}{2}\rho \ell_1^2 \dot{\psi}_{ij} \dot{\psi}_{ij}, \tag{2.3.1}$$

and the deformation energy density can be expressed as:

$$\mathcal{U} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}B_{ijkl}\gamma_{ij}\gamma_{kl} + \frac{1}{2}A_{ijklmn}\kappa_{ijk}\kappa_{lmn}$$
$$+ D_{ijklm}\gamma_{ij}\kappa_{klm} + F_{ijklm}\kappa_{ijk}\varepsilon_{lm} + G_{ijkl}\gamma_{ij}\varepsilon_{kl} \tag{2.3.2}$$

where $u_i, \varepsilon_{ij}, \psi_{ij}, \gamma_{ij}$ and κ_{ijk} are macroscopic displacement, macroscopic strain, macroscopic deformation, relative deformation (between micro and macro displacements) and the gradient of the microscopic deformation, respectively. Additionally, ρ would be mass density, and ℓ_1 is a parameter representing the size of the unit cell of the microstructure. In this regard macroscopic strain is defined as usual as:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and for the relative deformation one can write:

$$\gamma_{ij} = u_{j,i} - \psi_{ij}.$$

Furthermore, for the gradient of the microscopic deformation we can write as following:

$$\kappa_{ijk} = \psi_{jk,i}$$

The constitutive tensors $C_{ijkl}, B_{ijkl}, A_{ijklmn}, D_{ijklm}, G_{ijkl}$ represent 1765 coefficients, among which 903 are independent coefficients. As can be seen it is quite a number of parameters to be deal with in calculations. In the case of isotropic materials they are reduced to 18, yet still it would be a formidable task. In this case, the relation for the deformation energy density can be expressed as:

$$\begin{aligned}
\mathcal{U} = &\frac{1}{2}\lambda\varepsilon_{ij}\varepsilon_{ij} + \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}b_1\gamma_{ii}\gamma_{jj} + \frac{1}{2}b_2\gamma_{ij}\gamma_{ij} + \frac{1}{2}b_3\gamma_{ii}\gamma_{ji} + g_1\gamma_{ii}\varepsilon_{jj} \\
&+ g_2(\gamma_{ii} + \gamma_{ji})\varepsilon_{ij} + a_1(\kappa_{iik} + \kappa_{kjj}) + a_2\kappa_{iik}\kappa_{jkj} + \frac{1}{2}a_3\kappa_{iik}\kappa_{jjk} \\
&+ \frac{1}{2}a_4\kappa_{ijj}\kappa_{ikk} + a_5\kappa_{ijj}\kappa_{kik} + \frac{1}{2}a_8\kappa_{iji}\kappa_{kjk} + \frac{1}{2}a_{10}\kappa_{ijk}\kappa_{ijk} \\
&+ a_{11}\kappa_{ijk}\kappa_{jki} + \frac{1}{2}a_{13}\kappa_{ijk}\kappa_{ikj} + \frac{1}{2}a_{14}\kappa_{ijk}\kappa_{jik} + \frac{1}{2}a_{15}\kappa_{ijk}\kappa_{kji}.
\end{aligned} \quad (2.3.3)$$

where λ and μ are the Lame constants and a_i, b_i and g_i would be constitutive parameters. The difficulty associated with Mindlin's formulations originates from the fact that quantifying the large number of coefficients is not easy or straight forward task in practical situations and/or experimental settings. To cope with such a problem, Mindlin devised another version of his formulation by some simplifying assumptions, in that the deformation energy density was only expressed in terms of macroscopic displacements. This way parameters related to multi-scale considerations are simply dropped.

This approach led to development of three forms of this theory as following:

Form I: Micro scale gradient of deformation is defined as the second deformation gradient at macro scale, namely:

$$\kappa_{ijk} = u_{k,ij} \quad (2.3.4)$$

Form II: Micro scale gradient of deformation is expressed as the first gradient of the strain at macro level, meaning:

$$\kappa_{ijk} = \varepsilon_{jk,i}, \quad (2.3.5)$$

where $\varepsilon_{jk,i}$ is defined as:

$$\kappa_{ijk} = \frac{1}{2}(u_{k,ij} + u_{j,ik}). \quad (2.3.6)$$

Form III: In the last form microscopic deformations are expressed as the addition of two parts. First part would be gradient of microscopic rotation shown with χ and defined as:

$$\chi_{ij} = \frac{1}{2}e_{jlm}u_{m,il} \tag{2.3.7}$$

where e_{jlm} would be Levi-Civita permutation tensor.

The second term shaping the microscopic deformation would be symmetric part of the second gradient of macroscopic displacement field which is expressed as:

$$\kappa_{ijk} = \frac{1}{3}(u_{i,jk} + u_{j,ik} + u_{k,ij}). \tag{2.3.8}$$

Development of these forms were pursued further and were explained in terms of stress in another work by Mindlin and Eshel [44]. Nevertheless the equation of motion of all aforementioned three forms ends up to an identical expression.

Considering Form II, deformation energy density can be simplified a bit further as:

$$\mathcal{U} = \frac{1}{2}\lambda\varepsilon_{ij}\varepsilon_{ij} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1(\kappa_{iik} + \kappa_{kjj}) + a_2\kappa_{iik}\kappa_{jkj}$$
$$+ a_3\kappa_{iik}\kappa_{jjk} + a_4\kappa_{ijj}\kappa_{ikk} + a_5\kappa_{ijj}\kappa_{kik}. \tag{2.3.9}$$

Equations of motion can also be simplified and expressed only in terms of macro displacements as:

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj}$$
$$- \frac{4a_1 + 4a_2 + 3a_3 + 2a_4 + 3a_5}{2}u_{j,ijkk}$$
$$- \frac{a_3 + 2a_4 + a_5}{2}u_{i,jjkk} + b_i = \rho(\ddot{u}_i - \ell_1^2) \tag{2.3.10}$$

where b_i are body forces. It can further be written more succinctly when we note that distinction between a_i would be irrelevant in the case of participial implications. Therefore we will have:

$$(\lambda + \mu)(1 - \ell_2^2\frac{\partial^2}{\partial x_k^2})u_{i,jj} + \mu(1 - \ell_3^2\frac{\partial^2}{\partial x_k^2})u_{i,jj}$$
$$+ b_i = \rho(1 - \ell_1^2\frac{\partial^2}{\partial x_k^2})\ddot{u}_i \tag{2.3.11}$$

where

$$\ell_2 = \sqrt{\frac{4a_1 + 4a_2 + 3a_3 + 2a_4 + 3a_5}{2(\lambda + \mu)}} \qquad \text{and} \qquad \ell_3 = \sqrt{\frac{a_3 + 2a_4 + a_5}{2\mu}}.$$

This way there exist only three parameters need to be taken care of that include ℓ_1, ℓ_2 and ℓ_3. Looking at the equation more closely, it is revealed that these three additional parameters correspond the microscopic components of the model. Besides all higher order terms appear as the Laplacian of the related lower-order terms.

2.3.2 Aifantis' 1992 Theory

Aifantis' theory of gradient elasticity stems from his previous works on plasticity [57, 58] and nonlinear elasticity [59]. Those endeavors led the authors to enrich the constitutive relations with the Laplacian of strain [54–56] as:

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \ell^2 \varepsilon_{kl,mm}) \tag{2.3.12}$$

where similar to previous model ℓ stands for length scale coefficient.

Accordingly the equilibrium equation reads as:

$$C_{ijkl}(u_{k,jl} - \ell^2 u_{k,jlmm}) + b_i = 0, \tag{2.3.13}$$

and assuming an isotropic material, we can express the **C** tensor as:

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{il}\delta_{jk}. \tag{2.3.14}$$

Through operator splitting method, the above equation can be solved as a set of two uncoupled equations as:

$$C_{ijkl}u^c_{k,jl} + b_i = 0 \tag{2.3.15}$$

and

$$u^g_k - \ell^2 u^g_{k,mm} = u^c_k \tag{2.3.16}$$

Herein two different displacement fields are introduced: u^c_i which represents the role of classical elasticity and u^g_i which allows the involvement of gradient elasticity. Some authors have incorporated the effect of gradients through strain field rather than displacement field [60–62], as such one can write:

$$\varepsilon^g_{kl} - \ell^2 \varepsilon^g_{kl,mm} = \varepsilon^c_{kl} = \frac{1}{2}(u_k, l^c + u^c_{l,k}) \tag{2.3.17}$$

which upon multiplication with the tensor C_{ijkl} above relation can be expressed as:

$$C_{ijkl}(\varepsilon^g_{kl} - \ell^g \varepsilon^g_{kl,mm}) = C_{ijkl}u^c_{k,l}. \tag{2.3.18}$$

Herein,

$$\varepsilon^g_{kl} = \frac{1}{2}(u^g_{k,l} + u^g_{l,k}). \tag{2.3.19}$$

Either of the above equations can be employed along with the related equilibrium equation in so long as field equations are not altered. However from the standpoint of boundary conditions and their variational consistency they might result in slightly different outcomes [62] depending on the problem at hand.

2.4 Stress Gradient Theories

2.4.1 Eringen's Theory

Milestone works done by Eringen inspired a new rather independent theory in generalized continuum theories namely stress gradient theory. In Eringen formulation, integro-differential relations are transferred into differential forms, so that the constitutive stress-strain relation reads as:

$$\mathbf{C} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma} \tag{2.4.1}$$

in which ℓ similar to strain-gradient relations would be a material constant representing internal length scale, Δ is Laplacian operator, and $\mathbf{C}$ is the elasticity tensor of rank four. In nonlocal settings, this equation can be interpreted as a differential relationship between nonlocal stress field and local strain field $\boldsymbol{\varepsilon}$. In the original format the relationship between these two fields can be described through an integral relation as:

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_\Omega \alpha(|\mathbf{x}' - \mathbf{x}|) \mathbf{s}(\mathbf{x}') d\mathbf{v}(\mathbf{x}') \tag{2.4.2}$$

in which V is defined as the material domain and $\alpha(|\mathbf{x}'-\mathbf{x}|)$ would be the influence function [63] which is introduced to accounting for non-locality effects. This represents the distance between the field point $\mathbf{x}$ and the source po int $\mathbf{x}'$.

Nonlocal stress field obeys the usual equilibrium relation as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in } \Omega. \tag{2.4.3}$$

where $\mathbf{b}$ stands for body forces.

In terms of boundary conditions on $\partial \Omega$ we have:

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } \partial \Omega_f \tag{2.4.4}$$

where $\bar{\mathbf{t}}$ represents forces on free part of the boundary (i.e. no constraints are devised for that part).

Strain field is also required to follow compatibility conditions as:

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} \quad \text{in } \Omega \tag{2.4.5}$$

and

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial \Omega_c \tag{2.4.6}$$

where

$$\nabla^s \mathbf{u} = \frac{u_{i,j} + u_{j,i}}{2}$$

would be symmetric part of the gradient operator and $\partial \Omega_c$ represents the constrained part of the boundary.

Through the Eringen model, both equilibrium and compatibility conditions are introduced into the formulation through the differential form of stress-strain relation instead

of the integral type one. It means that nonlocality effects, inherently available in integral form, transferred into the differential form as the gradient effects of the Cauchy stress σ.

Combining the aforementioned set of equations and boundary conditions leads to an equation based on displacement which reads as:

$$\mathcal{L}\mathbf{u} = -\mathbf{b}^* \quad \text{in } \Omega \tag{2.4.7}$$

where

$$\mathbf{b}^* := \mathbf{b} - \ell^2 \Delta \mathbf{b} \tag{2.4.8}$$

Here $\mathcal{L}$ is an operator that represents second order partial-differential equations arising in isotropic elasticity, which through designating usual Lamé constants we can write:

$$\mathcal{L}\mathbf{u} := \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} \tag{2.4.9}$$

The resulting equation needs to be solved in conjunction with boundary conditions containing the stress field. This model has been widely used in the literature for addressing nano-scale objects, crack propagation at microscope, dislocations and so on [64–68].

2.4.2 Forest's Theory

Forest and Sab [69] argue that to consider a theory fully as stress gradient theory, the role of stress field and its gradients should be distinguished from stain gradient effects and needs to be distinctly expressed in the model. They claim that the presence of stress gradients in Aifantis' model merely stems from specific assumption in Mindlin's strain gradient theory [70] thereby it is still resides in the category of strain gradient models.

To construct a stress gradient theory, they opted for the variational methods and the method of virtual power. Consequently they considered an homogeneous elastic medium with clamping conditions at the boundaries. Constraints of the type of clamping means that the displacement field vanishes at the boundaries, i.e.

$$\mathbf{u}(\mathbf{x}) = 0 \quad \text{on } \partial \Omega.$$

and the domain is under the body forces. Variational formulation of this boundary value problem includes minimization of the complementary energy as:

$$\mathbf{W}^*(\boldsymbol{\sigma}) = \int_\Omega \frac{1}{2} \boldsymbol{\sigma} : \mathbf{S} : \boldsymbol{\sigma} dV \tag{2.4.10}$$

where $\mathbf{S}$ stands for elastic compliance tensor. In addition to the stress gradient, it is assumed that deviatoric part of the stress gradient (denoted by $\mathbf{R}$)contributes to the energy as well. Therefore two sets of conditions arise as following:

$$\boldsymbol{\sigma} \cdot \nabla + \mathbf{f} = 0 \tag{2.4.11}$$

$$\mathbf{R} = (\boldsymbol{\sigma} \otimes \nabla)^d \tag{2.4.12}$$

Now the solution is obtained through minimizing the following relation:

$$\mathbf{W}^{*SG}(\boldsymbol{\sigma}) = \int_\Omega \mathbf{w}^*(\boldsymbol{\sigma}, \mathbf{R}) dV, \tag{2.4.13}$$

where SG denotes stress gradient. Multiplying the equation 2.4.11 with displacement field and after integration by parts we have:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon} dv = \int_\Omega \mathbf{f} \cdot \mathbf{u} dv + \int_{\partial\Omega} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{u} da \qquad (2.4.14)$$

Based on the same approach, equation 2.4.12 is multiplied by a kinematic variable $\boldsymbol{\Phi}$ and integrated by parts over the domain, as a result of which it reads:

$$\int_\Omega \mathbf{R} \therefore \boldsymbol{\Phi} - (\boldsymbol{\sigma} \otimes \nabla)^d \therefore \boldsymbol{\Phi} dv = \int_\Omega \mathbf{R} \therefore \boldsymbol{\Phi} - (\boldsymbol{\sigma} \otimes \nabla) \therefore \boldsymbol{\Phi} dv$$
$$= 0 \qquad (2.4.15)$$

Integrating by parts would yield:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\Phi} \cdot \nabla + \mathbb{R} \therefore \boldsymbol{\Phi} dv = \int_{\partial\Omega} \boldsymbol{\sigma} : \boldsymbol{\Phi} \cdot \mathbf{n} da. \qquad (2.4.16)$$

Then summing up equations 2.4.14 and 2.4.14 results in:

$$\int_\Omega (\boldsymbol{\sigma} : (\boldsymbol{\varepsilon} + \boldsymbol{\Phi} \cdot \nabla) + \mathbf{R} \therefore \boldsymbol{\Phi}) dv = \int_\Omega \mathbf{f} \cdot \mathbf{u} dv + \int_{\partial\Omega} \boldsymbol{\sigma} : (\mathbf{u} \otimes \mathbf{n} + \boldsymbol{\Phi} \cdot \mathbf{n}) da \qquad (2.4.17)$$

The resulting equation in fact provides a conjugation between measures of strain and gradients of stress (in addition to stress itself) as following:

$$\mathbf{e} := \boldsymbol{\varepsilon} + \boldsymbol{\Phi} \cdot \nabla \qquad (2.4.18)$$

with a new kinematic variable, i.e. $\boldsymbol{\Phi}$ which is conjugate to $\mathbf{R}$. Thereby constitutive relation for conjugated stress and strain reads as:

$$\mathbf{e} = \frac{\partial w^*(\boldsymbol{\sigma}, \mathbf{R})}{\partial \boldsymbol{\sigma}} \qquad (2.4.19)$$

and

$$\boldsymbol{\Phi} = \frac{\partial w^*(\boldsymbol{\sigma}, \mathbf{R})}{\partial \mathbf{R}} \qquad (2.4.20)$$

Equation 2.4.17 when reformulated variationally leads to new clamping condition as following:

$$\mathbf{u} \otimes \mathbf{n} + \boldsymbol{\Phi} \cdot \mathbf{n} = \mathbf{0} \qquad (2.4.21)$$

As can be seen the outcome provides a defined well posed boundary value problem.

2.4.3 Polizzotto's Theory

In 2014 Polizzotto [71] contributed to the advancement of the stress gradient theory employing a similar approach to Forest and Sab [69] yet with fundamental differences in terms of the virtual power principle used and the final forms of boundary conditions obtained.

Polizzotto applied the complementary form of the principle of the virtual power through which the stress is given the chance to play the main role as the primary driving field. This is why in the work by Forest and Sab, where the principal of virtual power is used, kinematic variables including displacements and microdisplacments showed to have the

main effect; and stresses and stress gradients exhibited their effects as power conjugate variables.

Both theories exhibit the presence of higher order boundary conditions as a result of the application of the principle of virtual power. There is a slight difference though. In Forest and Sab, boundary conditions are expressed as:

$$\boldsymbol{\Phi} \cdot \mathbf{n} + \frac{1}{2}(\mathbf{n} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{n}) \qquad \text{assigned on } \partial\Omega \qquad (2.4.22)$$

wheres Polizzotto obtianed the boundary condition in the form of:

$$\boldsymbol{\Phi} \cdot \mathbf{n} \qquad \text{assigned on } \partial\Omega. \qquad (2.4.23)$$

In a more recent work by Polizzotto [72] he verifies the obtained higher order boundary condition by means of principle of virtual power as employed by Forest and Sab [69]. It arises that the latter form of boundary condition is obtained when the same method of virtual power is applied.

In terms of computation and numerical implementations it seems that the formulation developed by Polizzotto to be more feasible, as it involves the ordinary displacements and stresses, whereas in Forest and Sab formulation, three ordinary displacements, in conjunction with 15 microdisplacments, need to be calculated.

2.5 Variational Formulations

Variational principles as a method of analysis, offer a framework within which the well-posedness of the corresponding mathematical model can be investigated. They also serve as a foundation for numerical methods and to obtain resultant discrete forms. Eventually they have shown to be useful in assessing the reliability and effectiveness of the proposed numerical method [73]. Fortunately theories based on nonlocal elasticity and some other branches of generalized continuum theories have been shown to admit variational formulations analogous to classical continuum models [74]. Herein some variational formulations employed within the framework of gradient theories will be discussed.

2.5.1 Principle of Total Potential Energy

Assuming a material body occupying the domain V, under the body force $\bar{\mathbf{b}}$ and surface traction $\bar{\mathbf{t}}$ with imposed displacements $\bar{\mathbf{u}}$, and also assuming all these field variables are sufficiently smooth, according to [74] the boundary value problem in a nonlocal setting can be formulated through the following set of equations :

$$div\boldsymbol{\sigma} + \bar{\mathbf{b}} = \mathbf{0} \qquad (2.5.1)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \breve{\mathbf{t}} \qquad (2.5.2)$$

$$\varepsilon = \nabla^s \mathbf{u} \qquad (2.5.3)$$

$$\mathbf{u} = \bar{\mathbf{u}} \qquad (2.5.4)$$

$$\boldsymbol{\sigma} = \mathbf{D} : \mathcal{R}(\varepsilon) \qquad (2.5.5)$$

where ∇^s denotes the symmetric part of the gradient operator as:

$$\nabla^s \mathbf{u} = \frac{u_{i,j} + u_{j,i}}{2}$$

Total potential energy of a nonlocal elastic body is expressed as:

$$\Pi[\mathbf{u}] = \frac{1}{2} \int_V \int_V A(x,x')\nabla\mathbf{u}(x) : \mathbf{D} : \nabla\mathbf{u}(x')dV'dV$$
$$- \int_V \bar{\mathbf{b}} \cdot \mathbf{u}dV - \int_{S_t} \bar{\mathbf{t}} \cdot \mathbf{u} \tag{2.5.6}$$

where the field variable $\mathbf{u}(x)$ belongs to all kinematically admissible displacement fields so that $\mathbf{u}$ is $\mathbf{C}^1$ continuous, meaning that together with its first partial derivative, they belong to L_2 space. Additionally it satisfies the compatibility boundary condition. The following theorem then can be stated.

Theorem 1 *The displacement field $\mathbf{u} \in \mathscr{K}$ which is part of the solution $(\mathbf{u}, \varepsilon, \sigma)$ of the nonlocal elasticity problems as defined above, minimizes the total potential energy Eq. 2.5.6 in the class $\mathscr{K}$; converesely the field $\mathbf{u}$ minimizing 2.5.6 in $\mathscr{K}$ is part of the solution of the nonlocal elasticity problem.*

2.5.2 Complementary Energy Principle

If in the above defined boundary value problem, the constitutive equation is replaced by:

$$\sigma(x) = \mathscr{R}(s) := \int_V A(x,x')s(x')dv' \tag{2.5.7}$$

where $\mathscr{R}(.)$ denotes a linear integral operator that maps the local field of $\mathbf{s}(x)$ into the corresponding nonlocal one; and $\mathbf{s} = s_{ij}$ is the local stress with following relation to strain (which is in fact the classical Hooke's relationship):

$$\mathbf{s} = \mathbf{D} : \varepsilon \tag{2.5.8}$$

then the solution to the above equation could be parameterized by field variables of $(\mathbf{u}, \varepsilon, \sigma, \mathbf{s})$. It would be worthy of notice that here one more field variable is included within the solution space, i,e. local stress field $\mathbf{s}(x)$.

Accordingly, the complementary potential energy of the body is described by the following functional:

$$\Pi_c[\mathbf{s}] := \frac{1}{2} \int_V \int_V A(x,x')\mathbf{s}(x) : D^{-1} : \mathbf{s}(x')dV'dV$$
$$- \int_{S_u} \int_V A(x,x')\mathbf{n}(x') \cdot \mathbf{s}(x') \cdot \bar{\mathbf{u}}dV'dS \tag{2.5.9}$$

where $\mathbf{s}(x)$ belongs to the class $\mathscr{H}$ of all statistically admissible (local) stress fields and is defined as: every $\mathbf{s}$ is C^0 continuous (i.e. belonging to L_2) and the corresponding stress (nonlocal) $\sigma = \mathscr{R}(\mathbf{s})$, satisfies the field equations,as well as related boundary equations as declared in the definition of the defined boundary value problem. The following theorem then can be proved:

Theorem 2 *The local stress field $\mathbf{s} \in \mathscr{H}$, part of the unique solution $(\mathbf{u}, \varepsilon, \sigma, \mathbf{s})$ of the nonlocal elasticity problems,minimizes the complementary potential energy in $\mathscr{H}$; conversely the local stress field minimizing the above functional is part of the soluton to the same problem.*

It should be noted that the linear operator $\mathscr{R}$ is of type of self-adjoint operators, so that a Green-type equality as :

$$\int_V \mathscr{R}(G) \cdot F dV = \int_V G.\mathscr{R} \cdot F dv \tag{2.5.10}$$

holds for any pair of scalar and tensor fields.

2.5.3 Hu-Washizu Variational Principle

In this type of variational principle, the functional to be considered would be:

$$
\begin{aligned}
\Pi_H := & \frac{1}{2} \int_V \int_V A(x,x')\varepsilon(x) : \mathbf{D} : \varepsilon(x')dV'dV \\
& - \int_V [\boldsymbol{\sigma} : (\varepsilon - \nabla^s \mathbf{u}) + \bar{\mathbf{b}} \cdot \mathbf{u}]dV \\
& - \int_{S_u} \mathbf{t} \cdot (\mathbf{u} \cdot \bar{\mathbf{u}})dS - \int_{S_t} \bar{\mathbf{t}} \cdot \mathbf{u}dS
\end{aligned} \tag{2.5.11}
$$

where $\mathbf{u}, \varepsilon, \boldsymbol{\sigma}, \mathbf{t}$ are independent field variables. Accordingly the following theorem can be proved:

Theorem 3 *The solution $(\mathbf{u}, \varepsilon, \boldsymbol{\sigma})$ to the nonlocal elasticity problem as defined above, together with $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ on the boundary of the domain, make stationary the above functional; conversely the stationary solution to the above functional solves the nonlocal elasticity problem.*

Polizzotto advanced his method in employment of this principle in the case of strain gradient materials [72] by introducing the functional Ω_{HW} as following:

$$\Omega_{HW} := \int_V [\psi(\varepsilon^{(0)}, \varepsilon^{(1)}) - \boldsymbol{\sigma}^{(0)} : (\varepsilon^{(0)} - \varepsilon) \tag{2.5.12}$$

$$- \boldsymbol{\sigma}^{(1)} : (\varepsilon^{(1)} - \nabla\varepsilon) - \boldsymbol{\sigma} : (\varepsilon - \nabla_{(s)}\mathbf{u})]dv - W_{ext} \tag{2.5.13}$$

Herein W_{ext} is a scalar variable and stands for external potential and is expressed as:

$$
\begin{aligned}
W_{ext} = & \int_V \hat{\mathbf{b}} \cdot \mathbf{u}dv + \int_{S_f} (\hat{\mathbf{t}} \cdot \mathbf{u} + \hat{\mathbf{m}} \cdot \partial_n \mathbf{u})da \\
& + \int_{S_f} [\mathbf{t} \cdot (\mathbf{u} - \hat{\mathbf{u}}) + \mathbf{m} \cdot (\partial\mathbf{u} - \hat{\mathbf{g}})]da \\
& + \int_{\Gamma_f} \hat{\mathbf{f}} \cdot \mathbf{u}ds + \int_{\Gamma_c} \mathbf{f} \cdot (\mathbf{u} - \hat{\mathbf{u}})ds
\end{aligned} \tag{2.5.14}
$$

The symbols $\mathbf{t}, \mathbf{m}, \mathbf{f}$ denote unknown tractions on boundaries.

The stationary condition for the above functional including all the variables leads to the set of governing equations of the stress gradient boundary value problem. The first variation for the Hu-Washizu functional reads as:

$$\delta\Omega_{HW} = \int_V [(\psi_{,\varepsilon^{(0)}} - \boldsymbol{\sigma}^{(0)}) : \tilde{\varepsilon}^{(0)} + (\psi_{,\varepsilon^{(1)}} - \boldsymbol{\sigma}^{(0)}) : \tilde{\varepsilon}^{(1)}] dv$$

$$+ \int_V [\boldsymbol{\sigma}^{(0)} - \nabla \cdot \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}] : \tilde{\varepsilon} dv$$

$$- \int_V [(\varepsilon^{(0)} - \varepsilon) : \tilde{\boldsymbol{\sigma}}^{(0)} + (\varepsilon^{(1)} - \nabla\varepsilon) : \tilde{\boldsymbol{\sigma}}^{(1)} + (\varepsilon - \nabla\mathbf{u}) : \tilde{\boldsymbol{\sigma}}] dv$$

$$- \int_V [\nabla \cdot \boldsymbol{\sigma} + \hat{\mathbf{b}}] \cdot \tilde{\mathbf{u}} dv - \int_{S_f} [(\bar{\nabla}_S + \kappa\mathbf{n}) \cdot \Sigma + \hat{\mathbf{t}} - \mathbf{n} \cdot \boldsymbol{\sigma}] \cdot \tilde{\mathbf{u}} da$$

$$+ \int_{S_f} [\mathbf{n} \cdot \Sigma - \hat{\mathbf{m}}] \cdot \partial_n \tilde{\mathbf{u}} da + \int_{S_f} [\mathbf{n} \cdot \boldsymbol{\sigma} - (\bar{\nabla}_S + \kappa\mathbf{n}) \cdot \Sigma - \mathbf{t}] \cdot \tilde{\mathbf{u}} da$$

$$+ \int_{S_f} [\mathbf{n} \cdot \Sigma - \mathbf{m}] \cdot \tilde{\mathbf{u}} da - \int_{S_c} [(\mathbf{u} - \hat{\mathbf{u}}) \cdot \tilde{\mathbf{t}} + (\partial_n \mathbf{u} - \hat{\mathbf{g}})] \cdot \tilde{\mathbf{m}} da$$

$$+ \int_{\Gamma_f} [\Pi - \hat{\mathbf{f}}] \cdot \tilde{\mathbf{u}} da - \int_{\Gamma_c} [\Pi - \mathbf{f}] \cdot \tilde{\mathbf{u}} ds - \int_{\Gamma_c} [\mathbf{u} \cdot \hat{\mathbf{u}}] \cdot \tilde{\mathbf{f}} ds$$

$$= 0 \tag{2.5.15}$$

2.5.4 Hellinger-Reissner Variational Principle

In this type of variational principle we are concerned with two field variables instead of one. Two field variables in this case include stress and displacement fields [71, 75]. This principle reads as:

$$\Pi_H[\boldsymbol{\sigma}, \mathbf{u}] := \int_V [\boldsymbol{\sigma} : \nabla\mathbf{u} - G(\boldsymbol{\sigma}, \nabla\sigma) - \mathbf{b} \cdot \mathbf{u}] dv$$

$$+ \int_S \mathbf{r} : \boldsymbol{\sigma} da - \int_{S_t} t \cdot (\mathbf{u} - \bar{\mathbf{u}} da \tag{2.5.16}$$

Similar to other previous variational principles, this one is also formulated as a theorem and can be shown that if the solution to the corresponding boundary value problem exists, it makes Π_H to be stationary; and conversely that the stationary solution solves the boundary value problem. Worthy of mentioning that the solution will be unique.

2.6 Boundary-Value Problems (BVPs)

2.6.1 BVP for Stress Gradient Theories

Governing equations for the stress gradient elasticity, stemming form Eringen model, collectively reads as [71]:

$$\varepsilon = \mathbf{C}^{-1} : (\boldsymbol{\sigma} - \ell^2 \Delta\boldsymbol{\sigma}) \quad \text{in } V \tag{2.6.1}$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V \tag{2.6.2}$$

$$\varepsilon = \nabla\mathbf{u} \quad \text{in } V \tag{2.6.3}$$

where stress-strain relationship conveys the length scale parameter ℓ, and consequently equilibrium equation and kinematic restrictions are applied.

Accompanying boundary conditions include:

$$\partial_n \boldsymbol{\sigma} = \mathbf{0} \quad \text{on} \quad S \tag{2.6.4}$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on} \quad S_f \tag{2.6.5}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad S_c \tag{2.6.6}$$

where S denotes the boundary surface composed of two partitions as $S = S_f \cup S_c$, each of which convey specified imposition of traction and displacement.

When divergence theorem is applied to the relationship between the stress and the strain, the governing differential equation is obtained as:

$$\mathcal{L} = -\mathbf{b}^* \tag{2.6.7}$$

where $\mathbf{b}^*$ is defined as:

$$\mathbf{b}^* := \mathbf{b} - \ell^2 \Delta \mathbf{b} \tag{2.6.8}$$

Through the above governing equation the behavior of stress field in response to a predetermined strain field could be evaluated. According to variational principles discussed in the previous section, it can be shown that the solution to the mentioned problem obeys a variational form of a functional called $J[\boldsymbol{\sigma}]$ which is defined as:

$$J[\boldsymbol{\sigma}] := \int_V [\mathcal{G}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) - \boldsymbol{\varepsilon} : \boldsymbol{\sigma}]dv \tag{2.6.9}$$

Here $\boldsymbol{\varepsilon}$ is the strain field and $\mathcal{G}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})$ would be the Gibbs function defined as:

$$\mathcal{G} = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{A}\boldsymbol{\sigma} + \frac{1}{2}\ell^2 \mathbf{A} :: [(\nabla\boldsymbol{\sigma})^T \cdot \nabla\boldsymbol{\sigma}] \tag{2.6.10}$$

The problem can be restated in this way that in this model the corresponding stress field to the given strain, minimizes the functional J, and conversely the stress field that minimizes this functional would be the one that corresponds to the applied strain field.

Some mathematical treatments would yield the first variation of J as :

$$\delta J = \int_V [\mathbf{A} : (\boldsymbol{\sigma} - \ell^2 \Delta\boldsymbol{\sigma}) - \boldsymbol{\varepsilon}] : \delta\boldsymbol{\sigma} dv + \int_S \ell^2 \partial_n \boldsymbol{\sigma} : \mathbf{A} : \delta\boldsymbol{\sigma} da \tag{2.6.11}$$

Assuming that the second variation of J be computed based on the solution to the stress problem, then for any $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \delta\boldsymbol{\sigma}$ we have:

$$J[\boldsymbol{\sigma}'] = J[\boldsymbol{\sigma}] + \delta J[\boldsymbol{\sigma}] + \frac{1}{2}\delta^2 J[\boldsymbol{\sigma}] \tag{2.6.12}$$

where $\delta^2 J[\boldsymbol{\sigma}]$ is defined as:

$$\delta^2 J[\boldsymbol{\sigma}] = 2 \int_V \mathcal{G}(\delta\boldsymbol{\sigma}, \nabla\delta\boldsymbol{\sigma}) > 0 \tag{2.6.13}$$

Since $\delta^2 J[\boldsymbol{\sigma}]$ is greater than zero for any non trivial vanishing variation of stress, we can obtain:

$$J[\boldsymbol{\sigma}'] = J[\boldsymbol{\sigma}] + \frac{1}{2}\delta^2 J[\boldsymbol{\sigma}] \geq J[\boldsymbol{\sigma}] \tag{2.6.14}$$

The equality sign on the right holds if and only if $\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma}$ showing the solution is unique.

Boundary-value formulations for stress-gradient theories can be based on Hellinger-Reissner variational principle as well [71]. In this case the functional would be considered

the same as the one introduced in the previous section as $\mathcal{H}$. The first variation of $\mathcal{H}$ thus reads as:

$$
\begin{aligned}
\delta\mathcal{H} = &\int_V [\nabla\mathbf{u} - (\mathcal{G}_{,\boldsymbol{\sigma}} - \nabla\cdot\mathcal{G}_{,(\nabla\boldsymbol{\sigma})})] : \delta\boldsymbol{\sigma}\, dv \\
&- \int_S [\mathbf{n}\cdot\mathcal{G}_{,(\nabla\boldsymbol{\sigma})} - \mathbf{r}] : \delta\sigma\, da - \int_V [\nabla\cdot\boldsymbol{\sigma} + \mathbf{b}]\cdot\delta\mathbf{u}\, dv \\
&+ \int_{S_f} [\mathbf{n}\cdot\boldsymbol{\sigma} - \bar{\mathbf{t}}]\cdot\delta\mathbf{u}\, da + \int_{S_f} [\mathbf{n}\cdot\boldsymbol{\sigma} - \mathbf{t}]\cdot\delta\mathbf{u}\, da - \int_{S_f} [\mathbf{u} - \bar{\mathbf{u}}]\cdot\delta\mathbf{t}\, da \qquad (2.6.15)
\end{aligned}
$$

Assuming that the pair $(\boldsymbol{\sigma}, \mathbf{u})$ in conjunction with the strain field ε solve the aforementioned set of governing equations, all the square-bracketed terms should be vanishing and thus $\mathcal{H} = 0$ for any variations. It means that in this case $\mathcal{H}$ is stationary. On the other hand if for a certain pair of $(\boldsymbol{\sigma}, \mathbf{u})$ the calculated $\delta\mathcal{H}$ be vanishing for any variations of $\boldsymbol{\sigma}$ and $\mathbf{u}$ then all the bracketed terms should be vanishing within the corresponding domain. It demonstrates that the fields $(\boldsymbol{\sigma}, \mathbf{u})$ along with strain field ε are the solutions to the boundary-value problem as formulated for the gradient elasticity problem.

2.6.2 BVP for Strain Gradient Theories

Governing equations along with boundary conditions are well elaborated through recent decades in the literature [31, 47, 52, 76] and they can be stated as:

$$
\boldsymbol{\sigma} = \mathbf{C} : (\varepsilon - \ell^2\Delta\varepsilon) \qquad (2.6.16)
$$

$$
\tau = \ell^2\nabla\boldsymbol{\sigma} \qquad (2.6.17)
$$

$$
\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \qquad (2.6.18)
$$

$$
\varepsilon = \nabla^s\mathbf{u} \qquad (2.6.19)
$$

Consequently, the boundary conditions read as:

$$
(\bar{\nabla}_{(s)} + H\mathbf{n})\cdot(\mathbf{n}\cdot\tau) + \bar{\mathbf{t}} - \mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{0} \quad \text{on } S_f \qquad (2.6.20)
$$

$$
\mathbf{n}\cdot(\mathbf{n}\cdot\tau) = \bar{\mathbf{n}} \quad \text{on } S_f \qquad (2.6.21)
$$

$$
\mathbf{u} = \bar{\mathbf{u}}, \qquad \partial_n\mathbf{u} = \bar{\mathbf{g}} \quad \text{on } S_c \qquad (2.6.22)
$$

whereby $\bar{\nabla}_{(s)} := \nabla - \mathbf{n}\partial_n$ stands for the tangential operator over the surface at a position with the outward unit normal as $\mathbf{n}$, and $H := -\bar{\nabla}_{(s)}\cdot\mathbf{n}$ would be twice the mean curvature. Through combining the mentioned field equations a system of differential equations is obtained as:

$$
\mathcal{L}(\mathbf{u} - \ell^2\Delta\mathbf{u}) = -\mathbf{b} \quad \text{in } V \qquad (2.6.23)
$$

The obtained system of differential equations is of the order of 4 with respect to the displacement field $\mathbf{u}$ for which we need a set of boundary conditions to be appended to the body of the corresponding differential equations.

2.6.3 Comparisons

Comparing the two forms of differential equations derived for stress gradient and strain gradient theories reveals the differences between the two. For the stress gradient theory

the role of gradients is appeared through the body forces whereas in the case of strain gradient theory same effects manifest themselves through the order of the corresponding partial differential equation, which furthermore includes the coefficients carrying the internal length scale parameter ℓ.

But the more prominent difference raises throughout the boundary conditions. In strain gradient model of isotropic elastic materials, gradient effects show themselves through the surface effects in the sense that whereas particles within the bulk of material follow the classical Cauchy continuum, the particles near the boundary surface form a membrane like boundary which is governed by the principles of surface mechanics [77, 78]. In fact, the traction can be divided into two parts, i.e.

$$\bar{t} = \mathbf{t}_{GM} + \mathbf{t}_C \tag{2.6.24}$$

in which $\mathbf{t}_C = \mathbf{n} \cdot \boldsymbol{\sigma}$ would be the part which is transferred to the bulk material. However, the other part, i.e. $\mathbf{t}_{GM}$ which stands for Gurtin-Murdoch traction [78] is manifested through the boundary layer as a surface body force.

2.7 Thermodynamics in Nonlocal Setting

2.7.1 Thermodynamic Framework

To characterize the constitutive behavior of gradient materials, traditionally and similar to classical continuum mechanics, an internal energy function U as following is considered:

$$U = U(\boldsymbol{\varepsilon}, \mathbf{q}, s, \hat{\varepsilon}[\boldsymbol{\varepsilon}], \hat{\mathbf{q}}[\mathbf{q}]) \tag{2.7.1}$$

Again similar to local based formulations, U is a function of state variables that are strain tensor $\boldsymbol{\varepsilon}$, the internal variable vector field $\mathbf{q}$ and the entropy s. However when the nonlocality or gradient effects come into play additional variables such as $\hat{\varepsilon}$ and $\hat{\mathbf{q}}$ need also to be taken into account. These additional variables are usually called nonlocal variables and they can be defined as a functional of the related state variable. Since it incorporates the effects of nonlocality, the value of this functional at point $\mathbf{x}$ depends on the related state variable over a neighboring zone in the domain. It should be pointed out that logically nonlocal entropy should be considered in U, however dropping that means processes related to thermal effects inside the medium are assumed to happen in local fashion.

Nonlocal strain can be defined as:

$$\hat{\varepsilon}[\boldsymbol{\varepsilon}]|_x := (\nabla\boldsymbol{\varepsilon}(\mathbf{x}), \nabla^2\boldsymbol{\varepsilon}(\mathbf{x}), ..., \nabla^n\boldsymbol{\varepsilon}(\mathbf{x})) \qquad \text{for some } n = 1, 2, ... \tag{2.7.2}$$

as a comparison with nonlocal models, the same functional is defined as:

$$\hat{\varepsilon}[\boldsymbol{\varepsilon}]|_x = \mathcal{F}(\boldsymbol{\varepsilon})|_x := \int_V g(\|\mathbf{x}' - \mathbf{x}\|)\boldsymbol{\varepsilon}(\mathbf{x}')dV \tag{2.7.3}$$

In 2.7.3 g is called the influence function and is parametrized by r as $g = g(r)$. It is a function of Euclidean distance as $r = \|\mathbf{x}' - \mathbf{x}\|$ which can also be viewed as smallest distance between $\mathbf{x}$ and $\mathbf{x}'$ while not intersecting with the boundary [79–81], as denoted here by ∂V. As can be seen in the gradient model the functional $\hat{\varepsilon}$ is a function of a collection of k-th order gradients of strain $\nabla^k\boldsymbol{\varepsilon} = \partial_{p1}, \partial_{P2}...\partial_{pk}\varepsilon_{ij}$ $(k = 1, 2, ..., n)$. Similarly, the

functional for $\hat{\mathbf{q}}$ is expressed in terms of k-th order gradients. Having functional defined as above, alongside the internal energy potential function, one would be able to explain a variety of phenomena of interest within a material modeled either through gradient or nonlocal theories.

Assuming an isothermal and quasi-static deformation for the system under study, the first principle of thermodynamics (energy balance equation) globally can be written as:

$$\int_V^* \dot{U} dV = \int_V^* \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} dV \qquad \forall V^* : V_d \subseteq V^* \subseteq V \qquad (2.7.4)$$

where upper dot indicates the time rate, and diffusion domain is denoted as V_d. According to [67], diffusion domain is part of the domain V that nonlocality effects diffusion processes.It should be pointed out that long distance energy interchanges are neglected herein since it is assumed that they do not intersect the boundary of V_d as they completely exhaust within this domain.

By local enforcement of the energy balance of equation (2.7.4) where energy interchanges are given the chance to intervene, the integral relation is transformed into a differential form as following:

$$\dot{\mathbf{U}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + R \qquad \text{in } V \qquad (2.7.5)$$

with R named as nonlocality (energy) residual. In other words it is defined as energy density received by the material particle located at the generic point from all other particles with the domain V as a consequence of the nonlocality effects diffusion processes.

Insulation condition can be written as:

$$\int_V^* R dV = 0 \qquad \forall V^* : V_d \subseteq V^* \subseteq V \qquad (2.7.6)$$

which means no long distance energy is allowed to transport from the diffusion domain V_d outward.

Using the Legendre transform as $\psi = U - Ts$ and introducing Helmholtz free energy potential, entropy as a state variable is replaced by the absolute temperature $T > 0$, as a result of which Eq. (2.7.5) is transformed into the following form:

$$T\dot{s} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + R \geq 0 \qquad (2.7.7)$$

which relates the rate of the production of entropy to the deformation process. Second principle of thermodynamics (entropy production inequality)is manifested here through the non negativity sign assuming that its point wise form holds despite the existence of nonlocality effects. Not considering such an assumption would mean that there exists a deformation which satisfies $\int_V \dot{s} dV = 0$. Such a deformation would be reversible globally and at the same time irreversible locally which is illogical [82, 83].
Eq. (2.7.7) is known as Clausius-Duhem inequality and it differs from its local form only due to the presence of the nonlocality residual R. Classically [84, 85]this equation is used to extract thermodynamic restrictions pertaining to the constitutive relation of the material model at hand.

In the case of Helmholtz free energy, following typical forms are considered as for the nonlocal/gradient elasticity:

$$\psi = \psi(\boldsymbol{\varepsilon}, \hat{\boldsymbol{\varepsilon}}[\boldsymbol{\varepsilon}]); \qquad (2.7.8)$$

for nonlocal/gradient plasticity

$$\psi = \psi(\varepsilon - \varepsilon^p, \kappa, \hat{\kappa}[\kappa]); \tag{2.7.9}$$

and for nonlocal/gradient damage it reads

$$\psi = \psi(\varepsilon, \omega, \hat{\omega}[\omega]) \tag{2.7.10}$$

where parameters κ and ω pertain to measurement of plastic and elastic deformations, respectively.

The Toupin-Mindlin theory of polar-gradient elasticity [31, 42] still can be incorporated into (2.7.8), but as a local model since it takes the following form for a second strain gradient model:

$$\psi = \psi(\varepsilon^{(0)}, \varepsilon^{(1)}, \varepsilon^{(2)}), \tag{2.7.11}$$

where $\varepsilon^{(0)} = \varepsilon$ is the usual second-order strain tensor, and $\varepsilon^{(1)}, \varepsilon^{(2)}$ stand for third and fourth order strain tensors, respectively assumed with the same tonsorial properties as the strain gradients $\nabla\varepsilon, \nabla^2\varepsilon$.

In an analogous manner, Clausius-Duhem relation reads as

$$T\dot{s} = \boldsymbol{\sigma}^{(0)} : \dot{\varepsilon}^{(0)} + \boldsymbol{\sigma}^{(1)} \vdots \dot{\varepsilon}^{(1)} + \boldsymbol{\sigma}^{(2)} \vdots\vdots \dot{\varepsilon}^{(2)} - \dot{\psi} \geq 0, \tag{2.7.12}$$

with $R \equiv 0$ since the model is considered to be of the local type. Under such conditions, gradient effects are included into the formulation at the global level through the compatibility conditions which can be written as [31, 86]:

$$\dot{\varepsilon}^{(0)} = \nabla^s \mathbf{u}, \qquad \dot{\varepsilon}^{(1)} = \nabla^2 \mathbf{u}, \qquad \dot{\varepsilon}^{(2)} = \nabla^3 \mathbf{u} \tag{2.7.13}$$

2.7.2 Thermodynamic Restrictions

Following functional forms are assumed for free energy potential of a material with second gradient elasticity:

$$\psi = \psi(\varepsilon, \nabla\varepsilon, \nabla^2\varepsilon) \tag{2.7.14}$$

where ε is the classical strain tensor.

Considering the Clausius-Duhem inequality as bellowing:

$$T\dot{s} = \boldsymbol{\sigma} : \dot{\varepsilon} - \dot{\psi} + R \geq 0 \tag{2.7.15}$$

and expanding the time derivative of the mentioned functional would yield:

$$T\dot{s} = \sigma : \dot{\varepsilon} - \boldsymbol{\sigma}^{(0)} : \dot{\varepsilon} - \boldsymbol{\sigma}^{(1)} \vdots \nabla\dot{\varepsilon} - \boldsymbol{\sigma}^{(2)} \vdots\vdots \nabla^2\dot{\varepsilon} + R \geq 0 \tag{2.7.16}$$

where $\boldsymbol{\sigma}^n$ are stress tensors acting as thermodynamic forces here, that is

$$\boldsymbol{\sigma}^{(0)} := \frac{\partial\psi}{\partial\varepsilon}, \qquad \boldsymbol{\sigma}^{(1)} := \frac{\partial\psi}{\partial\nabla\varepsilon}, \qquad \boldsymbol{\sigma}^{(2)} := \frac{\partial\psi}{\partial\nabla^2\varepsilon} \tag{2.7.17}$$

These stress tensors exhibit the same symmetry properties as the corresponding work conjugate strain tensors, i.e. $\varepsilon = \varepsilon_{ij}, \nabla\varepsilon = \partial_p\varepsilon_{ij}$ and $\nabla^2\varepsilon = \partial_p\partial_q\varepsilon_{ij}$. According to [31]

all these tensors are symmetric with respect to the indices (i,j) and the third one is symmetric with respect to (p,q) as well.

Upon integration of equation 2.7.16 we will have:

$$T \int_V \dot{s} dV = \int_V [\boldsymbol{\sigma} : \dot{\varepsilon} - \boldsymbol{\sigma}^{(0)} : \dot{\varepsilon} - \boldsymbol{\sigma}^{(1)} \vdots \nabla \dot{\varepsilon} - \boldsymbol{\sigma}^{(2)} \vdots\vdots \nabla^2 \dot{\varepsilon}] dV \geq 0 \qquad (2.7.18)$$

After the application of the divergence theorem it reads as:

$$\int_V \boldsymbol{\sigma}^{(1)} \vdots \nabla \dot{\varepsilon} dV = - \int_V \nabla \cdot \boldsymbol{\sigma}^{(1)} : \dot{\varepsilon} dV + \int_S \mathbf{n} \cdot \boldsymbol{\sigma}^{(1)} : \dot{\varepsilon} dS, \qquad (2.7.19)$$

$$\int_V \boldsymbol{\sigma}^{(2)} \vdots\vdots \nabla^2 \dot{\varepsilon} dV = - \int_V \nabla^2 \cdot \boldsymbol{\sigma}^{(2)} : \dot{\varepsilon} dV - \int_S \mathbf{n} \nabla \cdot \boldsymbol{\sigma}^{(2)} : \dot{\varepsilon} dS + \int_S \mathbf{n} \cdot \boldsymbol{\sigma}^{(2)} \vdots \nabla \dot{\varepsilon} dS \qquad (2.7.20)$$

Through the surface integral transformation formula, the last surface integral in the above equation can be written as:

$$\int_S \mathbf{n} \cdot \boldsymbol{\sigma}^{(2)} \vdots \nabla \dot{\varepsilon} dS = \int_S \mathbf{G} \cdot (\mathbf{n} \cdot \boldsymbol{\sigma}^{(2)}) : \dot{\varepsilon} dS + \int_S \mathbf{nn} : \boldsymbol{\sigma}^{(2)} : \partial_n \dot{\varepsilon} dS. \qquad (2.7.21)$$

Having this, substituting obtained integral into the integral form of Clausius-Duhem inequality we can write:

$$T \int_V \dot{s} dV = \int_V (\boldsymbol{\sigma} - \boldsymbol{s}) : \dot{\varepsilon} dV - \int_S (\mathbf{P}^{(1)} : \dot{\varepsilon} + \mathbf{P}^{(2)} : \partial_n \dot{\varepsilon}) dS \geq 0, \qquad (2.7.22)$$

where s would be the total thermodynamic force related to the strain rate tensor $\dot{\varepsilon}$ that reads as:

$$\mathbf{s} := \boldsymbol{\sigma}^{(0)} - \nabla \cdot \boldsymbol{\sigma}^{(1)} + \nabla^2 : \boldsymbol{\sigma}^{(2)} \qquad \text{in } V \qquad (2.7.23)$$

Here $P^{(1)}$ and $P^{(2)}$ are defined as second rank symmetric tensor with the following expressions:

$$\mathbf{P}^{(1)} := \int_V \mathbf{n} \cdot \boldsymbol{\sigma}^{(1)} + \nabla^2 : \boldsymbol{\sigma}^{(2)} \qquad \text{in } V \qquad (2.7.24)$$

$$\mathbf{P}^{(2)} := \mathbf{nn} : \boldsymbol{\sigma}^{(2)} \qquad (2.7.25)$$

In contrast to gradient plasticity, in gradient elasticity the strain states, which are driven by strain fields $\boldsymbol{\varepsilon} \in C^4$, are not integrable with respect to displacement field $\mathbf{u}$ in V. Hence in the case of gradient elasticity, one should consider the strain states as being driven by displacement which is manifested through the constraint in the form $\dot{\varepsilon} = \nabla^s \dot{\mathbf{u}}$ with $\mathbf{u} \in C^5$ to be introduced into 2.7.22. Thereby, and after application of divergence theorem, we can rewrite the equation 2.7.22 as:

$$T \int_V \dot{s} dV = - \int_V \nabla \cdot (\boldsymbol{\sigma} - \mathbf{s}) \cdot \dot{\mathbf{u}} dV + \int_S \mathbf{n} \cdot (\boldsymbol{\sigma} - \mathbf{s}) \cdot \dot{\mathbf{u}} dS$$
$$- \int_S [\mathbf{P}^{(1)} : \nabla \dot{\mathbf{u}} + \mathbf{P}^{(2)} : \partial_n (\nabla \dot{\mathbf{u}})] dS \geq 0, \qquad (2.7.26)$$

It should be noted that in the above relation ∇^s has been replaced by $\nabla \dot{\mathbf{u}}$ as a result of symmetry properties of the microstress tensors.

Now again through the surface integral transformation we can write:

$$\int_S \mathbf{P}^{(1)} : \nabla \dot{\mathbf{u}} dS = \int_S \mathbf{G} \cdot \mathbf{P}^{(1)} \cdot \dot{\mathbf{u}} dS + \int_S \mathbf{n} \cdot \mathbf{P}^{(1)} \cdot \partial_n \dot{\mathbf{u}} dS. \qquad (2.7.27)$$

Having considered the following relation as noted in [31]:

$$
\begin{aligned}
\partial_n(\nabla \dot{\mathbf{u}}) &= \nabla(\partial_n \dot{\mathbf{u}}) - \nabla(\mathbf{n} \cdot \nabla)\dot{\mathbf{u}} \\
&= \nabla(\partial_n \dot{\mathbf{u}}) - (\nabla \mathbf{n})^T \cdot \nabla \dot{\mathbf{u}} \qquad (2.7.28) \\
&= \nabla(\partial_n \dot{\mathbf{u}}) - (\bar{\nabla} \mathbf{n})^T \cdot \nabla \dot{\mathbf{u}}. \qquad (2.7.29)
\end{aligned}
$$

we obtain:

$$
\begin{aligned}
\int_S \mathbf{P}^{(2)} : \partial_n(\nabla \mathbf{u}) dS &= \int_S \mathbf{P}^{(2)} : \nabla(\partial_n \dot{\mathbf{u}}) dS - \int_S [\mathbf{P}^{(2)} \cdot (\bar{\nabla} \mathbf{n})] : \nabla \dot{\mathbf{u}} dS \\
&= \int_S \mathcal{G} \cdot \mathbf{P}^{(2)} \cdot \partial_n \dot{\mathbf{u}} dS + \int_S \mathbf{n} \cdot \mathbf{P}^{(2)} \cdot \partial_n^2 \dot{\mathbf{u}} dS \\
&\quad - \int_S \mathcal{G} \cdot [(\bar{\nabla} \mathbf{n})^T \cdot \mathbf{P}^{(2)}] \cdot \dot{\mathbf{u}} dS - \int_S \mathbf{n} \cdot [(\bar{\nabla})^T \cdot \mathbf{P}^{(2)}] \cdot \partial_n \dot{\mathbf{u}} dS. \quad (2.7.30)
\end{aligned}
$$

Substituting obtained relations back into equation 2.7.26 we can write:

$$
\begin{aligned}
T \int_V \dot{s} dV &= - \int_V \nabla \cdot (\boldsymbol{\sigma} - \mathbf{s}) \cdot \dot{\mathbf{u}} dV \\
&\quad + \int_S [\mathbf{n} \cdot (\boldsymbol{\sigma} - \mathbf{s}) - T^{(0)}] \cdot \dot{\mathbf{u}} dS - \int_S [\mathbf{T}^{(1)} \cdot \partial_n \dot{\mathbf{u}} + \mathbf{T}^{(2)} \cdot \partial_n^2 \dot{\mathbf{u}}] dS \geq 0, \quad (2.7.31)
\end{aligned}
$$

with the following relations for $T^{(n)}$:

$$
\begin{aligned}
\mathbf{T}^{(0)} &:= [\mathbf{P}(1) - (\bar{\nabla} \mathbf{n}) \cdot \mathbf{P}^{(2)}], & (2.7.32) \\
\mathbf{T}^{(1)} &:= \mathbf{n} \cdot \mathbf{P}^{(1)} + \mathcal{G} \cdot \mathbf{P}^{(2)}, & (2.7.33) \\
\mathbf{T}^{(2)} &:= \mathbf{n} \cdot \mathbf{P}^{(2)}. & (2.7.34)
\end{aligned}
$$

Since $\dot{\mathbf{u}}$ along with its normal derivatives $\partial_n \dot{\mathbf{u}}$ and $\partial_n^2 \dot{\mathbf{u}}$ are free variables, inequality 2.7.31 looks to be a good choice for imposing required thermodynamic restrictions. Taking into account that equation 2.7.31 must be satisfied in the case of any form of displacement-based deformation, the necessary and sufficient conditions read as:

$$
\begin{aligned}
\nabla \cdot (\boldsymbol{\sigma} - \mathbf{s}) &= \mathbf{0} & \text{in } V, & \qquad (2.7.35) \\
\mathbf{n} \cdot (\boldsymbol{\sigma} - \mathbf{s}) - \mathbf{T}^{(0)} &= \mathbf{0} & \text{on } S & \qquad (2.7.36) \\
\mathbf{T}^{(1)} = \mathbf{T}^{(2)} &= \mathbf{0} & \text{on } S & \qquad (2.7.37)
\end{aligned}
$$

with vectors $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are named as nonlocality diffusion (Thermodynamic) forces.

Due to the fact that stress field $\boldsymbol{\sigma}$ is expressed as

$$\boldsymbol{\sigma} = \mathbf{s} + \rho \qquad (2.7.38)$$

where,

$$\nabla \cdot \boldsymbol{\rho} = \mathbf{0} \qquad \text{in} V \qquad (2.7.39)$$

$$\mathbf{n} \cdot \boldsymbol{\rho} = \mathbf{T}^{(0)} \qquad \text{on } S \tag{2.7.40}$$

it can not be determined uniquely through the thermodynamic forces.

This uncertainty in determination of stress can be vanished through conjecturing the existence of a surface layer containing suitable membrane stresses. $\boldsymbol{\rho} = \mathbf{0}$ can be set for all particles within the domain V, but on the surface, a condition of type 2.7.40 is applied that needs to be satisfied. Through the membrane equilibrium equations, stresses in membrane can be uniquely expressed in terms of surface layer traction $\mathbf{T}^{(0)}$. Since $\mathbf{T}^{(0)}$ depends on the displacement $\mathbf{u}$ on S, a surface energy which can be considered as $\psi^*(\mathbf{u})$ can be expressed for the surface layer via the following relation:

$$\psi^*(\mathbf{u}) := \int_{\mathbf{0}}^{\mathbf{u}} \mathbf{T}^{(0)}(\mathbf{u}) \cdot d\mathbf{u} \tag{2.7.41}$$

through which one can write:

$$\frac{\partial \psi^*}{\partial \mathbf{u}} = \mathbf{T}^{(0)}(\mathbf{u}) \tag{2.7.42}$$

The surface energy as defined above slightly differs from the one suggested by Mindlin [31] for polar-gradient based elasticity models. However it would still be possible that upon conjecturing, an initial value could be incorporated leading to the same Mindlin surface tension.

The surface layer encompassing membrane stresses can be viewed as a result of internal kinematic constraints imposed by the gradient model to the polar-gradient model through the requirement of displacement-driven strain states. Based on the mentioned reasoning, one can write the corresponding equation states as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(0)} - \nabla \cdot \boldsymbol{\sigma}^{(1)} + \nabla^2 : \boldsymbol{\sigma}^{(2)} \qquad \text{in } V \tag{2.7.43}$$

and as the boundary condition:

$$\mathbf{T}^{(1)} = \mathbf{T}^{(2)} = \mathbf{T} \qquad \text{on } S \tag{2.7.44}$$

No nonlocal energies are allowed to exit the domain through the surface S because the nonlocality diffusion forces, i.e. $\mathbf{T}^{(1)}, \mathbf{T}^{(2)}$ vanish on the surface. Additionally, as mentioned before, the surface layer traction, $\mathbf{T}^{(0)}$ is absorbed by the surface layer itself ($\mathbf{T}^{(0)}$ is resisted by the membrane states). Thereby in this configuration constitutive insulation of the domain is provided and by assuming that $\mathbf{n} \cdot \boldsymbol{\rho}$ represents membrane states, inequality 2.7.31 is satisfied as an equality. Followed by the fact that $\dot{s} = 0$ in V, one can write the constitutive relation for R as:

$$R = \dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \tag{2.7.45}$$

Chapter 3

Viscoelasticity

3.1 Introduction

Linear theory of viscoelasticity is employed and incorporated in development of the theory of gradient thermoviscoelasticity. Thereby in this chapter linear viscoelasticity is studied and integral and differential representations of constitutive relations between stress and strain are presented. Also some required mathematical tools in development of theory of viscoelasticity, such as the concept of pseudo-convolution, will be discussed.

Devising mathematical models capable of predicting material responses under different conditions, requires accurate description of stress and strain states, as well as determination of environmental parameters such as temperature. Stress and strain are at interplay with each other and their ultimate stable states define the overall state of body of material. Stress, strain and environmental parameters are related to each other through some fundamental equations namely equilibrium equation, kinematics relations, compatibility equation, and constitutive relations. To incorporate the role of external constraints, boundary conditions are also needed to be attached to aforementioned equations.

Equilibrium equations define the relationship between various components of stress locally (i.e. at any given point). Kinematic relations on the other hand relates the components of the strain by means of displacement. Compatibility equation in fact is a condition through which the continuity among strain components are granted. Constitutive relations reveal the complex interaction between material characteristics, states of stress and strain, and time. In the case of viscoelastic materials there exists a strong entanglement between stress, strain and time which will be described in following sections.

3.2 Classification of Materials Responses

3.2.1 Elastic Materials

For a majority of materials in small strains, a linear relationship exists between stress and stain which is defined through the Hooke's law (equation 3.2.1) in which stress and strain are linearly dependent to each other thorough a material constant named elastic module.

$$\sigma(t) = E\varepsilon(t) \tag{3.2.1}$$

In these materials, stress and strain follow a linear relation in tension, which upon unloading the strain turns back to its original location on the same line. Looking at the
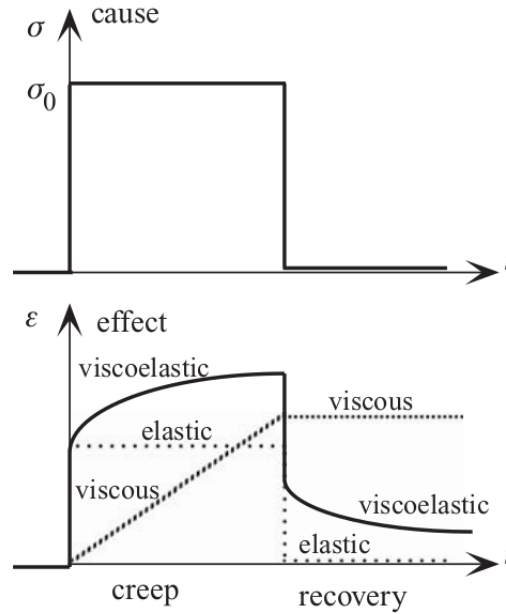
Figure 3.1: Creep and Recovery in Viscoelastic Materials though
Stress and Strain interplays [87]

strain graph, noteworthy to mention that immediate strain upon the applied stress could
be observed.

3.2.2 Plastic Materials

There are circumstances in which continuous application of stress leads the material to
cross the elastic limit and enter the plastic zone in which stress and strain do not exhibit
linear relationship any more. In these conditions, some amount of strain will be remained
under removal of stress, which is called inelastic strain. It has been seen that in some
materials, after removal of stress, the strain continues to increase then tends to a constant
value. The remaining strain in this case would be permanent and is called the plastic
strain. Usually plastic strain is considered as a time independent behavior, though time
dependent strain response accompanying the plastic strain is also observed.

3.2.3 Viscoelastic Materials

In viscoelastic materials we observe an almost immediate elastic response by applying the
load; however thereafter the strain starts to increase at a decreasing rate. With removal
of the stress, strain is also reduced to its initial state (see Figure 3.1).

Viscoelastic materials are highly influenced by the rate at which strain or stress is
applied. They can alternatively be described as materials which exhibit characteristics of
both elastic and viscous materials. There are various materials that can be mentioned as
examples, including plastics, wood, certain neutral fibers, as well as concrete and certain
metals when exposed to high temperatures.

The influence of time in viscoelastic responses is more noticeable via the constitutive
relation, in which time appears as an additional parameter (to stress and strain). Physi-
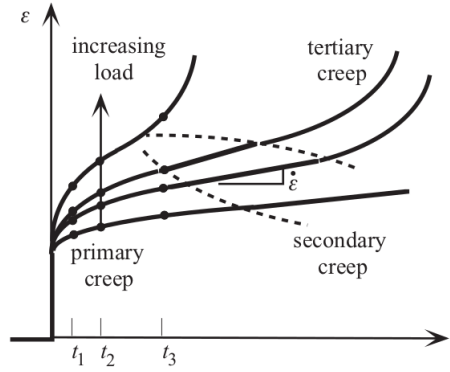
Figure 3.2: The effect of loading on creep behavior [87]

cally the significance of time in viscoelastic responses is observed through phenomena such as creep and relaxation.

3.3 Viscoelastic phenomena

3.3.1 Creep

Slow and continuous deformation of a body under constant stress is defined as creep. It encompasses three stages as illustrated in Fig. 3.2. As can be seen at the first stage, creep occurs at a decreasing rate and is called the primary creep. Geometrical wise, the curve is concave down in this region. In the secondary creep or second stage, deformation is proportional to time, or in another words creep proceeds at constant rate. At the last or tertiary stage, it proceeds at an increasing rate until finally the process ends up in fracture or rupture of the specimen. Total strain at each time would be the sum of elastic and creep strains.

$$\varepsilon = \varepsilon^e + \varepsilon^c. \tag{3.3.1}$$

3.3.2 Recovery

Recovery occurs when the load is removed. Then the elastic part of strain, in companion with a portion of creep strain, is recovered with a decreasing rate. Depending on the material the amount of time-dependent recoverable strain might vary. For example in metals, the contribution of creep strain is very small, while for plastics and polymers creep strain constitutes a large compartment of the total creep.

3.3.3 Relaxation

If a constant strain is applied to a viscoelastic material, it will relax with a gradual decrease in stress. Relaxation and recovery are depicted in Fig. 3.3.
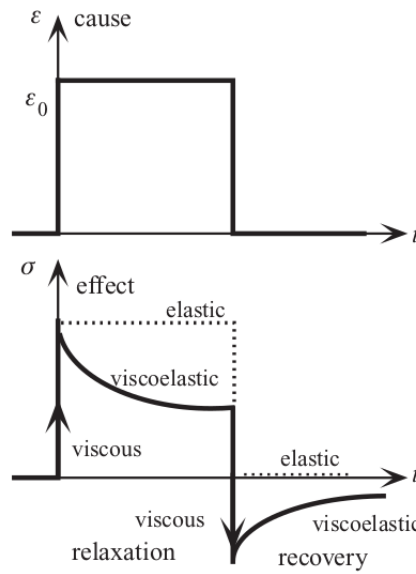
Figure 3.3: Relaxation and Recovery and in Viscoelastic
Materials though interplays of Strain and Stress [87]

3.4 Viscoelastic Models

3.4.1 Fundamental Compartments

Linear viscoelastic models are composed of two main parts namely spring and dashpot.
Assuming that inertial effects are negligible, a linear spring can be represented by:

$$\sigma = E\varepsilon \tag{3.4.1}$$

with E as a spring constant or Young's modulus. Spring obeys the instantaneous elasticity
and recovery, meaning that upon exertion or removal of stress it will immediately show
elongation or recovery respectively as shown in Fig. 3.4.
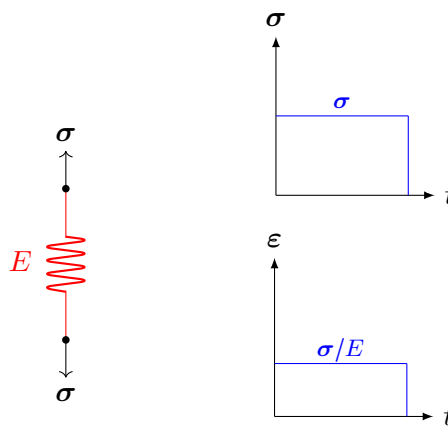
Figure 3.4: Stress and strain response of a linear spring.

On the other hand, dashpot in viscoelastic models represents the viscous nature of a

viscoelastic material. In a dashpot with linear properties, stress is related to the strain rate through a material parameter called coefficient of viscosity which is shown by μ:

$$\sigma = \mu \frac{d\varepsilon}{dt} = \mu \dot{\varepsilon} \tag{3.4.2}$$

From a physical standpoint, dashpot shows a response in which the material deforms continuously with a constant rate under a constant stress (Fig. 3.5).
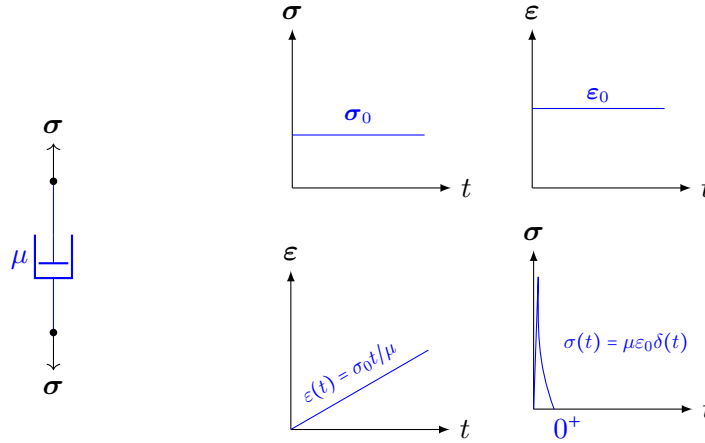
Figure 3.5: Stress and strain response of a linear dashpot [88]

As the graphs illustrate, at the immediate time of imposing a step strain, stress in dashpot tends to infinity, and at the time $t = 0$ the stress will disappear. Mathematically step behavior is modeled using the Dirac delta function $\delta(t)$ where $\delta(t) = 0$ when $t \neq 0+$ and $\delta(t) = \infty$ for $t = 0$. Having this function considered, and under the application of a step strain ε_0 the relationship 3.4.2 reads as:

$$\sigma(t) = \mu \varepsilon_0 \delta(t) \tag{3.4.3}$$

3.4.2 Maxwell Model

Maxwell model is a series combination of a spring and a dashpot as shown in Fig. 3.6.

According to the relationships 3.2.1 and 3.3.1 with imposed strains ε_1 and ε_2 for the dashpot and the spring respectively we can write:

$$\sigma = E\varepsilon_2 \tag{3.4.4a}$$
$$\sigma = \mu \dot{\varepsilon}_1. \tag{3.4.4b}$$

Observing that the elements are connected in series, then the total strain reads as:

$$\varepsilon = \varepsilon_1 + \varepsilon_2,$$

consequently the strain rate would be:

$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2. \tag{3.4.5}$$
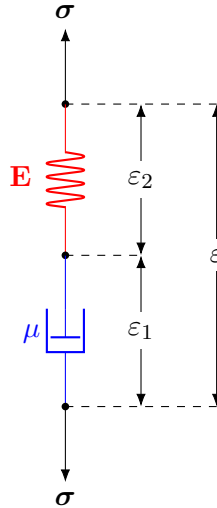
Figure 3.6: Maxwell's model for viscoelastic materials

In Equations (3.4.4) and (3.4.5) since ε_1 and ε_2 are considered to be as internal variables whereas σ and ε would be external variables, to derive the overall relation between the stress and strain one needs to eliminate internal variables. We can eliminate ε_1 and ε_2 by calculating the derivative of (3.4.4a) and inserting that along with (3.4.4b) into (3.4.5), as a result of which we obtain:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\mu} \tag{3.4.6}$$

The above differential equation describes stress and strain states in relationship with time under different loading conditions. For example with a constant stress $\sigma = \sigma_0$ at $t = 0$ equation (3.4.6) turns into a first order differential equation with respect to ε. Integrating that and defining initial conditions as $\sigma = \sigma_0$ at $t = t_0$ we reach to the following relation parameterized with time:

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\mu}t. \tag{3.4.7}$$

Considering another condition in which the constant strain ε_0 is applied at time $t = 0$ and with $\sigma = \sigma_0$ as the initial condition, the following relation describing the stress relaxation as a function of time in response of the applied (constant) strain is obtained:

$$\sigma(t) = \sigma_0 e^{-Et/\mu} = E\varepsilon_0 e^{-Et/\mu}, \tag{3.4.8}$$

where derivative with respect to time, or in other words the rate of stress can be given by:

$$\dot{\sigma} = -(\sigma_0 E/\mu)e^{-Et/\mu} \tag{3.4.9}$$

Figure 3.7 graphically represents such a material response.

Accordingly the initial strain rate at $t = 0+$ would be expressed as $\dot{\sigma} = -\sigma_0 E/\mu$. If a continuous decrease of stress is considered at this initial rate, the above equation reads as:

$$\sigma = -(\sigma_0 Et/\mu) + \sigma_0, \tag{3.4.10}$$

where it can be observed that at time $t_E = \mu/E$ stress would tend to zero; a phenomenon which is called the relaxation time of the Maxwell Model. Relaxation time varies for each viscoelastic material hence it is considered as a characteristic material property.
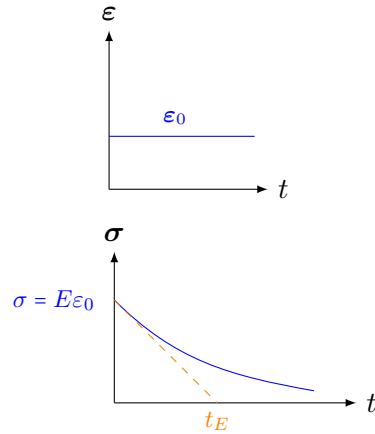
Figure 3.7: Stress Relaxation in Maxwell Model [88]
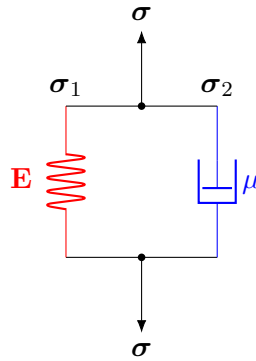
Figure 3.8: Kelvin model for a Viscoelastic Material

3.4.3 Kelvin Model

Kelvin model is composed of a parallel arrangement of a spring and a dashpot as shown in the figure 3.8.

The following relationships between stress and strain exist in this model:

$$\sigma_1 = E\varepsilon, \tag{3.4.11a}$$

$$\sigma_2 = \mu\dot{\varepsilon}. \tag{3.4.11b}$$

Considering that both elements are arranged in parallel, we will have the total stress as the sum of each stress as:

$$\sigma = \sigma_1 + \sigma_2 \tag{3.4.12}$$

Similar to what we observed for the Maxwell model, herein also there exist four unknowns among which σ_1 and σ_2 need to be eliminated through some straightforward mathematical treatments. As a result of that the following relation between stress and strain is obtained:

$$\dot{\varepsilon} + \frac{E}{\mu}\varepsilon = \frac{\sigma}{\mu} \tag{3.4.13}$$

When solved, the above differential equation yields the relationship for creep under a constant stress σ_0 as:

$$\varepsilon = \frac{\sigma_0}{E}(1 - e^{-Et/\mu}).\tag{3.4.14}$$

This behavior is graphically illustrated in figure 3.9. As can be seen the strain increases with a decreasing rate and reaches asymptotically σ_0/E when t tends to infinity. According to this model, when a material is subjected to a stress, first the viscous part of the model bears the response to the applied stress, then gradually transfers larger portions of the load to the elastic part of the model. In this situation the spring element starts to elongate more and more until the whole part of stress is handled by the spring element. This response is called delayed elasticity.
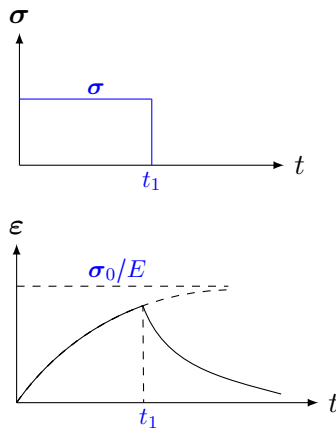
Figure 3.9: Creep and recovery in Kelvin model [88]

Herein under a constant stress σ_0, the strain rate $\dot{\varepsilon}$ in creep is obtained through differentiating (3.4.14) as:

$$\dot{\varepsilon} = \frac{\sigma_0}{\mu}e^{-Et/\mu}.\tag{3.4.15}$$

It is seen that the initial strain rate at $t = 0+$ wound yield a finite value as $\dot{\varepsilon}(0+) = \sigma_0/\mu$, and the strain rate asymptotically approaches to $\varepsilon(\infty) = 0$ as t tends to infinity.

Upon the increase of strain at the initial rate σ_0/μ, it would cross σ_0/E asymptotically at time $t_c = \mu/E$ which is called retardation time. On the other hand the accompanying strain at time t_1 can be determined through the superposition principle. In Kelvin model the resulting strain ε_a from the applied stress at time $t = 0$ is obtained from the equation (3.4.14) as:

$$\varepsilon_a = \frac{\sigma_0}{E}(1 - e^{-Et/\mu}).\tag{3.4.16}$$

The independent resulting strain ε_b from the applied stress at time $t = t_1$ would be:

$$\varepsilon_b = -\frac{\sigma_0}{E}(1 - e^{-E(t-t_1)/\mu}).\tag{3.4.17}$$
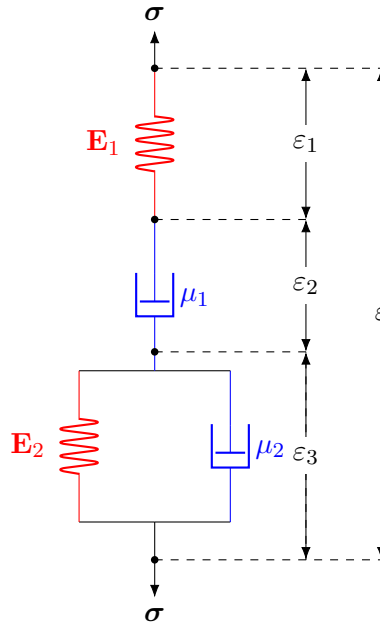
Figure 3.10: Burgers model

For $t > t_1$ strain during the recovery stage is obtained through application of the super-position principle, where assuming that the stress σ_0 is applied at $t = 0$ and removed at $t = t_1$ then one can write:

$$\varepsilon = \varepsilon_a + \varepsilon_b = \frac{\sigma_0}{E}e^{-Et/\mu}\big[e^{Et_1/\mu} - 1\big]. \tag{3.4.18}$$

As can be seen in equation (3.4.18) when time tends ot infinity, the recovery approaches zero. This behavior is also observable from figure 3.9 . Unfortunately both the Maxwell and Kelvin models suffer from significant deficiencies in describing the behavior of most viscoelastic materials. As for instance, time independency of strain during loading or unloading can not be expressed through the Kelvin model. This model is also unable to demonstrate the permanent strain after unloading.

In a similar manner, time-dependent recovery as well as decreasing strain with time-independent (constant) stress may not resolved through the Maxwell model. Both models yield a finite strain rate while for most materials it is very fast. So these limitations in presenting a more thorough model of viscoelastic behaviors, have led to development of more advanced models as discussed in the following.

3.4.4 Four-Element (Burgers) model

Burgers model is in fact a combination of a Kelvin and a Maxwell unit arranged in a configuration as shown in Fig. 3.10.

As can be seen in this model a Kelvin and a Maxwell compartments are joined together in series.To derive the constitutive relation for this model one should consider the strain response under an applied constant stress as shown. The total strain in this arrangement would be the sum of strains in three elements. Worthy to mention that the spring and dashpot elements in the Maxwell model are considered as two elements yielding the total

strain as:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{3.4.19}$$

where ε_1 is defined as:

$$\varepsilon_1 = \frac{\sigma}{E_1} \tag{3.4.20}$$

and the strain rate with ε_2 as the strain in dashpot reads as:

$$\dot{\varepsilon_2} = \frac{\sigma}{\mu_1}. \tag{3.4.21}$$

Here ε_3 would be the strain in the Kelvin unit which yields the corresponding strain rate as:

$$\dot{\varepsilon_3} + \frac{E_2}{\mu_2}\varepsilon_3 = \frac{\sigma}{\mu_2} \tag{3.4.22}$$
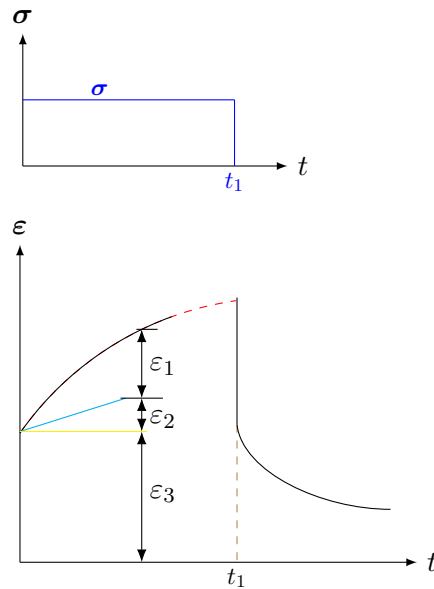
Figure 3.11: Creep and recovery in Kelvin model [88]

This model encompasses five unknowns including $\varepsilon, \sigma, \varepsilon_1, \varepsilon_2, \varepsilon_3$ where ε and σ belong to category of external variables and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ would be internal variables. Thereby to construct the constitutive relation between stress and strain, three strains from the aforementioned four equations need to be eliminated. The final constitutive relation between stress and strain would read as:

$$\sigma + \left(\frac{\mu_1}{E_1} + \frac{\mu_1}{E_2} + \frac{\mu_1}{E_2}\right)\dot{\sigma} + \frac{\mu_1\mu_2}{E_1E_2}\ddot{\sigma} = \mu_1\dot{\varepsilon} + \frac{\mu_1\mu_2}{E_2}\ddot{\varepsilon}. \tag{3.4.23}$$

One of the most recommended ways to solve the equation (3.4.23) is the Laplace transform method as a simple and consistent method. By applying the Laplace transform on equations (3.4.20)-(3.4.23) and assuming $\varepsilon_2 = \varepsilon_3 = 0$ at $t = 0-$ we will have:

$$\hat{\varepsilon} = \hat{\varepsilon}_1 + \hat{\varepsilon}_2 + \hat{\varepsilon}_3, \tag{3.4.24}$$

$$\hat{\varepsilon}_1 = \frac{\hat{\sigma}}{E_1}, \tag{3.4.25}$$

$$s\hat{\varepsilon}_2 = \frac{\hat{\varepsilon}_2}{\mu_1}, \tag{3.4.26}$$

$$(s + \frac{E_2}{\mu_2})\hat{\varepsilon}_3 = \frac{\hat{\sigma}}{\mu_2}, \tag{3.4.27}$$

where $\hat{(\,)}$ denotes the transformed quantity which is now parametrized by s (as a complex variable). Substituting equations (3.4.25)-(3.4.27) into (3.4.24) leads to:

$$\hat{\varepsilon} = \frac{\hat{\sigma}}{E_1} + \frac{\hat{\sigma}}{\mu_1 s} + \frac{\hat{\sigma}}{\mu_2(s + E_2/\mu_2)}. \tag{3.4.28}$$

If both sides of equation (3.4.28) are multiplied by $\frac{\mu_1\mu_2}{E_2}s(s + \frac{E_2}{\mu_2})$ and rearranged, we will have:

$$\hat{\sigma} + (\frac{\mu_1}{E_1} + \frac{\mu_1}{E_2} + \frac{\mu_2}{E_2})s\hat{\sigma} + \frac{\mu_1\mu_2}{E_1 E_2}s^2\hat{\sigma} = \mu_1 s\hat{\varepsilon} + \frac{\mu_1\mu_2}{E_2}s^2\hat{\varepsilon}. \tag{3.4.29}$$

Equation (3.4.23) when accompanied with suitable boundary conditions and solved under constant stress σ_0 could describe the creep behavior of a viscoelastic material as following:

$$\varepsilon = \varepsilon_1 = \frac{\sigma_0}{E_1}, \varepsilon_2 = \varepsilon_3 = 0, t = 0 \tag{3.4.30}$$

$$\dot{\varepsilon} = \frac{\sigma_0}{\mu_1} + \frac{\sigma_0}{\mu_2}, t = 0. \tag{3.4.31}$$

As a result the creep relation can be expressed as:

$$\varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\mu_1}t + \frac{\sigma_0}{E_2}(1 - e^{-E_2 t/\mu_2}). \tag{3.4.32}$$

According to Burgers model, the total creep would be the sum of the creep in Maxwell together with the kelvin unit. The first two terms on the right hand side are related to the contribution of elastic strain and viscosity. The last term denotes delayed elasticity within the kelvin part of the system.

Differentiating with respect to time gives the creep rate as following:

$$\dot{\varepsilon} = \frac{\sigma_0}{\mu_1} + \frac{\sigma_0}{\mu_2}e^{-E_2 t/\mu_2} \tag{3.4.33}$$

Accordingly creep at time $t = 0+$ would have a finite value as:

$$\dot{\varepsilon}(0+) = (\frac{1}{\mu_1} + \frac{1}{\mu_2})\sigma_0 = tan\alpha \tag{3.4.34}$$

As the graph in Fig. 3.11 illustrates, the creep rate starts at the initial time and asymptotically approaches to the value:

$$\dot{\varepsilon} = \sigma_0/\mu_1 = tan\beta \tag{3.4.35}$$

Upon removal of stress at t_1, the recovery could be modeled through the equation (3.4.32) in conjunction with superposition principle in a way that the recovery strain $\varepsilon(t)$, for $t > t_1$, would be the sum of these two independent actions as following:

$$\varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\mu_1}t + \frac{\sigma_0}{E_2}(1 - e^{-E_2 t/\mu_2})$$
$$- \left[\frac{\sigma_0}{E_1} + \frac{\sigma_0}{\mu_1}(t - t_1) + \frac{\sigma_0}{E_2}(1 - e^{-E_2(t-t_1)/\mu_1})\right]$$

or

$$\varepsilon(t) = \frac{\sigma_0}{\mu_1}t_1 + \frac{\sigma_0}{E_2}(e^{E_2 t_1/\mu_2} - 1)e^{-E_2 t/\mu_2} \tag{3.4.36}$$

In terms of recovery, an immediate elastic recovery which is followed by a creep recovery as appearing in equation (3.4.36) are observed. The second term in this relation is a result of contribution of the permanent strain stemming from the viscous flow of μ_1. Hence recovery also tends asymptotically to $\varepsilon(\infty) = (\sigma_0/\mu_1)t_1$ as time approaches to infinity.

Equation (3.4.23) represents the relaxation behavior in the Burgers model. For a step strain ε_0 at $t = 0+$ we will have $\varepsilon = \varepsilon_0 H(t), \dot{\varepsilon} = \varepsilon_0 \dot{\delta}(t), \ddot{\varepsilon} = \varepsilon_0 \frac{d\delta(t)}{dt}$, where $H(t)$ and $\delta(t)$ are the Heaviside and Dirac Delta functions receptively. Consequently equation (3.4.23) is reformed as:

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} = q_1\varepsilon_0\delta(t) + q_2\varepsilon_0\frac{d\delta(t)}{dt} \tag{3.4.37}$$

where

$$p_1 = \frac{\mu_1}{E_1} + \frac{\mu_1}{E_2} + \frac{\mu_2}{E_2}, p_2 = \frac{\mu_1\mu_2}{E_1 E_2}, q_1 = \mu_1, q_2 = \frac{\mu_1\mu_2}{E_2}$$

By applying Laplace transform we will have:

$$\hat{\sigma} + p_1 s\hat{\sigma} + p_2 s^2\hat{\sigma} = q_1\varepsilon_0 + q_2\varepsilon_0 s \tag{3.4.38}$$

which when solved for $\hat{\sigma}$ reads as:

$$\hat{\sigma} = \frac{\varepsilon_0(q_1 + q_2 s)}{1 + p_1 s + p_2 s^2}. \tag{3.4.39}$$

Expanding the above equation along side a Laplace transform leads to an expression for stress relaxation as

$$\sigma(t) = \frac{\varepsilon_0}{A}[(q_1 - q_2 r_1)e^{-r_1 t} - (q_1 - q_2 r_2)e^{-r_2 t}], \tag{3.4.40}$$

where

$$r_1 = (p_1 - A)/p_2, r_2 = (p_1 + A)/p_2, A = \sqrt{p_1^2 - 4p_2}.$$

Maxwell and Kelvin units can be arranged in several ways within a Burgers model that leads to five different subgroups. Each of these subgroups would represent a specific behavior ranging from solid-like and liquid-like responses to delayed elasticity response with retardation times.

Despite the apparent diversity of proposed models, these models are unable to accurately predict the behavior of most viscoelastic materials. Additionally some materials, such as concrete and some polymers, undergo the aging phenomenon which results in structural and compositional changes in relation to time and in the course of creep. These structural changes lead to alteration of material constants in the course of time. To cope with such a situation various complex models have been proposed in the form of Generalized Maxwell and Kelvin Models.

3.4.5   Generalized Maxwell and Kelvin models

If a number of Maxwell models are connected together in series, as shown in the figure 3.12, the related constitutive equation reads as:
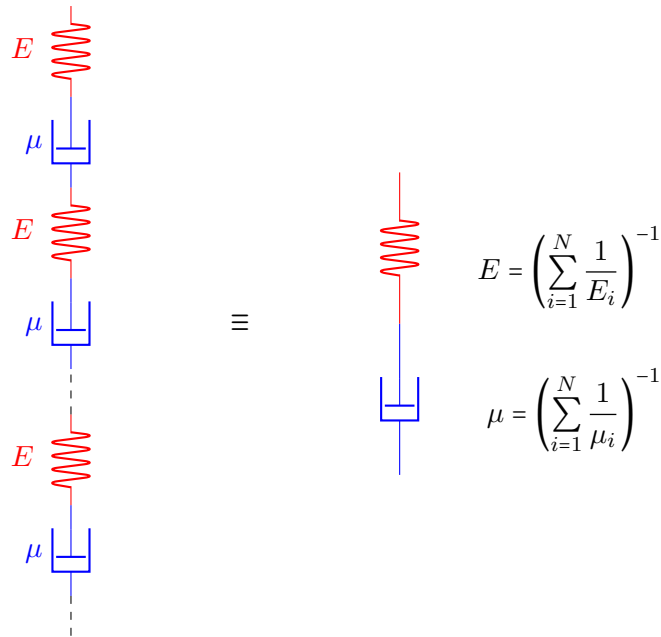
Figure 3.12: Generalized Maxwell Models in Series

$$\dot{\varepsilon} = \dot{\sigma} \sum_{i=1}^{N} \frac{1}{E_i} + \sigma \sum_{i=1}^{N} \frac{1}{\mu_i} \tag{3.4.41}$$

The above equation is identical to its counterpart namely equation (3.4.6) which represents the same mechanical response. Arranging a series of such elements each of which contains a spring and a dashpot would result in an identical model to the Maxwell model as can be seen in Fig. 3.12.

On the other hand, if a number of Kelvin models are arranged in a parallel form, they would show the same response as a Kelvin model, as in Figure 3.13. The related constitutive equation reads as:
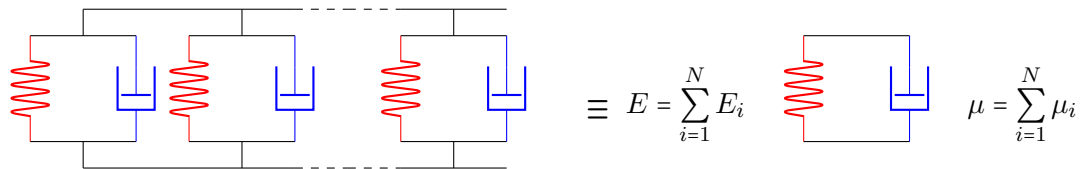
Figure 3.13: Generalized Kelvin Model in Parallel

$$\sigma = \varepsilon \sum_{i=1}^{N} E_i + \dot{\varepsilon} \sum_{i=1}^{N} \mu_i \tag{3.4.42}$$
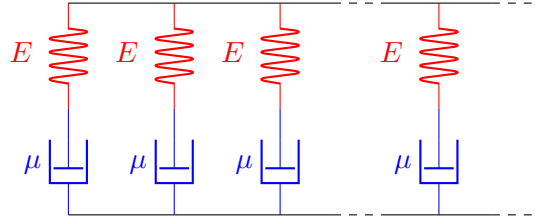
Figure 3.14: Generalized Maxwell Models in parallel

When several Maxwell models are connected together in parallel, the model will be able to exhibit instantaneous elasticity with retardation times, stress relaxation with different relaxation times as well as viscous flow. In this case, when Maxwell models are connected in parallel, describing the stress associated with applied variation in strain (Figure 3.14) would be a rather feasible task where the same prescribed strain is assigned to each element and the overall output stress is the sum of contributions from each individual element. The contribution of the first element reads as:

$$D_\varepsilon = \frac{D\sigma_1}{E_1} + \frac{\sigma_1}{\mu_1} \tag{3.4.43}$$

where D is the differential operator with respect to time, $D = \frac{d}{dt}$, based on which for stress one can write:

$$\sigma_i = \frac{D}{\frac{D}{E_i} + \frac{1}{\mu_1}} \varepsilon \tag{3.4.44}$$

Summing up the stress would give us:

$$\sigma = \sum_{i=1}^{a} = \left(\sum_{i=1}^{a} \frac{D}{\frac{D}{E_i} + \frac{1}{\mu_1}} \right) \varepsilon. \tag{3.4.45}$$

Multiplying both sides by $\Pi(\frac{D}{E_i} + \frac{1}{\mu_i})$, where Π indicates the product of a terms, the differential operator could be eliminated from the denominator of the relation as:

$$\left[\left(\frac{D}{E_1} + \frac{1}{\mu_1} \right) ... \right] \sigma = \left[\left(\frac{D}{E_1} + \frac{1}{\mu_1} \right) \left(\frac{D}{E_2} + \frac{1}{\mu_2} \right) ... \right] \left[\frac{D}{\frac{D}{E_1} + \frac{1}{\mu_1}} + \frac{D}{\frac{D}{E_2} + \frac{1}{\mu_2}} + ... \right] \tag{3.4.46}$$

There exists another generalized model in which several numbers of Kelvin models are connected together in series as shown in Figure 3.15.

In this model strain contribution of the first element expressed as:

$$\varepsilon_1 = \frac{1}{D\mu_1 + E_1} \sigma \tag{3.4.47}$$

The sum of the strain contribution of a-elements is described as:

$$\varepsilon = \sum_{i=1}^{a} \varepsilon_i = \left(\sum_{i=1}^{a} \frac{1}{D\mu_i + E_i} \right) \sigma \tag{3.4.48}$$
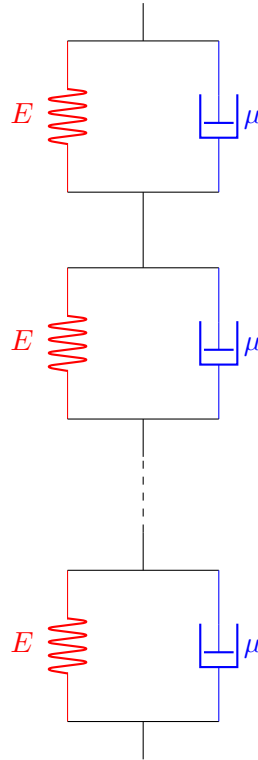
Figure 3.15: Kelvin models arranged in series

As before, if both sides are multiplied by $\Pi_{i=1}^{a}(D\mu_i + E_i)$ then we will have:

$$
\begin{aligned}
& [(D\mu_1 + E_1)(D\mu_2 + E_2)(D\mu_3 + E_3)...]\varepsilon] \\
& = [(D\mu_2 + E_2)(D\mu_3 + E_3)... + (D\mu_1 + E_1)(D\mu_3 + E_3)... + ...]\sigma
\end{aligned}
\tag{3.4.49}
$$

It has been seen that the generalized Kelvin model is more suitable in cases with prescribed stress history, however when we have prescribed strain history, generalized Maxwell model is preferred overt the Kelvin model. Due to the rich explanation of relaxation times both these models offer, they can describe behavior of viscoelastic materials in a more realistic manner and specially in a wider time spans, compared to simpler models were discussed before.

3.5   Differential and Integral Representations

3.5.1   Differential Form of Constitutive Relations

The constitutive relation of a viscoelastic material in the case of simple loading conditions can be described through a linear function of stress and strain fields as well as their time derivatives as

$$
f(\sigma, \dot{\sigma}, \ddot{\sigma}, ...; \varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}, ...) = 0,
\tag{3.5.1}
$$

where stress and strain are functions parametrized with time as $\sigma = \sigma(t)$ and $\varepsilon = \varepsilon(t)$, respectively. Dots indicate the derivatives with respect to time. Employing differential

operators, the above equation can be written as:

$$P\sigma = Q\varepsilon \qquad (3.5.2)$$

where P and Q as linear differential operators are defined as:

$$P = \sum_{r=0}^{a} p_r \frac{\partial^r}{\partial t^r}, Q = \sum_{r=0}^{b} q_r \frac{\partial^r}{\partial t^r}. \qquad (3.5.3)$$

When these equations are combined, a differential form of constitutive relation can be written as:

$$P\sigma = p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \dots + p_a\frac{\partial^a}{\sigma}\partial t^a$$
$$= q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \dots + q_b\frac{\partial^b\sigma}{\partial t^b} = Q_\varepsilon \qquad (3.5.4)$$

Herein again the dots represent differential with respect to time and $p_0, p_1, \dots$ as well as $q_0, q_1, q_2, \dots$ represent material constants. After application of a Laplace transform with zero initial conditions we obtain:

$$\hat{P}(s)\hat{\sigma}(s) = (p_0 + p_1 s + p_2 s^2 + \dots + p_a s^a)\hat{\sigma}(s)$$
$$= \hat{Q}(s)\hat{\varepsilon}(s) = (q_0 + q_1 s + q_2 s + \dots + q_b s^b)\hat{\varepsilon}(s) \qquad (3.5.5)$$

Here s indicates the transform variable originating from Laplace transform.
From the above relation it can be deduced that:

$$\frac{\hat{Q}(s)}{\hat{P}(s)} = \frac{\hat{\sigma}}{\hat{\varepsilon}} \qquad (3.5.6)$$

In the case of linear processes p_r and q_r would not depend on stress and strain, however they might depend on time.

In equation (3.5.4) terms can be arranged and combined in a way to represent different idealized viscoelastic processes. Differential forms can be employed fairly conveniently in connection with corresponding models composed basically of spring and dashpot. For example by choosing

$$p_0 = 1, p_1 = \frac{\mu}{E}, q_1 = \mu \qquad (3.5.7)$$

and setting other coefficients to zero, equation (3.5.4) would represent the Maxwell model. Upon choosing the coefficient as

$$p_0 = 1, q_0 = E, q_1 = \mu \qquad (3.5.8)$$

and setting the remaining coefficients to zero, the same equation leads to a mathematical representation of the Kelvin model. However, if only three coefficients either p_0, p_1, p_2 or p_0, q_0, q_1 are defined, the output constitutive relation would not reconstruct either Maxwell or Kelvin models accurately. For instance, in that case, Kelvin model would not be able to exhibit instantaneous deformation and transient creep under constant loading. Additionally, fixing those parameters would only result in a specific group of Burgers models.

It should be noted that if large number of parameters are needed to express the materials behavior, determining the corresponding coefficients would not be practical; in such cases other approaches need to be employed.

3.5.2 Integral form of Constitutive Relations

Differential forms are not the only available approach for describing stress-strain relationships. There exists another method for presenting the relationships mathematically which is based on the notion of integral. Integral representation of constitutive relations offer flexibility in introducing viscoelastic properties, and describing the aging process, as well as incorporating the influence of temperature. In this method Boltzmann superposition principle is employed to derive the constitutive relations between creep and relaxation modules.

Assuming a linear behavior for a material under study (e.g. creep test), a
step constant stress is imposed and the output strain as a function of time is measured accordingly through:

$$\varepsilon(t) = J(t)\sigma_0, \tag{3.5.9}$$

or

$$J(t) = \varepsilon(t)/\sigma_0 \tag{3.5.10}$$

where J would be the creep compliance and is defined as the creep per unit of imposed stress. As an example for a material response obeying a Maxwell model we have:

$$J(t) = \left(\frac{1}{E} + \frac{1}{\mu}t\right). \tag{3.5.11}$$

On the other hand throughout a relaxation test, a step of constant strain as defined as $\varepsilon = \varepsilon_0 H(t)$ is applied and the resulting stress is recorded. Assuming that the material responses linearly, one can write:

$$\sigma(t) = E(t)\varepsilon_0 \tag{3.5.12}$$

$$E(t) = \sigma(t)/\varepsilon_0 \tag{3.5.13}$$

where $E(t)$ is the relaxation function and as can be seen is a function of time. It is expressed as the stress per unit of applied strain and is of the material property meaning that it differs for each material. For a Maxwell model we have:

$$E(t) = E e^{(-Et/\mu)} \tag{3.5.14}$$

3.5.3 Boltzmann's Superposition Principle

The integral representation of viscoelastic solids is based on Boltzmann's superposition principle. Assuming that a constant stress σ_1 is applied at $t = \tau_1$, then $\sigma(t) = \sigma_1 H(t - \tau_1)$ and the corresponding strain as a result of introduced creep would read as:

$$\varepsilon(t) = \sigma_1 J(t - \tau_1) H(t - \tau_1) \tag{3.5.15}$$

where σ_0 and σ_1 would be imposed stresses at initial time $t = 0$ and time $t = \tau_1$ respectively. The overall strain as a result of imposed stress at any given time after $t = \tau_1$ would be the sum of strains at previous times. This is known as the Boltzmann superposition principle and illustrated in figure 3.16.
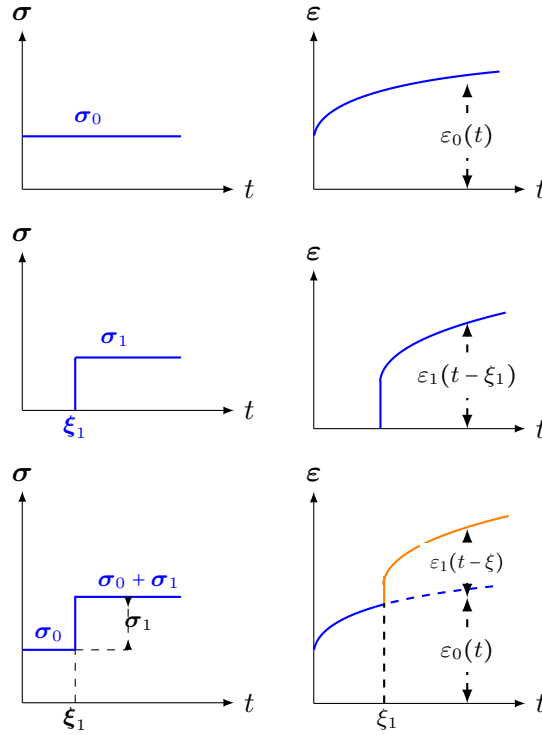
Figure 3.16: Boltzmann Superposition Principle [88]

Assuming an arbitrary behavior for input stress $\sigma(t)$ (opposed to previous constant value) it can be approximated by sum of constant stresses as depicted in Figure 3.17.

The varying stress in this case as a function of time can be described as:

$$\sigma(t) = \sum_{i=1}^{r} \Delta\sigma_i H(t - \tau_i). \tag{3.5.16}$$

Under these conditions and again bearing the Boltzmann principle one can express the strain as following:

$$\varepsilon(t) = \sum_{i=1}^{r}(t - \tau_i) = \sum_{i=1}^{r} \Delta\sigma_i J(t - \tau_i) H(t - \tau_i) \tag{3.5.17}$$

As the number of steps approaches to infinity, the total created strain in the material can be described as:

$$\varepsilon(t) = \int_0^t J(t - \tau) H(t - \tau) d[\sigma(\tau)] \tag{3.5.18}$$

The above relation should be seen in the form of Stieltjes integral; noting that stress history is differentiable and τ, as a dummy variable is always less than or equal to t. The Heaviside function $H(t-\tau)$ remains always unity. Therefore the integral relation (3.5.18) cab be reduced to :

$$\varepsilon(t) = \int_0^t J(t - \tau)\frac{\partial\sigma(\tau)}{\partial\tau}d\tau, \tag{3.5.19}$$
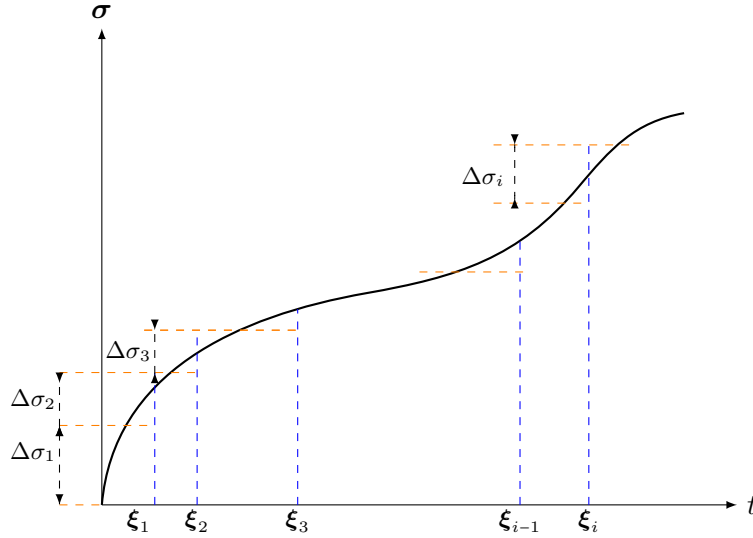
Figure 3.17: Varying input stress as approximated based on
Boltzmann superposition principle [88]

which expresses the creep under an arbitrary stress history given that the creep compliance
which is a material constant is a determined parameter.

An alternate form for (3.5.19) is devised by means of integration by parts and defining
$u = J(t - \tau)$ and $dv = \frac{\partial \sigma(\tau)}{\partial \tau}$ leading to:

$$\varepsilon(t) = \sigma(t)J(0) - \int_0^t J'(t - \tau)\sigma(\tau)d\tau \qquad (3.5.20)$$

where $J'(t - \tau) = \frac{\partial J(t-\tau)}{\partial \tau}$.

Creep compliance can be expressed as a combination of two components, one an elastic
compliance J_0 which is independent of time and the other, creep function $\varphi(t)$ which is a
function of time. In that case the equation (3.5.20) is converted into:

$$\varepsilon(t) = J_0\sigma(t) + \int_0^t \varphi(t - \varepsilon)\frac{\partial \sigma(\tau)}{\partial \tau}d\tau \qquad (3.5.21)$$

Equation 3.5.19 can be employed to calculate the stress based on the prescribed strain.
However, we need to solve an integral equation. The same scenario holds when the stress
needs to be acquired for example in the case where step strains are imposed and resul-
tant stress needs to be calculated. In these situations Boltzmann superposition principle
would be helpful. Using this principle, equations (3.5.19)-(3.5.21) can be reformulated by
substituting stress and strain. As a result the following relation can be obtained:

$$\sigma(t) = \int_0^t E(t - \tau)\frac{\partial \varepsilon(\tau)}{\partial \tau}d\tau \qquad (3.5.22)$$

Again dividing the modulus of relaxation into an elastic modulus E_0 (which is time-
independent), and stress relaxation $\psi(t)$ as a time dependent function would yield:

$$\sigma(t) = E_0\varepsilon(t) - \int_0^t \psi(t - \tau)\frac{\partial \varepsilon(\tau)}{\partial \tau}d\tau \qquad (3.5.23)$$

It can be seen that upon a known strain history, equation (3.5.22) is simpler to calculate the stress relaxation compared to its counterpart equation i.e. equation (3.5.19), as it offers the possibility of a direct integration. However $E(t)$ still needs to be determined from relaxation experiments.

3.6 Staverman-Schwarzl-Mandel-Brun Functional

Herein the expressions for free energy, free enthalpy and the dissipated power as suggested by Staverman & Schwarzl and Mandel & Brun (SSMB Theory) for isothermal linear viscoelasticity are recapitulated. The free energy in our formulation stands for Helmholtz free energy which is denoted by ψ and free enthalpy represents the thermodynamic density which is known as the Gibbs free energy and is shown by $\mathcal{G}$.

Constitutive equations for viscoelasticity in linear regime when expressed in form of Stieltjes integrals or Stieltjes convolution read as following:

$$\psi(t) = \int_{0^-}^{t} E(t-v) : d\epsilon(v) \tag{3.6.1}$$

$$\mathcal{G}(t) = \int_{0^-}^{t} J(t-v) : d\sigma(v) \tag{3.6.2}$$

where the kernels $E(t)$ and $J(t)$ respectively stand for relaxation and creep functions which mathematically are tensorial objects of rank fourth. Accordingly the free energy density per unit volume reads as:

$$
\begin{aligned}
\rho\psi(t) &= \frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} E(2t-\tau-\eta) : d\epsilon(\tau) : d\epsilon(\eta) \\
&= \frac{1}{2} \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \int_{0^-}^{2t-\tau} E(2t-\tau-\eta) : d\varepsilon(\tau) : d\varepsilon(\eta) \\
&= \frac{1}{2} \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \sigma(2t-\tau) : d\varepsilon(\eta) \\
&= \frac{1}{2} \int_{0^-}^{t} [\sigma(2t-\tau) : d\varepsilon(u) + \sigma(\tau) : d\varepsilon(2t-\tau)] \\
&= \sigma(t) : \varepsilon(t) - \frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} f(2t-\tau-\eta) : d\sigma(\tau) : d\sigma(\eta)
\end{aligned}
\tag{3.6.3}
$$

and free enthalpy density per unit volume would be

$$
\begin{aligned}
\rho\gamma(t) = \rho\phi(t) - \sigma(t) : \varepsilon(t) &= -\frac{1}{2} \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \int_{0^-}^{2t-\tau} f(2t-\tau-\eta) : d\sigma(\eta) : d\sigma(\tau) \\
&= -\frac{1}{2} \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \varepsilon(2t-\tau) : d\sigma(\tau) \\
&= -\frac{1}{2} \int_{0^-}^{t} [\varepsilon(2t-\tau) : d\sigma(\tau) + \varepsilon(\tau) : d\sigma(2t-\tau)]
\end{aligned}
\tag{3.6.4}
$$

and dissipated power density would be

$$
\begin{aligned}
\Lambda(t) &= -\int_{0^-}^{t}\int_{0^-}^{t}\dot{E}(2t-\tau-\eta):d\varepsilon(\eta):d\varepsilon(\tau)\\
&= -\left(\int_{0^-}^{t}-\int_{t}^{2t}\right)\int_{0^-}^{2t-\tau}\dot{E}(2t-\tau-\eta):d\varepsilon(\eta):d\varepsilon(\tau)\\
&= -\left(\int_{0^-}^{t}-\int_{t}^{2t}\right)\dot{\sigma}(2t-\tau):d\varepsilon(\tau)\\
&= -\int_{0^-}^{t}\left[\dot{\sigma}(2t-\tau):d\varepsilon(\tau)+\dot{\sigma}..d\varepsilon(2t-\tau)\right]\\
&= +\int_{0^-}^{t}\left[\dot{\epsilon}(2t-\tau):d\sigma(\tau)+\dot{\varepsilon}:d\sigma(2t-\tau)\right]\\
&= +\left(\int_{0^-}^{t}-\int_{t}^{2t}\right)\dot{\varepsilon}(2t-\tau):d\sigma(\tau)\\
&= +\left(\int_{0^-}^{t}-\int_{t}^{2t}\right)\int_{0^-}^{2t-\tau}\dot{f}(2t-\tau-\eta):d\sigma(\eta):d\sigma(\tau)\\
&= \int_{0^-}^{t}-\int_{0^-}^{t}\dot{f}(2t-\tau-\eta):d\sigma(\eta):d\sigma(\tau)\geq 0
\end{aligned}
\tag{3.6.5}
$$

The above equations were derived by Mandel and Brun through two different methods. Thereafter were generalized by Staverman and Schwarzl [89].

3.7   Concept of pseudo-convolution

As can be seen classical variational principles are derived through the concept of convolution and in particular Stieltjes convolutions and Stieltjes biconvolutions. So they are described here as following. Stieltjes convolution of two symmetric second rank tensors $a(t)$ and $b(t)$ are defined as

$$
c_a(b;t) = a*b = \int_{0^-}^{t}a(t-\tau):db(\tau) = \int_{0^-}^{t}b(t-\tau):da(u) = b*a
\tag{3.7.1}
$$

3.7.1   Stieltjes biconvolution

Stieltjes biconvolution is defined as

$$
a(t) = g*e = \int_{0^-}^{t}g(t-\eta):de(\eta) \Rightarrow
$$

$$
2c_e(e,b;t) = \int_{0^-}^{t}\int_{0^-}^{(t-\tau)}g(t-\tau-\eta):de(\eta):db(\tau)
\tag{3.7.2}
$$

$$
= g*e*b[g_{ijkl}*e_{kl}*b_{ij}] = g*b*e
$$

with symmetry conditions as

$$
g_{ijkl}(t) = g_{klij}(t) = g_{jikl}(t) = g_{ijlk}(t), \qquad \forall t.
\tag{3.7.3}
$$

Pseudo-convolution of two symmetric second rank tensor $a(t)$ and $b(t)$ would be

$$
2C_a(b,t) = a\,\square\, b = \left(\int_{0^-}^{t}-\int_{t}^{2t}\right)a(2t-\tau):db(\tau)
\tag{3.7.4}
$$

Pseudo-convolution form as denoted by $a \,\square\, b$ is observed in the above relationships written for the free energy and free enthalpy densities.

In this regard, pseudo-biconvolution $B_g(e, b, t)$ can be defined as the composition $g * e \,\square\, b$ of a pseudo-convolution $a \,\square\, b$ through the classical Stieltjes convolution in which a is expressed as $g * e$. For example bilinear pseudo-biconvolution can be written as

$$a_g(e; 2t - \tau) = g * e = \int_{0^-}^{2t-\tau} g(2t - \tau - \eta) : de(\eta) \Rightarrow \tag{3.7.5}$$

$$2B_g(e, b; t) = g * e \,\square\, b = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) g(2t - \tau - \eta) : de(\eta) : db(\tau) \tag{3.7.6}$$

$$a_g(e; 2t - \tau) = g * e = \int_{0^-}^{2t-\tau} g(2t - \tau - \eta) : de(\eta) \Rightarrow$$

$$2B_g(e, b; t) = g * e \,\square\, b = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \int_{0^-}^{2t-\tau} g(2t - \tau - \eta) : de(\eta) : db(\tau) \tag{3.7.7}$$

$$= \int_{0^-}^{t} \int_{0^-}^{t} g(2t - \tau - \eta) : de(\eta) : db(\tau) \neq g * b \,\square\, e = 2B_g(b, e; t)$$

For quadratic pseudo-biconvolution we have

$$2Q_g(b; t) = g * b \,\square\, b$$

$$= \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \int_{0^-}^{2t-\tau} g(2t - \tau - \eta) : db(\eta) : db(\tau) \tag{3.7.8}$$

$$= \int_{0^-}^{t} \int_{0^-}^{t} g(2t - \tau - \eta) : db(\eta) : db(\tau) \neq g * b \,\square\, e = 2B_g(b, b; t)$$

Chapter 4

Thermoelastic Damping

4.1   Introduction

Nowadays resonators constitute an important part of advanced technologies such as Microelectromechanical and Nanoelectromechanical systems (MEMS, NEMS). They are employed for instance as high precision actuators, micro-sensors, as well as in atomic force microscopy and semiconductors [90–93].

Theses structures are usually integrated into electronic systems encompassing semiconductors and other components which are designed based on the principle of minimized consumption of power. In these systems, efficiency is in direct relationship with the loss of energy. Dissipated energy within a structure could be controlled in design step of such a resonator. One of the main phenomena that is recognized to the source of such an energy loss would be thermoelastic damping [94–96].

Thermoelastic damping is a form of energy dissipation in thermoelastic solids. Considering a flexural vibration for a beam, generation of inhomogeneous stress field is observed. The part of the beam which undergoes tension, is expanded and is cooled down. On the other hand, the side that is undergoing compression, is heated up. In thermoelastic bodies, strain gradients are coupled to the temperature gradients; as a result of which an oscillating temperature gradient is created within the beam. To return to the equilibrium state, structure tries to reestablish the thermal equilibrium through the heat flow. This would be an irreversible process causing entropy generation and consequently dissipation of energy in the form of thermoelastic damping [97, 98]. The metric used for measuring thermoelastic damping is a parameter called the loss factor, which is in fact a dimensionless number used. In inverse form, sometimes it is called quality factor as well.

The theory of thermoelastic damping was first introduced through the studies of Zener in 1937-1938 [99, 100]. He endeavored to find out a relationship for loss factor though 1D modeling of flexural vibration of a beam. Zener's theory was developed to 3D by Alblas [101, 102] and Chadwick [103].

Since then many other studies have been conducted to extend the theory in various directions. Some researches were focused to study thermoelastic damping in composite and/or laminated structures as pioneered by Bishop and Kinra [104, 105] and pursued by others [106–108] until recently. Yi et al also examined different geometries and modes of vibration [109, 110].

Aside from classical theories in elasticity, non-classical and size dependent theories have also been utilized in deriving relationships for expressing quality factor in thermoelastic

solids. Kumar and Mukhopadhyay in a series of studies investigated size-dependent effects in thermoelastic damping in resonators [111–114]. Yang et al [115, 116] and more recently Wang et al [117] developed models based on stress couple theories. Gu et al [118, 119] investigated thermoelastic damping in microbeam resonators taking into account the effect of thermal relaxation dual-phase-lag in the framework of nonlocal strain gradient theory. Utilizing the same theory Ge et al [120] also Borjalilou and Asghari [121, 122] studied thermoelastic damping in micro-/nano-plates.

4.2   Zener's Theory

Traveling of an acoustic wave inside an infinitely large elastic body, or vibration of an elastic resonator with finite geometry, are accompanied with damping as a result of nonlinear interactions with phonons (or thermally-excited modes). The complex interactions between acoustic waves and thermal phonons are captured through a material property called thermal expansion coefficient, expressed as:

$$\alpha = \frac{1}{L}\frac{\partial L}{\partial T} \tag{4.2.1}$$

It is through this coefficient that the changes in length are coupled to changes in temperature.

The motion of an elastic body involves changes in kinetic and potential energies. While in a perfectly linear elastic material under isothermal conditions nonequilibrium condition will take forever, in a thermoelastic material due to existing coupling between the strain field and temperature field an energy dissipation will occur. As mentioned, pursuing the relaxation or equilibrium state would be achievable through irreversible heat flow stemming from temperature gradients. This process, i.e. thermoelastic damping, via the expansion coefficient imposes an upper limit to the quality factor of a resonator in spite of a perfect design.

The model Zener presented for approximating thermoelastic damping relies on the extension of Hooke's law where the stress and strain fields as well as their time derivatives are involved as following:

$$\sigma + \tau_\epsilon \dot{\sigma} = M_R(\epsilon + \tau_\sigma \dot{\epsilon}) \tag{4.2.2}$$

According to above relation, when the strain is constant, τ_σ would be the time through which stress is relaxed exponentially. In a similar fashion, when the stress is kept constant, strain will be relaxed with τ_σ as the stress relaxation time. Herein, M_R would be the relevant elastic modulus when all the relaxation has gone through. On the other hand M_U would be unrelaxed value of the elastic modulus which is defined as:

$$M_U = M_R(\tau_\sigma/\tau_\epsilon) \tag{4.2.3}$$

In periodic dynamical conditions amplitudes of stress and strain are in accordance with complex elastic modulus as:

$$\sigma(t) = \sigma_0 e^{i\omega t} \tag{4.2.4}$$

$$\epsilon(t) = \epsilon_0 e^{i\omega t} \tag{4.2.5}$$

Dissipation in a material which is also known as internal friction is represented as Q^{-1} and would be the fraction of ene rgy lost per radian of vibration. In case of small values of radiation of vibration, the dissipation would become the ratio of the imaginary and real parts of the complex modulus as:

$$Q^{-1} = \Delta_M \frac{\omega\tau}{1 + (\omega\tau)^2} \tag{4.2.6}$$

with $\tau = \sqrt{\tau_\sigma \tau_\epsilon}$ and

$$\Delta_M = \frac{M_U - M_R}{\sqrt{M_R M_U}} \tag{4.2.7}$$

where Δ_M is called relaxation strength and is dimensionless parameter.

In the case that the vibration frequency is much smaller than the effective relaxation rate, the system will hold its equilibrium, meaning that the amount of dissipation would be negligible. Even if it were much larger, then there would remain on time for the system to relax and again very limited dissipation will occur. However when these values are comparable, the body will experience energy dissipation.

For thermoelastic solids the relaxation strength would read as:

$$\Delta_E = \frac{E_{ed} - E}{E} = \frac{E\alpha^2 T_0}{C_p} \tag{4.2.8}$$

where $E_a d$ would be the adiabatic or unrelaxed Young's modulus and E would be isothermal or relaxed value. C_p would be heat capacity at constant pressure or stress.
Zener considered τ_Z as the relaxation time for a rectangular thin beam under flexure as:

$$\tau_Z = \frac{b^2}{\pi^2 \chi} \tag{4.2.9}$$

where b would be width of the beam and χ is the thermal diffusivity. Under these conditions Q^{-1} with a negligible error would be described as:

$$Q_Z^{-1} = \frac{E\alpha T_0}{C_p} \frac{\omega\tau_Z}{1 + (\omega\tau_Z)^2} \tag{4.2.10}$$

Zener's theory explains thermoelastic damping with negligible error, nevertheless Lifshitz and Roukes [123] later on developed an exact expression for beams with simple geometry which would be discussed in following sections.

4.3   Thermoelastic Damping in Thin Beams

Consider a beam with length L and a cross section with dimensions $b \times c$ undergoing bending oscillations along x. It is also assumed that the material is homogeneous and isotropic where no initial stress is applied. Herein normal stress and strain are related through the Hooke's law [124, 125] as:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) + \alpha\theta$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) + \alpha\theta \tag{4.3.1}$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) + \alpha\theta$$

Initial Temperature: T_O
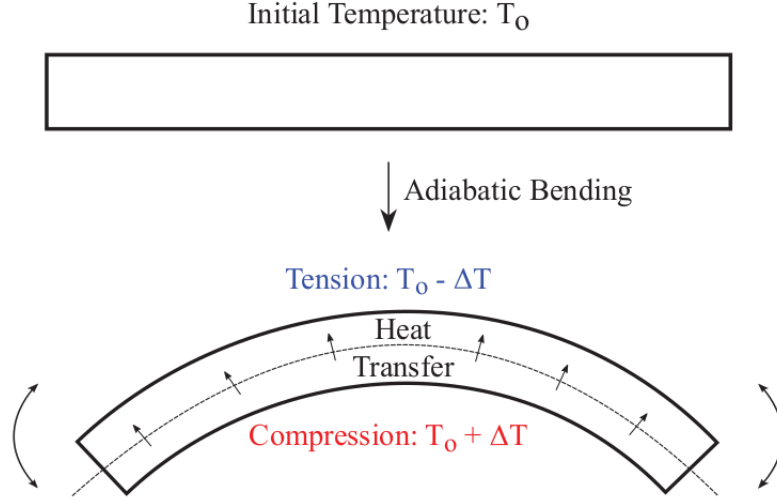
Figure 4.1: Irreversible heat flow within a beam undergoing bending [124]

Here θ would read as excess temperature as

$$\theta(x, y, z, t) = T(x, y, z, t) - T_0$$

In equilibrium the temperature of the beam which is homogeneously distributed within the beam would be T_0. With external stimulus, which here would be flexural displacement, the temperature departs from the equilibrium as represented through $T = T_0 + \theta$.

The change of temperature with time obeys the coupled relation of heat conduction as:

$$C\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta - \frac{E\alpha T}{(1 - 2\nu)} \frac{\partial}{\partial t}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \qquad (4.3.2)$$

where C would be the specific heat per unit volume, and k would be the thermal conductivity coefficient. It is common however to consider a linearized version of the above relation through replacing T with T_0 in the source term. Accordingly the linearized relation is read as:

$$C\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta - \frac{E\alpha T_0}{(1 - 2\nu)} \frac{\partial}{\partial t}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \qquad (4.3.3)$$

Under the above conditions, where the beam on its surface does not experience any stresses, all but the σ_{xx} component of the stress tensor is vanished leaving the strain and stress fields relations as:

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} + \alpha\theta \qquad (4.3.4)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} + \alpha\theta \qquad (4.3.5)$$

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \qquad (4.3.6)$$

where E would be the Young's modulus, ν is the Poisson's ratio and α reads as the coefficient of thermal expansion.

The beam curvature can be replaced with $-\frac{\partial^2 U}{\partial x^2}$ where U would be the potential energy of the system (a function of position). Thereby, strain components can be simplified as:

$$\varepsilon_{xx} = -y\frac{\partial^2 U}{\partial x^2}, \tag{4.3.7}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \nu y\frac{\partial^2 U}{\partial x^2} + (1+\nu)\alpha\theta \tag{4.3.8}$$

which leads to the following equation of motion for the beam:

$$\rho A\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 U}{\partial x^2} + E\alpha I_T) = 0 \tag{4.3.9}$$

where ρ is the density and A is the cross section area. I and I_T are defined as mechanical and thermal contribution to moment of inertia and defined as integrals over the cross section of the beam which are given by:

$$I = \int_A y^2 dydz \tag{4.3.10}$$

$$I_T = \int_A y\theta dydz \tag{4.3.11}$$

Heat conduction equation under the thermoelastic coupling is given as:

$$\frac{\partial\theta}{\partial t} = \kappa\nabla^2\theta - \frac{E\alpha T}{(1-2\nu)C_v}\frac{\partial}{\partial t}\sum_j \varepsilon_{jj} \tag{4.3.12}$$

The linearized heat equation in which $\theta \ll T_0$ and temperature variation is considered to be along the y direction, reads as:

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial y^2} + y\frac{\Delta_E}{\alpha}\frac{\partial}{\partial t}(\frac{\partial^2 U}{\partial x^2}) \tag{4.3.13}$$

Now substituting the strain field would lead to the final form as:

$$(1 + 2\Delta_E\frac{1+\nu}{1-2\nu})\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial t} + y\frac{\Delta_E}{\alpha}\frac{\partial}{\partial t}\frac{\partial^2 U}{\partial x^2} \tag{4.3.14}$$

Obtained thermoelastically coupled equation are solved for the case of simple harmonic vibrations by setting

$$U(x,t) = U_0(x)e^{i\omega t}, \qquad \theta(x,y,t) = \theta_0(x,y)e^{i\omega t} \qquad (4.3.15)$$

Temperature profile along the beam is calculated using the heat equation, accordingly substituted into the equation of motion leading to an expression for the normal modes of vibration and corresponding frequencies. The real part of frequencies provides eigenfrequencies under thermoelastic coupling condition and the imaginary part describes the attenuation of the vibration. Thermoelastic damping then alternatively could be given by

$$Q^{-1} = 2\left|\frac{Im(\omega)}{Re(\omega)}\right| \qquad (4.3.16)$$

where ω would be complex-valued eigenfrequency. This fraction is in fact the fraction of energy which is lost in relationship to the radian of vibration of the beam.
By substituting equation 4.3.15 into the heat equation we have:

$$\frac{\partial^2 \theta_0}{\partial y^2} = i\frac{\omega}{\kappa}\left(\theta_0 - \frac{\Delta_E}{\alpha}\frac{\partial^2 U_0}{\partial x^2}y\right), \qquad (4.3.17)$$

The solution of the above equation would be:

$$\theta_0 - \frac{\Delta_E}{\alpha}\frac{\partial^2 U_0}{\partial x^2}y = Asin(\zeta y) + Bcos(\zeta y), \qquad (4.3.18)$$

with

$$\zeta = \sqrt{i\frac{\omega}{\kappa}} = (1+i)\sqrt{\frac{\omega}{2\kappa}} \qquad (4.3.19)$$

Through suitable boundary conditions the coefficients of A and B could be determined. The profile of the temperature across the beam then would read as

$$\theta_0(x,y) = \frac{\Delta_E}{\alpha}\frac{\partial^2 U_0(x)}{\partial x^2}\left(y - \frac{sin(\zeta y)}{\zeta cos\left(\frac{b\zeta}{2}\right)}\right) \qquad (4.3.20)$$

Implementing the temperature profile into I_T integral as the thermal contribution to the moment of inertia one can obtain the following equation for the beam as:

$$\omega^2 U_0 = \frac{EI}{\rho A}\left\{1 + \Delta_E[1 + f(\omega)]\right\}\frac{\partial^4 U_0}{\partial x^4} \qquad (4.3.21)$$

Herein the complex function $f(\omega)$ is expressed as

$$f(\omega) = f(\zeta(\omega)) = \frac{24}{b^3\zeta^3}\left[\frac{b\zeta}{2} - tan\left(\frac{b\zeta}{2}\right)\right]. \qquad (4.3.22)$$

It should be noted that in the case of an isothermal beam with thermoelastic coupling, the Young's modulus is a function of frequency. That would be the only difference between

the beam with no thermoelastic coupling where the Young's modulus does not have any dependency on the frequency. Thereby in the presence of thermoelastic coupling the frequency dependent Young's modulus reads as

$$E_\omega = E \left\{ 1 + \Delta_E [1 + f(\omega)] \right\} \tag{4.3.23}$$

When ω becomes very large, $f(\omega)$ tends to zero. As a result of that, Young's modulus becomes inclined toward its adiabatic state. In the case of small values for ω, $f(\omega)$ tends to -1 and Young's modulus returns to its isothermal value.

Following expression describes the normal modes of vibration for a beam in isothermal state as

$$U_0(x) = Asin(qx) + Bcos(qx) + Csinh(qx) + Dcosh(qx) \tag{4.3.24}$$

Herein again the boundary conditions determine the coefficients A through D as well as suitable values for q.

The dispersion relation between ω and q_n for a thermoelastic beam is defined as

$$\omega = \sqrt{\frac{E_\omega I}{\rho A}} q_n^2 = \omega_0 \sqrt{1 + \Delta_E [1 + f(\omega)]} \tag{4.3.25}$$

with ω_0 as the isothermal eigenfrequency.

If corrections of order Δ_E^2 are ignored, $f(\omega)$ could be replaced by $f(\omega_0)$, therefore the dispersion relation becomes

$$\omega = \omega_0 \left[1 + \frac{\Delta_E}{2} [1 + f(\omega_0)] \right] \tag{4.3.26}$$

Accordingly real and imaginary parts would read as

$$Re(\omega) = \omega_0 \left[1 + \frac{\Delta_E}{2} \left(1 - \frac{6}{\xi^3} \frac{sinh\xi - sin\xi}{cosh\xi + cos\xi} \right) \right] \tag{4.3.27}$$

$$Im(\omega) = \omega_0 \frac{\Delta_E}{2} \left(\frac{6}{\xi^3} \frac{sinh\xi + sin\xi}{cosh\xi + cos\xi} - \frac{6}{\xi^2} \right) \tag{4.3.28}$$

where

$$\xi = b \sqrt{\frac{\omega_0}{2\kappa}} \tag{4.3.29}$$

The normalized frequency shift and normalized attenuation as function of ξ is depicted in 4.2. Finally and based on relation 4.3.16 the following relation for thermoelastic damping in a thin beam given by

$$Q^{-1} = \frac{E\alpha^2 T_0}{C} \left(\frac{6}{\xi^2} - \frac{6}{\xi^3} \frac{sinh\xi + sin\xi}{cosh\xi + cos\xi} \right) \tag{4.3.30}$$
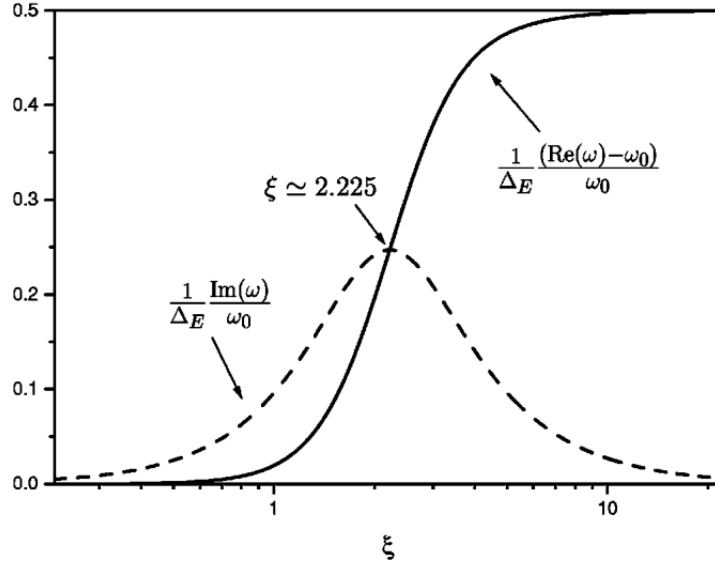
Figure 4.2: Frequency shift and attenuation as functions of ξ for small flexural vibrations [123]

4.4 Second-law analysis approach

4.4.1 Problem Formulation

The main mechanism responsible for thermoelastic damping is establishment of thermal currents within the material. And that is in fact a benefit compared to other types of damping in materials which are usually accompanied by lowering other desirable engineering properties such as stiffness and strength. Additionally, thermoelastic damping can be modified with respect to inhomogeneity properties in multiphase materials such as alloys and composites. This is while Zener's theory is more concerned with isotropic, homogeneous materials; thereby it is not well suited for description of thermoelastic damping in inhomogeneous materials. To cope with such a problem, Kinra and Milligan [98] extended the Zener's theory, choosing an approach based on second-law of thermodynamics.

The theory starts with field equations of classical thermoelasticity including the first law of thermodynamics:

$$\rho \frac{\partial u}{\partial t} = \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial t} - q_{i,j} \qquad (4.4.1)$$

Newton's law:

$$\sigma_{ji,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (4.4.2)$$

Kinematics relation:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (4.4.3)$$

Thermoelastic Hooke's law:

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) - \frac{E}{1-2\nu}\alpha\delta_{ij}(T - T_0) \qquad (4.4.4)$$

and Fourier law for conduction of heat which reads as:

$$q_i = -kT_{,i} \tag{4.4.5}$$

where σ_{ij} is stress tensor, ε_{ij} is strain tensor, u_i would be displacement vector, ν is the Poisson's ratio, E is the Young's modulus, T is the absolute temperature, q_i is heat flux, u is the internal energy, δ_{ij} is the Kronecker delta function, and the indices i, j, k span the range of 1, 2, 3.

As mentioned before, thermoelastic damping includes the irreversible flow of heat which generates entropy. Entropy production obeys the second law of thermodynamics as [126, 127],

$$\frac{\partial S_p}{\partial t} = \frac{k}{\rho T^2} T_{,i} T_{,i} \tag{4.4.6}$$

where S_p denotes the entropy produced per unit of mass. Upon generation of entropy, energy in its mechanical form is converted to the heat. This transformation is described through the Gouy-Stodola Theorem as

$$\Delta W = \rho T_0 S_p \tag{4.4.7}$$

Elastic energy stored in material per unit volume reads as

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \tag{4.4.8}$$

Assuming a time-harmonic deformation for the thermoelastic body, the term ΔW in fact would represent dissipation of mechanical energy in a single cycle, and W could be considered as maximum elastic energy stored in a cycle. Based on that a local thermoelastic damping is defined as

$$Q_L^{-1} = \frac{\Delta W}{W} \tag{4.4.9}$$

A volume averaged quantity for damping of a structure with finite geometry reads as

$$Q_{TED}^{-1} = \frac{\int_V \Delta W \, dV}{\int_V W \, dV} = \frac{\int_V Q_L^{-1} W \, dV}{\int_V W \, dV} \tag{4.4.10}$$

As a result, the two-way coupled heat flow relationship reads as

$$T_{,ii} = \frac{\rho c}{k} \frac{\partial T}{\partial t} + \frac{E\alpha}{k(1 - 2\nu)} T \frac{\partial \varepsilon_{kk}}{\partial t} \tag{4.4.11}$$

Usually for this class of problems, the temperature variations are considered very small, hence we can replace T with T_0 in the above formulation, resulting in a one-way coupled relation as

$$T_{,ii} = \frac{\rho c}{k} \frac{\partial T}{\partial t} + \frac{E\alpha}{k(1 - 2\nu)} T_0 \frac{\partial \varepsilon_{kk}}{\partial t} \tag{4.4.12}$$

Through the above set of equations a boundary value problem in thermoelastic damping can be solved.
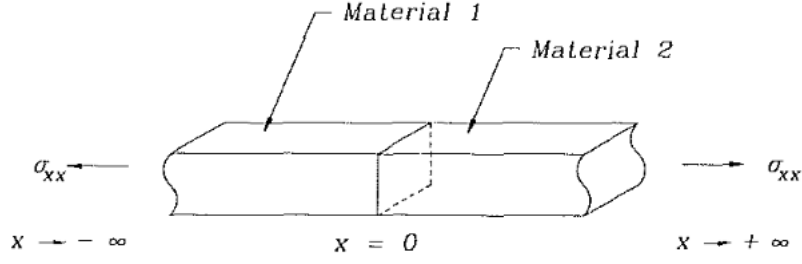
Figure 4.3: Thermoelastic interface at the junction of two pieces of materials, undergoing uniaxial stress [98]

4.4.2 Thermoelastic Interface

Let us consider inhomogeneous cases where two pieces of different metals (with different material properties)each of which with a semi-infinite dimension, are welded together. And the resulting structure is undergoing uniaxial stress $\sigma_{xx} = \sigma_0 e^{i\omega t}$ (Figure 4.3).

In this situation the effect of arising three dimensional stress field near the interface would be very small and could be neglected. As a result, we may write

$$\varepsilon_{xx} = \sigma_0 e^{i\omega t}/E \tag{4.4.13}$$

and

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} \tag{4.4.14}$$

thereby ε_{kk} reads as

$$\varepsilon_{kk} = (1 - 2\nu)\sigma_0 e^{i\omega t}/E \tag{4.4.15}$$

which results in a reduced form of a one-way coupled heat equation as

$$\frac{\partial^2 T}{\partial x^2} - \frac{\rho c}{k}\frac{\partial T}{\partial_t} = i\omega\frac{\alpha T_0}{k}e^{i\omega t} \tag{4.4.16}$$

As for the interface conditions, temperature field at interface is continuous:

$$T(0^-, t) = T(0^+, t). \tag{4.4.17}$$

The flux of heat is also continuous across the interface

$$k_1\frac{\partial T}{\partial x}(0^-, t) = k_2\frac{\partial T}{\partial x}(0^+, t). \tag{4.4.18}$$

Additionally, as $x \rightarrow -\infty$

$$T(x, t) \rightarrow T_0 - \frac{\alpha_1}{\rho_1 c_1}T_0\sigma_0 e^{i\omega t} \tag{4.4.19}$$

and as $x \rightarrow +\infty$

$$T(x, t) \rightarrow T_0 - \frac{\alpha_2}{\rho_2 c_2}T_0\sigma_0 e^{i\omega t} \tag{4.4.20}$$

Now we need to introduce normalized temperature as

$$\hat{V}^* = V^*/\Delta T \tag{4.4.21}$$

where V^* would be an unknown spatial variation of temperature and $\hat{V}^*$ would be normalized temperature. Then one can write

$$\hat{V}_1^* = V_1^*/(-\sigma_0 T_0 \alpha_1/\rho_1 c_1) \qquad x < 0 \tag{4.4.22}$$

$$\hat{V}_2^* = V_2^*/(-\sigma_0 T_0 \alpha_2/\rho_2 c_2) \qquad x > 0, \tag{4.4.23}$$

with $Z = \alpha/\rho c$ and $K = (\rho c k)^{1/2}$ as the coefficient of heat penetration. Geometrically, the defined problem does not introduce any characteristic length. In physics, characteristic length is an important dimension that defines the scale of of physical system. Since we do not have such a parameter, we introduce a diffusion length as $l = (2k/\omega \rho c)^{1/2}$. Utilizing that we can define two normalized coordinates as $\xi_1 = x/l_1$ for $x < 0$ and $\xi_2 = x/l_2$ for $x > 0$. Then the equation 4.4.12 can be reduced to

$$\frac{d^2 \hat{V}_1^*}{d\xi_1^2} - 2i\hat{V}_1^* = -2i \qquad \xi_1 < 0 \tag{4.4.24}$$

$$\frac{d^2 \hat{V}_2^*}{d\xi_2^2} - 2i\hat{V}_2^* = -2i \qquad \xi_2 < 0 \tag{4.4.25}$$

and consequently the interface conditions become

$$Z_1 \hat{V}_1^* = Z_2 \hat{V}_2^* \tag{4.4.26}$$

$$Z_1 K_1 \frac{d\hat{V}_1^*}{d\xi_1} = Z_2 K_2 \frac{d\hat{V}_1^*}{d\xi_2} \tag{4.4.27}$$

The solution to the arisen ordinary differential equation would be

$$\hat{V}_1^*(\xi_1) = 1 - \left(1 - \frac{Z_2}{Z_1}\frac{K_2}{K_1 + K_2}e^{(1+i)\xi_1}\right) \tag{4.4.28}$$

$$\hat{V}_2^*(\xi_2) = 1 - \left(1 - \frac{Z_1}{Z_2}\frac{K_1}{K_1 + K_2}e^{-(1+i)\xi_2}\right) \tag{4.4.29}$$

Having all the above formulation obtained, the thermoelastic damping can be written as

$$Q_{TED/L1}^{-1} = \Delta_{E1}\left[\frac{K_2}{K_1 + K_2}\left(1 - \frac{Z_2}{Z_1}\right)e^{\xi_1}\right]^2 \tag{4.4.30}$$

$$Q_{TED/L2}^{-1} = \Delta_{E2}\left[\frac{K_1}{K_1 + K_2}\left(1 - \frac{Z_1}{Z_2}\right)e^{-\xi_2}\right]^2 \tag{4.4.31}$$

where subscripts denote the materials 1 and 2.

4.4.3 Numerical Results

The numerical results of the above formulation have been depicted in Figures 4.4, 4.5, 4.6. In 4.4, profiles of V and ϕ with respect to x/l_1 for thermoelastically similar joined materials, namely Mg/Al with $Z_{Al}/Z_{Mg} = 0.64$, are shown. As was expected, when $x \to -\infty$, $V \to 1$ and $\phi \to 0$. At the interface $\hat{V}^*(0^-) = \hat{V}^*(0^+) = (Z_1 K_1 + Z_2 K_2)/(K_1 + K_2)$, noting as well that ϕ is zero here at the interface. It means that the temperature is always at the same phase with the stress. In next figure, i.e. Fig. 5 the profile of the same variables are plotted for thermoelastically dissimilar materials namely Mg/Tic with $Z_{TiC}/Z_{Mg} = 0.12$. It is worthy of noting that temperature field exhibits discontinuity across the interface. This originates from the necessity of continuity of heat flux $q = -k\partial T/\partial x$.
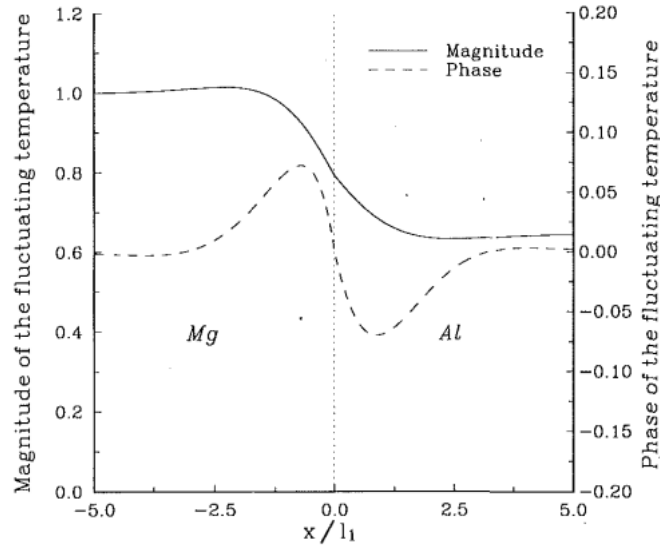
Figure 4.4: Magnitude and phase of fluctuating temperature as a function of space for a thermoelastically similar interface [98]

The influence of damping at the interface can be better understood by examining the specific damping capacity profile against position, as shown in Figure 6. When dealing with thermoelastically similar materials, such as $Z_{Mg} = 15.12$ and $Z_{Al} = 9.71$, the damping at the interface is not significantly pronounced since the values are relatively close to each other. However, in cases where the materials are thermoelastically different (e.g., Mg/TiC), the ratio $Z_{Mg}/Z_{TiC} = 15.1/1.85 \approx 10$, and the disparity in Z values becomes substantial. This difference in Z values and the ratio explain the high interface damping observed when joining materials that exhibit significant thermoelastic distinctions.

4.5 Laminated Structures

4.5.1 Problem Formulation

Bishop and Kinra [104, 105] investigated the thermoelastic damping in layered structures and developed a framework which was used in other works for studying specific types of layered structures [94, 95].

A typical cross-section for a three-layered Euler-Bernoulli is shown in Figure 4.7. Time
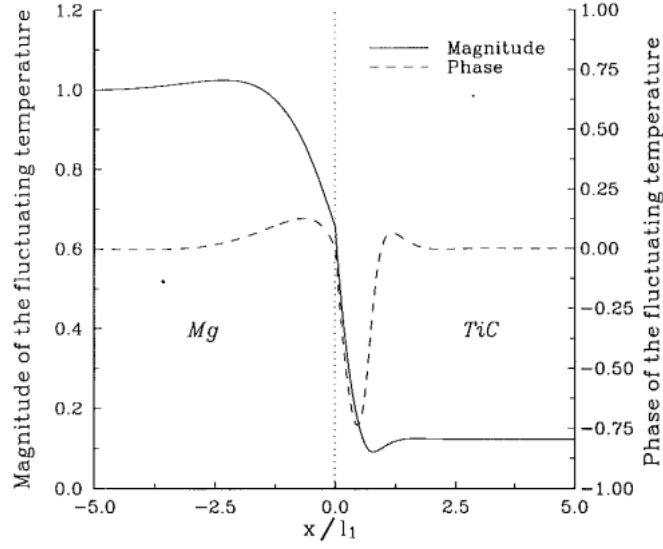
Figure 4.5: Magnitude and phase of fluctuating temperature as a
function of space for a thermoelastically different interface [98]
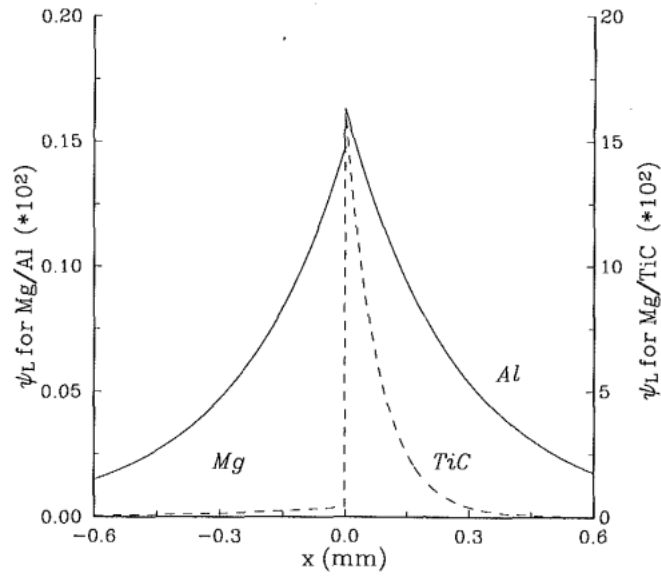
Figure 4.6: Specific damping capacity at the vicinity of interface [98]

constants for each layer are defined as

$$\tau_1 = \frac{C_1}{k_1}a^2 \qquad\qquad \tau_2 = \frac{C_2}{k_2}(b-a)^2 \qquad\qquad (4.5.1)$$

Accordingly the related normalized frequencies will be defined as

$$\Omega_1 = \omega\tau_1 \qquad\qquad \Omega_2 = \omega\tau_2 \qquad\qquad (4.5.2)$$
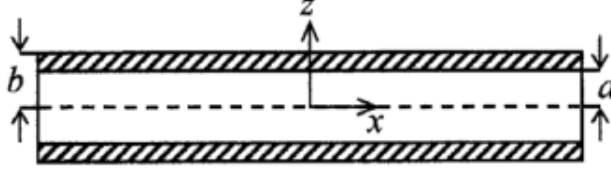
Figure 4.7: Cross-section of an Euler-Bernoulli beam with three layers [95]

Herein the subscripts 1 and 2 denote the central and outer layers. Next procedures are almost the same as before which include thermoelastic damping, dissipated energy and produced entropy alongside the coupled heat conduction equation that need to be defined. Thermoelastic damping for beams can be expressed as

$$Q^Z_{-1} = \Delta_E \frac{\sum_{n=1}^{\infty} \frac{\Omega_1 \gamma_n^2}{\Omega_1^2 + \gamma_n^4} Q_n}{\frac{1}{3} \left[\frac{a^3}{b^3} + \frac{E_2}{E_1} \left(1 - \frac{a^3}{b^3} \right) \right]} \tag{4.5.3}$$

where

$$\gamma_n = \beta_n \sqrt{\tau_1} \qquad\qquad \eta_n = \beta_n \frac{b}{b-a} \sqrt{\tau_2} \tag{4.5.4}$$

and β_n would represent eigenvalues resulted as the solution of the following equation

$$tan(\beta_n \sqrt{\tau_1}) tan(\beta_n \sqrt{\tau_2}) = K = \sqrt{\frac{k_1 C_1}{k_2 C_2}}. \tag{4.5.5}$$

In the above equation Q_n is expressed as

$$Q_n = \frac{\left[\frac{a^2}{b^2} I_i + \frac{E_2 \alpha_2}{E_1 \alpha_1} I_{ii} \right]^2}{\frac{a}{b} I_{iii} + \frac{C_2}{C_1} I_{iv}} \tag{4.5.6}$$

where

$$I_i = \frac{1}{\gamma_n^2} sin\gamma_n - \frac{1}{\gamma_n} cos\gamma_n \tag{4.5.7}$$

$$I_{ii} = \frac{1}{\eta_n} \left\{ \begin{array}{l} A_n \left[\frac{1}{\eta_n} \left(cos\eta_n - cos\frac{\eta_n a}{b} \right) + sin\eta_n - \frac{a}{b} sin\frac{\eta_n a}{b} \right] \\ + B_n \left[\frac{1}{\eta_n} \left(sin\eta_n - sin\frac{\eta_n a}{b} \right) - cos\eta_n + \frac{a}{b} cos\frac{\eta_n a}{b} \right] \end{array} \right\} \tag{4.5.8}$$

$$I_{iii} = \frac{1}{2} - \frac{1}{4\gamma_n} sin(2\gamma_n), \tag{4.5.9}$$

$$\begin{aligned} I_{iv} = & \frac{1}{2}(1 - \frac{a}{b})(A_n^2 - B_n^2) \\ & + \frac{1}{4\eta_n}(A_n^2 - B_n^2) \left[sin(2\eta_n) - sin(2\eta_n a/b) \right] \\ & - \frac{1}{2\eta_n} A_n B_n \left[cos(2\eta_n) - cos(2\eta_n a/b), \right] \end{aligned} \tag{4.5.10}$$

$$A_n = -\frac{K cos\gamma_n cos\eta_n}{sin\left(\eta_n\left(\frac{a}{b}-1\right)\right)} \qquad\qquad B_n = -\frac{K cos\gamma_n sin\eta_n}{sin\left(\eta_n\left(\frac{a}{b}-1\right)\right)} \qquad (4.5.11)$$

Above equations are able to model thermoelastic damping in symmetric, three-layered beams.

4.5.2   Numerical Results

Several types of structures can be designed to represent dependency of thermoelastic damping on the frequency. For instance compounds such as Al/Si/Al, Cu/Si/Cu, Ag/Si/Ag, Au/Si/Au, Al/SiC/Al, Cu/SiC/Cu, Ag/SiC/Ag, Au/SiC/Au and SiC/Si/SiC can be named.

Figure 4.8 shows the influence of the number of terms in the relation 4.5.3 on convergence behavior of thermoelastic damping as graphed for SiC/Si/SiC against the frequency. As can be seen with increase in the number of terms in series from 1 to 4, the peak damping values converges swiftly and after $n \geq 2$ it in fact remaines unchanged. Nevertheles at least four terms need to be included to achieve convergence for the high-frequency damping response.
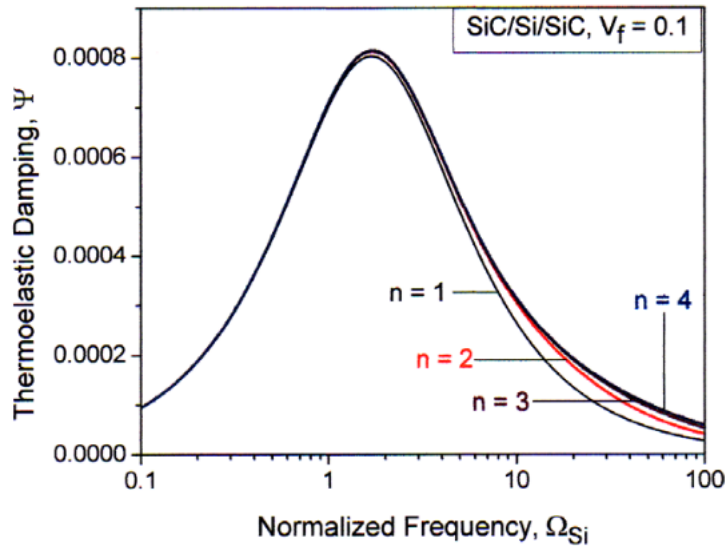
Figure 4.8: Thermoelastic damping as a function of frequency for different series expansions

The role of metallization on damping properties has been compared for different Si based beams in figure 4.9. It can be seen that metallization remarkably enhances the damping properties across a wide spectrum of frequency. In spite of that, no abrupt change in critical damping frequency is observed.

Similar comparisons have been performed for the case of SiC made structures investigating the damping against the frequency with modifications in metallization for four cases including Cu/SiC/Cu, Al/SiC/Al, Ag/SiC/Ag and Au/SiC/Au. As before, damping is enhanced in direct accordance with metallization over the wide range of frequencies. Values obtained for peak damping show the same trend as for the case of silicon-silver. Also
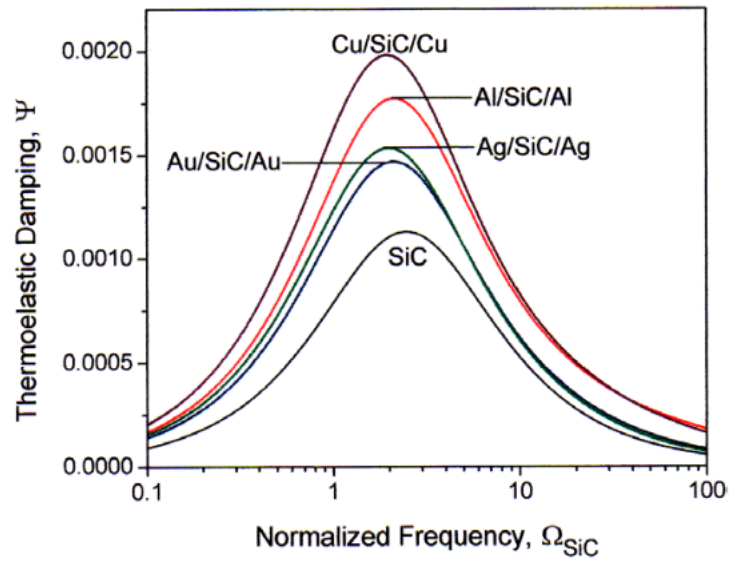
Figure 4.9: Comparison of thermoelastic damping in different
metalized silicon carbides beams [95]

apparently Al and Cu show to have more pronounced effects on damping in comparison
with gold.
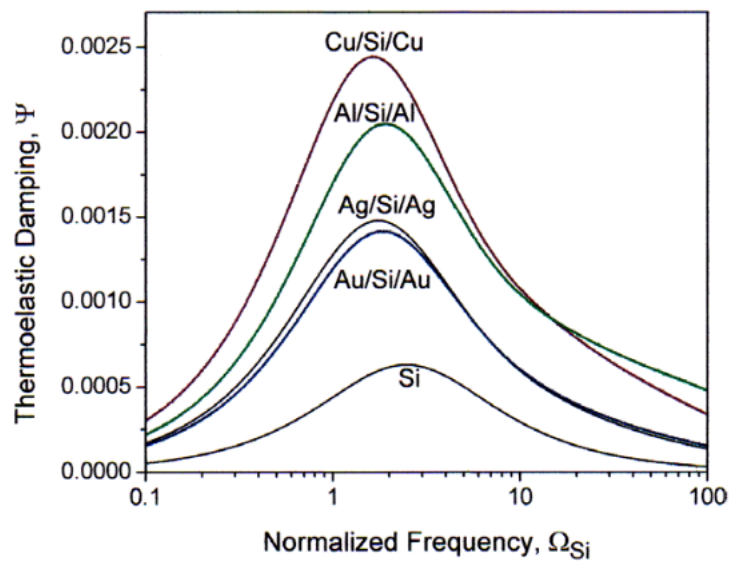
Figure 4.10: Comparison of thermoelastic damping in different
metalized silicon beams [95]

The effect of volume fraction is revealed through increasing the volume fraction of
silicon carbide in SiC/Si/SiC beams. Damping increased with increase in volume fraction.
Also a downward shift was observed for critical damping frequency. Surprisingly, however,
it was observed that damping in some of the cases become even less than that of the bare
silicon.
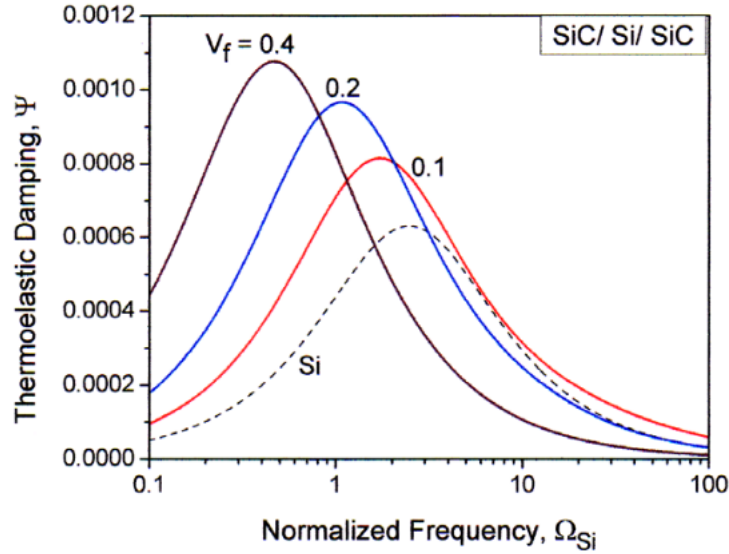
Figure 4.11: Variation of thermoelastic damping with respect to
volume fractions in SiC/Si/Sic [95]

4.6   TED and non-classical Theories

As mentioned, nowadays micro- and nano- resonators have found plenty of applications in micro-/ nano-electromechanical systems. And as a point of concern, developers of such systems always seek the reduced energy consumption in these technologies. Thermoelastic damping is one of the main intrinsic sources of energy dissipation in such systems, which needs to be minimized for higher levels of performance.

Quality factor is a well known criterion for assessing the performance which is defined as the ratio of stored energy to dissipated energy in the system.

Experimental results on behavior of materials at micro- nano-scales however show discrepancy with predictions made by classical elasticity theories. Contrarily, non-classical and generalized field theories such as couple stress, strain gradient and nonlocal elasticity show more capability in modeling size-dependent effects.

Non-classical theories have been recently utilized in developing relations for thermoelastic damping in micro- and nano-beams. Obtained results show considerable improvements compared to predictions made through classical elasticity theories. For example Kong et al. [128]compared the natural frequencies of microbeam as obtained by modified strain gradient theory (MSGT) to that resulted from modified couple stress theory (MCST) and classical elasticity theory (CT). It was observed that values given by MSGT were significantly bigger than those through MCST and CT.

4.6.1   Strain Gradient Elastic Bending Theory

Lam et al [129] developed a MSGT theory in which the strain energy density function for isotropic linear elastic bodies in small strain regime was developed as

$$U = \int_V \bar{u}dV = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^s\chi_{ij}^s)dV \tag{4.6.1}$$

where σ_{ij} represents components of the stress tensor, ε_{jj} are components of the classical strain tensor, $\boldsymbol{\gamma}$ would be the dilation gradient vector, the deviatoric part of stretch gradient tensor is shown by $\boldsymbol{\eta}^{(1)}$, $\boldsymbol{\chi}^s$ would be the symmetric part of the rotation gradient tensor, and $\boldsymbol{\theta}$ is the infinitesimal rotation vector. These terms for small deformations are defined as

$$\varepsilon_{jj} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4.6.2}$$

$$\gamma_i = \varepsilon_{mm,i}, \tag{4.6.3}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) - \frac{1}{15}\delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m})$$
$$- \frac{1}{15}\left[\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})\right], \tag{4.6.4}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \tag{4.6.5}$$

$$\theta_i = \frac{1}{2}(curl(u))_i = \frac{1}{2}e_{ijk}u_{k,j}, \tag{4.6.6}$$

Herein u_i represent the components of infinitesimal displacement vector and the comma sign (,) denotes the partial derivatives with respect to spatial coordinates. Accordingly, components of the classical stress tensor $\boldsymbol{\sigma}$ can be defined in terms of the above kinematic parameters $\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, \boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\chi}^s$ through following constitutive relations [129] as

$$\sigma_{ij} = \lambda tr(\varepsilon) + 2\mu\varepsilon_{ij}, \tag{4.6.7}$$

$$p_i = 2\mu l_0^2 \gamma_i, \tag{4.6.8}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2 \eta_{ijk}, \tag{4.6.9}$$

$$m_{ij}^s = 2\mu l_2^2 \chi_{ij}^s. \tag{4.6.10}$$

where λ and μ would be Lame constants which parametrized by Young's modulus and Poisson's ratio read as following:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{4.6.11}$$

Moreover, l_0, l_1 and l_2 are material length scale parameters that could be determined through typical size effect experiments such as nanoindentation and nanoscratching tests.

In the case of thermoelastic damping, when thermal effects come into play, the stress-strain relation read as

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \tag{4.6.12}$$

where e_{ij} as the the total strain field can be described as

$$\sigma_{ij} = \varepsilon_{ij} + \varepsilon_{ij}^{(T)} \tag{4.6.13}$$

where ε_{ij} and $\varepsilon_{ij}^{(T)}$ respectively represent mechanical and thermal strains. In the case of homogeneous, isotropic materials in linear regimes, the thermal strain is expressed as

$$\varepsilon_{ij}^{(T)} = \alpha\theta\delta_{ij} \tag{4.6.14}$$

with α as the thermal expansion coefficient and θ as temperature changes from the ambient temperature $(T_0): \theta = T - T_0$.

4.6.2   Non-Fourier Heat Conduction

Based on the strain gradient theory developed by Lam et al [129], Bostani and Karami Mohammadi [130] formulated a size dependent quality factor for thermoelastic damping. They implemented a non-Fourier heat conduction theory developed by Lord and Shulman [131] to develop a generalized thermoelasticity theory in which the effect of thermal relaxation time on the quality factor on thermoelastic damping in microbeams can be introduced and analyzed.

The non-Fourier heat flow relationship for a microbeam can be defined as:

$$k\frac{\partial^2\theta}{\partial x_i \partial x_i} = \rho c_v \frac{\partial\theta}{\partial t} + E\alpha T_0 \frac{\partial^2 u_i}{\partial t \partial x_i} + \tau_0\rho c_v \frac{\partial^2\theta}{\partial t^2} + \tau_0 E\alpha T_0 \frac{\partial^3 u_i}{\partial t^2 \partial x_i} \tag{4.6.15}$$

where thermal relaxation time τ_0 is expressed in terms of measurable parameters by Chester [132]as

$$\tau_0 = \frac{3k}{c_v\nu^2} \tag{4.6.16}$$

where ν is the phonon velocity which can be approximated by the elastic wave velocity [133].

Accordingly, the couple thermoelastic relationship based on non-Fourier heat conduction theory for microbeams can be expressed as

$$\chi\left(\frac{\partial^2\theta}{\partial z^2}\right) = (1+\Gamma) + \left(\frac{\partial\theta}{\partial t} + \tau_0\frac{\partial^2\theta}{\partial t^2}\right) - \frac{\Delta_E}{\alpha}z\left(\frac{\partial^3\omega}{\partial t \partial x^2} - \tau_0\frac{\partial^4}{\partial t^2 \partial x^2}\right), \tag{4.6.17}$$

in which Γ is of order of 10^{-4} and defined as following but can be neglected for simplicity:

$$\Gamma = 2\Delta_E\frac{1+\nu}{1-2\nu} \tag{4.6.18}$$

4.6.3   Quality Factor for thermoelastic damping

Two approaches can be implemented for deriving the quality factor: complex frequency approach and entropy generation approach.

In complex frequency approach thermal moments are defined as

$$M_T = \frac{EI\Delta_E}{(1+\Gamma)}\frac{d^2w_0}{dx^2}[1+f(\omega)], \qquad (4.6.19)$$

$$Q_T = \frac{EI\Delta_E}{1+\Gamma}\frac{dw_0}{dx^2}g(\omega), \qquad (4.6.20)$$

where

$$f(\omega) = \frac{24}{(\gamma h)^3}\left[\frac{\gamma h}{2} - tan\left(\frac{\gamma h}{2}\right)\right], \qquad g(\omega) = \frac{24}{(\gamma h)^3}\left[\frac{\gamma h}{2} - tan\left(\frac{\gamma h}{2}\right)\right] \qquad (4.6.21)$$

As before in complex frequency approach the quality factor for thermoelastic damping is defined as

$$Q^{-1} = 2\left|\frac{Im(\omega)}{Re(\omega)}\right| \qquad (4.6.22)$$

in which ω is replaced by ω_n as

$$\omega_n = \frac{1}{L^2}\sqrt{\frac{\beta_n^6(S_1 + H_1) + \beta_n^4(S_2 + H_2)}{\rho A}} \qquad (4.6.23)$$

Herein H_1 and H_2 are expressed as

$$H_1 = 3EI\Delta_E(1-2\nu)^2 l_0^2[1+f(\omega)]/(1+\Gamma)(1+\nu) \qquad (4.6.24)$$

$$H_2 = \frac{EI\Delta_E}{(1+\Gamma)}[1+f(\omega)+3l_0^2 g(\omega)(1-2\nu)/(1+\nu)] \qquad (4.6.25)$$

Quality factor of TED as approximated by Zener was defined already in equation 4.2.10. The differential equation explaining the relation between the heat flux and temperature gradient is however defined based on Lord-Shulmann theory [131] as

$$q_i + \tau_0\dot{q}_l = -kT_{,i} \qquad (4.6.26)$$

which through implementation of second law of thermodynamics would be transformed as

$$\tau_0\ddot{s}\left(1 + \tau_0\frac{2\dot{T}}{T_0} - \tau_0\frac{\dot{T}_{,i}}{T_{,i}}\right) = \frac{k(T_{,i})^2}{T_0^2}. \qquad (4.6.27)$$

Having calculated the total lost work as well as stored energy of the microbeam per time cycle of each vibration, quality factor for thermoelastic damping for classical formulations as well as strain gradient and stress couple theories reads as

$$Q_{CT}^{-1} = \frac{6\Delta_E}{\xi^2(1+\tau_0^2\omega^2)}2 \qquad (4.6.28)$$

$$Q_{CT}^{-1} = \frac{BE\Delta_E h^3}{4\xi^2(1+\tau_0^2\omega^2)W} + \left\{\left[1 + \frac{1+cos(\xi\eta)cosh(\xi/\eta)}{(cos(\xi\eta)+cosh(\xi/\eta))^2}\right.\right.$$

$$\left.\left.- \frac{3\eta}{\xi(\eta^4+1)}\frac{\eta^2 sin(\xi\eta)+sinh(\xi/\eta)}{cos(\xi\eta)+cosh(\xi/\eta)}\right]^2 + \left[-\frac{sin(\xi\eta)sinh(\xi/\eta)}{(cos(\xi\eta)+cosh(\xi/\eta))^2}\right.\right. \tag{4.6.29}$$

$$\left.\left.+ \frac{3\eta}{\xi(\eta^4+1)}\frac{\eta^2 sinh(\xi/\eta)-sin(\xi/\eta)}{cos(\xi\eta)+cosh(\xi/\eta)}\right]^2\right\}.$$

$$Q_{MCST}^{-1} = \frac{6\Delta_E}{\lambda\xi^2(1+\tau_0^2\Omega^2)}\left\{\left[1 + \frac{1+cos(\xi\eta)cosh(\xi/\eta)}{(cos(\xi\eta)+cosh(\xi/\eta))^2} - \frac{3\eta}{\xi(\eta^4+1)}\frac{\eta^2 sin(\xi\eta)+sinh(\xi/\eta)}{cos(\xi\eta)+cosh(\xi/\eta)}\right]^2\right.$$

$$\left.\left[-\frac{sin(\xi\eta)sinh(\xi/\eta)}{(cos(\xi\eta)+cosh(\xi/\eta))^2} + \frac{3\eta}{\xi(\eta^4+1)}\frac{\eta^2 sinh(\xi/\eta)-sin(\xi/\eta)}{cos(\xi\eta)+cosh(\xi/\eta)}\right]^2\right\}^{1/2}$$

$$\tag{4.6.30}$$

4.6.4   Numerical Results

Fig. 4.12 shows the TED profile against the ambient temperature as analyzed through classical (CT), modified couple stress (MCST) and modified strain gradient (MSGT) theories. It can be seen that the inverse of quality factor is enhanced with increases in the ambient temperature. Also thermoelastic damping obtained through MCST exhibit smaller values compared to those obtained by MSGT.

Analysis on the effect of thickness on TED is depicted in Fig. 4.13. As the thickness of microbeam tends to the critical value associated to the peak damping, the output shows considerable divergence between the three elasticity theories (CT, MCST and MSGT) and the two thermoelasticity theories (CT and GTE).
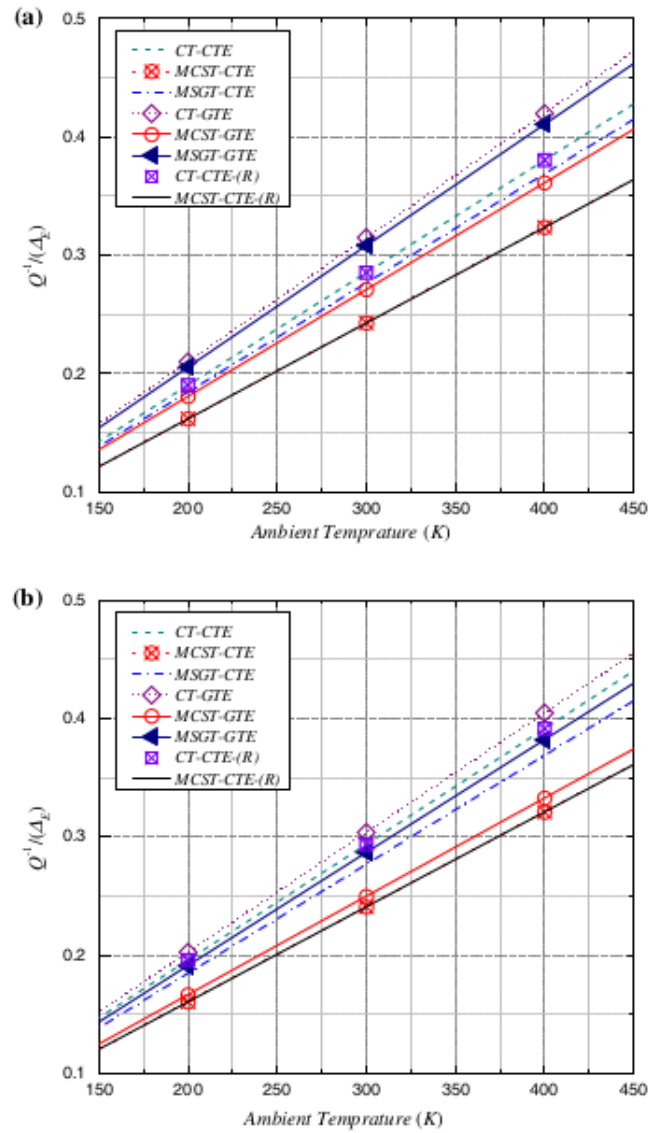
Figure 4.12: Variation of thermoelastic damping against the ambient
temperature as studied in a silicon microbeam in cases a)
clamped-clamped ends and b) cantilever microbeam [130].
(CT: Classical Elasticity Theory; MCST: Modified Couple Stress
Theory; MSGT: Modified Strain Gradient Theory; GTE: Generalized
Theromelasticity Theory)
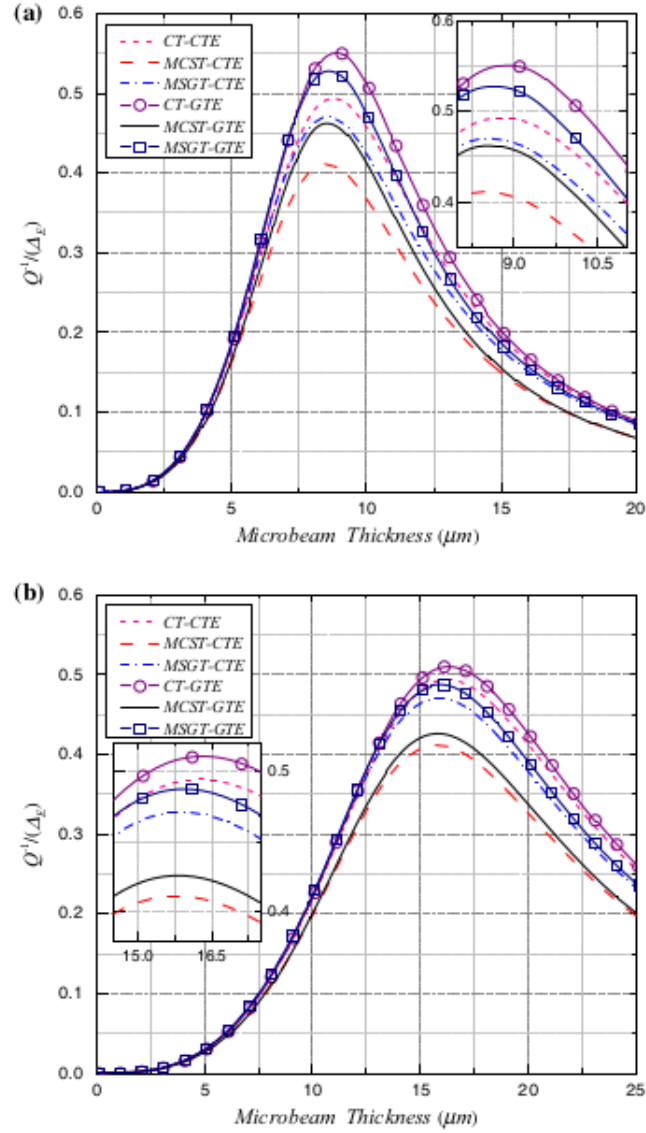
Figure 4.13: Profile of thermoelastic damping against the thickness as in a) Clamped–clamped situation, and b) cantilever microbeam [130] . (CT: Classical Elasticity Theory; MCST: Modified Couple Stress Theory; MSGT: Modified Strain Gradient Theory; GTE: Generalized Theromelasticity Theory) .

Chapter 5

Gradient Thermoviscoelasticity

5.1 Introduction

As explained in previous chapters, micro- and nano- resonators as implemented in MEMS and NEMS technologies exhibit loss of sensitivity as a result of internal energy dissipation. The mechanism involved for the dissipation of energy within these structures is known to be of thermoelastic damping type. This kind of damping involves coupling of thermal and mechanical fields. Deformation causes imposition of temperature gradients from the compressed areas toward dilated regions. Thermodynamically the system goes out of the equilibrium conditions and enters the nonequilibrium state. In seeking of its equilibrium, system undergoes the irreversible heat flux. The process of exerting the heat accompanies the production of entropy which ultimately leads to dissipation of energy within the thermodynamical system. This phenomenon is usually called the thermoelastic damping. The process of thermoelastic damping was first realized by Zener [99,100] as a result of internal friction in solids. Later on it was developed by others toward three dimensional models for expressing the quality factor.

One of the major inhibitors for achieving a good agreement between experimental measurements and theoretical predictions of energy loss within micro-resonators has been the incapability of classical elasticity theories for describing the arisen stress and strain fields of such structures at fine scales such as micro- or nano-meters. More recent theories dealing with nonlocality concepts as well as gradient based theories have shown to be very useful in reducing the distance between laboratory observations and theoretical models. As two of the most developed of such theories, one can mention stress and strain gradient theories, that have been implemented in this study.

As can be seen the problem of damping in microresonators resides at the interface of irreversible thermodynamics, thermoelastic damping, viscoelasticity and gradient theories. That is thereby the aim of this study that through employing and merging different theories builds up a new more comprehensive theory which is able to take into account various physical phenomena, notably internal friction in solids, and at the same time links it to observed behavior at micro and nano scales toward a better understanding of the mechanisms involved in energy loss in micro- and nanoresonators.

5.2 Linear viscoelasticity: Formulation & Variational Principles

The first remarkable variational formulation for viscoelastic materials might be ascribed to Gurtin [134] where he proposed a generalization to classical elasticity toward linear viscoelasticity through convolutive bilinear form. The significance of choosing a right bilinear form was elaborated by Tonti [135] where he demonstrated that the proper selection of a convolutive type bilinear form would yield a variational formulation for initial value problems which could bypass the necessity of their transformation into a totally boundary condition problem. Works of Gurtin and Tonti were further elaborated and developed by many others [136–139]. It was extended by Carini et al [140] toward the method of boundary integral equations and development of variational based formulations for linear viscoelastic problems [141, 142] . Later works were oriented toward development of extremum formulations [143–146] as well as minimum principles for incompressible viscoelastic solids [147].

Christensen [148, 149] employed free energy as a state function in development of extremum based variational formulation and imposed restriction on derived formulations. Using pseudo-convolutive and pseudo-biconvolutive bilinear forms, Huet [150] developed principles of total potential energy and complementary energy in relationship with linear elasticity.

Herein the model for linear viscoelastic problem is defined and a constitutive model is presented.

Let us assume a linear viscoelastic material occupying the domain $\Omega \subset \mathbb{R}^3$. This body can be heterogeneous and anisotropic.An orthogonal Cartesian reference system with coordinates $\mathbf{x}$ is considered. The time parameter is shown by t and the interval is considered between $[0, 2t]$. Let $\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ be displacement, strain and stress fields respectively. The stress field with $\boldsymbol{\sigma} = 0$ for $t < 0$ satisfies the equilibrium equations as:

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t) = \mathbf{0} \qquad \text{in } \Omega \times [0, 2t] \tag{5.2.1}$$

with boundary condition

$$\boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n} = \bar{\mathbf{t}}(\mathbf{x}, t) \qquad \text{on } \partial\Omega_t \times [0, 2t] \tag{5.2.2}$$

where $\mathbf{b}(\mathbf{x}, t)$ denotes the body forces, $\mathbf{t}(\mathbf{x}, t)$ the surface forces applied on $\partial\Omega_t$. The initial values for displacement field would be $\mathbf{u}(\mathbf{x}, t) = 0$ and for strain field is $\boldsymbol{\varepsilon}(\mathbf{x}, t) = 0$ for $t < 0$. Accordingly the compatibility relationships read as:

$$\boldsymbol{\varepsilon}(\mathbf{x}, t) = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \tag{5.2.3}$$

with boundary conditions as:

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on } \partial\Omega_u \times [0, 2t] \tag{5.2.4}$$

where $\bar{\mathbf{u}}$ is the displacement field on $\partial\Omega_u$.

Herein we consider only non-aging materials, meaning that the hereditary viscoelasticity would be the case, for which the relationship between stress and strain field is explained in the form of Boltzmann relations, which reads as:

$$\boldsymbol{\sigma} = \int_{0^-}^{t} \mathbf{E}(\mathbf{x}, t - \tau) d\boldsymbol{\varepsilon}(\mathbf{x}, \tau) \tag{5.2.5}$$

The above relation should stem from the Stieltjes integral and **E** would be the relaxation kernel which mathematically would be a fourth rank tensor. It is assumed that for $t < 0$, **E** = 0. The components of this tensor which are functions of time and space are defined for a unit-step strain history. The above integral expression is obtained from the Boltzmann superposition principle which yields the stress field as a function of space at each instant of time as a result of increments in strain $d\varepsilon$ for $\tau \in [0, t]$.

Assuming that the relaxation function exhibits symmetry properties as explained in index notation as:

$$E_{ijhk}(\mathbf{x}, t) = E_{jihk}(\mathbf{x}, t) = E_{ijkh}(\mathbf{x}, t) = E_{hkij}(\mathbf{x}, t) \qquad \forall \mathbf{x} \in \Omega, \forall t \in [0, 2t] \qquad (5.2.6)$$

also the following inequalities

$$E_{ijhk}^{0}(\mathbf{x})\gamma_{ij}\gamma_{hk} > 0, \quad E_{ijhk}^{\infty}(\mathbf{x})\gamma_{ij}\gamma_{hk} > 0 \qquad (5.2.7)$$

are valid for $\mathbf{x} \in \Omega$. In this regard $E_{ijhk}^{0}(\mathbf{x})$ and $E_{ijhk}^{\infty}(\mathbf{x})$ for all non-vanishing symmetric tensors of rank two $(\gamma_{ij})$are defined as

$$E_{ijhk}^{0}(\mathbf{x}, t) := \lim_{t \to 0} E_{ijhk}(\mathbf{x}, t), \quad E_{ijhk}^{\infty}(\mathbf{x}, t) := \lim_{t \to +\infty} E_{ijhk}(\mathbf{x}, t) \qquad (5.2.8)$$

Through integration by parts the expression for constitutive relation (5.2.6) can be rephrased in the Volterra form [141] as:

$$\sigma_{ij}(\mathbf{x}) = E_{ijhk}(\mathbf{x}, 0)\varepsilon_{hk}(\mathbf{x}, t) + \int_{0}^{t} \dot{E}_{ijhk}(\mathbf{x}, \tau)\varepsilon_{hk}(\mathbf{x}, t - \tau)d\tau \qquad (5.2.9)$$

where

$$\dot{E}_{ijhk}(\mathbf{x}, t) := \frac{\partial E_{ijhk}(\mathbf{x}, t)}{\partial t} \qquad (5.2.10)$$

would be locally integrated $\forall \mathbf{x} \in \Omega$ as a result of which it can be deduced that $E_{ijhk}(\mathbf{x}, t)$ would be continuous and bounded. Therefore we can have $R_{ijhk}^{0}(\mathbf{x}) = E_{ijhk}(\mathbf{x}, 0)$.

Using equation (5.2.9) one can see that once $\mathbf{x}$ is fixed, the stress field σ_{ij}becomes a function of strain ε_{ij} at time t, as well as the history of strain $\varepsilon_{ij}^{t}(\mathbf{x}, \tau) = \varepsilon(\mathbf{x}, t - \tau)$ for $\tau \in [0, t]$. Having $\varepsilon_{ij}^{t}(\mathbf{x}, \tau)$ relationship (5.2.9)the stress field σ_{ij} along with its history $\sigma_{ij}^{t}(\mathbf{x}, t) = \sigma_{ij}(\mathbf{x}, t - \tau)$ can be obtained for $\tau \in [0, t]$.
To be able to calculate the strain field from the stress field as

$$\varepsilon_{ij}(\mathbf{x}, t) = J_{ijhk}(\mathbf{x}, 0)\sigma_{hk}(\mathbf{x}, t) + \int_{0}^{t} \dot{J}_{ijhk}(\mathbf{x}, \tau)\sigma_{hk}(\mathbf{x}, t - \tau)d\tau \qquad (5.2.11)$$

the invertibility of the relation (5.2.9) needs to be assessed. In this regard let denote $\mathscr{D}_{\varepsilon}$ as the space of the strain histories and $\mathscr{D}_{\sigma}$ as the space the stress histories. Now we need to determine the conditions under which the strain field can be described in terms of stress field as in (5.2.11). In this relation $J_{ijhk}(\mathbf{x}, t)$ represents the creep kernel. Certainly establishment of such a relationship largely depends on the chosen space for $\mathscr{D}_{\varepsilon}$ as well as the properties of $\dot{E}_{ijhk}(\mathbf{x}, t)$. Assuming that

$$\mathscr{D}_{\varepsilon} = \left\{ \varepsilon_{ij}^{t}(\mathbf{x}, \tau); \varepsilon_{ij}^{t}(\mathbf{x}, \tau) \in L^{2}(0, 2t) \right\};$$

and

$$\dot{E}_{ijhk}(\mathbf{x}, t) \in L^1(0, 2t)$$

and conditions (5.2.7) and (5.2.8) hold, together with sine Fourier transform of $\dot{E}_{ijhk}(\mathbf{x}, t)$ which is

$$\hat{\dot{E}}_{ijhk}(\mathbf{x}, \omega) := \int_0^\infty \dot{E}_{ijhk}(\mathbf{x}, t) sin(\Omega t) dt$$

is defined such that

$$\hat{\dot{E}}_{ijhk}(\mathbf{x}, \omega) \gamma_{ij} \gamma_{hk} < 0 \qquad \Omega > 0; \tag{5.2.12}$$

(the above relation is called Graffi's inequality [151] and again γ_{ij} represents a symmetric second ranked tensor), then

$$\mathscr{D}_\sigma = \left\{ \sigma_{ij}^t(\mathbf{x}, \tau); \sigma_{ij}^t(\mathbf{x}, \tau) \in L^2(0, 2t) \right\}$$

and the inverse constitutive relation exist in the mentioned form.

5.2.1   Reformulation of Constitutive Relation

If the time interval $[0, 2t]$ is split into two equal subintervals as $[0, t]$ and $[t, 2t]$ then the stress and strain field can be expressed as:

$$\varepsilon_{ij}(\mathbf{x}, t) = \begin{cases} \varepsilon_{1_{ij}}(\mathbf{x}, t) & \text{for } t \in [0, t] \\ \varepsilon_{2_{ij}}(\mathbf{x}, t) & \text{for } t \in [t, 2t] \end{cases} \tag{5.2.13}$$

$$\sigma_{ij}(\mathbf{x}, t) = \begin{cases} \sigma_{1_{ij}}(\mathbf{x}, t) & \text{for } t \in [0, t] \\ \sigma_{2_{ij}}(\mathbf{x}, t) & \text{for } t \in [t, 2t] \end{cases} \tag{5.2.14}$$

where superscripts 1 refers to the first time interval i.e. $[0, t]$ and the superscript 2 refers the second time interval namely $[t, 2t]$.

Direct constitutive law in compact form would yield

$$\sigma_{ij}(\mathbf{x}, t) = \mathscr{L} \varepsilon_{ij}(\mathbf{x}, t)$$

where $\mathscr{L}(\cdot) := \int_{0^-}^t E_{ijhk}(t - \tau) d(\cdot)$. Then having (5.2.13) and (5.2.14) considered, and by virtue of the principle of Boltzmann superposition, (5.2.6) could be reshaped into

$$\sigma_{1_{ij}}(\mathbf{x}, t) = \int_{0^-}^t E_{ijhk}(t - \tau) d\varepsilon_{1_{hk}}(\tau) \qquad \text{for } t \in [0, t] \tag{5.2.15}$$

$$\begin{aligned} \sigma_{2_{ij}}(\mathbf{x}, t) &= \int_{0^-}^t E_{ijhk} d\varepsilon_{1_{hk}}(\tau) \\ &+ \int_t^{2t} E_{ijhk}(t - \tau) d\varepsilon_{hk}^2(\tau) \qquad \text{for } t \in [t, 2t] \end{aligned} \tag{5.2.16}$$

Setting

$$\mathbf{L} := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

$$:= \begin{bmatrix} \displaystyle\int_{0^-}^{t} E_{ijhk}(t-\tau)d(.) & \displaystyle\int_{t}^{2t} E_{ijhk}(t-\tau)d(.) \\[2ex] \displaystyle\int_{0^-}^{t} E_{ijhk}(t-\tau)d(.) & 0 \end{bmatrix} \quad \begin{array}{l} \text{for } t \in [t,2t] \\[3ex] \text{for } t \in [0,t] \end{array} \tag{5.2.17}$$

$$\boldsymbol{\varepsilon} := \begin{bmatrix} \varepsilon_{1_{ij}}(\mathbf{x},t) \\[1ex] \varepsilon_{2_{ij}}(\mathbf{x},t) \end{bmatrix} \tag{5.2.18}$$

$$\boldsymbol{\sigma} := \begin{bmatrix} \sigma_{2_{ij}}(\mathbf{x},t) \\[1ex] \sigma_{1_{ij}}(\mathbf{x},t) \end{bmatrix} \tag{5.2.19}$$

the operational formulation reads as

$$\mathbf{L}\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \tag{5.2.20}$$

which proves to be equivalent to the relations (5.2.15) and (5.2.16).

Now having the material point of the body fixed and for $T > 0$ the following convolutive bilinear form can be observed which is of the type of Stieltjes formulation and reads as:

$$\langle \sigma'_{ij}, \varepsilon''_{ij} \rangle_c := \sigma'_{ij}(2t) * \varepsilon''_{ij}(2t) := \int_{0^-}^{2t} \sigma'_{ij}(2t-t)d\varepsilon''_{ij}(t) \tag{5.2.21}$$

where $*$ denotes the time convolution product in Stieltjes integral over the time interval of $[0,2t]$, $\sigma'_{ij}(t) := \mathscr{L}\varepsilon'_{ij}(t)$ and ε'_{ij} and ε''_{ij} would be symmetric tensors whose histories lie in the space $\mathscr{D}_\varepsilon$.

Using decompositions (5.2.13) and (5.2.14) along with relationships (5.2.17) and (5.2.18) leads to

$$\begin{aligned} \langle \mathscr{L}\varepsilon'_{ij}, \varepsilon''_{ij} \rangle_c &= \langle A\varepsilon'_{1_{ij}}, \varepsilon''_{1_{ij}} \rangle_c + \langle B\varepsilon'_{2_{ij}}, \varepsilon''_{1_{ij}} \rangle_c + \langle C\varepsilon'_{1_{ij}}, \varepsilon''_{2_{ij}} \rangle_c \\ &= \langle \mathbf{L}\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'' \rangle_c \end{aligned} \tag{5.2.22}$$

Symmetry of the operator $\mathscr{L}$ in terms of bilinear form (5.2.21) leads to the symmetry of the operator $\mathbf{L}$. In this regard it is worthy to mention that the operator A is symmetric meaning

$$\begin{aligned} \langle A\varepsilon'_{1_{ij}}, \varepsilon''_{1_{ij}} \rangle_c &= \int_{0^-}^{t}\int_{0^-}^{t} E_{ijhk}(2t-t-\tau)d\varepsilon'_{1_{hk}}(\tau)d\varepsilon''_{1_{ij}}(t) \\ &= \langle A\varepsilon''_{1_{ij}}, \varepsilon'_{1_{ij}} \rangle_c \end{aligned} \tag{5.2.23}$$

and the adjoint operator of C which is shown as B would be

$$\langle B\varepsilon'_{2_{ij}}, \varepsilon''_{1_{ij}} \rangle_c = \int_{0^-}^{t} \int_{0^-}^{t} E_{ijhk}(2t - t - \tau) d\varepsilon'_{1_{hk}}(\tau) d\varepsilon''_{1_{ij}}(t)$$
$$= \int_{t}^{2t} \int_{0^-}^{2t-t} E_{ijhk}(2t - t - \tau) d\varepsilon''_{1_{hk}}(\tau) d\varepsilon'_{2_{ij}}(t) \qquad (5.2.24)$$
$$= \langle C\varepsilon''_{1_{ij}}, \varepsilon'_{2_{ij}} \rangle_c$$

To make it more adopted to thermodynamical context, it would be useful to illustrate its relation to the Helmholtz free energy (ψ). Several integral forms for the free energy have been suggested within the linear viscoelastic regime. Mainly and in a simplified case it is represented through a double-integral expression as

$$\psi(\varepsilon'_{1_{ij}}) = \frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} E_{ijhk}(2t - t - \tau) d\varepsilon'_{1_{hk}}(\tau) d\varepsilon'_{1_{ij}}(t) \qquad (5.2.25)$$

Accordingly a similar form for the power dissipated density reads

$$\phi(\varepsilon'_{1_{ij}}) = -\int_{0^-}^{t} \int_{0^-}^{t} \dot{E}_{ijhk}(2t - t - \tau) d\varepsilon'_{1_{hk}}(\tau) d\varepsilon'_{1_{ij}}(t) \qquad (5.2.26)$$

where $\varepsilon'_{1_{ij}}$ denotes a generic tensor representing strain history over the time interval $[0, t]$. Comparing with the bilinear form (5.2.23) yields the following form for Helmholtz free energy as:

$$\psi(\varepsilon'_{1_{ij}}) = \frac{1}{2} \langle A\varepsilon'_{1_{ij}}, \varepsilon'_{1_{ij}} \rangle_c \qquad (5.2.27)$$

Coleman [127] introduced the fee energy as a fundamental concept and adopted the second law of thermodynamics in the form of Clausius-Duhem inequality as the starting point. Assuming isothermal conditions when Clausius-Duhem inequality is combined with the local balance energy, we have

$$\phi = -\dot{\psi} + \sigma_{ij}(t)\dot{\varepsilon}_{ij}(t) \geq 0 \qquad (5.2.28)$$

The above relation indicates that the rate of dissipation energy ϕ should be non-negative. Integrated dissipation inequality is easily obtained through integrating (5.2.28) as:

$$\int_{t_0}^{t_1} \sigma_{ij}(\tau)\dot{\varepsilon}_{ij}(\tau) d\tau \geq \psi(t_1) - \psi(t_0) \qquad (5.2.29)$$

Restrictions imposed by the second law of thermodynamics imply that the integrated dissipation inequality (5.2.29) must hold [127].

5.2.2 Inverse Constitutive Law

Keeping the assumptions made in regard to the relaxation kernel E_{ijhk} and the strain histories it turns out the direct constitutive law would be invertible. Hence by virtue of relations (5.2.13) and (5.2.14) and utilizing the Boltzmann superposition one can write:

$$\varepsilon_{1_{ij}}(t) = \int_{0^-}^{t} J_{ijhk}(t - \tau) d\sigma^1_{hk}(\tau) d\sigma^1_{hk}(\tau) \qquad \text{for } t \in [0, t] \qquad (5.2.30)$$

$$\varepsilon_{2_{ij}}(t) = \int_{0^-}^{t} J_{ijhk}(t-\tau)d\sigma_{hk}^1(\tau)$$

$$+ \int_{t}^{2t} J_{ijhk}(t-\tau)d\sigma_{hk}^2(\tau) \qquad \text{for } t \in [t, 2t] \tag{5.2.31}$$

which in compact form reads as

$$\mathbf{L}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon} \tag{5.2.32}$$

where the operator $\mathbf{L}^{-1}$ is expressed as following:

$$\mathbf{L}^{-1} := \begin{bmatrix} 0 & \tilde{\mathscr{B}} \\ \mathscr{B} & \mathscr{A} \end{bmatrix}$$

$$:= \begin{bmatrix} 0 & \int_{0^-}^{t} J_{ijhk}(t-\tau)d(.) & \text{for } t \in [0, t] \\ \int_{t}^{2t} J_{ijhk}(t-\tau)d(.) & \int_{0^-}^{t} J_{ijhk}(t-\tau)d(.) & \text{for } t \in [t, 2t] \end{bmatrix} \tag{5.2.33}$$

New operators $\mathscr{A}$ and $\mathscr{B}$ have the following relationships with already introduced operators A and B as following:

$$\mathscr{B} = B^{-1}$$

$$\mathscr{A} = -B^{-1}A\tilde{B}^{-1} \tag{5.2.34}$$

Positive definiteness of the operator A in terms of bilinear form (5.2.21) implies that the operator $\mathscr{A}$ is symmetric and negative definite.

5.2.3 Reformulation of the Linear Viscoelastic Formulation

Assume that $\mathscr{E}$ and $\mathscr{C}$ denote equilibrium and kinematic operators respectively, meaning that $\mathscr{E}$ represents the divergence operator and $\mathscr{C}$ would be the symmetric part of the gradient operator. Hence, through the decompositions (5.2.13) and (5.2.14) the problem consisting of equilibrium equation, compatibility equation, along with strain-displacement relations and the constitutive law, can be reformed as:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\mathscr{E} \\ 0 & 0 & 0 & 0 & -\mathscr{E} & 0 \\ 0 & 0 & A & B & 0 & -1 \\ 0 & 0 & \tilde{B} & 0 & -1 & 0 \\ 0 & \mathscr{C} & 0 & -1 & 0 & 0 \\ \mathscr{C} & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1_i} \\ u_{2_i} \\ \varepsilon_{1_{ij}} \\ \varepsilon_{2_{ij}} \\ \sigma_{1_{ij}} \\ \sigma_{2_{ij}} \end{bmatrix} = \begin{bmatrix} b_{2_i} \\ b_{1_i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \text{in } \Omega \times [t, 2t] \\ \text{in } \Omega \times [0, t] \\ \text{in } \Omega \times [t, 2t] \\ \text{in } \Omega \times [0, t] \\ \text{in } \Omega \times [t, 2t] \\ \text{in } \Omega \times [0, t] \end{matrix} \tag{5.2.35}$$

Correspondingly, the related boundary conditions read as

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & n_j \\
0 & 0 & 0 & 0 & n_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -n_j & 0 & -1 & 0 & 0 \\
-n_j & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1_i} \\
u_{2_i} \\
\varepsilon_{1_{ij}} \\
\varepsilon_{2_{ij}} \\
\sigma_{1_{ij}} \\
\sigma_{2_{ij}}
\end{bmatrix}
=
\begin{bmatrix}
p_{2_i} \\
p_{1_i} \\
0 \\
0 \\
-n_{2_i}^0 \\
-n_j u_{1_i}^0
\end{bmatrix}
\begin{array}{l}
\text{on } \partial\Omega_p \times [t, 2t] \\
\text{on } \partial\Omega_p \times [0, t] \\
\\
\\
\text{on } \partial\Omega_u \times [t, 2t] \\
\text{on } \partial\Omega_u \times [0, t]
\end{array}
\tag{5.2.36}
$$

which in compact form can be written as

$$
\begin{aligned}
\mathbf{M}_I \mathbf{z}_I &= \mathbf{b}_I && \text{in } \Omega \times [0, 2t] \\
\mathbf{T}_I \mathbf{z}_I &= \mathbf{g}_I && \text{on } \partial\Omega \times [0, 2t]
\end{aligned}
\tag{5.2.37}
$$

If the inverse constitutive law (5.2.32) is considered, a four-fields formulation will be obtained in a similar fashion as:

$$
\begin{bmatrix}
0 & 0 & 0 & \mathscr{E} \\
0 & 0 & \mathscr{E} & 0 \\
0 & -\mathscr{C} & \mathscr{A} & \mathscr{B} \\
-\mathscr{C} & 0 & \mathscr{B} & 0
\end{bmatrix}
\begin{bmatrix}
u_{1_i} \\
u_{2_i} \\
\sigma_{1_{ij}} \\
\sigma_{2_{ij}}
\end{bmatrix}
=
\begin{bmatrix}
-b_{2_i} \\
-b_{1_i} \\
0 \\
0
\end{bmatrix}
\begin{array}{l}
\text{in } \Omega \times [t, 2t] \\
\text{in } \Omega \times [0, t] \\
\text{in } \Omega \times [t, 2t] \\
\text{in } \Omega \times [0, t]
\end{array}
\tag{5.2.38}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & -n_j \\
0 & 0 & -n_j & 0 \\
0 & -n_j & 0 & 0 \\
-n_j & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1_i} \\
u_{2_i} \\
\sigma_{1_{ij}} \\
\sigma_{2_{ij}}
\end{bmatrix}
=
\begin{bmatrix}
-p_{2_i} \\
-p_{1_i} \\
-n_j u_{2_i}^0 \\
-n_j u_{1_i}^0
\end{bmatrix}
\begin{array}{l}
\text{on } \partial\Omega_p \times [t, 2t] \\
\text{on } \partial\Omega_p \times [0, t] \\
\text{on } \partial\Omega_u \times [t, 2t] \\
\text{on } \partial\Omega_p \times [0, t]
\end{array}
\tag{5.2.39}
$$

or, in a compact from we have:

$$
\begin{aligned}
\mathbf{M}_{II} \mathbf{z}_{II} &= \mathbf{b}_{II} && \text{in } \Omega \times [0, 2t] \\
\mathbf{T}_{II} \mathbf{z}_{II} &= \mathbf{g}_{II} && \text{on } \partial\Omega \times [0, 2t]
\end{aligned}
\tag{5.2.40}
$$

The two problems (5.2.37) and (5.2.40) can be condensed as

$$
\begin{aligned}
\mathbf{M}_i \mathbf{z}_i &= \mathbf{b}_i && \text{in } \Omega \times [0, 2t] \\
\mathbf{T}_i \mathbf{Z}_i &= \mathbf{g}_i && \text{on } \partial\Omega \times [0, 2t]
\end{aligned}
\tag{5.2.41}
$$

which in compact form reads as

$$
\mathbf{N}_i \mathbf{z}_i = \mathbf{f}_i
\tag{5.2.42}
$$

where $i = I, II$.

Considering the following bilinear form:

$$
\langle\langle \mathbf{N}_i \mathbf{z}_i', \mathbf{z}_i'' \rangle\rangle_c = \int_\Omega \mathbf{M}_i \mathbf{z}_i'(2t) * \mathbf{z}_i''(2t) d\Omega + \int_{\partial\Omega} \mathbf{T}_i \mathbf{z}_i'(2t) * \mathbf{z}_i''(2t) * \mathbf{z}_i''(2t) d\Gamma
\tag{5.2.43}
$$

where $\mathbf{z}'_i$ and $\mathbf{z}''_i$ denote arbitrary vectors residing in the domain of the operator $\mathbf{N}_i$, $i = I, II, III$. The symmetry of the operator $\mathbf{N}_i$ with respect to (5.2.43) can be proved. Then the equivalency compact form (5.2.42) can be shown to be equivalent to the following variational formulation:

$$\mathcal{F}_i(\mathbf{z}_i) = \operatorname*{stat}_{\mathbf{z}'_i} \mathcal{F}^c_i(\mathbf{z}'_i) \tag{5.2.44}$$

where

$$
\begin{aligned}
\mathcal{F}_i(\mathbf{z}'_i) = & \frac{1}{2} \langle\langle \mathbf{N}_i \mathbf{z}'_i, \mathbf{z}'_i \rangle\rangle_c - \langle\langle \mathbf{f}_i, \mathbf{z}'_i \rangle\rangle_c \\
= & \frac{1}{2} \int_\Omega \mathbf{M}_i \mathbf{z}'_i(2t) * \mathbf{z}'_i(2t) d\Omega + \frac{1}{2} \int_{\partial\Omega} \mathbf{T}_i \mathbf{z}'_i(2t) * \mathbf{z}'_i(2t) d\Gamma \\
& - \int_\Omega \mathbf{b}_i(2t) * \mathbf{z}'_i(2t) d\Omega - \int_{\partial\Omega} \mathbf{g}_i(2t) * \mathbf{z}'_i(2t) d\Gamma
\end{aligned}
\tag{5.2.45}
$$

with $\mathbf{z}'_i$ denoting any admissible vector in the domain of $\mathbf{N}_i$ and $\mathbf{z}_i$ being the solution of the problem. Particularly, choosing $i = I$ an Hu-Washizu type of variational formulation is obtained, while for $i = II$ we obtain a variational formulation of type Hellinger-Reissner. And finally in the case of $i = III$ a new variational formulation is obtained.

Now incorporating strain-displacement relations into the Hu-Washizu formulation, leads to the following functional form of the Total Potential Energy as

$$
\begin{aligned}
\text{TPE}(u'_{1_i}, u'_{2_i}) = & \frac{1}{2} \int_\Omega \left(\mathcal{A} \mathscr{C} u'_{1_i}(2t) * \mathscr{C} u'_{1_i}(2t) + 2\tilde{\mathcal{B}} \mathscr{C} u'_{1_i}(2t) * \mathscr{C} u'_{2_i}(2t) \right) d\Omega \\
& - \int_\Omega \left(b_{2_i}(2t) * u'_{1_i}(2t) + b_{1_i}(2t) * u'_{2_i}(2t) \right) d\Omega \\
& - \int_{\partial\Omega_p} \left(p_{2_i}(2t) * u'_{1_i}(2t) + p_{1_i}(2t) * u'_{2_i}(2t) \right) d\Gamma
\end{aligned}
\tag{5.2.46}
$$

If we choose to introduce the equilibrium equations into Hellinger-Reissner functional, then we can have the following functional of the Total Complementary Energy type as

$$
\begin{aligned}
\text{TCE}(\sigma'_{i_{ij}}, \sigma'_{2_{ij}}) = & \frac{1}{2} \int_\Omega \left(\mathcal{A}\sigma'_{1_{ij}}(2t) * \sigma'_{1_{ij}}(2t) + 2\tilde{\mathcal{B}} \sigma'_{1_{ij}}(2t) * \sigma'_{2_{ij}}(2t) \right) d\Gamma \\
& - \int_{\partial\Omega_u} \left(n_j u^0_{2_i}(2t) * \sigma'_{1_{ij}}(2t) + n_j u^0_{1_i}(2t) * \sigma'_{2_{ij}}(2t) \right) d\Gamma
\end{aligned}
\tag{5.2.47}
$$

Both the above functionals are of the saddle-point type and respectively fulfill the following variational principles:

$$\text{TPE}(u_{1_i}, u_{2_i}) = \operatorname*{min}_{u'_{1_i}} \operatorname*{stat}_{u'_{2_i}} \text{TPE}(u'_{1_i}, u'_{2_i}) \tag{5.2.48}$$

where u'_{1_i} and u'_{2_i} denote compatible displacement fields, and

$$\text{TCE}(\sigma_{1_{ij}}, \sigma_{2_{ij}}) = \operatorname*{min}_{\sigma'_{1_i}} \operatorname*{stat}_{\sigma'_{2_i}} \text{TCE}(\sigma'_{1_{ij}}, \sigma'_{2_{ij}}) \tag{5.2.49}$$

where $\sigma'_{1_{ij}}$ and $\sigma'_{2_{ij}}$ being the equilibrated stress fields. The TPE functional is minimum with respect to u'_{1_i} and the TCE functional is minimum with respect to $\sigma'_{1_{ij}}$ because of the positive definiteness of the operator A.

5.3   The Linear Transient Heat Conduction: Formulation & Variational Principle

We consider a body $\Omega \subset \mathbb{R}^3$ and assume that it is thermally isotropic. In a Cartesian coordinates system it means that when a spatial point $\mathbf{x} = (x_i)_{i=1,2,3} \in \Omega$ is heated, the heat spreads equally in all directions. In addition the material might be thermally inhomogeneous which implies the conditions of conduction change from point to point within the material.

   The aim in this section is to provide a spatial description of temperature field $\theta(\mathbf{x}, t)$ as a result of external actions for every point $\mathbf{x} \in \Omega$ at each instant of time t in the interval $[0, 2t]$ with $t > 0$. For $t < 0$ the body is undisturbed and is in steady state of conduction, meaning that the temperature difference is constant and the spatial distribution of the temperature within the region does not change with time, with $\theta(\mathbf{x}, t) = \theta_0(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ and for $t < 0$.

   For $t \in [0, 2t]$ the heat conduction problem is governed by following set of relations as:

$$\rho c \frac{\partial \theta(\mathbf{x}, t)}{\partial t} + \frac{\partial q_i(\mathbf{x}, t)}{\partial x_i} = b(\mathbf{x}, t) \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.1}$$

$$p_i(\mathbf{x}, t) = -\frac{\partial \theta(\mathbf{x}, t)}{\partial x_i} \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.2}$$

$$q_i(\mathbf{x}, t) = k(\mathbf{x}, t) p_i(\mathbf{x}, t) \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.3}$$

where c denotes the specific heat in $JK^{-1}kg^{-1}$, ρ is the density of the constituent material in kgm^{-3}, the function $b(\mathbf{x}, t)$ is the rate of heat production per unit of volume in $Jm^{-3}s^{-1}$ and $k(\mathbf{x}, \theta)$ represents the thermal conductivity in $Jm^{-1}s^{-1}K^{-1}$ which is parametrized in terms of the position within the body and the temperature. In situations where the range of temperature is limited, as in the present case, the dependence upon $\theta(\mathbf{x}, t)$ can be neglected and thereby $k(\mathbf{x})$ is considered constant. The balance equation (5.3.1) which represents the principle of conservation of energy is applied to a differential control volume through which the energy is transferred exclusively by conduction.

   The compatibility equation (5.3.2) asserts that the vector $p_i(\mathbf{x}, t)$ is in the opposite direction to the gradient of the temperature. Lastly, the constitutive law (5.3.3) relates the heat flux $q_i(\mathbf{x}, t)$ with the temperature gradient. Combining relations (5.3.2) and (5.3.3) yields the following well-known equation

$$q_i(\mathbf{x}, t) = -k \frac{\partial \theta(\mathbf{x}, t)}{\partial x_i} \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.4}$$

derived by Fourier observing that heat transfer is in the direction of decreasing temperature.

   The combination of Fourier's law with the balance equation (5.3.1) leads to the so called heat equation which is known as prototypical example of a parabolic partial differential equation:

$$\rho c \frac{\partial \theta(\mathbf{x}, t)}{\partial t} - \frac{\partial}{\partial x_i} \left(k(\mathbf{x}) \frac{\partial \theta(\mathbf{x}, t)}{\partial x_i} \right) = b(\mathbf{x}, t) \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.5}$$

Equations (5.3.1), (5.3.2) and (5.3.3) are equipped with boundary conditions as below. On the portion $\partial\Omega_\theta$ of the boundary $\partial\Omega(\partial\Omega = \partial\Omega_\theta \cup \partial\Omega_q), \partial\Omega_q$ and $\partial\Omega_\theta$ being complementary parts of $\partial\Omega(\partial\Omega_\theta \cap \partial\Omega_q = \varnothing))$, the surface temperature $\bar\theta(\mathbf{x}, t)$, that may be constant or a function of time or position or both, is imposed as

$$\theta(\mathbf{x}, t) = \bar\theta(\mathbf{x}, t) \qquad \text{on } \partial\Omega_\theta \times [0, 2t] \qquad\qquad \text{Dirichlet boundary condition} \qquad (5.3.6)$$

whereas on $\partial\Omega_q$ the flux of heat $q(\mathbf{x}, t)$ across the surface is assigned as:

$$q(\mathbf{x}, t) = q_i(\mathbf{x}, t)n_i(\mathbf{x}) = \bar q(\mathbf{x}, t) \qquad \text{on } \partial\Omega_q \times [0, 2t] \qquad\qquad \text{Neumann boundary condition}$$
$$(5.3.7)$$

For simplicity, linear radiation boundary condition for which the flux across the surface is proportional to the temperature difference between the surface and the surrounding medium is not considered. Also, non-linear boundary conditions such as block-body radiation where the fourth power of the temperature field is imposed on the boundary or the natural convention regarding powers of the temperature field are neglected.

Ultimately, the initial conditions read as

$$\theta(\mathbf{x}, t) = \theta_0(\mathbf{x}) \qquad \text{in } \Omega, t = 0 \qquad\qquad (5.3.8)$$

Having all the required components of a boundary value problem defined as above, one can reformulate the heat conduction problem in a three-field form as

$$\begin{bmatrix} \rho c \frac{\partial(\cdot)}{\partial t} & 0 & \frac{\partial(\cdot)}{\partial x_i} \\ 0 & k & -I \\ -\frac{\partial(\cdot)}{\partial x_i} & -I & 0 \end{bmatrix} \begin{bmatrix} \theta \\ p_i \\ q_i \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} \quad \text{in } \Omega \times [0, t] \qquad\qquad (5.3.9)$$

$$\begin{bmatrix} 0 & 0 & -n_i \\ 0 & 0 & 0 \\ n_i & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ p_i \\ q_i \end{bmatrix} = \begin{matrix} \begin{bmatrix} -\bar q \\ 0 \\ n_i\bar\theta \end{bmatrix} \end{matrix} \begin{matrix} \text{on } \partial\Omega_q \times [0, 2t] \\ \\ \text{on } \partial\Omega_\theta \times [0, 2t] \end{matrix} \qquad (5.3.10)$$

$$\begin{bmatrix} \rho c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ p_i \\ q_i \end{bmatrix} = \begin{bmatrix} \rho c\theta_0 \\ 0 \\ 0 \end{bmatrix} \quad \text{in } \Omega, t = 0 \qquad\qquad (5.3.11)$$

where I denotes the identity operator. The above relations can be written in compact form as

$$\mathbf{M}_I \mathbf{z}_I = \mathbf{b}_I \qquad \text{in } \Omega \times [0, 2t] \qquad\qquad (5.3.12)$$
$$\mathbf{B}_I \mathbf{z}_I = \mathbf{g}_I \qquad \text{on } \partial\Omega \times [0, 2t] \qquad\qquad (5.3.13)$$
$$\mathbf{T}_I \mathbf{z}_I = \mathbf{h}_I \qquad \text{on } \Omega, t = 0 \qquad\qquad (5.3.14)$$

which in condensed form would be:
$$\mathbf{N}_I \mathbf{z}_I = \mathbf{f}_I \qquad\qquad (5.3.15)$$

Through elimination of the $p_i(t)$ the following two-field relation in operator form can be written as:

$$\begin{bmatrix} -\rho c \frac{\partial(\cdot)}{\partial t} & \frac{\partial(\cdot)}{\partial x_i} \\ \frac{\partial(\cdot)}{\partial x_i} & \frac{1}{k} \end{bmatrix} \begin{bmatrix} \theta \\ q_i \end{bmatrix} = \begin{bmatrix} -b \\ 0 \end{bmatrix} \quad \text{in } \Omega \times [0, 2t] \tag{5.3.16}$$

$$\begin{bmatrix} 0 & n_i \\ -n_i & 0 \end{bmatrix} \begin{bmatrix} \theta \\ q_i \end{bmatrix} = \begin{bmatrix} \bar{q} \\ -n_i \bar{\theta} \end{bmatrix} \quad \begin{array}{l} \text{on } \partial\Omega_q \times [0, 2t] \\ \text{on } \partial\Omega_\theta \times [0, 2t] \end{array} \tag{5.3.17}$$

$$\begin{bmatrix} -\rho c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ q_i \end{bmatrix} = \begin{bmatrix} -\rho c\theta \\ 0 \end{bmatrix} \quad \text{in } \Omega, t = 0 \tag{5.3.18}$$

which in compact form reads as

$$\mathbf{M}_{II}z_{II} = b_{II} \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.19}$$
$$\mathbf{B}_{II}z_{II} = g_{II} \qquad \text{on } \partial\Omega \times [0, 2t] \tag{5.3.20}$$
$$\mathbf{T}_{II}z_{II} = h_{II} \qquad \text{in } \Omega, t = 0 \tag{5.3.21}$$

which could be condensed as

$$\mathbf{N}_{II}\mathbf{z}_{II} = \mathbf{f}_{II} \tag{5.3.22}$$

Finally, the following one-field operator can be formulated as:

$$\rho c \frac{\partial\theta}{\partial t} - \frac{\partial}{\partial x_i}\left(k\frac{\partial\theta}{\partial x_i}\right) = b \qquad \text{in } \Omega \times [0, 2t] \tag{5.3.23}$$

$$-k\frac{\partial\theta}{\partial x_i}n_i = \bar{q} \qquad \text{on } \partial\Omega_q \times [0, 2t] \tag{5.3.24}$$

$$\theta = \theta_0 \qquad \text{in } \Omega, t = 0 \tag{5.3.25}$$

where the temperature field satisfies a priori the boundary condition on $\partial\Omega_\theta$ i.e.

$$\theta = \bar{\theta} \qquad \text{on } \partial\Omega_\theta \times [0, 2t] \tag{5.3.26}$$

One-field problem in a compact form reads:

$$\mathbf{M}_{III}\mathbf{z}_{III} = \mathbf{b}_{III} \qquad \text{in } \partial\Omega \times [0, 2t] \tag{5.3.27}$$
$$\mathbf{B}_{III}\mathbf{z}_{III} = \mathbf{g}_{III} \qquad \text{in } \partial\Omega \times [0, 2t] \tag{5.3.28}$$
$$\mathbf{T}_{III}\mathbf{z}_{III} = \mathbf{h}_{III} \qquad \text{in } \Omega, t = 0 \tag{5.3.29}$$

or, in a condensed form:

$$\mathbf{N}_{III}\mathbf{z}_{III} = \mathbf{f}_{III} \tag{5.3.30}$$

5.3.1 Variational Model for Heat Transport

It can be shown that the operator $N_i, i = I, II, III$ is symmetric in correlation to the following non-degenerate convolutive bilinear form:

$$\langle z_i', \mathbf{N}_i z_i'' \rangle_c = \int_\Omega z_i' * \mathbf{M}_i z_i'' d\omega + \int_{\partial\Omega} z_i' * \mathbf{B}_i z_i'' d\Gamma + \int_{\partial\Omega} z_i'(0)\mathbf{T}_i z_i''(2t)d\Omega \tag{5.3.31}$$

where $\mathbf{z}_i'$ and $\mathbf{z}_i''$ represent arbitrary vectors in the domain N_i, i, and the symbol $*$ denotes the convolution product with respect to time. Henceforth it follows that the compact form as

$$\mathbf{N}_i \mathbf{z}_i = \mathbf{f}_i \tag{5.3.32}$$

is equivalent to the following variational formulation as:

$$\mathcal{F}_i^c(z_i) = \underset{z_i'}{\text{stat}} \, \mathcal{F}_i^c(z_i') \tag{5.3.33}$$

with

$$
\begin{aligned}
\mathcal{F}_i^c(z_i') =\ & \frac{1}{2}\langle z_i', \mathbf{N}_i z_i'\rangle_c - \langle \mathbf{z}_i', \mathbf{f}_i\rangle_c \\
=\ & \frac{1}{2}\int_\Omega \mathbf{z}_i' * \mathbf{M}_i z_i' d\Omega + \frac{1}{2}\int_{\partial\Omega} z_i' * \mathbf{B}_i z_i' d\Gamma + \frac{1}{2}\int_\Omega \mathbf{z}_i'(0)\mathbf{T}_i z_i'(2t) d\Omega \\
& - \int_\Omega \mathbf{a}_i * z_i' d\Omega - \int_{\partial\Omega} \mathbf{g}_i * z_i' d\Gamma - \int_\Omega \mathbf{h}_i \mathbf{z}' i(2t) d\Omega
\end{aligned}
\tag{5.3.34}
$$

where $\mathbf{z}_i'$ denotes a vector in $\mathbf{N}_i$ domain and $\mathbf{z}_i$ would be the solution of the problem. Also it can be shown that the operator $\mathbf{N}_i$ is symmetric with respect to the following non-degenerate biconvolutive bilinear form:

$$f(x_i, t) * y * g(x_i, t) = \int_0^t \int_0^{t-\tau} r(t - \tau - \eta) g(x_i, \eta) f(x_i, \tau) d\tau dt \tag{5.3.35}$$

where $y(t)$ represents a relaxation function.

In a similar fashion, the compact form shows to be equivalent to the following variational formulation:

$$\mathcal{F}_i^{cc}(\mathbf{z}_i) = \underset{\mathbf{z}_i'}{\text{stat}} \, \mathcal{F}_i^{cc}(\mathbf{z}_i') \tag{5.3.36}$$

where

$$
\begin{aligned}
\mathcal{F}_i^{cc}(\mathbf{z}_i') =\ & \frac{1}{2}\langle \mathbf{z}_i', \mathbf{N}_i \mathbf{z}_i'\rangle_{cc} - \langle \mathbf{z}_i', \mathbf{f}_i\rangle_{cc} \\
=\ & \frac{1}{2}\int_\Omega \mathbf{z}_i' * y * \mathbf{M}_i \mathbf{z}_i' d\Omega + \frac{1}{2}\int_{\partial\Omega} \mathbf{z}_i' * y * \mathbf{B}_i \mathbf{z}_i' d\Gamma + \frac{1}{2}\int_\Omega \mathbf{z}_i'(0)\mathbf{T}_i z_i'(2t) d\Gamma \\
& - \int_\Omega \mathbf{a}_i * y * \mathbf{z}_i' d\Omega - \int_{\partial\Omega} \mathbf{g}_i * y * \mathbf{z}_i' d\Gamma - \int_\Omega \mathbf{h}_i \mathbf{z}_i'(2t) d\Omega
\end{aligned}
\tag{5.3.37}
$$

where $\mathbf{z}_i'$ is a vector that belongs to $\mathbf{N}_i$ domain and $\mathbf{z}_i$ represents the solution vector.

Gurtin's Formulation

In the spirit of the convolution bilinear form, the Gurtin's functional (as a function of temperature) of type of the Total Potential Energy reads as

$$
\begin{aligned}
\mathcal{F}_{cc}^G(\theta') =\ & \frac{1}{2}\int_\Omega \theta' * \rho c \frac{\partial\theta'}{\partial t} d\Omega + \frac{1}{2}\int_\Omega k \frac{\partial\theta'}{\partial x_i} * \frac{\partial\theta'}{\partial x_i} d\Omega + \frac{1}{2}\int_\Omega \rho c\theta' \cdot \theta'(0) d\Omega \\
& - \int_\Omega \theta' * a\, d\Omega + \int_{\partial\Omega_p} \theta' * \bar{q}\, d\Gamma - \int_\Omega \theta' \cdot \partial\Omega \rho \theta_0 d\Omega
\end{aligned}
\tag{5.3.38}
$$

In a similar fashion and in the spirit of biconvolutive bilinear form, Gurtin's functional reads as

$$
\begin{aligned}
\mathcal{F}_{cc}^{G}(\theta') = {} & \frac{1}{2}\int_{\Omega}\theta'*y*\rho c\frac{\partial\theta'}{\partial t}d\Omega + \frac{1}{2}\int_{\Omega}k\frac{\partial\theta'}{\partial x_i}*y*\frac{\partial\theta'}{\partial x_i}d\Omega + \frac{1}{2}\int_{\Omega}(\rho c\theta'*y)\cdot\theta'(0)d\Omega \\
& - \int_{\Omega}\theta'*y*a\,d\Omega + \int_{\partial\Omega_p}\theta'*y*\bar{q}\,d\Gamma - \int_{\Omega}(\theta'*y)\cdot\rho c\theta_0\,d\Omega \\
= {} & \frac{1}{2}\int_{\Omega}\rho c\theta'*(y(0)\theta'+\dot{y}*\theta')d\Omega + \frac{1}{2}\int_{\Omega}k\frac{\partial\theta'}{\partial x_i}*y*\frac{\partial\theta'}{\partial x_i}d\Omega \\
& - \int_{\Omega}\theta'*y*a\,d\Omega + \int_{\partial\Omega_p}\theta'*y*\bar{q}\,d\Gamma - \int_{\Omega}(\theta'*y)\cdot\rho c\theta_0\,d\Omega
\end{aligned}
$$
$$(5.3.39)$$

where $\bar{\theta}$ represent boundary values on $\partial\Omega_\theta$.

5.3.2   Reformulation of the model

Through decomposition of time domain into two equal subintervals as explained before, a min-max variational formulation using the convolutive bilinear form is obtained.

Time domain decomposition

Carini and Mattei [141] developed five forms of variational formulations for modeling viscoelastic problems. The concept of time decomposition employed there is incorporated here in a similar manner. Accordingly the time interval $[0,2t]$ is decomposed into two sub-intervals as $[0,t]$ and $[t,2t]$. Field variables of the problem consequently decompose as

$$
\theta(t) = \begin{cases} \theta_1(T) & \text{for } t\in[0,t] \\ \theta_2(T) & \text{for } t\in[t,2t] \end{cases}
\tag{5.3.40}
$$

$$
p_i(t) = \begin{cases} p_{1_i}(t) & \text{for } t\in[0,t] \\ p_{2_i}(t) & \text{for } t\in[t,2t] \end{cases}
\tag{5.3.41}
$$

$$
q_i(t) = \begin{cases} q_{1_i}(t) & \text{for } t\in[0,t] \\ q_{2_i}(t) & \text{for } t\in[t,2t] \end{cases}
\tag{5.3.42}
$$

where subscript 1 refers to subinterval $[0,t]$ while subscript 2 denotes subinterval $[t,2t]$.

Min-Max variational Formulation of Tonti's type

In order to propose a functional form for $\mathcal{F}_i$ with $i=I,II,III$ as defined by (5.3.34) we first consider the case in which $i=I$. Starting with $\int_{\Omega}z'_I*\mathbf{M}_I\mathbf{z}'_I d\Omega$, and its time decomposition one can write

$$
\begin{aligned}
\int_{\Omega}\mathbf{z}'_I*\mathbf{M}_I\mathbf{z}'_I d\Omega = {} & \rho c\int_{\Omega}\frac{\partial\theta'}{\partial t}*\theta'd\Omega + \int_{\Omega}\frac{\partial q'_i}{\partial x_i}*\theta'd\Omega + \int_{\Omega}p'_i*kp'_i d\Omega \\
& - 2\int_{\Omega}p'_i*q'_i d\Omega - \int_{\Omega}\frac{\partial\theta'}{\partial x_i}*q'_i d\Omega
\end{aligned}
$$
$$(5.3.43)$$

The integrand containing the derivative with respect to time when decomposed could get various forms. Two of them can be written as

$$\rho c \int_\Omega \frac{\partial \theta'}{\partial t} * \theta' d\Omega = 2\rho c \int_\Omega \int_0^t \frac{\partial \theta'(t)}{\partial t} \theta'(2t) dt d\Omega - D \qquad (5.3.44a)$$

$$\rho c \int_\Omega \frac{\partial \theta'}{\partial t} * \theta' d\Omega = 2\rho c \int_\Omega \int_t^{2t} \frac{\partial \theta'(t)}{\partial t} \theta'(2t) dt d\Omega + D \qquad (5.3.44b)$$

with D defined as

$$D = \rho c \int_\Omega \int_0^t \frac{\partial \theta'(t)}{\partial t} \theta'(2t) dt d\Omega - \rho c \int_\Omega \int_t^{2t} \frac{\partial \theta'(t)}{\partial t} \theta'(2t) dt d\Omega \qquad (5.3.45)$$

Integration by parts transform the latter term into

$$D = \rho c \int_\Omega \theta'^2(T) - \rho c \int_\Omega \theta'(2t)\theta'(0) d\Omega \qquad (5.3.46)$$

Therefore the decomposition forms as introduced in (5.3.44a) and (5.3.44b) respectively turn into

$$\rho c \int_\Omega \frac{\partial \theta'}{\partial t} * \theta' d\Omega = 2\rho c \int_\Omega \int_0^t \frac{\partial \theta'_1(t)}{\partial t} \theta'_2(2t) dt d\Omega - \rho c \int_\Omega \theta'^2_1(T) d\Omega$$
$$+ \rho c \int_\Omega \theta'_1(2t)\theta'_1(0) d\Omega \qquad (5.3.47a)$$

$$\rho c \int_\Omega \frac{\partial \theta'}{\partial t} * \theta' d\Omega = 2\rho c \int_\Omega \int_t^{2t} \frac{\partial \theta'_2(t)}{\partial t} \theta'_1(2t) dt d\Omega + \rho c \int_\Omega \theta'^2_2(T) d\Omega$$
$$- \rho c \int_\Omega \theta'_1(2t)\theta'_1(0) d\Omega \qquad (5.3.47b)$$

The advantage of the introduced forms (5.3.47a) and (5.3.47b) resides in the fact that they lead to negative quadratic form for θ_1 and positive quadratic form for θ_2 which inherently motivate application of maximum and minimum principles for variational formulations, respectively.

Looking back to the remaining terms in (5.3.43) those not involving time differentiation can easily be decomposed through splitting the time integrals. For example the second term could be converted into

$$\int_\Omega \frac{\partial q'_i}{\partial x_i} * \theta' d\Omega = \int_\Omega \int_0^t \frac{\partial q'_{1_i}}{\partial x_i} \theta'_2(2t) d\Omega + \int_\Omega \int_t^{2t} \frac{\partial q'_{2_i}(t)}{\partial x_i} \theta'_1(2t) d\Omega \qquad (5.3.48)$$

As a result, one can write the two following variational formulations:

$$\mathcal{F}_I^a(\theta_1, \theta_2, p_{1_i}, p_{2_i}, q_{i_1}, q_{2_i}) = \max_{\theta'_1} \operatorname*{stat}_{\theta'_2, p'_{1_i}, p'_{2_i}, q'_{i_1}, q'_{i_2}} \mathcal{F}_I^a(\theta'_1, \theta'_2, p'_{1_i}, p'_{2_i}, q'_{i_1}, q'_{i_2}) \qquad (5.3.49)$$

$$\mathcal{F}_I^b(\theta_1, \theta_2, p_{1_i}, p_{2_i}, q_{i_1}, q_{2_i}) = \max_{\theta'_2} \operatorname*{stat}_{\theta'_1, p'_{1_i}, p'_{2_i}, q'_{1_i}, q'_{2_i}} \mathcal{F}_I^b(\theta'_1, \theta'_2, p'_{1_i}, p'_{2_i}, q'_{1_i}, q'_{2_i}) \qquad (5.3.50)$$

where superscripts a and b respectively refer to decompositions (5.3.47a) and (5.3.47b). Following the same procedure for functional $\mathcal{F}_{II}$ one can write

$$\mathcal{F}_{II}^a(\theta_1, \theta_2, q_{1_i}, q_{2_i}) = \min_{\theta'_1} \operatorname*{stat}_{\theta'_2, q'_{1_i}, q'_{2_i}} \mathcal{F}_{II}^a(\theta'_1, \theta'_2, q'_1, q'_{2_i}) \qquad (5.3.51)$$

$$\mathcal{F}_{II}^b(\theta_1, \theta_2, q_{1_i}, q_{2_i}) = \max_{\theta'_2} \operatorname*{stat}_{\theta'_1, q'_{1_i}, q'_{2_i}} \mathcal{F}_{II}^b(\theta'_1, \theta'_2, q'_1, q'_{2_i}) \qquad (5.3.52)$$

And lastly for $\mathcal{F}_{III}$ following forms of variational principles hold:

$$\mathcal{F}_{III}^a(\theta_1, \theta_2) = \max_{\theta'1} \text{stat}_{\theta'2} \, \mathcal{F}_{III}(\theta'1, \theta'2) \tag{5.3.53}$$

$$\mathcal{F}_{III}^b(\theta_1, \theta_2) = \min_{\theta'2} \text{stat}_{\theta'1} \, \mathcal{F}_{III}(\theta'1, \theta'2) \tag{5.3.54}$$

where

$$
\begin{aligned}
\mathcal{F}_{III}^a(\theta_1', \theta_2') = {} & \rho c \int_\Omega \frac{\partial \theta_1'}{\partial t} \circ \theta_2' d\Omega + \frac{1}{2} \int_\Omega \frac{\partial \theta_1'}{\partial x_i} \circ k \frac{\partial \theta_1'}{\partial x_i} \circ k \frac{\partial \theta_2'}{\partial x_i} d\Omega + \frac{1}{2} \int_\Omega \frac{\partial \theta_2'}{\partial x_i} \circ k \frac{\partial \theta_1'}{\partial x_i} d\Omega \\
& - \int_\Omega \theta_1' \circ a_2 d\Omega - \int_\Omega a_1 \circ \theta_2' d\Omega + \int_{\partial\Omega_q} \theta_1' \circ \bar{q} d\Gamma \\
& + \int_{\partial\Omega_p} \bar{q} \circ \theta_2' d\Gamma - \rho c \int_\Omega \theta_0 \theta_2'(2t) d\Omega + \rho c \int_\Omega \theta_1'(0) \theta_2'(2t) d\Omega \\
& - \frac{1}{2} \rho c \int_\Omega (\theta_1'(T))^2 d\Omega
\end{aligned}
\tag{5.3.55}
$$

and

$$
\begin{aligned}
\mathcal{F}_{III}^b(\theta_1', \theta_2') = {} & \rho c \int_\Omega \theta_1' \circ \frac{\partial \theta_2'}{\partial t} d\Omega + \frac{1}{2} \int_\Omega \frac{\partial \theta_1'}{\partial x_i} \circ k \frac{\partial \theta_2'}{\partial x_i} d\Omega + \frac{1}{2} \int_\Omega \frac{\partial \theta_1'}{\partial x_i} \circ k \frac{\partial \theta_1'}{\partial x_i} d\Omega \\
& - \int_\Omega \theta_1' \circ b_2 d\Omega - \int_\Omega b_1 \circ \theta_2' d\Omega + \int_{\partial\Omega_p} \theta_1' \circ \bar{q}_2 d\Gamma \\
& + \int_{\partial\Omega_p} \bar{q}_1 \circ \theta_2' d\Gamma - \rho c \int_\Omega \theta_0 \theta_2'(2t) d\Omega + \frac{1}{2} \rho c \int_\Omega (\theta_2'(T))^2 d\Omega
\end{aligned}
\tag{5.3.56}
$$

Herein θ_1' and θ_2' represent two admissible fields that satisfy prescribed boundary conditions, and the symbol $\circ$ denotes the convolution product operator; for instance:

$$\theta_1' \circ a_2 = \int_0^t \theta_1'(t) a_2(2t) dt = \int_t^{2t} \theta_1'(2t) a_2(t) dt \tag{5.3.57}$$

Interestingly, stationarity of $\mathcal{F}_i^a$ with $i = I, II, III$ with respect to θ_1' leads to the continuity condition as following:

$$\theta_1(T) = \theta_2(T). \tag{5.3.58}$$

The same can be achieved for $\mathcal{F}_i^b$ after imposing the stationarity this time with respect to θ_2'.

Considering the sum of the functionals $\mathcal{F}_{III}^a$ and $\mathcal{F}_{III}^b$ and following the same sort of procedures one can obtain a min-max variational principle. In fact,

$$
\begin{aligned}
\mathcal{F}_{III}^c = \mathcal{F}_{III}^a + \mathcal{F}_{III}^b = {} & \rho c \int_\Omega \frac{\partial \theta_1'}{\partial t} \circ \theta_2' d\Omega + \rho c \int_\Omega \theta_1' \circ \frac{\partial \theta_2'}{\partial t} d\Omega + 2 \int_\Omega \frac{\partial \theta_1'}{\partial x_i} \circ k \frac{\partial \theta_2'}{\partial x_i} d\Omega \\
& - 2 \int_\Omega \theta_1' \circ a_2 d\Omega - 2 \int_\Omega a_1 \circ \theta_2' d\Omega + 2 \int_{\partial\Omega_p} \theta_1' \circ \bar{q}_{2_i} n_i d\Gamma \\
& + 2 \int_{\partial\Omega_q} \bar{q}_{1_i} n_i \circ \theta_2' d\Gamma - 2\rho c \int_\Omega \theta_0 \theta_2'(2t) d\Omega + \rho c \int_\Omega \theta_1'(0) \theta_2'(2t) d\Omega \\
& - \frac{1}{2} \rho c \int_\Omega (\theta_1'(T))^2 d\Omega + \frac{1}{2} \rho c \int_\Omega (\theta_2'(T))^2 d\Omega
\end{aligned}
\tag{5.3.59}
$$

comply with the following variational principle:

$$\mathcal{F}_{III}^c(\theta_1, \theta_2) = \min_{\theta_2'} \max_{\theta_1'} \mathcal{F}_{III}^c(\theta_1', \theta_2') \tag{5.3.60}$$

The stationary of the functional (5.3.60)with respect to θ_1' and θ_2' respectively reads as

$$\delta_{\theta_1'} \mathcal{F}_{III}^c = 0 \Rightarrow \begin{cases} \rho c \frac{\partial \theta_2'}{\partial t} - \frac{\partial}{\partial x_i}(k \frac{\partial \theta_2'}{\partial x_i}) = a_2 & \text{in } \Omega \times [t, 2t] \\ -k \frac{\partial \theta_2'}{\partial x_i} n_i = \bar{q}_2 & \text{on } \partial \Omega_q \times [t, 2t] \end{cases} \tag{5.3.61}$$

$$\delta_{\theta_2'} \mathcal{F}_{III}^c = 0 \Rightarrow \begin{cases} \rho c \frac{\partial \theta_1'}{\partial t} - \frac{\partial}{\partial x_i}(k \frac{\partial \theta_1'}{\partial x_i}) = a_1 & \text{in } \Omega \times [0, 2t] \\ -k \frac{\partial \theta_1'}{\partial x_i} n_i = \bar{q}_1 & \text{on } \partial \Omega_q \times [0, t] \\ \theta_1'(0) = \theta_0 & \text{on } \partial \Omega \\ \theta_1'(T) = \theta_2'(T) & \text{in } \Omega \end{cases} \tag{5.3.62}$$

5.4 Linear Thermoviscoelasticity

In this section thermodynamical derivation of constitutive law for thermoviscoelastic materials under isothermal conditions are presented. Different approaches for such a derivation have been employed such as those by Biot [18],Eringen [21], Hunter [152], Schapery [136]. The method presented by Christensen [149] as opted here is based on two fundamental thermodynamical postulates namely the balance of energy and the entropy production inequality.

The balance of energy in local form reads as

$$\rho r - \rho[\dot{\psi} + \dot{T}s + T\dot{s}] + \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \nabla \cdot \mathbf{q} = 0 \tag{5.4.1}$$

where ρ denotes the mass density, r would be the heat supply per unit mass, ψ represents the Helmholtz free energy per unit mass, T is the absolute temperature, s denotes the entropy per unit mass and q_i represent Cartesian components of the heat flux which is defined per unit area and per unit time. The function r offer the possibility of adding or removing exerted heat through external sources.

The balance of energy can be written either in terms of internal energy or the free energy. These two however can be interrelated through Legendre transformation as

$$\rho u = \rho \psi + T \rho s \tag{5.4.2}$$

where u denotes the internal energy per unit mass. From that standpoint, the entropy production inequality in local form reads as

$$\rho T \dot{s} - \rho r + \nabla \cdot \mathbf{q} - \mathbf{q} \frac{\nabla T}{T_0} \geq 0 \tag{5.4.3}$$

which as already mentioned is referred as Clausius-Duhem inequality.

We consider $\varepsilon_{ij}(t)$ and $T(t)$ as continuous functions on the interval $-\infty < t < \infty$ and $\varepsilon_{ij}(T)$ tend to zero and $T(t) \to T_0$ as $t \to -\infty$. According to Stone-Weierstrass theorem, a real continuous scalar functional of $\varepsilon_{ij}(\tau)$ and $T(\tau)$, $-\infty < \tau \leq t$ can be approximated by a polynomial in a set of continuous and linear functional of $\varepsilon_{ij}(\tau)$ and $T(\tau)$. Employing the Riesz representation theorem these linear functional can be formulated in terms of

Stieltjes integrals as explained in previous chapter. Henceforth introducing $\theta(t)$ as the excess temperature from T_0 and assuming that $\varepsilon_{ij}(\tau)$ and $\theta(\tau)/T_0$ are infinitesimals of $\mathscr{O}(\varepsilon)$, the polynomial expansion of $\rho\psi$ based on these linear functionals reads in the following form as

$$
\begin{aligned}
\rho\psi = \rho\psi_0 &+ \int_{-\infty}^{t} D_{ij}(t-\tau)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}d\tau - \int_{-\infty}^{t}\beta(t-\tau)\frac{\partial T(\tau)}{\partial\tau}d\tau \\
&+ \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} E_{ijkl}(t-\tau,t-\eta)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\frac{\partial\varepsilon_{kl}(\eta)}{\partial\eta}d\tau d\eta \\
&- \int_{-\infty}^{t}\int_{-\infty}^{t}\phi_{ij}(t-\tau,t-\eta)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\frac{\partial T(\eta)}{\partial\eta}d\tau d\tau \\
&- \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} m(t-\tau,t-\tau)\frac{\partial\theta(\tau)}{\partial\tau}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta + \mathcal{O}(\varepsilon^3)
\end{aligned}
\tag{5.4.4}
$$

where ψ_0 denotes the mean free energy, and the integrands are assumed to be continuous for $\tau_i \geq 0$ and to vanish for $\tau_i \leq 0$ meaning

$$
\begin{aligned}
\beta(\tau_1) = 0, \qquad & D_{ij}(\tau_1) = 0, \qquad && E_{ijkl}(\tau_1,\tau_2) = 0 \\
\phi_{ij}(\tau_1,\tau_2) = 0, \qquad & m(\tau_1,\tau_2) = 0, \qquad && \text{for } \tau_1 < 0, \tau_2 < 0.
\end{aligned}
\tag{5.4.5}
$$

Herein the terms of $\mathcal{O}(\varepsilon^3)$ in (5.4.4) are neglected and integrands in (5.4.4) are independent of strain and temperature.

Heat source function r can be removed between (5.4.1) and (5.4.3) as a result of which the following relation is obtained as

$$
-\rho s\dot{\theta} - \rho\dot{\psi} + \boldsymbol{\sigma}\!:\!\dot{\boldsymbol{\varepsilon}} - \mathbf{q}\frac{\nabla\theta}{T_0} \geq 0
\tag{5.4.6}
$$

where $T = T_0 + \theta$ is opted and only terms of $\mathcal{O}(\varepsilon)$ have been kept in $T_{,i}/T$.

If $\rho\psi$ from (5.4.4) is substituted into (5.4.6) and if the indicated time differentiation is carried out, after applying the Leibnitz's rule we have

$$
\begin{aligned}
&\left\{-D_{ij}(0) - \int_{-\infty}^{t} E_{ijkl}(t-\tau,0)\frac{\partial\varepsilon_{kl}(\tau)}{\partial\tau}\right. \\
&\left. + \int_{-\infty}^{t}\phi_{ij}(0,t-\tau)\frac{\partial\theta(\tau)}{\partial\tau}d\tau + \sigma_{ij}\right\}\dot{\varepsilon_{ij}}(t) \\
&+ \left\{\beta(0) + \int_{-\infty}^{t} m(t-\tau,0)\frac{\partial\theta(\tau)}{\partial\tau}d\tau\right. \\
&\left. + \int_{-\infty}^{t}\phi_{ij}(t-\tau,0)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}d\tau - \rho S\right\}\dot{\theta}(t) \\
&+ \left\{-\int_{-\infty}^{t}\frac{\partial}{\partial t}D_{ij}(t-\tau)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}d\tau + \int_{-\infty}^{t}\frac{\partial}{\partial t}\beta(t-\tau)\frac{\partial\theta(\tau)}{\partial\tau}d\tau\right. \\
&\left. + \Lambda - q_i\frac{\theta_{,i}}{T_0} \geq 0\right\}
\end{aligned}
\tag{5.4.7}
$$

where

$$\Lambda = -\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}E_{ijkl}(t-\tau,t-\eta)\frac{\partial\varepsilon_{ij}\tau}{\partial\tau}\frac{\partial\varepsilon_{kl}(\eta)}{\partial\tau}d\tau d\eta$$
$$+\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\phi_{ij}(t-\tau,t-\eta)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta \qquad (5.4.8)$$
$$+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}m(t-\tau,t-\eta)\frac{\partial\theta(\tau)}{\partial\tau}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta$$

where following symmetrical properties applied:

$$E_{ijkl}(t-\tau,t-\tau)=E_{klij}(t-\eta,t-\tau)$$
$$m(t-\tau,t-\eta)=m(t-\tau,t-\eta) \qquad (5.4.9)$$

The inequality (5.4.7) must be valid for all $\dot{\varepsilon}_{ij}(t)$ and $\bar{\theta}(t)$. As a result it demands the coefficients of $\dot{\varepsilon}_{ij}(t)$ and $\dot{\theta}(t)$ in (5.4.7) to vanish. Therefore,

$$\sigma_{ij}=D_{ij}(0)+\int_{-\infty}^{t}E_{ijkl}(t-\tau,0)\frac{\partial\varepsilon_{kl}\tau}{\partial\tau}-\int_{-\infty}^{t}\phi_{ij}(0,t-\tau)\frac{\partial\theta(\tau)}{\partial\tau} \qquad (5.4.10)$$

and

$$\rho s=\beta(0)+\int_{-\infty}^{t}\phi_{ij}(t-\tau,0)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\frac{\partial\varepsilon_{ij}\tau}{\partial\tau}d\tau+\int_{-\infty}^{t}m(t-\tau,0)\frac{\partial\theta(\tau)}{\partial\tau}d\tau \qquad (5.4.11)$$

leading inequality (5.4.7) to be read as

$$-\int_{-\infty}^{t}\frac{\partial}{\partial t}D_{ij}(t-\tau)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}+\int_{-\infty}^{t}\frac{\partial}{\partial t}\beta(t-\tau)\frac{\partial\theta(\tau)}{\partial\tau}d\tau$$
$$+\Lambda-q_{i}\frac{\theta_{,i}}{T_{0}}\geq0. \qquad (5.4.12)$$

Expressions (5.4.10) and (5.4.11) respectively represent the constitutive relations for stress and entropy. It can be deduced that $D_{ij}(0)$ would be the initial stress and $\beta(0)$ would denote the initial entropy ρS_{0}. The integrands including $E_{ijkl}(t-\tau,0)$, $\phi_{ij}(0,t-\tau)$, $\phi_{ij}(t-\tau,0)$ and $m(t-\tau,0)$ represent relaxation functions forms of the mechanical properties. In this regard and from the thermodynamical point of view, if $E_{ijkl}(\tau,\eta)$ is considered as a surface in τ and η space, then the relaxation functions in (5.4.10) and (5.4.11) can be imagined as curves on these surfaces. Additionally, the relaxation function $E_{ijkl}(t,0)$ in the current development could be a representation to the relaxation function $E_{ijkl}(t)$ in isothermal theory.

The first two terms in (5.4.12) are of first order while the last two are of the second order. The proposed inequality to be satisfied for all processes it is required that

$$\frac{\partial D_{ij}(t)}{\partial t}=0, \qquad \frac{\partial\beta(t)}{\partial t}=0 \qquad (5.4.13)$$

also it demands

$$\Lambda-q_{i}(\theta_{,i}/T_{0})\geq0. \qquad (5.4.14)$$

In specific processes where for example $\theta_{,i}=0$ and the temperature field is uniform then fulfillment of (5.4.14) implies

$$\Lambda\geq0. \qquad (5.4.15)$$

Above expression is usually called the dissipation inequality where Λ yields the rate of dissipation of energy. Relation (5.4.15) results in

$$q_i(\theta_{,i}/T_0) \leq 0. \tag{5.4.16}$$

As a next step in the development of the theory, we need to postulate a constitutive expression for the heat flux q_i. It can be defined as

$$q_i = -\int_{-\infty}^{t} k_{ij}(t-\tau)\frac{\partial\theta_{,j}(\tau)}{\partial\tau}d\tau \tag{5.4.17}$$

where q_i is a linear function of the history of the temperature gradient $\theta_{,j}$. Substituting (5.4.17) in (5.4.16) one can write

$$\theta_{,i}\int_{-\infty}^{t} k_{ij}(t-\tau)\frac{\partial\theta_{,j}(\tau)}{\partial\tau}d\tau \geq 0. \tag{5.4.18}$$

For a given t and $\theta_{,i}$ only if k_{ij} is positive definite and be constant with respect to time then the expression (5.4.17) is reduced in the following form as

$$q_i = -k_{ij}\theta_{.j} \tag{5.4.19}$$

where the entries of k_{ij} are constant and the tensor is symmetric with respect to i and j. Now using the relations (5.4.4),(5.4.10),(5.4.11),(5.4.13) and (5.4.19) the original balance of energy expression can be rewritten as

$$\rho r + \Lambda - T_0\frac{\partial}{\partial t}\left[\int_{-\infty}^{t}\phi_{ij}(t-\tau,0)\frac{\varepsilon_{ij}(\tau)}{\partial\tau}d\tau\right.$$
$$\left. + \int_{-\infty}^{t}m(t-\tau,0)\frac{\partial\theta(\tau)}{\partial\tau}d\tau\right] + (k_{ij}\theta_{,j})_{,i} = 0 \tag{5.4.20}$$

where Λ is defined as in (5.4.8). It is a term of the second order and therefore must be eliminated from (5.4.20) for the sake of keeping consistency with the first order development theory. Henceforth, the equation (5.4.20) can be rewritten as

$$\rho r - T_0\frac{\partial}{\partial t}\left[\int_{-\infty}^{t}\phi_{ij}(t-\tau,0)\frac{\varepsilon_{ij}(\tau)}{\partial\tau}d\tau\right.$$
$$\left. + \int_{-\infty}^{t}m(t-\tau,0)\frac{\partial\theta(\tau)}{\partial\tau}d\tau\right] + (k_{ij}\theta_{,j})_{,i} = 0 \tag{5.4.21}$$

In (5.4.21) the term involving the strain history serves as a coupling term between mechanical and thermal effects. Neglecting this term, equation (5.4.21) is reduced to the heat conduction problem.

In isotropic theory ϕ_{ij} must be defined as

$$\phi_{ij}(\tau,\eta) = \delta_{ij}\phi(\tau,\eta). \tag{5.4.22}$$

Similarly, E_{ijkl} would be defined as

$$E_{ijkl}(t) = \frac{1}{3}[E_2(t) - E_1(t)]\delta_{ij}\delta_{kl}\frac{1}{2}[E_1(t)](\delta_{ik}\delta_{jl} + \delta_{il} + \delta_{jk}) \tag{5.4.23}$$

where $E_1(t)$ and $E_2(t)$ would represent independent relaxation functions and δ_{ij} would be the Kronecker delta function. Defining deviatoric components of stress s_{ij} and strain e_{ij} respectively as

$$s_{ij} = \int_{-\infty}^{t} E_1(t-\tau)\frac{de_{ij}(\tau)}{d\tau}d\tau \tag{5.4.24}$$

and

$$\sigma_{kk} = \int_{-\infty}^{t} E_2(t-\tau)\frac{d\varepsilon_{kk}(\tau)}{d\tau}d\tau \tag{5.4.25}$$

where

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \qquad\qquad s_{ii} = 0 \tag{5.4.26}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}, \qquad\qquad e_{ii} = 0. \tag{5.4.27}$$

Thereby the free energy for isotropic bodies can be formulated as

$$\begin{aligned}
\rho\psi = &\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} E_1(t-\tau, t-\tau)\frac{\partial e_{ij}(\tau)}{\partial\tau}\frac{\partial e_{ij}(\eta)}{\partial\eta}d\tau d\eta \\
&+ \frac{1}{6}\int_{-\infty}^{t}\int_{-\infty}^{t} E_2(t-\tau, t-\tau)\frac{\partial\varepsilon_{kk}(\tau)}{\partial\tau}\frac{\partial\varepsilon_{ij}(\eta)}{\partial\eta}d\tau d\eta \\
&- \int_{-\infty}^{t}\int_{-\infty}^{t} \phi(t-\tau, t-\tau)\frac{\partial\varepsilon_{kk}(\tau)}{\partial\tau}\frac{\partial\theta_{ij}(\eta)}{\partial\eta}d\tau d\eta \\
&- \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} m(t-\tau, t-\eta)\frac{\partial\theta(\tau)}{\partial\tau}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta
\end{aligned} \tag{5.4.28}$$

where by comparison with (5.4.4) we notice that the initial stress and initial entropy have been neglected.

Consequently, the constitutive relations in the case of isotropic bodies read as

$$s_{ij} = \int_{-\infty}^{t} E_1(t-\tau, 0)\frac{\partial e_{ij}(\tau)}{\partial\tau}d\tau \tag{5.4.29}$$

and

$$\sigma_{kk} = \int_{-\infty}^{t} E_2(t-\tau, 0)\frac{\partial\varepsilon_{kk}(\tau)}{\partial\tau}d\tau - 3\int_{-\infty}^{t} \phi(0, t-\tau)\frac{\partial\theta(\tau)}{\partial\tau}d\tau. \tag{5.4.30}$$

The constitutive relation for entropy in the case of isotropic material would be the same as the one introduced in (5.4.11) except the fact that ϕ_{ij} and ε_{ij} are respectively replaced by ϕ and ε_{kk}.

The rate of dissipation of energy then reads as

$$\begin{aligned}
\Lambda = &-\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}E_1(t-\tau, t-\eta)\frac{\partial e_{ij}(\tau)}{\partial\tau}\frac{\partial e_{ij}(\eta)}{\partial\eta}d\tau d\eta \\
&- \frac{1}{6}\int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}E_2(t-\tau, t-\eta)\frac{\partial\varepsilon_{kk}(\tau)}{\partial\tau}\frac{\partial\varepsilon_{ij}(\eta)}{\partial\eta}d\tau d\eta \\
&- \frac{1}{6}\int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}\phi(t-\tau, t-\eta)\frac{\varepsilon_{kk}(\tau)}{\partial\tau}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta \\
&+ \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}m(t-\tau, t-\eta)\frac{\partial\theta(\tau)}{\partial\eta}\frac{\partial\theta(\eta)}{\partial\eta}d\tau d\eta
\end{aligned} \tag{5.4.31}$$

Ultimately, the coupled heat conduction relation reads as

$$
\begin{aligned}
\rho r + k\theta_{,ii} - T_0 \frac{\theta}{\partial t} &\left[\int_{-\infty}^{t} \phi(t-\tau,0) \frac{\partial \varepsilon_{kk}(\tau)}{\partial \tau} d\tau \right. \\
&\left. + \int_{-\infty}^{t} m(t-\tau,0) \frac{\partial \theta(\tau)}{\partial \tau} d\tau \right] = 0.
\end{aligned}
\tag{5.4.32}
$$

Obtaining the heat conduction relation as done above completes the development of constitutive formulation for the linear thermoviscoelasticity theory.

5.5 Gradient Thermoviscoelasticity: Constitutive Modeling

Notation

In this study a compact from of notation is used where necessary. Accordingly vectors and tensors (of any order) are shown through bold face letters and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system. The scalar product between vectors or tensors are represented with as many dots as the contracted index pairs. For example $\mathbf{u} = u_i$, $\boldsymbol{\sigma} = \sigma_{ij}$, $\boldsymbol{\tau} = \tau_{ijk}$ and $\mathbf{E} = E_{ijkh}$ respectively denote first, second, third and fourth rank tensors. Consequently the following scalar products between vectors and tensors can be expected as $\mathbf{u} \cdot \mathbf{v} = u_i v_i$, $\boldsymbol{\sigma}{:}\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, $\mathbf{A}{:}\boldsymbol{\varepsilon} = A_{ijhk}\varepsilon_{kh}$ and $\mathbf{A}{\therefore}\boldsymbol{\tau} = A_{ijhk}\tau_{jkh}$. The summation rule applies for repeated indexes. Finally convolution product of two functions is represented through the $*$ sign. Other symbols will be defined in the text upon their first appearance.

5.5.1 Thermodynamic Premises

Let us consider a viscoelastic body $\Omega \subset \mathbb{R}^3$, with gradient elasticity, and boundary $\partial\Omega$. The material might be visco-elastically and thermally inhomogeneous.

We also assume that for $t \geq 0$ the body is in the temperature T_0 and be in an undeformed state. Due to the heat sources and heating of the surface of the body, the medium undergoes deformation and change in temperature for $t \geq 0$.

We denote by $\mathbf{u}$ the displacement vector of a point $\mathbf{x}$ of the body; $\theta = T - T_0$ the temperature change; T the absolute temperature; $\boldsymbol{\epsilon}$ the strain tensor and $\boldsymbol{\sigma}$ represents the stress tensor.

Herein the aim is to adjust small displacements linear regimes (in the case of viscoelastic materials turns to be linear viscoelasticity), thereby small values for θ and $\boldsymbol{\epsilon}$ are considered.

In the case of stress gradient materials, the mechanical and thermal states of the medium is, at a given instant, completely described by the distribution of stress $\boldsymbol{\sigma}$ and temperature change θ. Where the process evolution remains isothermal, thermodynamically is considered as a reversible process. However processes which involve the change in the temperature, two interrelated phenomena namely reversible elastic processes and irreversible thermal processes are observed.

Herein we denote u as the internal energy density and s as the entropy density with r as the specific heat supply. Following Polizzotto [71], we define the stress gradient material as the material in which the usual strain power $W = \boldsymbol{\epsilon} : \dot{\boldsymbol{\sigma}}$ it is substitute by the more

general one as:

$$W = \boldsymbol{e} : \dot{\boldsymbol{\sigma}} + \boldsymbol{\eta} \therefore \nabla\dot{\boldsymbol{\sigma}} \tag{5.5.1}$$

where the strain tensors $\boldsymbol{e}$ and $\boldsymbol{\eta}$ are strains power-conjugate to $\dot{\boldsymbol{\sigma}}$ and $\nabla\dot{\boldsymbol{\sigma}}$, respectively. In the following we will consider material for which the expression (5.5.1) of the strain power holds. Then according to the first law of thermodynamics the local internal energy balance equation reads as:

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{e}} + \nabla\boldsymbol{\sigma} \therefore \dot{\boldsymbol{\eta}} + r - \nabla \cdot \boldsymbol{q} \tag{5.5.2}$$

The Clausius-Duhem inequality (second law of thermodynamics) reads as:

$$\dot{s}^{pr} = \dot{s} + \nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) - \frac{r}{T} \geq 0 \tag{5.5.3}$$

or

$$T\dot{s}^{pr} = T\dot{s} + \nabla \cdot \boldsymbol{q} - \frac{\mathbf{q}}{T} \cdot \nabla T - r \geq 0 \tag{5.5.4}$$

where $\dot{s}^{pr}$ denotes the internal entropy production and $-\frac{\mathbf{q}}{T} \cdot \nabla T$ represents the thermal dissipation by conduction.

Gibbs function per unit volume is considered as a function of the stress, its first gradient and of the temperature field, $\mathcal{G} = \mathcal{G}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}, T, t)$

$$\mathcal{G} = u - \boldsymbol{\sigma} : \boldsymbol{e} - \nabla\boldsymbol{\sigma} : \boldsymbol{\eta} - Ts \tag{5.5.5}$$

Then the local internal energy balance can be written as:

$$\dot{\boldsymbol{\sigma}} : \boldsymbol{e} + \nabla\dot{\boldsymbol{\sigma}} : \boldsymbol{\eta} + \dot{T}s + T\dot{s} + \dot{\mathcal{G}} = r - \nabla \cdot \boldsymbol{q} \tag{5.5.6}$$

Substituting the last equation into the Clausius-Duhem inequality, we obtain

$$T\dot{s}^{pr} = -\dot{\boldsymbol{\sigma}} : \boldsymbol{e} - \nabla\dot{\boldsymbol{\sigma}} : \boldsymbol{\eta} - \dot{T}s - \dot{\mathcal{G}} - \boldsymbol{q} \cdot \frac{\nabla T}{T} \geq 0 \tag{5.5.7}$$

Within the framework of solid mechanics, the Helmholtz free energy $\psi(\boldsymbol{\varepsilon}, T)$ is frequently used in place of u, to which it is related through the Legendre Transform.

In the case of *strain gradient material*, the mechanical and thermal state of the medium is, at a given instant, completely described by the distribution of the strain $\boldsymbol{\epsilon}$ and the temperature change θ.

In this case, the local internal energy balance equation is

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r - \nabla \cdot \boldsymbol{q} + R \tag{5.5.8}$$

where the additional thermodynamic variable R is the so-called nonlocality residual.

Introducing Helmholtz free energy as a function of strain tensor, gradient of strain tensor, temperature and time,

$$\psi = \psi(\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}, T, t) \tag{5.5.9}$$

by means of the Legendre transformation we have

$$\psi = u - Ts, \tag{5.5.10}$$

and the local internal energy balance equation (5.5.8) can be written

$$\dot{u} = \dot{\psi} + \dot{T}s + T\dot{s} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r - \nabla \cdot \boldsymbol{q} + R \tag{5.5.11}$$

Substituting this equation into the Clausius-Duhem inequality (5.5.3), we obtain

$$T\dot{s}^{pr} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{T}s - \dot{\psi} - \boldsymbol{q} \cdot \frac{\nabla T}{T} + R \geq 0 \tag{5.5.12}$$

In the case of linear viscoelastic solids in isothermal conditions, as shown by Staverman and Schwarzl (1952a,b), the specific Gibbs function $\mathcal{G}$, and the specific Helmholtz free energy ψ take, respectively, the following form:

$$\mathcal{G} = -\frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{J}(2t - \tau - \eta) :: d\boldsymbol{\sigma}(\tau) \, d\boldsymbol{\sigma}(\eta) \tag{5.5.13}$$

$$\psi = \frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{E}(2t - \tau - \eta) :: d\varepsilon(\tau) \, d\varepsilon(\eta) \tag{5.5.14}$$

where $\mathbf{J}$ and $\mathbf{E}$ are the creep and the relaxation tensors generalization of the usual compliance and stiffness elastic tensors, respectively, where the integrals are used in Stieltjes way. If the derivatives $\dot{\boldsymbol{\sigma}}$ or $\dot{\boldsymbol{\epsilon}}$ exist (no jumps), the Stieltjes integral can be transformed to the usual Riemann integral by setting $d\boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}} \, dt$ or $d\boldsymbol{\epsilon} = \dot{\boldsymbol{\epsilon}} \, dt$. The minus label at the lower limit of the integration must be just before t=0, which is necessary when the loading starts at time $t = 0$ with a jump.

Under hypotheses of small strains and small variations of the temperature with respect to the given reference configuration, we assume the existence of a specific Gibbs energy density depending upon stress, gradient of stress and temperature difference histories in a quadratic manner, generalizing the simple viscoelastic case (5.5.13):

$$\begin{aligned}
\mathcal{G}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}, \theta, t) = &-\frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{J}^{(0)}(2t - \tau - \eta) :: d\boldsymbol{\sigma}(\tau) \, d\boldsymbol{\sigma}(\eta) \\
&-\frac{1}{2}\ell^2 \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{J}^{(1)}(2t - \tau - \eta) :: \left[d\nabla\boldsymbol{\sigma}^{\mathrm{T}}(\tau) \cdot d\nabla\boldsymbol{\sigma}(\eta) \right] \\
&-\frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} c(2t - \tau - \eta) \, d\theta(\tau) \, d\theta(\eta) \\
&- \int_{0^-}^{t} \int_{0^-}^{t} \boldsymbol{\alpha}(2t - \tau - \eta) : d\boldsymbol{\sigma}(\tau) \, d\theta(\eta)
\end{aligned} \tag{5.5.15}$$

and the existence of a specific Helmholtz free energy depending upon strain, gradient strain and temperature difference histories in a quadratic manner, generalizing the simple viscoelastic case (5.5.14)

$$\begin{aligned}
\psi(\boldsymbol{\epsilon}, \nabla\boldsymbol{\epsilon}, \theta, t) = &\frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{E}^{(0)}(2t - \tau - \eta) :: d\boldsymbol{\epsilon}(\tau) \, d\boldsymbol{\epsilon}(\eta) \\
&+\frac{1}{2}l^2 \int_{0^-}^{t} \int_{0^-}^{t} \mathbf{E}^{(1)}(2t - \tau - \eta) :: \left[d\nabla\boldsymbol{\epsilon}^{T}(\tau) \cdot d\nabla\boldsymbol{\epsilon}(\eta) \right] \\
&+\frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} m(2t - \tau - \eta) \, d\theta(\tau) \, d\theta(\eta) \\
&+ \int_{0^-}^{t} \int_{0^-}^{t} \boldsymbol{\phi}(2t - \tau - \eta) : d\boldsymbol{\epsilon}(\tau) \, d\theta(\eta)
\end{aligned} \tag{5.5.16}$$

where ℓ is a material constant with meaning of internal length scale parameter. The integrating function (mechanical properties) are assumed to be continuous for $\tau \geq 0$.

5.5.2   Constitutive law for the stress gradient model

Now we calculate the time derivative of the Gibbs function as:

$$
\begin{aligned}
\dot{\mathcal{G}} = &- \dot{\boldsymbol{\sigma}}(t) : \int_{0^-}^{t} \mathbf{J}^{(0)}(t-\tau) : \mathrm{d}\boldsymbol{\sigma}(\tau) \\
&- \ell^2 \nabla\dot{\boldsymbol{\sigma}}(t) \therefore \int_{0^-}^{t} \mathbf{J}^{(1)}(t-\tau) : \mathrm{d}\nabla\boldsymbol{\sigma}(\tau) \\
&- \dot{\theta}(t) \int_{0^-}^{t} c(t-\tau)\,\mathrm{d}\theta(\tau) \\
&- \dot{\boldsymbol{\sigma}}(t) : \int_{0^-}^{t} \boldsymbol{\alpha}(t-\tau)\,\mathrm{d}\theta(\tau) \\
&- \dot{\theta}(t) \int_{0^-}^{t} \boldsymbol{\alpha}(t-\tau) : \mathrm{d}\boldsymbol{\sigma}(\tau) \\
&+ \dot{\Lambda} \\
= &- \dot{\boldsymbol{\sigma}} : \mathbf{E} * \boldsymbol{\sigma} \\
&- \ell^2 \nabla\dot{\boldsymbol{\sigma}} : \mathbf{E} * \nabla\sigma \\
&- \dot{\theta} c * \theta \\
&- \dot{\boldsymbol{\sigma}} : \alpha * \theta \\
&- \dot{\theta}\boldsymbol{\alpha} * \boldsymbol{\sigma} \\
&+ \dot{\Lambda}
\end{aligned}
$$

where $\dot{\Lambda}$ is the dissipation function. Substituting the above relation into the Clausius-Duhem inequality, we obtain:

$$
\begin{aligned}
&\dot{\boldsymbol{\sigma}} : \left[-\boldsymbol{e} + \mathbf{J} * \boldsymbol{\sigma} + \boldsymbol{\alpha} * \theta \right] \\
&+ \nabla\dot{\boldsymbol{\sigma}} \therefore \left[-\boldsymbol{\eta} + \ell^2 \mathbf{J} * \nabla\boldsymbol{\sigma} \right] \\
&+ \dot{\theta}\left[-s + c * \theta + \boldsymbol{\alpha} * \boldsymbol{\sigma} \right] \\
&+ \dot{\Lambda} - \mathbf{q} \cdot \frac{\nabla\theta}{T_0} \geq 0
\end{aligned}
$$

For this relations to be satisfied for all values of $\dot{\boldsymbol{\sigma}}$, $\nabla\dot{\boldsymbol{\sigma}}$ and $\dot{\theta}$, it is necessary that the coefficients in brackets vanish, that is:

$$
\mathbf{e} = \mathbf{J}^{(0)} * \boldsymbol{\sigma} + \boldsymbol{\alpha} * \theta \tag{5.5.17}
$$

$$
\boldsymbol{\eta} = \ell^2 \mathbf{J}^{(1)} * \nabla\boldsymbol{\sigma} \tag{5.5.18}
$$

$$
s = c * \theta + \boldsymbol{\alpha} * \boldsymbol{\sigma} \tag{5.5.19}
$$

These relations are the constitutive laws for the conjugate strains and entropy, respectively and represent the reversible part of the constitutive law. As shown by Polizzotto [71], the total strain $\boldsymbol{\varepsilon}$ reads

$$
\boldsymbol{\varepsilon} = \mathbf{e} - \nabla \cdot \boldsymbol{\eta} = \nabla^s \mathbf{u}, \tag{5.5.20}
$$

where $\nabla^s \mathbf{u}$ denotes the symmetric part of the gradient operator ∇.

Substituting (5.5.17) and (5.5.18) into (5.5.20) we obtain:

$$\boldsymbol{\varepsilon} = \mathbf{J}^{(0)} * \boldsymbol{\sigma} + \boldsymbol{\alpha} * \theta - \nabla \cdot \left(\ell^2 \mathbf{J}^{(1)} * \nabla \boldsymbol{\sigma} \right) \tag{5.5.21}$$

If the system is homogeneous and if $\mathbf{J}^{(0)} = \mathbf{J}^{(1)} = \mathbf{J}$, the above equation becomes

$$\boldsymbol{\varepsilon} = \mathbf{J} * \left(\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma} \right) + \boldsymbol{\alpha} * \theta \tag{5.5.22}$$

For isothermal conditions and neglecting the viscosity, the above constitutive law reduces to the well known one of stress gradient elasticity as:

$$\boldsymbol{\varepsilon} = \mathbf{C} : \left(\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma} \right) \tag{5.5.23}$$

It remains to consider the dissipative part of the entropy production:

$$\dot{\Lambda} - \mathbf{q} \cdot \frac{\nabla \theta}{T_0} \geq 0 \tag{5.5.24}$$

where the change of T with respect to T_0 would be valid for small values of θ. Also $\dot{\Lambda}$ reads as

$$\begin{aligned}
\dot{\Lambda} = &\left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left(\dot{\mathbf{J}}^{(0)} * \boldsymbol{\sigma} \right) (2t - \tau) \, \mathrm{d}\boldsymbol{\sigma}(\tau) \\
&+ l^2 \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left(\dot{\mathbf{J}}^{(1)} * \nabla \boldsymbol{\sigma} \right) (2t - \tau) \, \mathrm{d}\nabla \boldsymbol{\sigma}(\tau) \\
&+ \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left(\dot{c} * \theta \right) (2t - \tau) \, \mathrm{d}\theta(\tau) \\
&+ \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left(\dot{\boldsymbol{\alpha}} * \boldsymbol{\sigma} \right) (2t - \tau) \, \mathrm{d}\theta(\tau)
\end{aligned}$$

In the above equation we have used the following formula due to Brun (1969)

$$\left(\int_{0^-}^{t} - \int_{t}^{2t} \right) y(2t - \tau) \, \mathrm{d}x(\tau) = \int_{0^-}^{t} \int_{0^-}^{t} r(2t - \tau - \eta) \, \mathrm{d}x(\tau) \, \mathrm{d}x(\eta) \tag{5.5.25}$$

where

$$y(t) = \int_{0^-}^{t} r(t - \tau) x(\tau) \, \mathrm{d}\tau = r * x(t) \tag{5.5.26}$$

Using another formula due to Brun (1969)

$$\left(\int_{0^-}^{t} - \int_{t}^{2t} \right) x(2t - \tau) \, \mathrm{d}x(\tau) = x^2(t) \quad \text{if } x(0^-) = 0 \tag{5.5.27}$$

and using the Fourier's law, we can write

$$\begin{aligned}
-\mathbf{q} \cdot \frac{\nabla \theta}{T_0} = \frac{k}{T_0} \nabla \theta \cdot \nabla \theta &= \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \frac{k}{T_0} \nabla \theta (2t - \tau) \cdot \mathrm{d}\nabla \theta(\tau) \\
&= -\frac{1}{T_0} \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \mathbf{q}(2t - \tau) \cdot \mathrm{d}\nabla \theta(\tau)
\end{aligned} \tag{5.5.28}$$

In the isotropic case, $\boldsymbol{\alpha}$ becomes a scalar α_v and then remains only the volumetric part of $\dot{\boldsymbol{\alpha}} * \boldsymbol{\sigma}(2t - \tau)\mathrm{d}\theta(\tau)$. The dissipative part of the entropy production becomes:

$$
\begin{aligned}
\left(\int_{0-}^{t} - \int_{t}^{2t}\right) \Big\{ & \dot{j}_v^{(0)} * \sigma_v(2t - \tau)\mathrm{d}\sigma_v(\tau) \\
& + \dot{j}_d^{(0)} * \boldsymbol{\tau}(2t - \tau){:}\,\mathrm{d}\boldsymbol{\tau}(\tau) \\
& + \ell^2 \dot{j}_v^{(1)} * \nabla\sigma_v(2t - \tau) \cdot \mathrm{d}\nabla\sigma_v(\tau) \\
& + \ell^2 \dot{j}_d^{(1)} * \nabla\boldsymbol{\tau}(2t - \tau){\therefore}\,\mathrm{d}\nabla\boldsymbol{\tau}(\tau) \\
& + \dot{c} * \theta(2t - \tau)\mathrm{d}\theta(\tau) \\
& + \dot{\alpha} * \sigma_v(2t - \tau)\mathrm{d}\theta(\tau) \\
& + \frac{1}{T_0} k\nabla\theta(2t - \tau) \cdot \mathrm{d}\nabla\theta(\tau) \Big\}
\end{aligned}
\tag{5.5.29}
$$

where $\boldsymbol{\tau}$ is the deviatoric part of $\boldsymbol{\sigma}$.

We now have to write constitutive equations relating the thermodynamic fluxes to the thermodynamic forces. According to Curie's theorem, in an isotropic medium, fluxes and forces of different tensorial rank cannot be coupled (see for instance, deGroot and Mazur, 1962 [153]). Hence, scalar, vectorial and tensorial terms in the above equation represent dissipation processes of different physical nature and, accordingly, should be non-negative independently of the others, that is

$$
\dot{s}_{scalar}^{pr} = \left(\int_{0-}^{t} - \int_{t}^{2t}\right) \Big\{ \dot{j}_v^{(0)} * \sigma_v(2t - \tau)\mathrm{d}\sigma_v(\tau) + \dot{c} * \theta(2t - \tau)\mathrm{d}\theta(\tau) \\
+ \dot{\alpha} * \sigma_v(2t - \tau)\mathrm{d}\theta(\tau) \Big\}
\tag{5.5.30}
$$

$$
\dot{s}_{vector}^{pr} = \left(\int_{0-}^{t} - \int_{t}^{2t}\right) \Big\{ \ell^2 \dot{j}_v^{(1)} * \nabla\sigma_v(2t - \tau) \cdot \mathrm{d}\nabla\sigma_v(\tau) - \frac{1}{T_0}\mathbf{q}(2t - \tau) \cdot \mathrm{d}\nabla\theta(\tau) \Big\}
\tag{5.5.31}
$$

$$
\dot{s}_{s.o.tensor}^{pr} = \left(\int_{0-}^{t} - \int_{t}^{2t}\right) \Big\{ \dot{j}_d^{(0)} * \boldsymbol{\tau}(2t - \tau) : \mathrm{d}\boldsymbol{\tau}(\tau) \Big\}
\tag{5.5.32}
$$

$$
\dot{s}_{t.o.tensor}^{pr} = \left(\int_{0-}^{t} - \int_{t}^{2t}\right) \Big\{ l^2 \dot{j}_d^{(1)} * \nabla\boldsymbol{\tau}(2t - \tau){\therefore}\,\mathrm{d}\nabla\boldsymbol{\tau}(\tau) \Big\}
\tag{5.5.33}
$$

According to the Onsager's theory, for sufficient small deviations from equilibrium, a constitutive equation gives the thermodynamic flux as a linear function of the thermodynamic force for each dissipation [20, 153]

$$
J_i = \sum_k L_{ik} X_k
\tag{5.5.34}
$$

with symmetric coefficient matrix $L_{ik} = L_{ki}$.

In Onsager theory the response of a system to an applied force is simultaneous with the application of the force. There are situations where there is no simultaneity, for example in the case of the deformation of viscoelastic solids. Zwanzig (1961) considered situations in which the rate of entropy production $\dot{s}^{pr}$ has the following form:

$$
\dot{s}^{pr} = \int_0^t K_{ij}(\tau)F_j(t - \tau)\,\mathrm{d}\tau
\tag{5.5.35}
$$

and proved the following reciprocity relationship:

$$
K_{ij}(\tau) = K_{ji}(\tau)
\tag{5.5.36}
$$

related to memory functions K_{ij}.

In our case, we can choose the following fluxes and forces:

$$J_1 = \dot{J}_v^{(0)} * \sigma_v + \frac{1}{2}\dot{\alpha} * \theta \qquad\qquad X_1 = \sigma_v$$

$$J_2 = \dot{c} * \theta + \frac{1}{2}\dot{\alpha} * \sigma_v \qquad\qquad X_2 = \theta$$

$$J_3 = l^2 \dot{J}_v^{(1)} * \nabla\sigma_v \qquad\qquad X_3 = \nabla\sigma_v$$

$$J_4 = -\mathbf{q} = \frac{k}{T_0}\nabla\theta \qquad\qquad X_4 = \nabla\theta$$

$$J_5 = \dot{J}_d^{(0)} * \boldsymbol{\tau} \qquad\qquad X_5 = \boldsymbol{\tau}$$

$$J_6 = l^2 \dot{J}_d^{(1)} * \nabla\boldsymbol{\tau} \qquad\qquad X_6 = \nabla\boldsymbol{\tau} \qquad (5.5.37)$$

Thus, the coupled constitutive laws assume the following form:

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{66} \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} \qquad (5.5.38)$$

5.5.3 Constitutive law for the strain gradient model

Integrating of the equation (5.5.12) over Ω, expanding the time derivative of ψ, gives

$$\int_\Omega T\dot{s}^{pr}\,\mathrm{d}\Omega = \int_\Omega \left[\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\nabla\boldsymbol{\varepsilon}} \therefore \nabla\dot{\boldsymbol{\varepsilon}}\right]\mathrm{d}\Omega - \int_\Omega \left[\left(\frac{\partial\psi}{\partial T} + s\right)\dot{T} + \dot{\psi} + \mathbf{q}\cdot\frac{\nabla T}{T}\right]\mathrm{d}\Omega \geq 0$$

$$(5.5.39)$$

We denote

$$\boldsymbol{\sigma}^{(0)} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} \quad;\qquad \boldsymbol{\sigma}^{(1)} = \frac{\partial\psi}{\partial\nabla\boldsymbol{\varepsilon}} \qquad (5.5.40)$$

where $\boldsymbol{\sigma}^{(0)} = \left\{\sigma_{ij}^{(0)}\right\}$ is a Cauchy-like stress tensor, whereas $\boldsymbol{\sigma}^{(1)} = \left\{\sigma_{pij}^{(1)}\right\}$ is a third order stress tensor. $\boldsymbol{\sigma}^{(0)}$ and $\boldsymbol{\sigma}^{(1)}$ are thermodynamic forces associated, respectively, with $\boldsymbol{\epsilon}$ and $\nabla\boldsymbol{\epsilon}$, considered independent of one another.

Applying the divergence theorem, one can write:

$$\int_\Omega \boldsymbol{\sigma}^{(1)} \therefore \nabla\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega = -\int_\Omega \nabla\cdot\boldsymbol{\sigma}^{(1)} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\Omega + \int_{\partial\Omega} \mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\Gamma \qquad (5.5.41)$$

Substituting (5.5.41) into (5.5.39), and using the compatibility relation

$$\dot{\boldsymbol{\varepsilon}} = \nabla^s\mathbf{u}, \qquad (5.5.42)$$

the latter equation becomes

$$\int_\Omega T\dot{s}^{pr}\,\mathrm{d}\Omega = \int_\Omega \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)} + \nabla\cdot\boldsymbol{\sigma}^{(1)}\right) : \nabla\dot{\mathbf{u}}\,\mathrm{d}\Omega$$

$$- \int_\Omega \left[\left(\frac{\partial\psi}{\partial T} + s\right)\dot{T} + \dot{\psi} + \mathbf{q}\cdot\frac{\nabla T}{T}\right]\mathrm{d}\Omega - \int_{\partial\Omega} \mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} : \nabla\dot{\mathbf{u}}\,\mathrm{d}\Gamma \geq 0$$

$$(5.5.43)$$

Applying the surface integral transformation formula (see Appendix A)one can write

$$\int_{\partial\Omega} \mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} : \nabla\dot{\mathbf{u}}\,\mathrm{d}\Gamma = \int_{\partial\Omega} \mathbf{G}\cdot\left(\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)}\right)\cdot\dot{\mathbf{u}}\,\mathrm{d}\Gamma + \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)}\cdot\partial_n\dot{\mathbf{u}}\,\mathrm{d}\Gamma \qquad (5.5.44)$$

where $\mathbf{G}$ denotes the surface gradient. The derivation of $\mathbf{G}$ has been explained in the appendix at the end of the current chapter. Also $\partial_n\dot{\mathbf{u}} \coloneqq \frac{\partial\dot{\mathbf{u}}}{\partial\mathbf{n}}$ ($\mathbf{n}$ in the direction of the vector n on $\partial\Omega$).

Using the above equation and applying the divergence theorem, inequality (5.5.43) becomes:

$$\begin{aligned}
\int_\Omega T\dot{s}^{pr}\,\mathrm{d}\Omega = &-\int_\Omega \nabla\cdot\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)} + \nabla\cdot\boldsymbol{\sigma}^{(1)}\right)\cdot\dot{\mathbf{u}}\,\mathrm{d}\Omega \\
&- \int_\Omega \left[\left(\frac{\partial\psi}{\partial T} + s\right)\dot{T} + \frac{\partial\psi}{\partial t} + \mathbf{q}\cdot\frac{\nabla T}{T}\right]\mathrm{d}\Omega \\
&+ \int_{\partial\Omega}\left[\mathbf{n}\cdot\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)} + \nabla\cdot\boldsymbol{\sigma}^{(1)}\right) - \mathbf{G}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)}\right]\cdot\dot{\mathbf{u}}\,\mathrm{d}\Gamma \\
&- \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)}\cdot\partial_n\dot{\mathbf{u}}\,\mathrm{d}\Gamma \geq 0
\end{aligned} \qquad (5.5.45)$$

Consider that inequality (5.5.45) must hold true for any displacement-driven deformation mechanism and for any $\dot{T}$, it follows, as necessary and sufficient conditions, that

$$\begin{aligned}
\nabla\cdot\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)} + \nabla\cdot\boldsymbol{\sigma}^{(1)}\right) &= \mathbf{0} \qquad \text{in } \Omega \\
\mathbf{n}\cdot\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)} + \nabla\cdot\boldsymbol{\sigma}^{(1)}\right) &= \mathbf{G}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} \qquad \text{on } \partial\Omega \\
\mathbf{n}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} &= \mathbf{0} \qquad \text{on } \partial\Omega
\end{aligned} \qquad (5.5.46)$$

that is

$$\begin{aligned}
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^{(0)} - \nabla\cdot\boldsymbol{\sigma}^{(1)} \qquad \text{in } \Omega \\
\mathbf{n}\cdot\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)} &= \mathbf{0} \qquad \text{on } \partial\Omega
\end{aligned} \qquad (5.5.47)$$

Since heat conduction is, by assumption, a local-type phenomenon, the evaluation of R can be achieved by considering isothermal deformation processes. Then

$$\mathbf{R} = \dot{h}|_{T=\text{const}} - \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} \qquad \text{in } \Omega \qquad (5.5.48)$$

Thus, expanding the time derivative of ψ at constant T, gives:

$$\mathbf{R} = \nabla\cdot\boldsymbol{\sigma}^{(1)}:\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{(1)}\therefore\nabla\dot{\boldsymbol{\varepsilon}} + \frac{\partial\psi}{\partial t} = \nabla\cdot\left[\boldsymbol{\sigma}^{(1)}:\dot{\boldsymbol{\varepsilon}}\right] + \mathscr{D} \qquad \text{in } \Omega \qquad (5.5.49)$$

where $\mathscr{D} = \frac{\partial\psi}{\partial t}$.

Therefore, taking a quadratic form for ψ

$$\begin{aligned}
\psi(\boldsymbol{\varepsilon},\nabla\boldsymbol{\varepsilon},\theta,t) = \frac{1}{2}\int_{0^-}^t\int_{0^-}^t \Big\{ &\mathbf{J}^{(0)}(2t-\tau-\eta)::\mathrm{d}\boldsymbol{\varepsilon}(\tau)\,\mathrm{d}\boldsymbol{\varepsilon}(\eta) \\
&+ \ell^2\mathbf{J}^{(1)}(2t-\tau-\eta)::\left(\mathrm{d}\nabla\boldsymbol{\varepsilon}^T(\tau)\cdot\mathrm{d}\nabla\boldsymbol{\varepsilon}(\eta)\right) \\
&+ m(2t-\tau-\eta)\,\mathrm{d}\theta(\tau)\,\mathrm{d}\theta(\eta) \\
&+ 2\boldsymbol{\phi}(2t-\tau-\eta):\mathrm{d}\boldsymbol{\varepsilon}(\tau)\,\mathrm{d}\theta(\eta)\Big\}
\end{aligned} \qquad (5.5.50)$$

the last contribution in cross term.

$$
\begin{aligned}
\mathscr{D} = \dot{\psi}(\varepsilon, \nabla\varepsilon, \theta, t) = \int_{0^-}^{t} \int_{0^-}^{t} \Big\{ & \dot{\mathbf{J}}^{(0)}(2t - \tau - \eta) :: \mathrm{d}\varepsilon(\tau)\,\mathrm{d}\varepsilon(\eta) \\
& + \dot{\mathbf{J}}^{(1)}(2t - \tau - \eta) :: (\mathrm{d}\nabla\varepsilon(\tau) \cdot \mathrm{d}\nabla\varepsilon(\eta)) \\
& + m(2t - \tau - \eta)\,\mathrm{d}\theta(\tau)\,\mathrm{d}\theta(\eta) \\
& + 2\boldsymbol{\phi}(2t - \tau - \eta) : \mathrm{d}\varepsilon(\tau)\,\mathrm{d}\theta(\eta) \Big\}
\end{aligned}
\tag{5.5.51}
$$

Using Brun's formulae, the irreversible part of the entropy production reads:

$$
\begin{aligned}
\left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \Big\{ & \dot{\mathbf{J}}^{(0)} * \varepsilon(2t - \tau) : \mathrm{d}\varepsilon(\tau) \\
& + \dot{\mathbf{J}}^{(1)} * \nabla\varepsilon(2t - \tau) \cdot \mathrm{d}\nabla\varepsilon(\tau) \\
& + m * \theta(2t - \tau)\,\mathrm{d}\theta(\tau) \\
& + \boldsymbol{\phi} * \varepsilon(2t - \tau)\,\mathrm{d}\theta(\tau) + \boldsymbol{\phi} * \theta(2t - \tau) : \mathrm{d}\varepsilon(\tau) \\
& - \frac{1}{T_0}\mathbf{q}(2t - \tau) \cdot \mathrm{d}\nabla\theta(\tau) \Big\}
\end{aligned}
\tag{5.5.52}
$$

that is

$$
\begin{aligned}
\left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \Big\{ & \big[\dot{\mathbf{J}}^{(0)} * \varepsilon(2t - \tau) + \boldsymbol{\phi} * \theta(2t - \tau) \big] : \mathrm{d}\varepsilon(\tau) \\
& + \big[\dot{\mathbf{J}}^{(1)} * \nabla\varepsilon(2t - \tau) \big]\,\mathrm{d}\nabla\varepsilon(\tau) \\
& + \big[m * \theta(2t - \tau) + \boldsymbol{\phi} * \varepsilon(2t - \tau) \big]\,\mathrm{d}\theta(\tau) \\
& - \frac{1}{T_0}\mathbf{q}(2t - \tau) \cdot \mathrm{d}\nabla\theta(\tau) \Big\}
\end{aligned}
\tag{5.5.53}
$$

5.5.4   A particular class of strain gradient thermoviscoelasticity

We now consider only isotropic solids with the following simplified free energy density:

$$
\begin{aligned}
\psi = \psi(\varepsilon, \nabla\varepsilon, \theta, t) = \frac{1}{2} \int_{0^-}^{t} \int_{0^-}^{t} \Big\{ & \mathbf{J}^{(0)}(2t - \tau - \eta) : \mathrm{d}\varepsilon(\tau) : \mathrm{d}\varepsilon(\eta) \\
& + \phi(2t - \tau - \eta)d\varepsilon_v(\tau)d\theta(\eta) \\
& + \mathbf{J}^{(1)}(2t - \tau - \eta) :: \big[\mathrm{d}\nabla\varepsilon(\tau) \cdot \mathrm{d}\nabla\varepsilon(\eta) \big] \\
& + m(2t - \tau - \eta)\,\mathrm{d}\theta(\tau)\,\mathrm{d}\theta(\eta) \Big\}
\end{aligned}
\tag{5.5.54}
$$

Splitting ε in the volumetric and deviatoric parts, ε_v and ε_d respectively, the dissipation part of the entropy production becomes:

$$
\begin{aligned}
\dot{\psi} - \frac{\mathbf{q}}{T_0} \cdot \nabla\theta = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \Big\{ & j_v^{(0)} * \varepsilon_v(2t - \tau) + j_d^{(0)} * \varepsilon_d(2t - \tau) : d\varepsilon_d(\tau) \\
& + \phi * \varepsilon_v(2t - \tau)d\theta(\tau) \\
& + j_v^{(1)} * \nabla\varepsilon_v(2t - \tau) \cdot d\nabla\varepsilon_v(\tau) \\
& + m * \theta(2t - \tau)d\theta(\tau) \\
& - \frac{1}{T_0}\mathbf{q}(2t - \tau) \cdot d\nabla\theta(\tau) \Big\} \geq 0
\end{aligned}
\tag{5.5.55}
$$

According to Gurtin's theorem, in an isotropic medium, fluxes and forces of different tensorial ranks cannot be coupled. Hence, scalar, vectorial and tensorial terms in the above equation represent dissipation processes of different physical nature and, accordingly, would be non-negative independently of the others, that is

$$\dot{s}_{scalar}^{pr} = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left\{ \dot{J}_v^{(0)} * \varepsilon_v(2t - \tau) d\varepsilon_v(\tau) \right.$$
$$+ m * \theta(2t - \tau) d\theta(\tau)$$
$$\left. + \phi * \varepsilon_v(2t - \tau) d\theta(\tau) \right\} \geq 0$$

$$(5.5.56)$$

and

$$\dot{s}_{vector}^{pr} = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \left\{ \dot{J}_v^{(1)} * \nabla\varepsilon_v(2t - \tau) \cdot d\nabla\varepsilon_v(\tau) - \frac{1}{T_0}\mathbf{q}(2t - \tau) \cdot d\nabla\theta(\tau) \right\} \geq 0, \quad (5.5.57)$$

$$\dot{s}_{tensor}^{pr} = \left(\int_{0^-}^{t} - \int_{t}^{2t} \right) \dot{J}_d^{(0)} * \boldsymbol{\varepsilon}_d(2t - \tau) : d\boldsymbol{\varepsilon}_d(\tau) \geq 0 \quad (5.5.58)$$

Obtained equations can be arranged in matrix-vector format as:

$$J_1 = \dot{J}_v^{(0)} * \varepsilon_v + \frac{1}{2}\phi * \theta \qquad\qquad X_1 = \varepsilon_v$$

$$J_2 = m * \theta + \frac{1}{2}\phi\varepsilon_v \qquad\qquad X_2 = \theta$$

$$J_3 = \dot{J}_v^{(1)} * \nabla\varepsilon_v \qquad\qquad X_3 = \nabla\varepsilon_v$$

$$J_4 = -\frac{1}{T_0}\mathbf{q} = \frac{k}{T_0}\nabla\theta \qquad\qquad X_4 = \nabla\theta$$

$$J_5 = \dot{J}_d^{(0)} * \boldsymbol{\varepsilon}_d \qquad\qquad X_5 = \boldsymbol{\varepsilon}_d \qquad\qquad (5.5.59)$$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 & 0 \\ 0 & 0 & L_{33} & 0 & 0 \\ 0 & 0 & 0 & L_{44} & 0 \\ 0 & 0 & 0 & 0 & L_{55} \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} \qquad (5.5.60)$$

or

$$J_i = \sum_k L_{ik}X_K \qquad\qquad (5.5.61)$$

5.6   Case Studies

As a proof of concept, and to show the applicability of the developed theory, a few subcases are reported herein. Mathematica software, version 13.3 [154], is used to numerically calculate inverse Laplace transforms and plot the resultant functions.

5.6.1 Gradient Elasticity

A one-dimensional bar model under extension with length L is considered. At one end $(x = 0)$ it is clamped and at the other end $(x = L)$ the displacement $u = \bar{u}$ is prescribed. The bar is made of a gradient elastic material and obeys the constitutive relation (Aifantis Model [54]) as

$$\sigma = E(\varepsilon - \ell^2 \varepsilon'') \qquad \forall x \in (0, L), \tag{5.6.1}$$

and boundary conditions as

$$\varepsilon' = 0 \ \text{ at } x = 0 \text{ and } x = L, \tag{5.6.2}$$

where E denotes the Young modulus, and prime and double prime signs respectively stand for first and second degree spatial derivatives (here in 1d problem with respect to x). The letter ℓ represents the length scale parameter.

Two different conditions, namely a homogeneous and a nonhomogeneous bar are considered.

Homogeneous bar

To derive the relationship for displacement response of a one-dimensional gradient elastic bar with constant cross section (A) under an axial load (F) one can start with the constitutive relation and equilibrium equation. In this case, we have $\sigma = \frac{F}{A}$, and the constitutive relation $\sigma = E(\varepsilon - \ell^2 \varepsilon'')$, where strain and its first and second derivatives are defined respectively as

$$\varepsilon = \frac{du}{dx}$$

$$\varepsilon' = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d^2 u}{dx^2}$$

$$\varepsilon'' = \frac{d}{dx}\left(\frac{d^2 u}{dx^2}\right) = \frac{d^3 u}{dx^3}.$$

Thereby one can deduce

$$\sigma = \frac{F}{A} = E\left(\frac{du}{dx} - \ell^2 \frac{d^3 u}{dx^3}\right), \tag{5.6.3}$$

and

$$\frac{d}{dx}\left(\frac{F}{A}\right) = \frac{d}{dx}\left(E\frac{du}{dx} - E\ell^2 \frac{d^3 u}{dx^3}\right) = 0, \tag{5.6.4}$$

which leads to

$$E\left(\frac{d^2 u}{dx^2} - \ell^2 \frac{d^4 u}{dx^4}\right) = 0, \tag{5.6.5}$$

or

$$\frac{d^2 u}{dx^2} - \ell^2 \frac{d^4 u}{dx^4} = 0. \tag{5.6.6}$$

So, with a change of notation, under the aforementioned conditions, displacement response of a homogeneous bar in 1D reads as

$$(u - \ell^2 u'')'' = 0 \qquad in\ (0, L) \tag{5.6.7}$$

which involves the following boundary conditions:

$$u(0) = 0, \quad u(L) = \bar{u} \tag{5.6.8}$$

and

$$u''(0) = u''(L) = 0. \tag{5.6.9}$$

The general solution for displacement reads as [86]:

$$u(x) = A_1 x + A_2 + B_1 \sinh \frac{x}{\ell} + B_2 \cosh \frac{x}{\ell} \tag{5.6.10}$$

where A_1, A_2, B_1 and B_2 would be constants which are determined through (5.6.8) and (5.6.9). Using (5.6.9) one can write

$$B_2 = 0, \quad B_1 \sinh \frac{L}{\ell} = 0 \rightarrow B_1 = 0. \tag{5.6.11}$$

Consequently (5.6.8) yields

$$u(x) = \bar{u}x/L, \quad \sigma = E\bar{u}/L. \tag{5.6.12}$$

It can be observed that in this case, the solution for the gradient problem would collapse to the classical elasticity formulation.

Nonhomogeneous bar

Consider ξ to be a coordinate axis in alignment with the x axis such that $x = \xi + L/2$ meaning $-L/2 \leq \xi \leq L/2$. For $\xi \leq 0$ let $E^- = E$ and for $\xi \geq 0$ let $E^+ = \mu E$. Displacement responses for two halves of the bar read as

$$u^+(\xi) = A_1^+ \xi + A_2^+ + B_1^+ \sinh \frac{\xi}{\ell} + B_2^+ \cosh \frac{\xi}{\ell} \quad (\xi \geq 0), \tag{5.6.13a}$$

$$u^-(\xi) = A_1^- \xi + A_2^- + B_1^- \sinh \frac{\xi}{\ell} + B_2^- \cosh \frac{\xi}{\ell} \quad (\xi \leq 0). \tag{5.6.13b}$$

where A_1, A_2, B_1 and B_2 would be constants that are be obtained using boundary conditions. Boundary conditions would be

$$[\![u(0)]\!] = [\![u'(0)]\!] = [\![u''(0)]\!] = 0, \tag{5.6.14}$$

$$[\![\sigma(0)]\!] = 0, \tag{5.6.15}$$

$$u(-L/2) = 0, \quad u(L/2) = \bar{u} \tag{5.6.16}$$

where $[\![(\cdot)(x)]\!]$ represents the jump of $(\cdot)$ at x. Through (5.6.14) one can obtain

$$A_2^+ = A_2^- := A_2, \quad B_2^+ = B_2^- := B_2, \tag{5.6.17}$$

$$A_1^+ - A_1^- + (B_1^+ - B_1^-)/\ell = 0. \tag{5.6.18}$$

By means of (5.6.15) we have

$$A_1^- = \mu A_1^+, \tag{5.6.19}$$

and through (5.6.9), and observing that $u''(-L/2) = u''(L/2) = 0$, we can write

$$B_1^+ = -B_1^- := B_1, \quad B_2 = -B_1 \tanh \frac{L}{2\ell}. \tag{5.6.20}$$

Having all the required constants determined, the following describing the displacement responses of each side of the bar are obtained as

$$u^+(\xi) = \frac{\bar{u}}{(1+\mu)}\left[2\frac{\xi}{L} + \mu - (1-\mu)\frac{\ell}{L}\left(\sinh\frac{\xi}{\ell} - \tanh\frac{L}{2\ell}\cosh\frac{\xi}{\ell}\right)\right] \qquad (\xi > 0), \quad (5.6.21a)$$

$$u^-(\xi) = \frac{\bar{u}}{(1+\mu)}\left[2\mu\frac{\xi}{L} + \mu + (1-\mu)\frac{\ell}{L}\left(\sinh\frac{\xi}{\ell} + \tanh\frac{L}{2\ell}\cosh\frac{\xi}{\ell}\right)\right] \qquad (\xi < 0). \quad (5.6.21b)$$

Consequently, the stress response reads as

$$\sigma = \frac{2E\bar{u}}{(1+\mu)L}, \qquad (5.6.22)$$

and strains on each side of the bar are expressed as

$$\varepsilon^+(x) = \frac{\bar{u}}{(1+\mu)L}\left[2 - (1-\mu)\left(\cosh\frac{2x-L}{2\ell} - \tanh\frac{L}{2\ell}\sinh\frac{2x-L}{2\ell}\right)\right] \qquad \left(x \geq \frac{L}{2}\right) \; (5.6.23a)$$

$$\varepsilon^-(x) = \frac{\bar{u}}{(1+\mu)L}\left[2\mu + (1-\mu)\left(\cosh\frac{2x-L}{2\ell} + \tanh\frac{L}{2\ell}\sinh\frac{2x-L}{2\ell}\right)\right] \qquad \left(x \leq \frac{L}{2}\right).$$
$$(5.6.23b)$$

Strain responses of homogeneous and nonhomogeneous bars for both gradient and local elasticity theories have been plotted in Figures 5.1 and 5.2 respectively. As mentioned earlier, the Young modulus of the left side of the bar is E and of the right side is μE. The influence of μ is observed clearly in both cases.

In the case of gradient elastic bar (Fig. 5.1) where the constitutive law obeys by the gradient elasticity theory through the introduction of the length scale parameter (ℓ) the strain response shows to be continuous on either sides of the interface. However, when the same bar is defined through the local elasticity theory, i.e. when $\ell \to 0$ (hyperbolic expressions are vanished), strains responses at the either sides of the bar are expressed as

$$\varepsilon^+ = \frac{2\bar{u}}{(1+\mu)L}, \qquad \left(x \geq \frac{L}{2}\right) \qquad (5.6.24)$$

and

$$\varepsilon^- = \frac{2\mu\bar{u}}{(1+\mu)L}, \qquad \left(x \leq \frac{L}{2}\right) \qquad (5.6.25)$$

and we observe a discontinuity (jump) in strain response of the bar (Fig. 5.2).

5.6.2 Gradient Viscoelasticity

As the second example, a one-dimensional gradient viscoelastic bar under extension is considered. Based on gradient elasticity theory [56, 155], and after adapting it to one-dimensional form, the total stress σ can be described as the combined effect of the Cauchy stress τ and double stress ω:

$$\sigma = \tau - \omega', \qquad (5.6.26)$$

where

$$\tau = Eu' = E\varepsilon, \qquad (5.6.27)$$

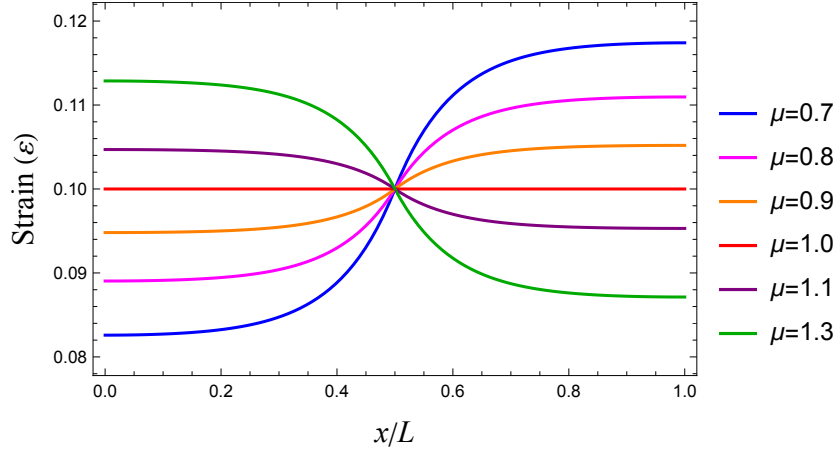$$\omega = \ell^2 Eu'' = \ell^2 E\varepsilon'. \qquad (5.6.28)$$

Figure 5.1: Gradient elasticity response ($\ell \neq 0$) of a nonhomogeneous bar ($\mu \neq 1$) with Young modulus E in the left half and Young modulus μE in the right half; $\mu = 1$ represents homogeneous bar.

Substituting (5.6.28) in (5.6.26) one can get

$$\sigma = Eu' - \ell^2 Eu''' = E\varepsilon - \ell^2 E\varepsilon''. \tag{5.6.29}$$

According to Christensen [149], for linear viscoelastic materials, we can write

$$\sigma(t) = \int_0^t E(t - \tau)[d\varepsilon(t)/d\tau]d\tau = E(t) * \dot{\varepsilon}(t), \tag{5.6.30}$$

where $E(t)$ would be the relaxation function.

Following the same reasoning of Christensen [149], Papargyri-Beskou et al. [155] concluded that the equations (5.6.26) and (5.6.28) can be extended to gradient viscoelastic form as

$$\sigma(t) = \tau(t) - \omega'(t), \tag{5.6.31}$$

$$\tau(t) = E(t) * \bar{u}(t)' = E(t) * \dot{\varepsilon}(t), \tag{5.6.32}$$

$$\omega(t) = \ell^2 E(t) * \dot{u}(t)'' = \ell^2 E(t) * \dot{\varepsilon}(t)'. \tag{5.6.33}$$

Equilibrium equation of a bar with uniform cross section A and density ρ under an axial, symmetric load $q = q(x,t)$ follows as

$$A\sigma' + q = \rho\ddot{u}. \tag{5.6.34}$$

The above equation, when written for a gradient elastic bar, leads to the following expression for displacement

$$AE[u(x,t)'' - \ell^2 u(x,t)''''] + q(x,t) = \rho\ddot{u}(x,t). \tag{5.6.35}$$

For a gradient viscoelastic bar (5.6.31) - (5.6.33), equilibrium equation (5.6.34) when written for displacement reads as

$$AE(t) * [\dot{u}(x,t)'' - \ell^2 \dot{u}(x,t)''''] + q(x,t) = \rho\ddot{u}(x,t). \tag{5.6.36}$$

Neglecting the inertial effects, $\rho\ddot{u}(x,t)$, the last two equations, respectively represent governing equations for static and quasi-static conditions.
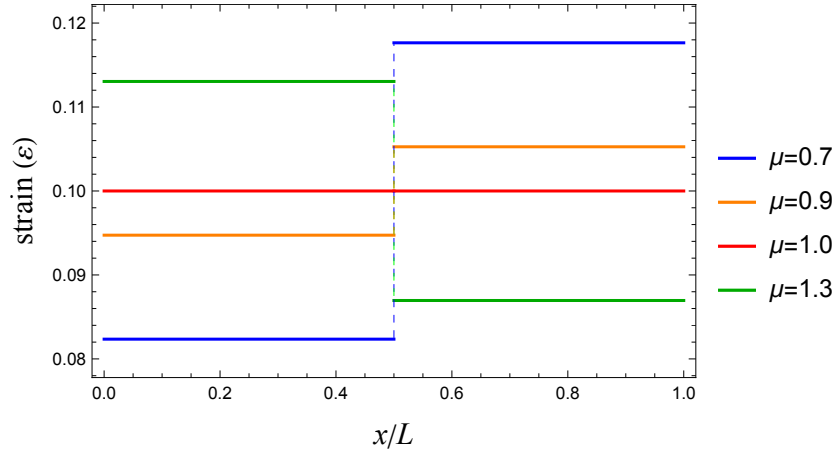
Figure 5.2: Local elasticity response ($\ell = 0$) of a nonhomogeneous bar ($\mu \neq 1$)with Young modulus E in the left half and Young modulus μE in the right half; $\mu = 1$ represents homogeneous bar.

Correspondence Principle

The Laplace transform with respect to time t of function $f(x,t)$ is denoted by $\bar{f}(x,s)$ and is defined as

$$\bar{f}(x,s) = \int_0^\infty f(x,t)e^{-st}dt, \tag{5.6.37}$$

where s is the Laplace transform parameter.

According to the correspondence principle, solution to the gradient viscoelastic problem can be expressed in the form of time domain Laplace transformed solution of the corresponding gradient elastic problem when the elastic modulus E is replaced by $s\bar{E}(s)$ [155]. Thereby, for a gradient viscoelastic bar we can write:

$$\bar{\sigma} = \bar{\tau} - \bar{\omega}',$$
$$\bar{\tau} = s\bar{E}\bar{u}',$$
$$\bar{\omega} = \ell^2 s\bar{E}\bar{u}'',$$
$$As\bar{E}(\bar{u}'' - \ell^2\bar{u}'''') + \bar{q} = \rho s^2\bar{u}. \tag{5.6.38}$$

1D Bar under Quasi-Static Loading

Assuming a straight bar with cross-section area A and length L, clamped at one end $x = 0$ and loaded with axial force P at the other end $x = L$ we write the classical boundary conditions as

$$u(0,t) = 0, \qquad AE(t) * [u'(L,t) - \ell^2 u'''(L,t)] = P, \tag{5.6.39}$$

and the non-classical boundary conditions as

$$A\ell^2 E(t) * u''(0,t) = 0, \qquad u'(L,t) = \varepsilon_0 \tag{5.6.40}$$

with ε_0 denoting a constant. Initial conditions are assumed to be zero.

i	$E_i(\text{lbs/in}^2)$	$t_i(\text{sec.})$
0	1300.0	
1	2592.2	1.5×10^{-5}
2	1398.8	1.5×10^{-4}
3	1284.4	1.5×10^{-3}
4	1019.2	1.5×10^{-2}
5	795.6	1.5×10^{-1}
6	400.4	1.5
7	309.4	1.5×10
8	52.0	1.5×10^2

Table 5.1: Values of relaxation function, $E(t)$, at different times for a polyurethane based nanocomposite [149].

We also consider that the bar is composed of a polymeric nanocomposite (polyurethane matrix with salt crystals and aluminum powder as nano-fillers) material. For this composite under a creep test, Christensen [149] suggests the values for relaxation function as presented in the table 5.1.

Accordingly, he assumes the following relation for the relaxation function:

$$E(t) = E_0 + \sum_{i=1}^{8} E_i e^{-t/t_i}, \tag{5.6.41}$$

where E_0, E_i and t_i are given in the above table. The elastic modulus is obtained from Eq. (5.6.41) for $t = 0$ in the form of:

$$E_e = \sum_{i=0}^{8} E_i. \tag{5.6.42}$$

The corresponding gradient elastic problem is already solved in [156], according to which the displacement u and strain ε are expressed in the following forms:

$$u(\xi) = C_1 e^{\xi/\bar{\ell}} + C_2 e^{-\xi/\bar{\ell}} + C_3 \xi + C_4,$$
$$\varepsilon(\xi) = (C_1/\bar{\ell}) e^{\xi/\bar{\ell}} - (C_2/\bar{\ell}) e^{-\xi/\bar{\ell}} + C_3, \tag{5.6.43}$$

where $\xi = x/L$, $\bar{\ell} = \ell/L$ and the constants $(C_1, C_2, C_3 \text{ and } C_4)$ defined through the boundary conditions as

$$C_1 = \frac{\bar{\ell}[\varepsilon_0 - (P/AE_e)]}{2\cosh(1/\bar{\ell})},$$
$$C_2 = -\frac{\bar{\ell}[\varepsilon_0 - (P/AE_e)]}{2\cosh(1/\bar{\ell})},$$
$$C_3 = P/AE_e,$$
$$C_4 = 0. \tag{5.6.44}$$

For $\bar{\ell} = 0$ the above gradient elasticity solution collapses to classical elastic solution as

$$u_e(\xi) = (P/AE_e)\xi,$$
$$\varepsilon_e(\xi) = P/AE_e. \tag{5.6.45}$$

Employing the correspondence principle, one can obtain the Laplace transformed solutions of gradient viscoelastic problem in time domain as

$$\bar{u}(\xi, s) = C_1(s)e^{\xi/\bar{\ell}} + C_2(s)e^{-\xi/\bar{\ell}} + C_3(s)\xi + C_4(s),$$
$$\bar{\varepsilon}(\xi, s) = (C_1(s)/\bar{\ell})e^{\xi/\bar{\ell}} - (C2(s)/\bar{\ell})e^{-\xi/\bar{\ell}} + C_3(s), \tag{5.6.46}$$

where ε_0 is replaced by ε_0/s, P by P/s and E_e by $s\bar{E}(s)$. The transformed form of relaxation function $\bar{E}(s)$ is expressed as

$$\bar{E}(s) = \frac{E_0}{s} + \sum_{i=1}^{8} \frac{E_i}{s + (1/t_i)} \tag{5.6.47}$$

The Laplace transformed solutions as defined above, are then transformed into the time domain solution through numerical Laplace inversion. Figs. 5.3 to 5.6 demonstrate normalized displacement $u(\xi, \tau)$ and strain $\varepsilon(\xi, \tau)$ responses versus the dimensionless time variable τ where

$$\tau = (t/L)\sqrt{E_e/\rho}.$$

As can be seen, in Fig. 5.3a to 5.3c changing $\bar{\ell}$ from 0.1 to 0.5 significantly alters the strain responses at each point of the bar. It is also observed that increasing the $\bar{\ell}$, lowers the displacement response. The influence of gradient coefficient on displacement response is further investigated through graphs 5.4a to 5.4c. In accordance with previous graphs, moving on the bar, starting from the clamped end toward free end where the load is applied, the magnitude of displacement increases. Again, comparing these three graphs, we notice that the effect of length scale parameter is conveyed throughout the bar.

In terms of strain responses, as shown in Figs. 5.5a to 5.5c, the influence of gradient effect on the strain is clearly demonstrated. Additionally, the analyses in Figs. 5.6 reveal that with increasing the value of the gradient coefficient, this effect would become more pronounced. For instance, in Fig. 5.6a where $\bar{\ell} = 0.1$ the strain responses at $\xi = 0.1 - 0.3$ almost overlapped. It means that at this range, and with this $\bar{\ell}$, these points are not kinematically differentiated. This is while, with increase in the value of the gradient coefficient from 0.3 to 0.5 (Figs. 5.6b and 5.6c), strain responses are well separated and differentiated. Moreover, a closer look shows that this differentiation is elevated as $\bar{\ell}$ increases from 0.3 to 0.5. These interesting results illustrate the influence of considering the length scale effect (and gradient based theories) in advanced materials and/or complex phenomena.

5.6.3 Gradient Thermoelasticity

A one-dimensional bar with length $2L$ clamped at both ends is considered. Temperature reduction along the x axis of this bar is expressed in the form of $\theta(x) = \bar{\theta}(1 - x^2/L^2)$ where $\bar{\theta} \geq 0$ represents the maximum temperature reduction. The resultant thermal strain reads as $\varepsilon^\theta = -\alpha\bar{\theta}(1 - x^2/L^2)$ with α as the thermal expansion coefficient. Assuming the stress
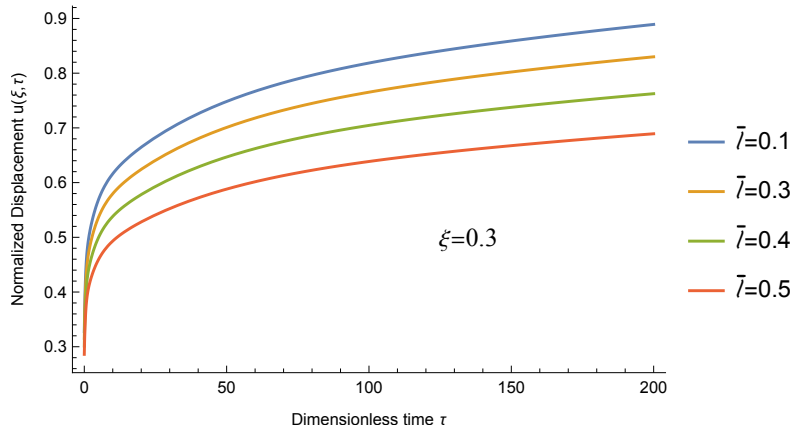
(a) Displacement versus time at $\xi = 0.3$ of the bar.
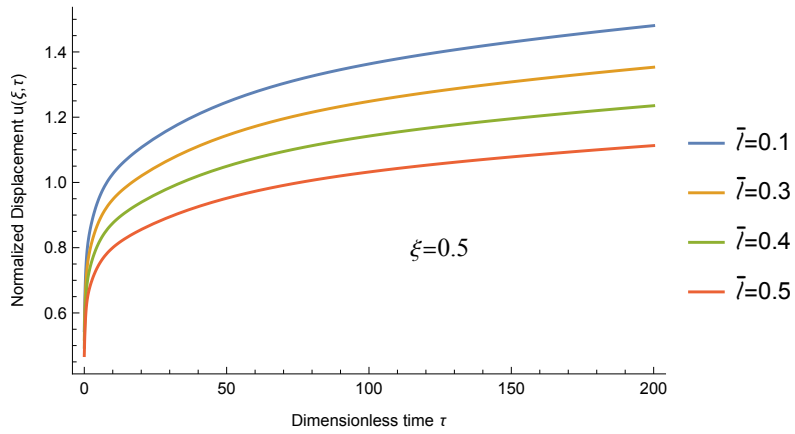
(b) Displacement versus time at the middle ($\xi = 0.5$) of the bar.
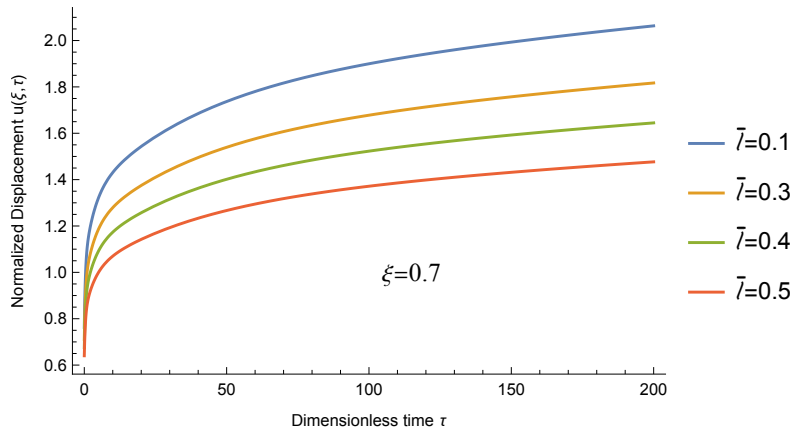
(c) Displacement versus time at $\xi = 0.7$ of the bar.

Figure 5.3: Displacement response of the quasi-statically loaded gradient viscoelastic bar versus dimensionless time variable for various values of $\bar{\ell}$ at $\xi = 0.3$, 0.5 and 0.7 of the bar.
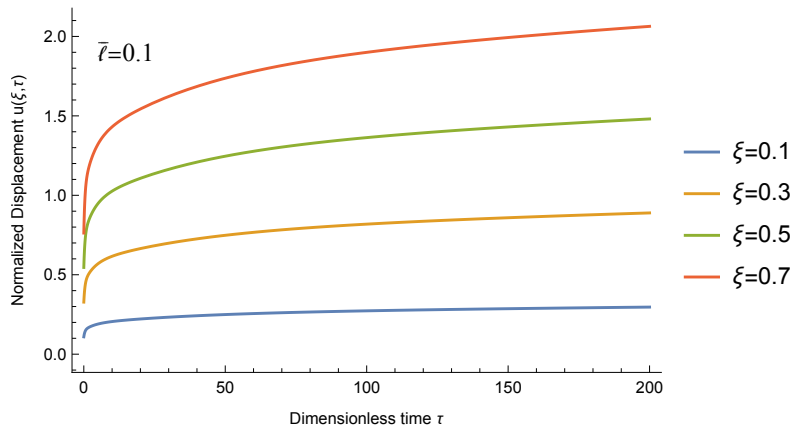
(a) Displacement at various positions (ξ) of the bar with $\bar{\ell} = 0.1$.
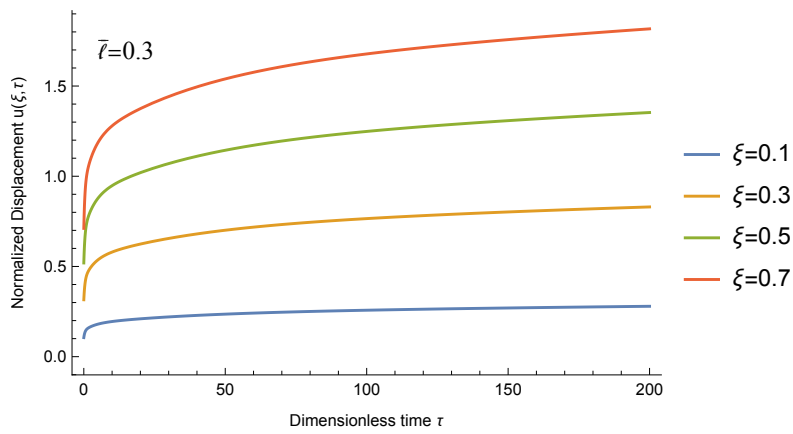
(b) Displacement at various positions (ξ) of the bar with $\bar{\ell} = 0.3$.
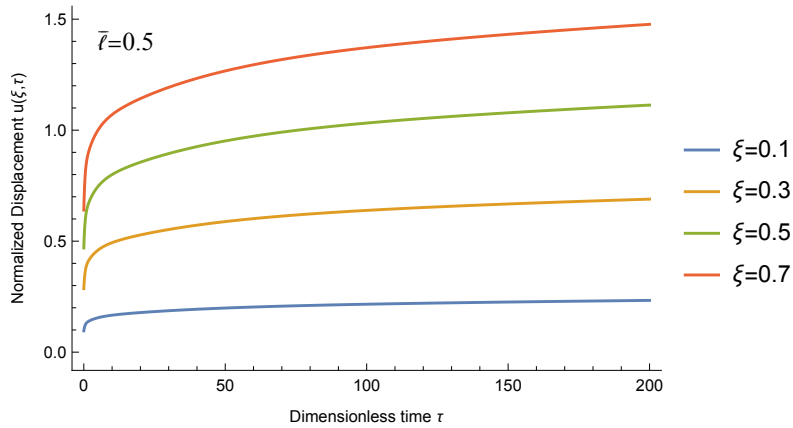
(c) Displacement at various positions (ξ) of the bar with $\bar{\ell} = 0.5$.

Figure 5.4: Influence of gradient coefficient on displacement response at various positions (ξ = 0.1, 0.3, 0.5 and 0.7) on the gradient viscoelastic bar .
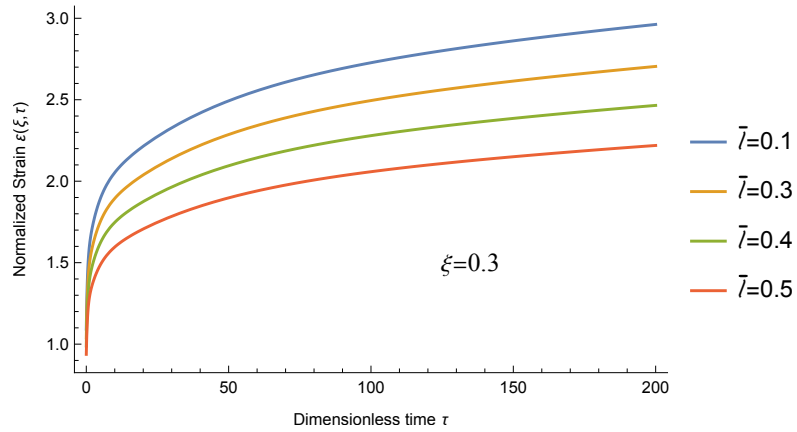
(a) Strain versus time at $\xi = 0.3$ of the bar.
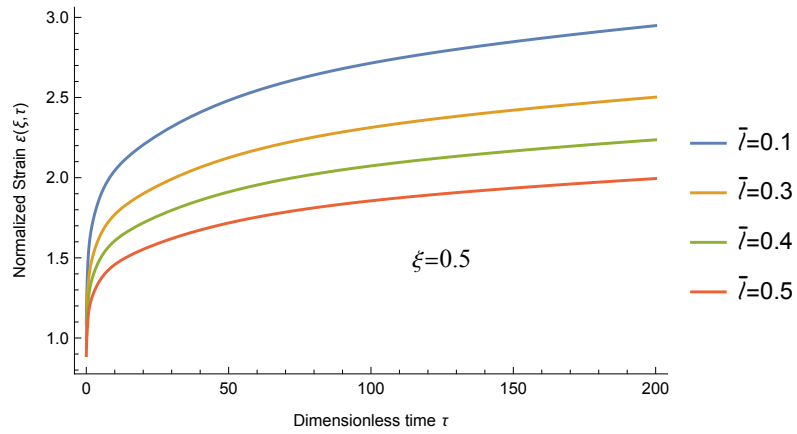
(b) Strain versus time at the middle ($\xi = 0.5$) of the bar.
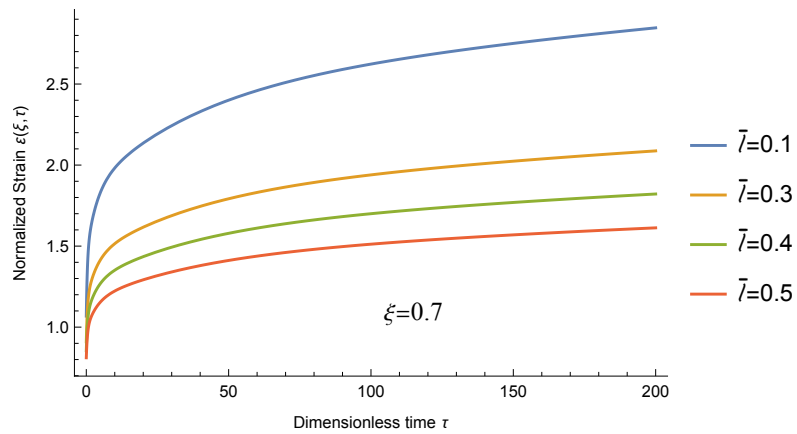
(c) Strain versus time at $\xi = 0.7$ of the bar.

Figure 5.5: Strain response of the quasi-statically loaded gradient viscoelastic bar versus dimensionless time variable for various values of $\bar{\ell}$ at $\xi = 0.3$, 0.5 and 0.7 of the bar.
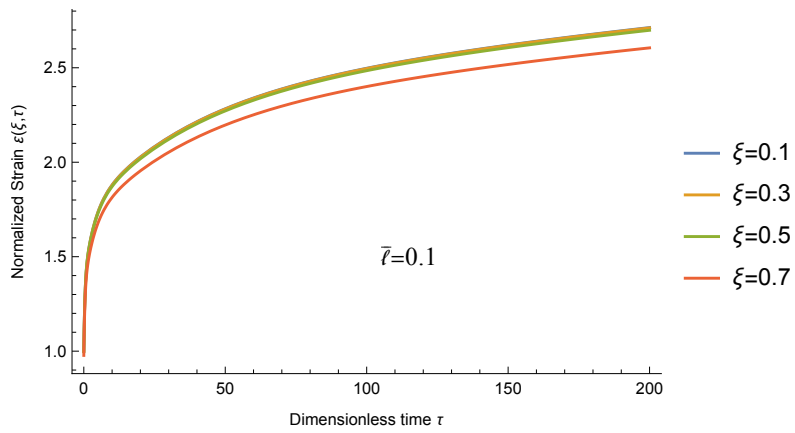
(a) Strain at various positions (ξ) of the bar with $\bar{\ell} = 0.1$.
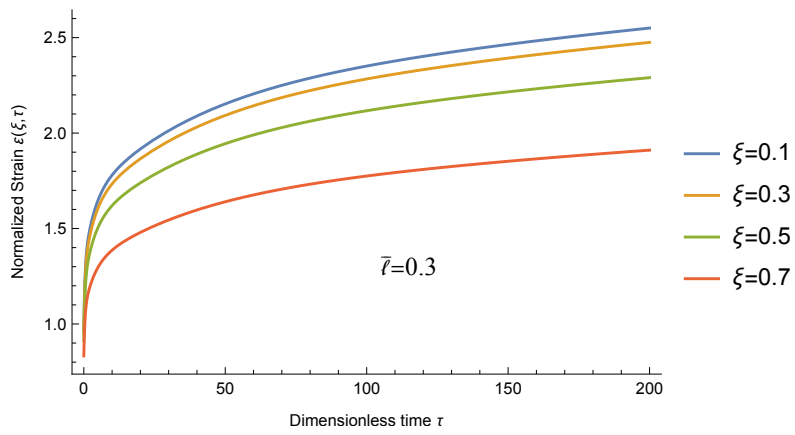
(b) Strain at various positions (ξ) of the bar with $\bar{\ell} = 0.3$.
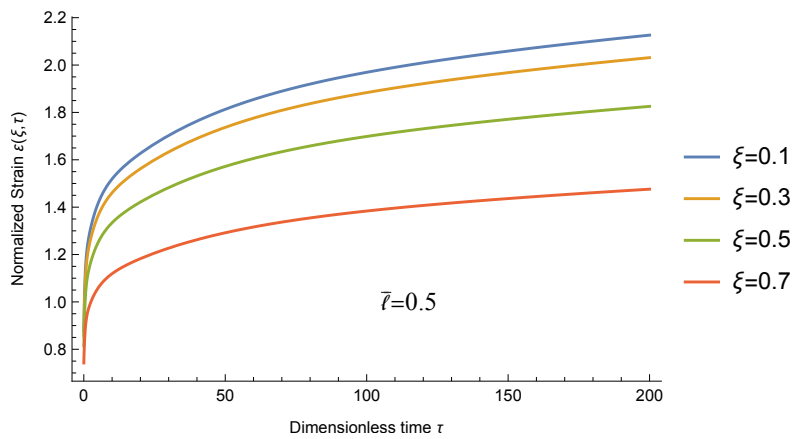
(c) Strain at various positions (ξ) of the bar with $\bar{\ell} = 0.5$.

Figure 5.6: Influence of gradient coefficient on strain at various positions ($\xi = 0.1$, 0.3, 0.5 and 0.7) of the gradient viscoelastic bar .

to be a function of strain and its second gradient, the governing differential equation to be solved would be [83]

$$\sigma = E[(\varepsilon - \varepsilon^\theta) - \ell^2(\varepsilon - \varepsilon^\theta)''], \tag{5.6.48}$$

that is equipped with nonstandard boundary conditions $\varepsilon' = u'' = 0$ at $x = \pm L$. The following general solution is obtained as

$$u' - \varepsilon^\theta = \frac{A}{\ell}\cosh\frac{x}{\ell} + \frac{B}{\ell}\sinh\frac{x}{c} + \frac{\sigma}{E}, \tag{5.6.49}$$

with A and B as constants. Integrating the last equation, we obtain an expression for displacement

$$u = A\sinh\frac{x}{\ell} + B\cosh\frac{x}{\ell} + \frac{\sigma x}{E} + u^\theta(x) + C, \tag{5.6.50}$$

where C is introduced as another constant and $u^\theta(x)$ is expressed as

$$u^\theta(x) := -\alpha\bar{\theta}L(\frac{x}{L} - \frac{x^3}{3L^3}). \tag{5.6.51}$$

Through the nonstandard boundary conditions, $u''(-L) = u''(L) = 0$, A and B are obtained

$$A = -\frac{2\alpha\bar{\theta}L}{\sinh\frac{L}{\ell}}\frac{c^2}{L^2}, \qquad B = 0. \tag{5.6.52}$$

Using standard boundary conditions, $u(-L) = u(L) = 0$, C and σ are formulated as

$$C = 0, \qquad \sigma = 2\alpha E\bar{\theta}(\frac{1}{3} + \zeta^2) \tag{5.6.53}$$

with $\zeta = \ell/L$.

Defining $\xi = x/L$, the relations for displacement and strain respectively reads as:

$$\frac{u}{L} = \alpha\bar{\theta}\left[\frac{1}{3}(\xi^2 - 1)\xi + 2\zeta^2\left(\xi - \frac{\sinh(\xi/\zeta)}{\sinh(1/\zeta)}\right)\right], \tag{5.6.54}$$

$$\varepsilon = \alpha\bar{\theta}\left[\xi^2 - \frac{1}{3} + 2\zeta^2 - 2\zeta\frac{\cosh(\xi/\zeta)}{\sinh(1/\zeta)}\right]. \tag{5.6.55}$$

It is noticed that for $\zeta = 0$, the local type solution is obtained as

$$\sigma = \frac{2}{3}\alpha E\bar{\theta},$$
$$\frac{u}{L} = \frac{1}{3}\alpha\bar{\theta}(\xi^2 - 1),$$
$$\varepsilon = \alpha\bar{\theta}\left(\xi^2 - \frac{1}{3}\right). \tag{5.6.56}$$

Local type ($\zeta = 0$) and gradient type solutions for strain and displacement are plotted in Fig. 5.7 and 5.8, respectively. The influence of length scale parameter is observed in both solutions; besides a distinguishable difference with local type solutions.
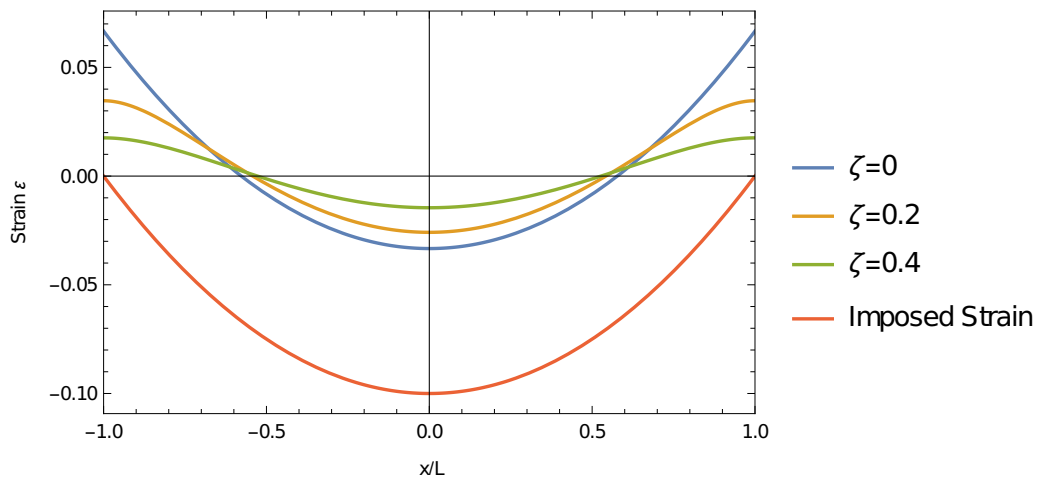
Figure 5.7: Strain response of a gradient elastic bar under thermal loading with different length scale parameters ($\zeta = 0.2, 0.4$) as compared to classical (local) elasticity ($\zeta = 0$) solution and imposed strain($\frac{\sigma}{E} = 74.666 \times 10^{-3}, \zeta = 0.2$; $\frac{\sigma}{E} = 66.666 \times 10^{-3}, \zeta = 0$).
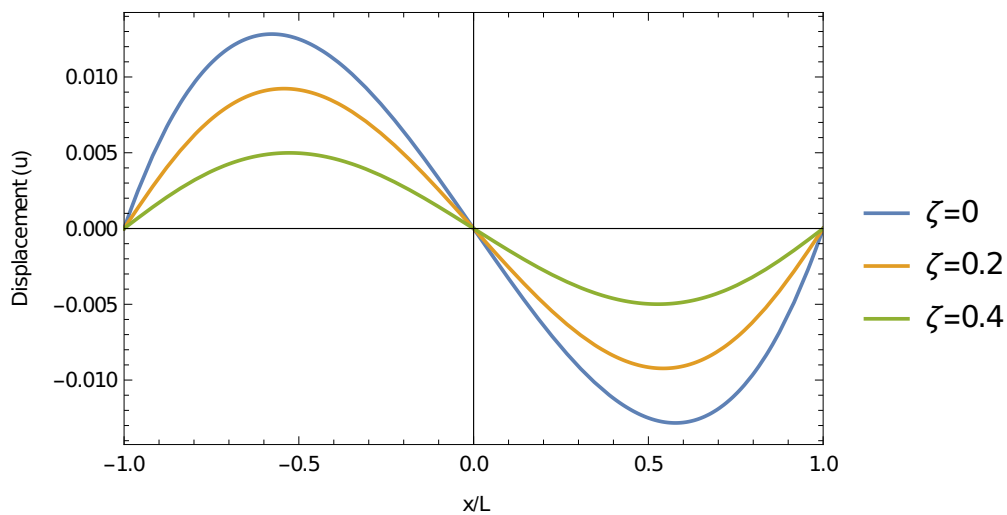
Figure 5.8: Displacement response of a gradient elastic bar under thermal loading with two length scale parameters ($\zeta = 0.2, 0.4$) as compared to classical elasticity ($\zeta = 0$) solution.

A Boundary Integral Transformation

Let $\mathbf{A} = A_{pi_1i_2\ldots i_m}$ and $\mathbf{B} = B_{i_1i_2\ldots i_m}$ denoting tensors of ranks $m+1$ and m respectively with $m \geq 0$; where $m = 0$ would represent $\mathbf{A}$ as a vector and $\mathbf{B}$ as a scalar. $\mathbf{A}$ is defined over the boundary surface $\Gamma = \partial\Omega$ of domain Ω and $\mathbf{B}$ is defined over Ω. Furthermore it is assumed that both tensors are sufficiently regular to support surface integral:

$$\int_\Gamma \mathbf{A} \odot (\nabla \mathbf{B}) d\Gamma = \int_\Gamma A_{pi_1i_2\ldots i_m} \partial_p \mathbf{B} = B_{i_1i_2\ldots i_m} d\Gamma. \tag{A.1}$$

The symbol $\odot$ represents a scalar product with complete index contraction for the factor of lower order.

Due to regularity assumption of surface Γ we may decompose the gradient $\nabla = \partial_p$ into its tangential and normal components as

$$\nabla = \bar{\nabla} + \mathbf{n}\partial_n \tag{A.2}$$

where with $\mathbf{I} = \delta_{ij}$ denoting the unit second-order tensor we can write the tangential part as

$$\bar{\nabla} := (\mathbf{I} - \mathbf{nn}) \cdot \nabla, \qquad \partial_n := \mathbf{n} \cdot \nabla \tag{A.3}$$

which in index notation reads as

$$\bar{\nabla}_p = (\delta_{pq} - n_p n_q)\partial_q, \qquad \partial_n = n_p \partial_q. \tag{A.4}$$

Substituting A.2 into A.1 leads to

$$\int_\Gamma \mathbf{A} \odot (\nabla \mathbf{B}) d\Gamma = \int_\Gamma \bar{\nabla} \odot (\bar{\nabla}\mathbf{B}) d\Gamma + \int_\Gamma \mathbf{A} \odot (\mathbf{n}\partial_n \mathbf{B}) d\Gamma. \tag{A.5}$$

when transformed, the first integral on the right-hand side of A.5 can be rewritten as

$$\int_\Gamma \mathbf{A} \odot (\nabla \mathbf{B}) d\Gamma = \int_\Gamma \bar{\nabla} \cdot (\mathbf{A}^T \odot \mathbf{B}^T) d\Gamma - \int_\Gamma (\bar{\nabla} \cdot \mathbf{A}) \odot \mathbf{B} d\Gamma. \tag{A.6}$$

Through the so called *surface divergence theorem* and in the case of sufficient regularity (as in here) one can transform the first integral on the right hand side according to identity as

$$\int_\Gamma \bar{\nabla} \cdot (\mathbf{A}^T \odot \mathbf{B}^T) d\Gamma = \int_\Gamma K\mathbf{n} \cdot (\mathbf{A}^T \odot \mathbf{B})^T d\Gamma, \tag{A.7}$$

where K equals twice the mean curvature of Γ at the integration point, that considering r_1 and r_2 as the principal curvature radii,

$$\mathbf{K} = \bar{\nabla} \cdot \mathbf{n} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{A.8}$$

Henceforth, substituting A.7 into A.6 then A.6 into A.5, and introducing $\mathbf{G} = G_p$ for the sake of compactness one can write

$$\mathbf{G} := \mathbf{Kn} - \bar{\nabla}, \tag{A.9}$$

and ultimately equation A.5 takes the form

$$\int_\Gamma \mathbf{A} \odot (\nabla \mathbf{B}) d\Gamma = \int_\Gamma (\mathbf{G} \cdot \mathbf{A}) \odot \mathbf{B} d\Gamma + \int_\Gamma (\mathbf{n} \cdot \mathbf{A}) \odot \partial_n \mathbf{B} d\Gamma. \tag{A.10}$$

Worthy of notice that the surface integrals on the right-hand side involves only values of $\mathbf{B}$ and its normal derivatives. The formula A.10 offers a general applicability and can be referred as *surface integral transformation formula*.

Chapter 6

Conclusion

This research aimed to study and better understand the phenomenon of thermoelastic damping in tiny scales and in the case of micro-/nano-resonators. Such small structures are widely employed in various branches of science and technology where their performance (which is measured through their sensitivity) largely depends on the magnitude of energy dissipated within their structures. Unfortunately there is a gap between mathematical models describing the dynamic behavior of such structures and what is actually measured in the lab.

It turns out that at this level (namely micro/nano scales) classical elasticity theories are not very well equipped to describe and resolute the mechanical fields such as deformation and strain. In other words they are not sufficient to fully (or at least to an acceptable level) characterize the kinematics of the deformed body. In fact a parameter through which the length scale effects can be introduced into the model is missing in classical theories. However more recent theories such as nonlocal elasticity and gradient theories have been emerged to fill such a gap. This study employed gradient elasticity theory and developed constitutive models for both stress gradient and strain gradient materials.

On the other side, in thermoelastic damping, thermal and mechanical fields are coupled leading to exertion of heat and production of entropy which eventually causes dissipation of energy within the structure of micro and nano beams. Henceforth another influencing factor, namely thermodynamics of irreversible processes, was needed to be taken into account in order to devise a more comprehensive theory where all influencing parties collide.

Irreversible thermodynamics can be divided into two categories, namely Classical Irreversible Thermodynamics (CIT) and Extended Irreversible Thermodynamics (EIT). In EIT the space of state variables is formed by the union of the space of classical variables such as mass, momentum, energy, etc., and the space of corresponding fluxes. In the case of problems with internal friction many calculations based on CIT have been performed showing excellent agreement with experimental data. It implies that the assumption of the CIT would be valid for these problems. Thereby this study opted to stick with principles of CIT though room for improvements always exists.

In the context of irreversible thermodynamics, a well known theory namely Onsager's reciprocal theory was used for introducing interference effects of various fields on each other. Additionally the Zwanzig's generalization (to Onsager's theory) was incorporated as well to incorporate causal effects.

As a generalized class of materials, thermoviscoelasticity was chosen to be combined

with gradient theories. As a result, for the first time, a general framework obeying the laws of thermodynamics as well as principles of irreversible thermodynamics was developed in which the length scale effects were incorporated as well.

However there is always room for improvement. In the case of current endeavor several other milestones can be suggested as future outlook. For instance, in this study the development of theory was grounded on linear theory of viscoelasticity. It can definitively be advanced through modeling materials exhibiting nonlinear viscoelasticity.

Also, simple boundary conditions for heat transport as employed in the current study can be extended toward more advanced boundary conditions. In a linear regime, boundary conditions can be divided into three main categories. First kind involves the prescription of temperature on the surface of the boundary. The other type involves prescription of the heat flux on the surface. And the third type would be the convection boundary condition. Depending on the problem any of these or certain combination of these boundary conditions could be imposed.

Fourier model was used to describe the heat transport in this study. There are studies in which Non-Fourier model for heat is considered. So as an extension to the current theory, Fourier heat transport equation can be advanced toward non-Fourier models.

As already mentioned current theory is based on principles of CIT and it has been shown to be sufficient for the problem at hand. However since the EIT offers a broader space in terms of thermodynamical state variables, it is suggested that this branch of non-equilibrium thermodynamics to be exploited for specialized classes of problems in which a larger thermodynamical space would be necessary and leads to better resolution of the problem.

Finally constitutive models as developed in the current study for the case of stress and strain gradient materials can be used and incorporated within a well structured boundary value problem describing the deflection or oscillation of cantilevered micro/nano beams through which the quality factor of the beam can be predicted and compared to experimental observation and numerical computations.

Appendix A

Mathematica Scripts

```
1   (*
2   Author: Behrooz Karami
3   Department of Civil & Environemental Engineering
4   University of Brescia
5   Date: 2023-06-15
6   Strain Response of a nonhomogeneous 1D Bar with
7   Gradient  Elasticity Theory
8   *)
9   (*
10  Strain relation at the right half of the bare
11  *)
12
13  \[CurlyEpsilon]p =
14  uboL/(1 + \[Mu]) (2 - (1 - \[Mu]) (Cosh[(2 xoL - 1)/(2 coL)] -
15  Tanh[1/(2 coL)]*Sinh[(2 xoL - 1)/(2 coL)]));
16
17  (*
18  Strain relaiton at the left half of the bare
19  *)
20  \[CurlyEpsilon]n =
21  uboL/(1 + \[Mu]) (2 \[Mu] + (1 - \[Mu]) (Cosh[(2 xoL - 1)/(2 coL)] +
22  Tanh[1/(2 coL)]*Sinh[(2 xoL - 1)/(2 coL)]));
23
24  (*
25  Plotting the strain of the right half
26  *)
27  p1 = Plot[
28  {
29    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.7},
30    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.8},
31    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.9},
32    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1},
33    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1.1},
34    \[CurlyEpsilon]p /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1.3}
35  },
36  {xoL, 1/2, 1},
37  PlotStyle ->
38  {
39    {Blue, Thickness[0.005]},
40    {Magenta, Thickness[0.005]},
41    {Orange, Thickness[0.005]},
```

```
42       {Red, Thickness[0.005]},
43       {Purple, Thickness[0.005]},
44       {Darker[Green], Thickness[0.005]}}];
45
46    (*
47    Plotting the strain of the left half
48    *)
49    p2 = Plot[
50    {
51       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.7},
52       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.8},
53       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 0.9},
54       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1},
55       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1.1},
56       \[CurlyEpsilon]n /. {uboL -> 0.1, coL -> 0.1, \[Mu] -> 1.3}
57    },
58    {xoL, 0, 1/2},
59    PlotStyle ->
60    {
61       {Blue, Thickness[0.005]},
62       {Magenta, Thickness[0.005]},
63       {Orange, Thickness[0.005]},
64       {Red, Thickness[0.005]},
65       {Purple, Thickness[0.005]},
66       {Darker[Green], Thickness[0.005]}},
67    PlotLegends -> {"\[Mu]=0.7", "\[Mu]=0.8", "\[Mu]=0.9",
68       "\[Mu]=1.0", "\[Mu]=1.1", "\[Mu]=1.3"}];
69
70    (*
71    Showing the graph
72    *)
73    p0 = Show[{p1, p2},
74    PlotRange -> {{0, 1}, {0.08, 0.12}},
75    Frame -> {True, True, True, True},
76    Axes -> False,
77    FrameStyle -> {{Black, Thickness[0.001]}},
78    FrameLabel -> {Style["\!\(\*
79       StyleBox[\"x\",\nFontSlant->\"Italic\"]\)/\!\(\*
80       StyleBox[\"L\",\nFontSlant->\"Italic\"]\)",
81       FontFamily -> "Times New Roman", FontSize -> 16, Black],
82       Style["Strain (\[CurlyEpsilon])", FontFamily -> "Times New Roman",
83       FontSize -> 16, Black]} ]
84
85    (*
86    Exporting the file.
87    *)
88    Export["Gradient.pdf", p0]
89
```

```mathematica
(*
Author: Behrooz Karami
Department of Civil & Environemental Engineering
University of Brescia
Date: 2023-06-15
Strain Response of a nonhomogeneous 1D Bar with
Local  Elasticity Theory
*)
(* Strain relation at the right half of the bare*)
\[CurlyEpsilon]p = (2 uboL)/(1 + \[Mu]);

(* Strain relaiton at the left half of the bare*)
\[CurlyEpsilon]n = (2 \[Mu] uboL)/(1 + \[Mu]);

(* Plotting the strain of the right half*)
p1 = Plot[
{
  \[CurlyEpsilon]p /. {uboL -> 0.1, \[Mu] -> 0.7},
  \[CurlyEpsilon]p /. {uboL -> 0.1, \[Mu] -> 0.9},
  \[CurlyEpsilon]p /. {uboL -> 0.1, \[Mu] -> 1},
  \[CurlyEpsilon]p /. {uboL -> 0.1, \[Mu] -> 1.3}
},
{xoL, 1/2, 1},
PlotStyle ->
{
  {Blue, Thickness[0.005]},
  {Orange, Thickness[0.005]},
  {Red, Thickness[0.005]},
  {Darker[Green], Thickness[0.005]}}];

(* Plotting the strain of the left half*)
p2 = Plot[
{
  \[CurlyEpsilon]n /. {uboL -> 0.1, \[Mu] -> 0.7},
  \[CurlyEpsilon]n /. {uboL -> 0.1, \[Mu] -> 0.9},
  \[CurlyEpsilon]n /. {uboL -> 0.1, \[Mu] -> 1},
  \[CurlyEpsilon]n /. {uboL -> 0.1, \[Mu] -> 1.3}
},
{xoL, 0, 1/2},
PlotStyle ->
{
  {Blue, Thickness[0.005]},
  {Orange, Thickness[0.005]},
  {Red, Thickness[0.005]},
  {Darker[Green], Thickness[0.005]}},
PlotLegends -> {
  "\[Mu]=0.7",
  "\[Mu]=0.9",
  "\[Mu]=1.0",
  "\[Mu]=1.3"}];


(*
Showing the graph
*)
p0 = Show[{p1, p2},
PlotRange -> {{0, 1}, {0.08, 0.12}},
Frame -> {True, True, True, True},
```

```
59    Axes -> False, FrameStyle -> {{Black, Thickness[0.001]}},
60    FrameLabel -> {Style["\!\(\*
61      StyleBox[\"x\",\nFontSlant->\"Italic\"]\)/\!\(\*
62      StyleBox[\"L\",\nFontSlant->\"Italic\"]\)",
63      FontFamily -> "Times New Roman", FontSize -> 16, Black],
64      Style["strain (\[CurlyEpsilon])",
65      FontFamily -> "Times New Roman",
66      FontSize -> 16, Black]} ];
67
68    (*
69    Exporting the file.
70    *)
71    Export["Local.pdf", p0]
```

```mathematica
(*
Gradient Viscoelasticity Code for Displacement Response of 1D bar
*)

(*
Author: Behrooz Karami
Department of Civil & Environemental Engineering
University of Brescia
Date: 2023-06-15
1D Bar Modeling: Gradient Viscoelasticity
*)


(*
Defining material properties and experimental data.
*)
E0 = 1300;
Ej = {2592.2, 1398.8, 1284.4, 1019.2, 795.6, 400.4, 309.4, 52.0};
tj = {1.5*10^-5, 1.5*10^-4, 1.5*10^-3, 1.5*10^-2, 1.5*10^-1, 1.5,
   1.5*10, 1.5*10^2};

(*
Defining the parameters.
*)
\[CurlyEpsilon]0 = 0.6;
L = 0.1; \[Rho] = 50;

(*
Relaxation function relationships.
*)
Ebar = E0/s + \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(j = 1\), \(8\)]
\*FractionBox[\(Ej[[j]]\), \(s +
\*FractionBox[\(1\), \(
\*FractionBox[\(tj[\([\)\(j\)\(]\)]\), \(L\)]
\*SqrtBox[
FractionBox[\(Ee\), \(\[Rho]\)]]\)]\)]\);
Ee = E0 + \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(j = 1\), \(8\)]\(Ej[[j]]\)\);
(*
Defining constants
*)
c1 = (lbar (\[CurlyEpsilon]0/s - P/(s A s Ebar)))/(2 Cosh[1/lbar]);
c2 = -((lbar (\[CurlyEpsilon]0/s - P/(s A s Ebar)))/(
2 Cosh[1/lbar]));
c3 = P/(s A s Ebar);
c4 = 0;


(* Laplace transform solution for the displacement
*)
ubar = c1 Exp[\[Xi]/lbar] + c2 Exp[-\[Xi]/lbar] + c3 \[Xi] + c4;
(*
Laplace transform solution for the strain
*)
\[CurlyEpsilon]bar =
c1/lbar Exp[\[Xi]/lbar] - c2/lbar Exp[-\[Xi]/lbar] + c3;
(*
```

```
59    Time domain displacement solution:
60    *)
61    u\[Xi]t = InverseLaplaceTransform[ubar, s, t];
62
63    (*
64    Time domain strain solution:
65    *)
66    \[CurlyEpsilon]\[Xi]t =
67    InverseLaplaceTransform[\[CurlyEpsilon]bar, s, t];
68
69
70    (*
71    Introducing dimensionless \[Tau] parameter for displacement.
72    *)
73    u\[Xi]\[Tau] := u\[Xi]t  /. t -> (L \[Tau] )/Sqrt[Ee/\[Rho]];
74    (*
75    Introducing dimensionless \[Tau] parameter for strain.
76    *)
77    \[CurlyEpsilon]\[Xi]\[Tau] := \[CurlyEpsilon]\[Xi]t /.
78    t -> (L \[Tau] )/Sqrt[Ee/\[Rho]];
79
80    (*
81    Normalized displacement soluiton
82    *)
83    u\[Xi]\[Tau]Nrm = u\[Xi]\[Tau] /((P L)/(A Ee));
84
85    (*
86    Normalized strain solution.
87    *)
88    \[CurlyEpsilon]\[Xi]\[Tau]Nrm  = \[CurlyEpsilon]\[Xi]\[Tau] /(P /(
89    A Ee));
90
91
92    (* =============== PLOTTING THE DISPLACEMENT =========== *)
93
94    (* ===== Varying Overscript[\[ScriptL], _] & \[Xi] =  0.3 ==== *)
95
96    pu3 = Plot[
97    {
98      u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.3, P -> 10000, A -> 1} //
99      Chop // Simplify,
100     u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.3, P -> 10000, A -> 1} //
101     Chop // Simplify,
102     u\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.3, P -> 10000, A -> 1} //
103     Chop // Simplify,
104     u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.3, P -> 10000, A -> 1} //
105     Chop // Simplify
106   },
107   {\[Tau], 0, 2*10^2},
108   Frame -> {True, True, False, False},
109   FrameLabel -> {"Dimensionless time \[Tau]",
110     "Normalized Displacement u(\[Xi],\[Tau])"},
111   Axes -> False,
112   (*
113   PlotStyle\[Rule]
114   {
115     {Darker[Green],Thickness[0.005]},
116     {Red,Thickness[0.005]},
```

```
117        {Darker[Blue],Thickness[0.005]},
118        {Orange,Thickness[0.005]}
119      },
120      *)
121      PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
122        "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
123        "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
124        "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
125
126      (* ================== Optimizing the graphics =================== *)
127      gu3 = Graphics[
128      {
129        Text
130        [
131        Style["\[Xi]=0.3",
132        FontFamily -> "Times New Roman",
133        FontSize -> 14], {135, .5}
134        ](*,
135        Thickness[0.005],
136        Green,
137        Line
138        [
139        {{110,1},{120,1}}
140        ]*)
141      }
142      ];
143
144
145      (* ================== Showing the graphics =================== *)
146      pu3 = Show[{pu3, gu3}]
147
148      (* ================== Exporting the graphics =================== *)
149      Export["GV_Displacement_lbar_changing_xi_.3.pdf", pu3]
150      (* ============================================= *)
151
152
153
154      (* = Varying Overscript[\[ScriptL], _] & \[Xi] = 0.5 = *)
155
156      pu5 = Plot[
157      {
158        u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.5, P -> 10000, A -> 1} //
159        Chop // Simplify,
160        u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.5, P -> 10000, A -> 1} //
161        Chop // Simplify,
162        u\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.5, P -> 10000, A -> 1} //
163        Chop // Simplify,
164        u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.5, P -> 10000, A -> 1} //
165        Chop // Simplify
166      },
167      {\[Tau], 0, 2*10^2},
168      Frame -> {True, True, False, False},
169      FrameLabel -> {"Dimensionless time \[Tau]",
170        "Normalized Displacement u(\[Xi],\[Tau])"},
171      Axes -> False,
172      (*
173      PlotStyle\[Rule]
174      {
```

```mathematica
175      {Darker[Green],Thickness[0.005]},
176      {Red,Thickness[0.005]},
177      {Darker[Blue],Thickness[0.005]},
178      {Orange,Thickness[0.005]}
179    },
180    *)
181    PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
182      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
183      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
184      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
185
186    (* ================= Optimizing the graphics ================= *)
187    gu5 = Graphics[
188    {
189      Text
190      [
191      Style["\[Xi]=0.5",
192      FontFamily -> "Times New Roman",
193      FontSize -> 14], {125, .8}
194      ](*,
195      Thickness[0.005],
196      Green,
197      Line
198      [
199      {{110,1},{120,1}}
200      ]*)
201    }
202    ];
203
204
205    (* ================= Showing the graphics ================= *)
206    pu5 = Show[{pu5, gu5}]
207
208    (* ================= Exporting the graphics ================= *)
209    Export["GV_Displacement_lbar_changing_xi_.5.pdf", pu5]
210    (* ========================================== *)
211
212
213
214    (* ======= Varying Overscript[\[ScriptL], _] & \[Xi] =  0.7========= *)
215
216    pu7 = Plot[
217    {
218      u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.7, P -> 10000, A -> 1} //
219      Chop // Simplify,
220      u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.7, P -> 10000, A -> 1} //
221      Chop // Simplify,
222      u\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.7, P -> 10000, A -> 1} //
223      Chop // Simplify,
224      u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.7, P -> 10000, A -> 1} //
225      Chop // Simplify
226    },
227    {\[Tau], 0, 2*10^2},
228    Frame -> {True, True, False, False},
229    FrameLabel -> {"Dimensionless time \[Tau]",
230      "Normalized Displacement u(\[Xi],\[Tau])"},
231    Axes -> False,
232    (*
```

```
233   PlotStyle\[Rule]
234   {
235     {Darker[Green],Thickness[0.005]},
236     {Red,Thickness[0.005]},
237     {Darker[Blue],Thickness[0.005]},
238     {Orange,Thickness[0.005]}
239   },
240   *)
241   PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
242     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
243     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
244     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
245
246   (* =============== Optimizing the graphics ================= *)
247   gu7 = Graphics[
248   {
249     Text
250     [
251     Style["\[Xi]=0.7",
252     FontFamily -> "Times New Roman",
253     FontSize -> 14], {115, 1}
254     ](*,
255     Thickness[0.005],
256     Green,
257     Line
258     [
259     {{110,1},{120,1}}
260     ]*)
261   }
262   ];
263
264
265   (* ================== Showing the graphics ================== *)
266   pu7 = Show[{pu7, gu7}]
267
268   (* ================== Exporting the graphics ================== *)
269   Export["GV_Displacement_lbar_changing_xi_.7.pdf", pu7]
270   (* ============================================== *)
271
272
273   (* ========================================== *)
274   (* ========================================== *)
275   (* ===Fixing Overscript[\[ScriptL], _] = 0.1 &  Varying \[Xi]  ==== *)
276
277   pul1 = Plot[
278   {
279     u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.1, P -> 10000, A -> 1} //
280     Chop // Simplify,
281     u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.3, P -> 10000, A -> 1} //
282     Chop // Simplify,
283     u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.5, P -> 10000, A -> 1} //
284     Chop // Simplify,
285     u\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.7, P -> 10000, A -> 1} //
286     Chop // Simplify
287   },
288   {\[Tau], 0, 200},
289   Frame -> {True, True, False, False},
290   FrameLabel -> {"Dimensionless time \[Tau]",
```

```
291      "Normalized Displacement u(\[Xi],\[Tau])"},
292    Axes -> False,
293    (*
294    PlotStyle\[Rule]
295    {
296      {Darker[Green],Thickness[0.005]},
297      {Red,Thickness[0.005]},
298      {Darker[Blue],Thickness[0.005]},
299      {Orange,Thickness[0.005]}
300    },
301    *)
302    PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
303      "\[Xi]=0.7"}];
304
305    (* ================== Optimizing the graphics =================== *)
306    gul1 = Graphics[
307    {
308      Text
309      [
310      Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
311      FontFamily -> "Times New Roman",
312      FontSize -> 14], {15, 1.9}
313      ](*,
314      Thickness[0.005],
315      Green,
316      Line
317      [
318      {{110,1},{120,1}}
319      ]*)
320    }
321    ];
322
323
324    (* ================== Showing the graphics =================== *)
325    pul1 = Show[{pul1, gul1}]
326
327    (* ================== Exporting the graphics =================== *)
328    Export["GV_Displacement_lbar_.1_xi_changing.pdf", pul1]
329    (* ================================================= *)
330
331    (* === Fixing Overscript[\[ScriptL], _] = 0.3 & Varying \[Xi]  === *)
332
333    pul3 = Plot[
334    {
335      u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.1, P -> 10000, A -> 1} //
336      Chop // Simplify,
337      u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.3, P -> 10000, A -> 1} //
338      Chop // Simplify,
339      u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.5, P -> 10000, A -> 1} //
340      Chop // Simplify,
341      u\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.7, P -> 10000, A -> 1} //
342      Chop // Simplify
343    },
344    {\[Tau], 0, 200},
345    Frame -> {True, True, False, False},
346    FrameLabel -> {"Dimensionless time \[Tau]",
347      "Normalized Displacement u(\[Xi],\[Tau])"},
348    Axes -> False,
```

```
349   (*
350   PlotStyle\[Rule]
351   {
352     {Darker[Green],Thickness[0.005]},
353     {Red,Thickness[0.005]},
354     {Darker[Blue],Thickness[0.005]},
355     {Orange,Thickness[0.005]}
356   },
357   *)
358   PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
359     "\[Xi]=0.7"}];
360
361   (* ================== Optimizing the graphics ================== *)
362   gul3 = Graphics[
363   {
364     Text
365     [
366     Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
367     FontFamily -> "Times New Roman",
368     FontSize -> 14], {15, 1.7}
369     ](*,
370     Thickness[0.005],
371     Green,
372     Line
373     [
374     {{110,1},{120,1}}
375     ]*)
376   }
377   ];
378
379
380   (* ================== Showing the graphics ================== *)
381   pul3 = Show[{pul3, gul3}]
382
383   (* ================== Exporting the graphics ================== *)
384   Export["GV_Displacement_lbar_.3_xi_changing.pdf", pul3]
385   (* ================================================ *)
386
387   (* ===== Fixing Overscript[\[ScriptL], _] = 0.5 & Varying \[Xi]  ===== *)
388
389   pul5 = Plot[
390   {
391     u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.1, P -> 10000, A -> 1} //
392     Chop // Simplify,
393     u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.3, P -> 10000, A -> 1} //
394     Chop // Simplify,
395     u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.5, P -> 10000, A -> 1} //
396     Chop // Simplify,
397     u\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.7, P -> 10000, A -> 1} //
398     Chop // Simplify
399   },
400   {\[Tau], 0, 200},
401   Frame -> {True, True, False, False},
402   FrameLabel -> {"Dimensionless time \[Tau]",
403     "Normalized Displacement u(\[Xi],\[Tau])"},
404   Axes -> False,
405   (*
406   PlotStyle\[Rule]
```

```
407    {
408      {Darker[Green],Thickness[0.005]},
409      {Red,Thickness[0.005]},
410      {Darker[Blue],Thickness[0.005]},
411      {Orange,Thickness[0.005]}
412    },
413    *)
414    PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
415      "\[Xi]=0.7"}];
416
417    (* =================== Optimizing the graphics =================== *)
418    gul5 = Graphics[
419    {
420      Text
421      [
422      Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5",
423      FontFamily -> "Times New Roman",
424      FontSize -> 14], {15, 1.4}
425      ](*,
426      Thickness[0.005],
427      Green,
428      Line
429      [
430      {{110,1},{120,1}}
431      ]*)
432    }
433    ];
434
435
436    (* =================== Showing the graphics =================== *)
437    pul5 = Show[{pul5, gul5}]
438
439    (* =================== Exporting the graphics =================== *)
440    Export["GV_Displacement_lbar_.5_xi_changing.pdf", pul5]
441    (* ============================================= *)
```

```mathematica
(* Gradient Viscoelasticity: Strain Response of 1D bar *)

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Date: 2023-06-15
1D Bar Modeling: Gradient Viscoelasticity
*)


(*
Defining material properties and experimental data.
*)
E0 = 1300;
Ej = {2592.2, 1398.8, 1284.4, 1019.2, 795.6, 400.4, 309.4, 52.0};
tj = {1.5*10^-5, 1.5*10^-4, 1.5*10^-3, 1.5*10^-2, 1.5*10^-1, 1.5,
  1.5*10, 1.5*10^2};

(*
Defining the parameters.
*)
\[CurlyEpsilon]0 = 0.6;
L = 0.1; \[Rho] = 50;

(*
Relaxation function relationship.
*)
Ebar = E0/s + \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(j = 1\), \(8\)]
\*FractionBox[\(Ej[[j]]\), \(s +
\*FractionBox[\(1\), \(
\*FractionBox[\(tj[\([\)\(j\)\(]\)]\), \(L\)]
\*SqrtBox[
FractionBox[\(Ee\), \(\[Rho]\)]]\)]\)]\);
Ee = E0 + \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(j = 1\), \(8\)]\(Ej[[j]]\)\);
(*
Defining constants
*)
c1 = (lbar (\[CurlyEpsilon]0/s - P/(s A s Ebar)))/(2 Cosh[1/lbar]);
c2 = -((lbar (\[CurlyEpsilon]0/s - P/(s A s Ebar)))/(
2 Cosh[1/lbar]));
c3 = P/(s A s Ebar);
c4 = 0;


(* Laplace transform solution for the displacement
*)
ubar = c1 Exp[\[Xi]/lbar] + c2 Exp[-\[Xi]/lbar] + c3 \[Xi] + c4;
(*
Laplace transform solution for the strain
*)
\[CurlyEpsilon]bar =
c1/lbar Exp[\[Xi]/lbar] - c2/lbar Exp[-\[Xi]/lbar] + c3;
(*
Time domain displacement solution:
*)
```

```
59    u\[Xi]t = InverseLaplaceTransform[ubar, s, t];
60
61    (*
62    Time domain strain solution:
63    *)
64    \[CurlyEpsilon]\[Xi]t =
65    InverseLaplaceTransform[\[CurlyEpsilon]bar, s, t];
66
67
68    (*
69    Introducing dimensionless \[Tau] parameter for displacement.
70    *)
71    u\[Xi]\[Tau] := u\[Xi]t  /. t -> (L \[Tau] )/Sqrt[Ee/\[Rho]];
72    (*
73    Introducing dimensionless \[Tau] parameter for strain.
74    *)
75    \[CurlyEpsilon]\[Xi]\[Tau] := \[CurlyEpsilon]\[Xi]t /.
76    t -> (L \[Tau] )/Sqrt[Ee/\[Rho]];
77
78    (*
79    Normalized displacement soluiton
80    *)
81    u\[Xi]\[Tau]Nrm = u\[Xi]\[Tau] /((P L)/(A Ee));
82
83    (*
84    Normalized strain solution.
85    *)
86    \[CurlyEpsilon]\[Xi]\[Tau]Nrm  = \[CurlyEpsilon]\[Xi]\[Tau] /(P /(
87    A Ee));
88
89
90    (* ==================== PLOTTING THE STRAIN ==================== *)
91    (* ====== Varying Overscript[\[ScriptL], _] & \[Xi] = 0.3 ====== *)
92
93    ps3 = Plot[
94    {
95      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.3,
96        P -> 10000, A -> 1} // Chop // Simplify,
97      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.3,
98        P -> 10000, A -> 1} // Chop // Simplify,
99      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.3,
100        P -> 10000, A -> 1} // Chop // Simplify,
101      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.3,
102        P -> 10000, A -> 1} // Chop // Simplify
103    },
104    {\[Tau], 0, 2*10^2},
105    Frame -> {True, True, False, False},
106    FrameLabel -> {"Dimensionless time \[Tau]",
107      "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
108    Axes -> False,
109    (*
110    PlotStyle\[Rule]
111    {
112      {Darker[Green],Thickness[0.005]},
113      {Red,Thickness[0.005]},
114      {Darker[Blue],Thickness[0.005]},
115      {Orange,Thickness[0.005]}
116    },
```

```
117   *)
118   PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
119     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
120     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
121     "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
122
123   (* =================== Optimizing the graphics =================== *)
124   gs3 = Graphics[
125   {
126     Text
127     [
128     Style["\[Xi]=0.3",
129     FontFamily -> "Times New Roman",
130     FontSize -> 14], {135, 1.5}
131     ](*,
132     Thickness[0.005],
133     Green,
134     Line
135     [
136     {{110,1},{120,1}}
137     ]*)
138   }
139   ];
140
141
142   (* =================== Showing the graphics =================== *)
143   ps3 = Show[{ps3, gs3}]
144
145   (* =================== Exporting the graphics =================== *)
146   Export["GV_Strain_lbar_changing_xi_.3.pdf", ps3]
147
148
149
150   (* ======= Varying Overscript[\[ScriptL], _] & \[Xi] = 0.5 ======= *)
151
152   ps5 = Plot[
153   {
154     \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.5,
155       P -> 10000, A -> 1} // Chop // Simplify,
156     \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.5,
157       P -> 10000, A -> 1} // Chop // Simplify,
158     \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.5,
159       P -> 10000, A -> 1} // Chop // Simplify,
160     \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.5,
161       P -> 10000, A -> 1} // Chop // Simplify
162   },
163   {\[Tau], 0, 2*10^2},
164   Frame -> {True, True, False, False},
165   FrameLabel -> {"Dimensionless time \[Tau]",
166     "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
167   Axes -> False,
168   (*
169   PlotStyle\[Rule]
170   {
171     {Darker[Green],Thickness[0.005]},
172     {Red,Thickness[0.005]},
173     {Darker[Blue],Thickness[0.005]},
174     {Orange,Thickness[0.005]}
```

```mathematica
175    },
176    *)
177    PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
178      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
179      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
180      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
181
182    (* ================== Optimizing the graphics ================== *)
183    gs5 = Graphics[
184    {
185      Text
186      [
187      Style["\[Xi]=0.5",
188      FontFamily -> "Times New Roman",
189      FontSize -> 14], {125, 1.5}
190      ](*,
191      Thickness[0.005],
192      Green,
193      Line
194      [
195      {{110,1},{120,1}}
196      ]*)
197    }
198    ];
199
200
201    (* ================== Showing the graphics ================== *)
202    ps5 = Show[{ps5, gs5}]
203
204    (* ================== Exporting the graphics ================== *)
205    Export["GV_Strain_lbar_changing_xi_.5.pdf", ps5]
206
207
208
209
210    (* ======= Varying Overscript[\[ScriptL], _] & \[Xi] = 0.7 ======= *)
211
212    ps7 = Plot[
213    {
214      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.1, \[Xi] -> 0.7,
215        P -> 10000, A -> 1} // Chop // Simplify,
216      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.3, \[Xi] -> 0.7,
217        P -> 10000, A -> 1} // Chop // Simplify,
218      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.4, \[Xi] -> 0.7,
219        P -> 10000, A -> 1} // Chop // Simplify,
220      \[CurlyEpsilon]\[Xi]\[Tau] /. {lbar -> 0.5, \[Xi] -> 0.7,
221        P -> 10000, A -> 1} // Chop // Simplify
222    },
223    {\[Tau], 0, 2*10^2},
224    Frame -> {True, True, False, False},
225    FrameLabel -> {"Dimensionless time \[Tau]",
226      "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
227    Axes -> False,
228    (*
229    PlotStyle\[Rule]
230    {
231      {Darker[Green],Thickness[0.005]},
232      {Red,Thickness[0.005]},
```

```mathematica
233      {Darker[Blue],Thickness[0.005]},
234      {Orange,Thickness[0.005]}
235    },
236    *)
237    PlotLegends -> {"\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
238      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
239      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.4",
240      "\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5"}];
241
242    (* ================== Optimizing the graphics =================== *)
243    gs7 = Graphics[
244    {
245      Text
246      [
247      Style["\[Xi]=0.7",
248      FontFamily -> "Times New Roman",
249      FontSize -> 14], {115, 1.1}
250      ](*,
251      Thickness[0.005],
252      Green,
253      Line
254      [
255      {{110,1},{120,1}}
256      ]*)
257    }
258    ];
259
260
261    (* ================== Showing the graphics =================== *)
262    ps7 = Show[{ps7, gs7}]
263
264    (* ================== Exporting the graphics =================== *)
265    Export["GV_Strain_lbar_changing_xi_.7.pdf", ps7]
266
267
268    (* =================================================================
269    *)
270    (* =================================================================
271    *)
272    (* ==== Fixing Overscript[\[ScriptL], _] = 0.1 & Varying \[Xi]  ====== *)
273
274    psl1 = Plot[
275    {
276      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.1, \[Xi] -> 0.1,
277        P -> 10000, A -> 1} // Chop // Simplify,
278      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.1, \[Xi] -> 0.3,
279        P -> 10000, A -> 1} // Chop // Simplify,
280      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.1, \[Xi] -> 0.5,
281        P -> 10000, A -> 1} // Chop // Simplify,
282      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.1, \[Xi] -> 0.7,
283        P -> 10000, A -> 1} // Chop // Simplify
284    },
285    {\[Tau], 0, 200},
286    Frame -> {True, True, False, False},
287    FrameLabel -> {"Dimensionless time \[Tau]",
288      "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
289    Axes -> False,
290    (*
```

```
291    PlotStyle\[Rule]
292    {
293      {Darker[Green],Thickness[0.005]},
294      {Red,Thickness[0.005]},
295      {Darker[Blue],Thickness[0.005]},
296      {Orange,Thickness[0.005]}
297    },
298    *)
299    PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
300      "\[Xi]=0.7"}];
301
302    (* ================== Optimizing the graphics ================== *)
303    gsl1 = Graphics[
304    {
305      Text
306      [
307      Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.1",
308      FontFamily -> "Times New Roman",
309      FontSize -> 14], {115, 1.5}
310      ](*,
311      Thickness[0.005],
312      Green,
313      Line
314      [
315      {{110,1},{120,1}}
316      ]*)
317    }
318    ];
319
320
321    (* ================== Showing the graphics ================== *)
322    psl1 = Show[{psl1, gsl1}]
323
324    (* ================== Exporting the graphics ================== *)
325    Export["GV_Strain_lbar_.1_xi_changing.pdf", psl1]
326    (* ================================================================
         *)
327
328    (* ===== Fixing Overscript[\[ScriptL], _] = 0.3 & Varying \[Xi]  ==== *)
329
330    psl3 = Plot[
331    {
332      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.3, \[Xi] -> 0.1,
333        P -> 10000, A -> 1} // Chop // Simplify,
334      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.3, \[Xi] -> 0.3,
335        P -> 10000, A -> 1} // Chop // Simplify,
336      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.3, \[Xi] -> 0.5,
337        P -> 10000, A -> 1} // Chop // Simplify,
338      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.3, \[Xi] -> 0.7,
339        P -> 10000, A -> 1} // Chop // Simplify
340    },
341    {\[Tau], 0, 200},
342    Frame -> {True, True, False, False},
343    FrameLabel -> {"Dimensionless time \[Tau]",
344      "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
345    Axes -> False,
346    (*
347    PlotStyle\[Rule]
```

```
348    {
349      {Darker[Green],Thickness[0.005]},
350      {Red,Thickness[0.005]},
351      {Darker[Blue],Thickness[0.005]},
352      {Orange,Thickness[0.005]}
353    },
354    *)
355    PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
356      "\[Xi]=0.7"}];
357
358    (* ================== Optimizing the graphics =================== *)
359    gsl3 = Graphics[
360    {
361      Text
362      [
363      Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.3",
364      FontFamily -> "Times New Roman",
365      FontSize -> 14], {115, 1.3}
366      ](*,
367      Thickness[0.005],
368      Green,
369      Line
370      [
371      {{110,1},{120,1}}
372      ]*)
373    }
374    ];
375
376
377    (* ================== Showing the graphics =================== *)
378    psl3 = Show[{psl3, gsl3}]
379
380    (* ================== Exporting the graphics =================== *)
381    Export["GV_Strain_lbar_.3_xi_changing.pdf", psl3]
382    (* =============================================================== *)
383
384    (* === Fixing Overscript[\[ScriptL], _] = 0.5 & Varying \[Xi]  === *)
385
386    psl5 = Plot[
387    {
388      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.5, \[Xi] -> 0.1,
389        P -> 10000, A -> 1} // Chop // Simplify,
390      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.5, \[Xi] -> 0.3,
391        P -> 10000, A -> 1} // Chop // Simplify,
392      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.5, \[Xi] -> 0.5,
393        P -> 10000, A -> 1} // Chop // Simplify,
394      \[CurlyEpsilon]\[Xi]\[Tau]Nrm /. {lbar -> 0.5, \[Xi] -> 0.7,
395        P -> 10000, A -> 1} // Chop // Simplify
396    },
397    {\[Tau], 0, 200},
398    Frame -> {True, True, False, False},
399    FrameLabel -> {"Dimensionless time \[Tau]",
400      "Normalized Strain \[CurlyEpsilon](\[Xi],\[Tau])"},
401    Axes -> False,
402    (*
403    PlotStyle\[Rule]
404    {
405      {Darker[Green],Thickness[0.005]},
```

```
406      {Red,Thickness[0.005]},
407      {Darker[Blue],Thickness[0.005]},
408      {Orange,Thickness[0.005]}
409    },
410    *)
411    PlotLegends -> {"\[Xi]=0.1", "\[Xi]=0.3", "\[Xi]=0.5",
412      "\[Xi]=0.7"}];
413
414    (* =================== Optimizing the graphics =================== *)
415    gsl5 = Graphics[
416    {
417      Text
418      [
419      Style["\!\(\*OverscriptBox[\(\[ScriptL]\), \(_\)]\)=0.5",
420      FontFamily -> "Times New Roman",
421      FontSize -> 14], {115, 1}
422      ](*,
423      Thickness[0.005],
424      Green,
425      Line
426      [
427      {{110,1},{120,1}}
428      ]*)
429    }
430    ];
431
432
433    (* =================== Showing the graphics =================== *)
434    psl5 = Show[{psl5, gsl5}]
435
436    (* =================== Exporting the graphics =================== *)
437    Export["GV_Strain_lbar_.5_xi_changing.pdf", psl5]
438    (* ===================================
439    *)
440
```

```mathematica
(*
Author: Behrooz Karami
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Date: 2023-06-15
Gradient Thermoelasticity: Strain and Displacement Responses of 1D Bar.
*)


(*
Defining displacement responses.
*)
uoL1 = 1/3 \[Alpha]  (\[Xi]^2 - 1) \[Xi];
uoL2 = \[Alpha]  ((1/3 (\[Xi]^2 - 1) \[Xi]) +
2 *k^2 *(\[Xi] - Sinh[\[Xi]/k]/Sinh[1/k]));

p0 = Plot
[
{
  uoL1 /. \[Alpha] -> 0.1,
  uoL2 /. {\[Alpha] -> 0.1, k -> 0.2 },
  uoL2 /. {\[Alpha] -> 0.1, k -> 0.4}
},
{\[Xi], -1.0, 1.0},
Frame -> {True, True, True, True},
PlotRange -> {{-1, 1}, Automatic},
FrameLabel -> {" x/L", "Displacement (u)"},
PlotLegends -> {"\[Zeta]=0", "\[Zeta]=0.2", "\[Zeta]=0.4"}
]

Export["displacement.pdf", p0]


(*
Defining strain responses.
*)

\[CurlyEpsilon]Theta = - \[Alpha] (1 - \[Xi]^2 );
\[CurlyEpsilon]1 = \[Alpha]  (\[Xi]^2 - 1/3);
\[CurlyEpsilon]2 = \[Alpha] (\[Xi]^2 - 1/3 + 2 k^2 -
2 k Cosh [\[Xi]/k]/Sinh[1/k]);

(*
Specifying the parameters and plotting the strains.
*)

p1 = Plot
[
{
  \[CurlyEpsilon]1 /. \[Alpha] -> 0.1,
  \[CurlyEpsilon]2 /.  {\[Alpha] -> 0.1, k -> 0.2},
  \[CurlyEpsilon]2 /.  {\[Alpha] -> 0.1, k -> 0.4},
  {\[CurlyEpsilon]Theta /. \[Alpha] -> 0.1}
},
{\[Xi], -1.0, 1.0},
Frame -> {True, True, True, True},
PlotRange -> {{-1, 1}, Automatic},
FrameLabel -> {" x/L", "Strain \[CurlyEpsilon]"},
```

```
59   PlotLegends -> {"\[Zeta]=0", "\[Zeta]=0.2", "\[Zeta]=0.4",
60     "Imposed Strain"}
61   ]
62
63
64   p1 = Plot[
65   \[CurlyEpsilon]2 /.  {\[Alpha] -> 0.1,
66     k -> 0.2}, \[CurlyEpsilon]2 /.  {\[Alpha] -> 0.1, k -> 0.4},
67   \[CurlyEpsilon]Theta /. {\[Alpha] ->
68     0.1, \[CurlyEpsilon]1 /. \[Alpha] -> 0.1},
69   PlotLegends -> {"\[Zeta]=0", "\[Zeta]=0.2", "\[Zeta]=0.4",
70     "Imposed Strain"}, {\[Xi], -1, 1}]
71
72   Export["strain.pdf", p1];
```

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