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Cycle XXXIV

AN ANALYSIS OF EQUILIBRIA FOR NASH PROBLEMS, RADNER PROBLEMS AND MULTI-LEADER-FOLLOWER GAMES

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Abstract

This thesis aims to analyse some classes of non-cooperative games under different perspectives: the existence of competitive equilibria, stability of equilibria under parameter perturbations, and definition of promising strategies of players under different game settings.

In particular, the thesis will focus on generalised Nash equilibrium problems (GNEP), Radner equilibrium problems (REP) and bi-level optimisation models, such as multi-leader-single-follower games (MLSF) and single-leader-multi-follower games (SLMF), from both a theoretical and an applicative point of view. Several applications of these problems can be found in economics such as electricity market behaviour, industrial eco-parks, sequential trading exchange under uncertainty.

The first topic of this thesis is devoted to the generalisation of the paper by Bernhard von Stengel [92] to the case of $n+1$ players. In this paper, B. von Stengel provides optimal strategies for a duopoly game. In this thesis, the author defines an $(n+1)$-player game where a group of n players interacts in a non-cooperative way through a GNEP and a new player, namely player $n+1$, wants to enter the game.

More precisely, the author intends to evaluate the gap of the pay-off of $n+1$ players between two possible models: on the one hand, a non-cooperative model in which this player is one of the players of a Nash game (one-level game) and on the other hand a bi-level game in which this player plays the role of a common follower, so that the game will be a MLSF or player $n+1$ becomes a single leader and the game will be a SLMF. So the discussion on the best strategy to adopt will be focused on player $n+1$.

Indeed, this player can not influence the strategy of the group of other n players, but by choosing to play in the first or second period, he can change the nature of the game to improve his pay-off. The problem is first tackled in a general framework with the introduction of the new concept of *weighted generalised Nash game*, useful to obtain uniqueness of the equilibrium solution under mild conditions. Then it is applied to the case of the quadratic utility function with a feasible set composed by inequality constraints.

The second part of the thesis is devoted to the study of stability properties of parametrised Nash problems. This class of problems corresponds to a Nash problem in which parameters represent perturbations on the different players' objective functions and their strategy sets. It is well-known that variational inequality is a helpful tool to approach this problem. This work aims to analyse the impact of these perturbations on the set of Nash equilibria. A qualitative analysis is carried out, and properties of closedness and upper semi-continuity of the solution map are established.

Three distinct approaches are used, leading to three types of closedness results (direct, component-wise and an alternative approach). Comparisons between the three types of assumptions are stated, and an application to the more complex model of a single-leader-multi-follower game is also included.

Finally, the last part of the thesis deals with the conditions for the existence of equilibria of a Radner equilibrium problem. In this part, the target is to prove the existence of a Radner equilibrium for a sequential trading exchange with two

periods for which the players' utility functions are quasi-concave. This type of problem arises, for instance, when dealing with the stock exchange in the spot and future markets.

This problem can be treated by proving that the solution of the ad-hoc quasi-variational inequality problem is also a solution of REP. The problem is studied by using only component-wise assumptions for each agent and with the use of the property of "net-lower-sign" continuity, recently introduced by D. Aussel and co-authors in [11, 12]. Sufficient conditions to verify the recent concept of net-lower-sign continuity are also presented under some separability properties.

Riassunto

Lo scopo della presente tesi è quello di analizzare alcune categorie di giochi non cooperativi sotto diversi aspetti: esistenza di equilibri, stabilità degli stessi in presenza di perturbazioni e ricerca delle strategie ottimali per ogni categoria di giocatore.

In particolare vengono studiati giochi di Nash generalizzati (GNEP), problemi di equilibrio alla Radner (REP), problemi di ottimizzazione bilivello quali multi-leader single-follower games (MLSF) e single-leader multi-follower game (SLMF), sia da un punto di vista teorico che applicativo. Molte sono, infatti, le applicazioni reali di tali problemi, tra cui lo studio del comportamento del mercato elettrico, lo studio di giochi sequenziali in condizioni di incertezza.

La prima tematica affrontata nella tesi è una generalizzazione del lavoro di Bernhard von Stengel [92], che definisce le strategie ottimali in situazioni di duopolio, al caso di $n+1$ giocatori. In particolare si ipotizza che nel mercato vi siano n giocatori che stiano già interagendo in maniera non cooperativa e che un nuovo giocatore debba decidere se entrare nel mercato.

Più precisamente, si intende valutare il diverso pay-off del giocatore $n+1$ a seconda del tipo di modello individuato: da un lato un gioco non cooperativo in cui il giocatore è membro di un Nash (gioco ad un solo livello), dall'altro un gioco bilivello in cui il giocatore agisce da follower, quindi si avrà un gioco MLSF oppure da leader, generando un gioco SLMF. Il punto di vista che verrà utilizzato è quello, pertanto, del giocatore $n+1$, di cui si determinano le strategie ottimali.

Chiaramente il giocatore $n+1$ non può influenzare le strategie degli altri giocatori ma può decidere in quale periodo giocare cambiando la natura del gioco per migliorare il suo pay-off. Il problema viene prima affrontato in un contesto generale con l'introduzione di un nuovo concetto di "Weighted GNEP" utile per ottenere unicità dell'equilibrio del GNEP sotto condizioni non restrittive, poi declinato in un problema specifico con funzione di utilità quadratica e insieme di definizione costituito da disuguaglianze.

Il corpo centrale della tesi è invece dedicato allo studio delle proprietà di stabilità di problemi di Nash parametrizzati. Tale classe di problemi corrisponde ad un problema di Nash in cui i parametri rappresentano perturbazioni sulle funzioni obiettivo dei diversi giocatori e sui loro insiemi di strategie. È noto che le disuguaglianze variazionali sono uno strumento utile per affrontare questo problema. In questa tesi si propone di analizzare l'impatto di queste perturbazioni sull' insieme degli equilibri di Nash. In particolare si dimostrano proprietà di chiusura e upper semicontinuity della mappa delle soluzioni del problema di Nash parametrizzato.

Vengono, infine, utilizzati tre approcci distinti, che portano a tre tipi di risultati di chiusura (approccio diretto, per componenti e alternativo). Vengono condotti confronti tra i tre tipi di ipotesi e viene inclusa un'applicazione al modello più complesso di un gioco single-leader-multi-follower.

Nell'ultima parte della tesi, infine, viene analizzato un gioco sequenziale (REP) che considera due istanti di tempo e viene investigata l'esistenza di soluzioni di equilibrio con funzioni di utilità dei giocatori quasiconcave. Questo tipo di problemi sorgono, ad esempio, nel caso di negoziazione di titoli nel mercato spot e future.

Tale problema può essere studiato tramite l'utilizzo di disuguaglianze variazionali e quasi-variazionali, dimostrando che la soluzione di uno specifico problema variazionale è anche la soluzione di un REP. A tal fine viene utilizzata la proprietà di "net-lower-sign" continuity introdotta nel lavoro di Aussel et al [11, 12] e sotto alcune proprietà di separabilità vengono presentate anche condizioni sufficienti per verificarne la sua applicabilità.

Résumé

L'objectif de cette thèse est l'étude de certaines classes de jeux non coopératifs. Elle est réalisée selon différentes approches: existence d'équilibres, stabilité des équilibres soumis à perturbations des paramètres, stratégies optimales de joueurs placés dans différents contextes de jeu.

En particulier, la thèse se concentrera sur les problèmes d'équilibre de Nash généralisés (GNEP), les problèmes d'équilibre de Radner (REP) et les modèles d'optimisation à deux niveaux, tels que les jeux multi-leader-single-follower (MLSF) et les jeux single-leader-multi-follower (SLMF), aussi bien d'un point de vue théorique que d'un point de vue des applications. Plusieurs applications de ces problèmes peuvent être trouvées en économie telles que le comportement du marché de l'électricité, les éco-parcs industriels, les échanges commerciaux séquentiels dans l'incertitude.

Le premier thème abordé est celui de la généralisation des travaux de Bernhard von Stengel [92] au cas de $n+1$ joueurs. Dans cet article, B. von Stengel fournit des stratégies optimales pour un jeu de duopole. Dans cette thèse, l'auteur définit un jeu à $(n+1)$ joueurs où un groupe de n joueurs interagit de manière non-coopérative à travers un GNEP et un nouveau joueur, à savoir le joueur $n+1$, veut entrer dans le jeu.

L'auteur réalise une analyse de l'écart de gain des $n+1$ joueurs entre deux modèles possibles : d'une part un modèle non coopératif dans lequel ce nouveau joueur est l'un des joueurs d'un jeu de Nash (jeu à un niveau) et d'autre part un jeu à deux niveaux dans lequel ce joueur joue le rôle soit d'un suiveur commun, -de sorte que le jeu sera un MLSF- soit d'un leader unique et le jeu sera un SLMF. L'analyse sur la meilleure stratégie à adopter se concentrera sur le joueur $n+1$.

Ce joueur ne peut pas influencer la stratégie du groupe des n autres joueurs, mais en choisissant de jouer en première ou en deuxième période, il est capable de changer la nature du jeu pour améliorer son propre gain. Le problème est d'abord abordé dans un cadre général avec l'introduction du nouveau concept de *jeu de Nash généralisé pondéré*, utile pour obtenir l'unicité de la solution d'équilibre. Le cas où les fonctions d'utilité sont quadratiques et l'ensemble de contraintes est défini par des inégalités en ensuite considéré.

La deuxième partie de la thèse est consacrée à l'étude des propriétés de stabilité des problèmes de Nash paramétrés. Cette classe de problèmes correspond à un problème de Nash dans lequel les paramètres représentent des perturbations sur la fonction objectif des différents joueurs et sur leur ensemble de stratégies. Ce travail vise à analyser l'impact de ces perturbations sur l'ensemble des équilibres de Nash. Une analyse qualitative est effectuée et les propriétés de fermeture et de semi-continuité supérieure de la multi-application solution sont établies. Une approche par inégalité variationnelle est utilisé pour aborder ce problème.

Trois approches distinctes sont utilisées, conduisant à trois types de résultats de fermeture (directe, par composantes et une approche alternative). Des comparaisons entre les trois types d'hypothèses sont réalisées ainsi qu'une application au modèle plus complexe de jeu SLMF.

Enfin, la dernière partie de la thèse traite des conditions d'existence des équilibres de type Radner. Dans cette partie, l'objectif est de prouver l'existence d'un

équilibre de Radner pour un marché séquentiel à deux périodes pour lequel les fonctions d'utilité des joueurs sont quasi-concaves. Ce type de problème se pose, par exemple, pour la modélisation d'échanges boursiers sur les marchés au comptant et à terme.

Ce problème est traité grace à l'utilisation d'une inégalité quasi-variationnelle. Contrairement à des approches précédentes, les hypothèses portent uniquement sur les données de chaque agent (fonction objectif et contraintes) et en utilisant la propriété de continuité "net-lower-sign", récemment introduite par D. Aussel et al. dans [11, 12]. Sous certaines hypothèses de séparabilité, des conditions suffisantes assurant la continuité net-lower-sign sont également présentées.

Acknowledgement

Pursuing academic and scientific research, indeed, is not an easy path. However, it has a special meaning for me and is worth trying to step forward. What I heard, I saw, I learned and experienced during the past time has been honed through every word in this achievement.

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This moment has come, whether that...

BRESCIA, December 16th, 2021

It is a sincere thank those
who have contributed to my human forged.

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Chapter 1
Introduction

The topic of this thesis is, as already stated, *An analysis of equilibria for Nash problems, Radner problems and Multi-leader-follower games*. This thesis work has been done throughout the learning and research process of the Analytics of Economics and Management program, cycle XXXIV and with the special support of the doctoral schools at the University of Brescia in Italy(1) and the University of Perpignan in France(2), along with the strong cooperation with the research laboratory PROMES(3).

The clear common line of research of this thesis is non-cooperative games. Our aim and motivation have been to explore different aspects and models describing a non cooperation interaction between players/agents. Following the pioneering work of J.F. Nash [75], the so-called Nash approach is clearly the most well-known. Each player's optimization problem depends on the strategies of the other players in its objective function as well as in its strategy sets. To take into account non-cooperative situation where the execution of the decisions of the players is done in a second period and to integrate some incertainty on the state of the world at this second period, R. Radner [79] proposed another approach which is also considered in this thesis. Finally when some of the players have a hierarchical interaction with the other players then one leads to models where non cooperation Nash interactions mix with bilevel optimization problems. Such complex models are called Multi-Leader-Follower games.

This common research line is explored through different analytical aspects of non-cooperative game first by comparing the payoffs in different bi-level configurations (and thus to identify the best strategy for players), second to prove some stability/sensibility results for Nash and multi-leader-follower game, and finally to prove the existence of solution for Radner problem. In these problems, the focus has been put on determining existence and properties of equilibria. All of these problems have implications in practice, most clearly in economic competitive markets. As examples, electricity markets and industrial ecological parks are perfect applications for these models. With the goal of defining a "most acceptable" strategy for players, we try to clarify the settings of these problems for applying mathematical methods to find out solutions, and to state related properties as well. Obviously, finding a general solution is not always easy to be done, but we can define narrower problems and try to weaken the assumptions as much as possible.

This chapter gives a brief introduction to the thesis work. Section 1.1 describes some "real life" motivations. Section 1.2 contains three subsections in which the first topic of decision concept for two-period game is acquainted in Subsection 1.2.2. Then, Section 1.2.3 reveals the qualitative stability for the Nash game using an approach under component assumptions for the Nash game. Section 1.2.4 focus on the proof of existence of solution

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(3) Le laboratoire PROMES (Procédés, Matériaux Et Énergie Solaire) est une Unité Propre du CNRS (UPR 8521)

for sequential trading exchange model through Radner equilibrium concept. Each content points out the existing studies and related literature to emphasize the corresponding scope and innovation of this scientific research. Finally, the author summarizes the thesis structure in Section 1.3 to overview the entire work presented.

1.1 Real life motivation

Currently, the demand for energy consumption is one of the crucial issues of modern societies. In economics and industry, energy models are gaining much attention because of their urgency and importance. The energy domain is currently facing intense transformation. The energy crisis is a global challenge when it comes to harmonizing economic, environmental and administrative standards. Historically, many innovative approaches have been considered, creating a competitive advantage and increasing energy use efficiency. In addition to the old ways of interacting between producers and end consumers, new markets have been implemented for energy distribution on a larger and more diverse scale. Manufacturers do not always focus on building production facilities. Instead, these units can decide to buy/sell energy according to the plan to profit. On the other hand, they must also follow policies to regulate the energy flow. The most significant purpose is still to balance between the production and consumption needs of clients. Many models have emerged to deal with this, typically models of non-cooperative situations such as:

- Bidding process in day-ahead, energy price auction and adjustment electricity markets [8, 9, 13, 43, 53];
- Industrial eco-parks [80, 81];
- Demand-side management for energy exchanges (see, e.g. [10]);
- Economic models and transportation (see, e.g. [23, 24], [93]);
- Sequential trading exchange (see, e.g. [16]);
- and many other applications (see, e.g. [51] or [78]).

An essential feature of these models is the interaction of the different actors/players in the model. For example, companies interact with each other, producers and consumers, businesses and agents, etc. One can use non-cooperative games to represent that interaction, such as Nash games, multi-leader-follower games (MLF in short, see, e.g. [54, 30]) or Radner problems. These models are appropriate and very useful way for formulating mentioned scenarios to find a suitable solution for non-cooperative contextual participants in a situation where players must deploy independent strategies, trying to come up with a solution that is acceptable to all players is a challenge. In addition, sequential trading is often described thanks to Radner equilibrium problem (REP) (see, e.g. [37, 79, 82, 83]) when the decisions come up in different time-periods.

In many cases of mentioned references, the Nash game or Nash equilibrium problem (NEP) (see, e.g. [8, 9, 50, 88]) or its more general form known as generalized Nash equilibrium problem (GNEP) (see, e.g. [12, 13, 34, 44]) can be a bi-level or tri-level (thus hierarchical) game. Whenever the bi-level problem involves only one leader and several followers it leads to a single-leader-multi-follower (SLMF) game (see, e.g. [93]). And vice versa, it will be a multi-leader-single-follower (MLSF) game (see, e.g. [62]) when there is only one player on the lower level in the interaction with many players on the upper level. If there are multiple players on both levels in a hierarchical game, that definitely would be the multi-leader-multi-follower (MLMF) game (see, e.g. [59]).

1.2 Research scope

1.2.1 Non-cooperative game

One of the cornerstones among various applications, a non-cooperative game is a competition between individual players/participants. In general, they must compete independently and cannot group into alliances. This feature distinguishes non-cooperative games from cooperative ones. Let us briefly make some comparisons with cooperative games to see the difference between these two classes of games.

- Non-cooperative games generally analyse a framework that predicts individual strategies and payoffs of players with purpose to find Nash equilibria. It is the opposite of cooperative games that focus on predicting which groups/coalitions will be created and which will be the best joint actions for gaining collective payoffs.
- A distinction between the two sorts of games is the non-reaching a binding agreement in the non-cooperative game while it is possible in a cooperative game.
- In a non-cooperation framework, players will have to anticipate what their opponents will do in each case to react for stating their best strategy. Oppositely, a cooperation implies that agents cooperate to achieve a common goal in a coalition; each participant has own skills/potential to contribute strength to the group.
- Further, it has been supposed that the non-cooperative game is purported to survey the effect of independent determinations on community. In contrast, the cooperative theory focuses only on the effects of participants in a particular coalition attempting to improve collective welfare.

A Nash game is a non-cooperative game, but the opposite is not true. The Radner problem is also a non-cooperative game, but it considers additional constraints that simulate the players' contracts prepared in advance and will be executed in the future. It is helpful to use it to describe sequential processes. The term "sequential" highlights that the time factor is a feature in these problems. Depending on how the situation is set up, the more time points exist, the more complicated the situation will be. Besides the well-known Nash games and their derivatives, the Radner problem was introduced in this thesis as an adjunction to enrich aspects of non-cooperative games.

The generalized Nash game is likewise an enhancement of the Nash game, where the set of player strategies is a set-valued map. In other words, each player's decision at any point depends on decisions of the other players. Therefore, the difficulty also increases significantly. With the desire to understand the interaction among the players when they intend to behaviour independently, many kinds of games are investigated in different situations. These manner will be demonstrated precisely in the incoming subsections.

1.2.2 Decision concept for two-period game

Historically, the earliest *duopoly model* was introduced by the French economist A. A. Cournot (see [36], 1838). It states that if, in a two-player game, each player's strategy is completely independent on the other's strategy, they will get a lower profit than considering the opponent's choice through periods. Therefore, players should consider their interdependence which is defined as the best response to the opponent for improving their profits. Besides, the concept of the *leadership game* was introduced by H. von Stackelberg ([90, 91], 1932). With the same payoff functions, the game introduced is a sequential pattern in which the first mover is called the leader with a strategy based on the best response of the player playing after (follower). In the case that each player, regardless leader of follower, is playing a best response to the other player, the solution to this problem is a sub-game perfect equilibrium.

Many recent papers show that in order to solve *sequential games* they have to consider an endogenous timing problem. This is important because it determines the role of players being leaders or followers based on the point of time they join the game/make their decision. For instance, [3, 4] compare explicitly the follower payoff to the payoff the player would get as a leader in sequential play or as Nash player in simultaneous play. In the paper of Bernard von Stengel (see [92], 2010), a comparison of the leader and follower payoff in a duopoly game, whose payoffs arise in sequential play, is observed with the Nash payoff in simultaneous play. If the Nash game is symmetric, it will admit a unique symmetrical equilibrium and each player's payoff is monotonic in the opponent's choice along with their own best reply function. The conclusion is that the player's payoff as a follower is either higher than the leader payoff or even lower than in a simultaneous game. This gap for the possible follower payoff has not been observed in earlier duopoly models of endogenous timing. Then, these payoffs are compared with the conclusion that it is either better to be a leader in a sequential game or to play in a Nash game than becoming a follower.

In Chapter 3 of this thesis, the author develops von Stengel's idea for the scheme of multi-leader-follower games in order to analyse non-cooperative/hierarchical games between $n+1$ players. More precisely we intend to evaluate the gap of payoff of a player between two possible models: on one hand a non-cooperative model in which this player is one of the players of a Nash game (one-level game) and on the other hand a bi-level game in which this player plays the role of a common follower or common leader.

By defining this model as a two-period game, we have two situations similar to [92], that is sequential play and simultaneous play. Apart from the order of play, we distinguish two types of players, one is a group of n players, the other is the $(n+1)^{\text{th}}$ independent player. Here, there appears three specific cases, that are:

- generalized Nash equilibrium problem for $n+1$ players, together giving the strategy in same phase/period;
- secondly, the group of n players choose phase 1 to go and the $(n+1)^{\text{th}}$ player chooses phase 2 (in other words, this is the multi-leader-single-follower game);
- and finally a single-leader-multi-follower game when the $(n+1)^{\text{th}}$ player chooses to

play in phase 1 and becomes the leader while the rival group plays in the other phase.

Based on this, possible outcomes for the payoff of the $(n+1)^{\text{th}}$ player have been studied. The aim is to propose a decision making policy in order to advice the new player in taking a strategy to play in period 1 or 2 by which it generates the above three situations (SLMF, GNEP and MLSF). Of course, this strategic decision of player $n+1$ must be chosen without knowing which period the group will prefer to play.

In other words, the main contribution of this work is to define the most favourable plan for player $n+1$ without knowing the strategies of the other players, namely to play in period 1 or 2. The cases in which player $n+1$ has perfect information about how the group will react is also considered and results are provided in extended corollaries at the end of this topic.

The problem examined here is not only controlled by the endogenous timing factor (meaning that the chronological order of the strategy also affects the outcome of the player) but also the predicted choice between one-level Nash and bi-level game. Different from von Stengel's orientation, which only focuses on the behaviour of the follower (due to symmetry), we try to clarify the possibilities of the $(n+1)^{\text{th}}$ player depending on the situations. Player $n+1$, indeed, can not influence the strategy of the group of other n players, but by choosing to play in first or second period he is able to change the nature of the game that becomes SLMF, MLSF or GNEP for improving further his own payoff. And thus, it is a determinant for strategic decisions of player $n+1$ when he decides to take part in the game. In order to do so and assuming that the utility function of each player i is strictly concave, the author investigates under which conditions they can guarantee the optimal payoff to player $n+1$. It depends on the order of playing (1 or 2) and on the information that one can get from the cost structure of the opponent group since a generalized Nash game was already played between them.

It is crucial to mention that in the duopoly presented in [92], it is necessary to require symmetry and monotonicity assumptions. In the case of $n+1$ players, these assumptions cannot be used to guarantee the uniqueness of the equilibrium in a Nash game. First, the symmetry property will not be preserved when extended to the multi-player problem, namely that $n+1$ players are divided into two unequal parts, including one group and one individual. The group and the individual will behave differently. Furthermore, the way a single player reacts to a group will be different from how one group member has to respond to the other $n-1$ members in the same group and the new player $n+1$ at the same time. Second, von Stengel assumed monotonicity to the best reply functions for two players. As a consequence, a Nash equilibrium of duopoly, if existing, would be unique. It is unlikely in our situation because the solution of an n-player Nash game is rarely unique, so are the multi-leader-follower games. This fact causes a lot of ambiguity and thus leads the author to introduce the concept of weighted Nash equilibrium. Herein, the weighted approach will consider a particular case, achieving the uniqueness of equilibrium for $(n+1)$-player games.

So as to present the obtained results in a systematic way and in an *-as far as possible-easy* approach, the topic is divided into two parts: at first, the general settings is defined, then a classification of decisions for player $n+1$ is proposed and moreover the new concept of weighted generalized Nash equilibrium is introduced. This notion allows to obtain uniqueness of generalized Nash equilibrium under mild hypothesis; The second part of this chapter is devoted to the study of a particular setting where the utility function is a concave quadratic function and the constraint set is defined by inequalities. In this context, a complete decision making policy is developed.

1.2.3 Qualitative stability in direct approach for Nash problem

A parametrized Nash equilibrium problem is a non-cooperative game with multiple players and with parameters affecting objective functions and strategy sets. These "parametrized terms" can be understood as perturbations coming from exogenous causes: fluctuation of prices, of initial endowments, etc. Thus the solution set which is the set of Nash equilibria, naturally depends on the parameters. Studying how this set evolutes when the parameters variate is known as the stability analysis of the parametrized. *Quantitative stability* consists in evaluating upper bounds of the "distance" between two solution sets in terms of the norm of the difference between the corresponding parameters, for example through Lipschitz or Hölder-type analysis. Interested readers can consult [1] for a case of quantitative stability for variational inequalities. Specifically, for a parametrized Nash game $\text{NEP}_{\lambda,\gamma}$ associated to perturbed parameters λ and γ, by locating the Hausdorff distance between 2 sets, one can apply for estimating the distance between 2 solution sets on changing parameters. With this approach, a solution map of $\text{NEP}_{\lambda,\gamma}$ is concluded "quantitative stable" if the distance is bounded by the norm with respect to perturbations λ and γ.

In Chapter 4 of this thesis, we focus on the so-called *qualitative stability* of parametrized Nash games. The aim is therefore to consider the solution (set-valued) map of the parametrized Nash equilibrium problem through a variational inequality setting and its semi-continuity properties, namely closedness and upper semi-continuity.

In [2, 15, 14], stability results for the solution set of the parametrized problems were not only developed for variational inequality VI but also extended to quasi-variational inequality (QVI). In these references, monotonicity-type hypotheses play an important role, such as quasi-monotonicity and quasi-convexity. While in [63, 64, 65, 66, 72, 73] stability analysis has been conducted using vast of lower/upper semi-continuity and pseudo-convexity assumptions. Particularly, in [66], B. Morgan et al. introduced numerous forms of parametric Nash games and parametric variational inequalities, and clearly showed the connection in transforming the two types of problems.

Three different approaches leading to three sets of hypotheses for observing the closedness of solution set are proposed. First, the qualitative stability analysis is operated through a reformulation of the parametrized Nash equilibrium problem into a parametrized Stampacchia variational inequality. Quasi-monotonicity and quasi-convexity assumptions are applied for the following two different approaches: the first one assuming the quasi-monotonicity of the product of "generalized derivatives" coming from players' cost functions, while in the second one the key-role hypothesis is the semi-strict quasi-convexity to the cost function of each player. Afterwards, an alternative approach, which is used to obtain the closedness by defining a new style of solution map, is reformulated as an intersection of specific sub-maps. Eventually, using results of parametrized NEP, the author proves the upper semi-continuity of the marginal function for the leader in a single-leader-multi-follower game.

The first sections of this subject provides some conventions and basic notions regarding parametrized Nash game. Next, the first qualitative stability result is proved by using a variational inequality reformulation. The main stability result (Theorem 4.3.1) is shown in the subsequent section using a component-wise set of assumptions. Then, an alternative type of closedness result for the solution map of the parametrized game is concerned and there is a comparisons between the three approaches and other existing results. Finally, it comes up an application to single-leader-multi-follower game basing on claimed consequences.

1.2.4 An existence of solution for Radner problem

In the long history of the development of competitive games, the most famous is probably the Nash game published in John Nash's article, "Non-cooperative games" in 1951 (see [75]). Thanks to this platform, a vast series of surveys on non-cooperative situations emerge. In a Nash game, the key is to find an equilibrium that satisfies the acceptable requirements of all players, whose strategies are separated but can be influenced by the decisions of other players. It is this feature that makes the Nash equilibria suitable for describing competitive situations.

The Arrow-Debreu model (1954, see [5]), named after the Nobel laureates Kenneth Arrow and Gerard Debreu, is also a well-known pattern in the economic framework. It is a formalized Walrasian economic equilibrium system and it is used to prove the existence of competitive equilibria (see, e.g. [67, 57, 60]). By starting from standard Arrow-Debreu general equilibrium, Radner equilibrium problem (REP) has been introduced as "Competitive equilibrium under uncertainty" (see [79]) by the microeconomic theorist Roy Radner in 1968 to add some extra conditions intended to reflect the real-world economy. It is a consistent approach to tackle the context of general equilibria for incomplete markets framework. Namely, at the time of making a decision, people gain imperfect information about their own outcome and the outcome of the other players. It is usually used to examine the competitive problem under uncertainty to detect equilibrium conditions and describe the real-world existence of financial institutions and markets, for instance, stock or money exchanges. In REP, players are divided into producers and consumers, who make production plans and make consumption plans, respectively. All plans are established in an initial (current) time under imperfect information with respect to the strategies of other participants. Players all know the external conditions that may lead to distinct scenarios of their goals and the preferences for those outcomes in a second (future) time. Roughly speaking, participants need to prepare ahead for all future circumstances, but they have to make simultaneous decisions right now. Understandably, this is a sequential process and is often referred to as sequential trading (exchange). Some examples of sequential trading can be found in the following papers, [48, 47].

Let us define the Radner equilibrium problem more in detail. Considers a situation with multiple assets, goods/commodities, periods and multiple states. The assets can be real estate, money, agricultural products or any valuable possession. It is often treated as a special kind of commodity for exchanging. Commodities can be anything and not necessarily of the same type. Regarding periods, it is theoretically possible to construct a multi-period problem, however for simplicity, a two-period structure is preferred to use, thus time $t \in \{0, 1\}$. In parallel, multiple states imply what is called uncertainty.

Note that the markets will close or open depending on different situations of distinct commodities. Indeed, they are not all available to be traded at any period and most state-contingent commodities cannot be traded. Yet, as claimed by Arrow and proved by Radner, at $t = 0$, even if not all state-contingent commodities are available for trading, an equilibrium can be attained when markets re-opened at time $t = 1$. The state would be reactivated, and trading occurs if players predict accurately at $t = 0$. It is not required to have a complete set of state-contingent forward contracts but only its subset.

In [16, Aussel et al.], an application to the Radner problem was presented. Accordingly, the existence of equilibrium has been proven for two cases by assuming differentiability and non-differentiability. A technique by utilizing an existence result of VI in a very specific case to deal with QVI is deployed to obtain equilibrium existence in REP. Instead of using the VI approach, in Chapter 5 of this thesis, a different method on QVI is provided along with a more general form of utility function. A new-type assumption introduced freshly in [11], which is the *net-lower-sign continuity* is adopted for dealing with a weaker assumption. Nonetheless, the assumption is constructed in terms of “flexibly adapted” but is quite technical and not so easily verified. Thanks to [12], some “more natural” conditions at which the net-lower-sign continuity at least can be confirmed, are also provided.

The structure of this chapter is as follows. A complete description of sequential trading exchange is provided. Then a link between the sequential exchange and a particular type of QVI for obtaining a solution is stated. Under mild assumptions for each player, the main result (Theorem 5.3.3) is devoted to the core proof for the existence of Radner equilibrium. In the end, a proposal of a set of hypotheses allows handling the net-lower-sign continuity property more convincingly.

1.3 Thesis structure

As introduced in the previous section, the content of the whole work covers different aspects all dealing with the main topic of the non-cooperative game. The results presented in manuscript have been also submitted for publication in the following papers.

1) The couple of papers on *Strategic decision in a two-period game using a multi-leader-follower approach*
Part 1 - General setting and weighted Nash equilibrium
Part 2 - Decision making for the new player
which have been submitted in 2021 [23, 24]).

2) *Qualitative stability for solution map of parametrized Nash equilibrium problem and application* which have been submitted in 2021 ([21]).
3) *Existence of Radner equilibrium for a sequential trading problem with quasi-concave utility functions* which have been submitted in 2021 ([22]).

The rest of the thesis is organized as follows. Chapter 2 specifies notations and concept used throughout the manuscript.

Thereafter, Chapter 3 focuses on multi-player two-period game. For the sake of comparison with above referred articles, Sections 3.1 to 3.2 is based on [23] and is called Part 1. Sections 3.3 - 3.5 are Part 2 and is based on [24]. The concepts of weighted bounds and weighted Nash game for this extended model is introduced to establish the uniqueness of equilibrium. Thereby, a notion of decision choice follows to find the safest outcome for the new player. Some simulations are included to clarify the upshot and to draw some remarks from the theory.

Chapter 4 continues with a qualitative analysis of the solution set of Nash equilibrium problem in a parametrized form and of single-leader-multi-follower game. Finding sufficient conditions, splitting of assumptions for each component, using different approaches and giving counterexamples constitute the aims of this chapter. In addition, a comparison of these results with related studies is also proposed to evaluate the advantages and disadvantages of the proposed methods.

In Chapter 5, an economic problem of sequential trading exchange is considered in a market with two time points using the concept of Radner equilibrium problem. This chapter investigates the existence of solution for quasi-concave objective function. With the same method as in Chapter 4, that is assuming component-wise hypotheses, solution of Radner equilibrium is proven to exist under an assumption of net-lower-sign continuous of a specific pair of set-valued maps. A particular separability structure of both objective function and constraint set is also considered.

Finally, the last Chapter 6 is a complete synthesis of the presented results. This thesis is closed by conclusions and suggestions for future work.

Chapter 2

Preliminary and existing result

Before getting further, let us introduce some fundamental concepts and used notations.

2.1 Basic notations

In this section, we collect the foundational notions which will be used throughout the thesis. For any $x,\ y \in \mathbb{R}^p$ the notation $[x, y]$ stands for $\{tx + (1-t)y : t \in [0,1]\}$ and the segments $]x, y],\ [x, y[,\ ]x, y[$ are defined similarly with $t \in]0,1], t \in [0,1[, t \in]0,1[$, respectively.

For any given non-empty subset $A \subset \mathbb{R}^p$ and arbitrary point $x \in \mathbb{R}^p$, the distance from the point to the subset is denoted by $\mathrm{d}(x, A) = \inf\{\|x - y\| : y \in A\}$. For $x \in \mathbb{R}^p$ and $\rho > 0$, we denote by $\mathcal{B}(x, \rho)$, $\bar{\mathcal{B}}(x, \rho)$ and $\mathcal{S}(x, \rho)$ respectively, the open ball, the closed ball and the sphere of centre x and radius ρ. Also, we denote the closed unit ball and the unit sphere of $\mathbb{R}^p$ by $\bar{\mathcal{B}}_\rho$ and $\mathcal{S}_\rho$, respectively.

For any given subsets X and Y of $\mathbb{R}^p$, set-valued maps will be denoted by $T : X \longrightarrow 2^Y$, where 2^Y is the set of all subsets of the vector space Y. In other words, for any given $x \in X$ then $T(x)$ is a subset of Y. For flexibility, we can use a variant notation of set-valued map T as $X \rightrightarrows Y$ for the same meaning without confusion. Naturally, whenever the set-valued map is single valued, that means $T(x)$ is a singleton for any x, then T becomes a function. We use $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ and $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ to represent classical functions and set-valued maps. The domain, graph and sets of minimums/maximums for set-valued maps and functions are described as follows.

$$\operatorname{dom} T = \{x \in X : T(x) \neq \emptyset\}, \qquad \operatorname{gr} T = \{(x, y) \in X \times Y : y \in T(x)\},$$
$$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}, \qquad \operatorname{gr} f = \{(x, y) \in X \times Y : y = f(x)\},$$
$$\operatorname*{argmin}_X f = \{x \in X : f(x) = \min_X f\}, \qquad \operatorname*{argmax}_X f = \{x \in X : f(x) = \max_X f\}.$$

Given a non-empty subset $A \subset \mathbb{R}^p$, the subsets $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$, $\mathrm{conv}(A)$ and $\mathrm{cone}(A)$ denote respectively for the closure, the interior, the boundary, the convex hull and the conical hull of a subset A, that is, more precisely:

- $\mathrm{cl}(A) = \{x \in A : \mathcal{B}(x, \epsilon) \cap A \neq \emptyset,\ \text{for any } \epsilon > 0\}$,
- $\mathrm{int}(A) = \{x \in A : \mathcal{B}(x, \epsilon) \subset A,\ \text{for some } \epsilon > 0\}$,
- $\mathrm{bd}(A) = \mathrm{cl}(A) \setminus \mathrm{int}(A)$,
- $\mathrm{conv}(A) = \left\{\sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in [0,1],\ x_i \in A,\ n \in \mathbb{N}\right\}$,
- $\mathrm{cone}(A) = \left\{\sum_{i=1}^{n} \lambda_i x_i :\ \lambda_i \geq 0,\ x_i \in A,\ n \in \mathbb{N}\right\} = \mathbb{R}_+(A)$.

2.2 Continuity and set convergence

2.2.1 Continuity

Let us now recall some well-known continuity concepts for both function and set-valued map.

Definition 2.2.1 ([7], Continuity for functions)**.** A function $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be

i) *continuous* at $x \in \operatorname{dom} f$ if for any neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(u) \in V$, for any $u \in U$;

ii) *lower semi-continuous* at $x \in \operatorname{dom} f$ if for any sequence $(x_n)_n \subset \operatorname{dom} f$ converging to x, one has $f(x) \leq \liminf_{n\to\infty} f(x_n)$;

iii) *upper semi-continuous* at $x \in \operatorname{dom} f$ if for any sequence $(x_n)_n \subset \operatorname{dom} f$ converging to x, one has $f(x) \geq \limsup_{n\to\infty} f(x_n)$.

Definition 2.2.2 ([7], Continuity for set-valued map)**.** A set-valued map $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is said to be

i) *lower semi-continuous* at $x^0 \in \operatorname{dom} T$ if for any sequence $(x^k)_k$ of $\mathbb{R}^p$ converging to x^0, and any element y^0 of $T(x^0)$, there exists a sequence $(y^k)_k$ of $\mathbb{R}^p$ converging to y^0 such that $y^k \in T(x^k)$, for any k;

ii) *upper semi-continuous* at $x^0 \in \operatorname{dom} T$ if for any neighbourhood V of $T(x^0)$, there exists a neighbourhood U of x^0 such that $T(U) \subset V$;

iii) *closed* at $x^0 \in \operatorname{dom} T$ if for any sequence $(x^k, y^k)_k \subset \operatorname{gr} T$ converging to (x^0, y^0), one has $(x^0, y^0) \in \operatorname{gr} T$.

Nevertheless it is well known that upper semi-continuity is not adapted to conical-valued set-valued maps. A pertinent and weak concept of semi-continuity (global and local form) for such conical-valued set-valued map has been defined in [49] and specifically used in [15].

Definition 2.2.3 ([11], Upper-sign continuity)**.** Let K be a non-empty convex subset of $\mathbb{R}^p$ and let $T : K \rightrightarrows \mathbb{R}^p$ be a set-valued map with non-empty values. We say that T is

i) *upper-sign continuous* on K if for every $u,\ v \in K$, the following implication holds,

$$\left(\forall t \in]0,1[, \quad \inf_{u_t^* \in T(u_t)} \langle u_t^*, v - u\rangle \geq 0\right) \Longrightarrow \sup_{u^* \in T(u)} \langle u^*, v-u\rangle \geq 0,$$

where $u_t := (1-t)u + tv$;

ii) *locally upper-sign continuous* on K if for every $u \in K$, there exists a convex neighbourhood V_u and an upper-sign continuous map $\varphi_u : V_u \cap K \rightrightarrows \mathbb{R}^N$ with non-empty convex compact values satisfying that $\varphi_u(v) \subseteq T(v)\backslash\{0\}$, for all $v \in V_u \cap K$.

Let us observe that, due to the condition that 0 is not element of the sub-map $\varphi_u(v)$, upper sign-continuity of a set-valued map does not imply in general its locally upper sign-continuity. Nevertheless, if $0 \notin T(u)$ for each $u \in K$ and if T has non-empty convex values, then upper sign-continuity implies locally upper sign-continuity. It is also important to emphasize that these two concepts of semi-continuity are weak, that is clearly implied by upper semi-continuity and even by hemi-continuity.

2.2.2 Mosco convergence

The concept of Mosco convergence of a sequence of subsets has been introduced in [74] to study the convergence property of the solution sets of various variational inequalities and quasi-variational inequalities.

Let us first recall, for any sequence $(S^k)_k$ of subsets of $\mathbb{R}^p$, the definitions of *lower limit* and *upper limit* in the sense of Kuratowski (see, e.g. [6]),

$$\operatorname*{Liminf}_{k\to\infty} \; S^k = \big\{x \in X \;:\; \lim_{k\to\infty} \; \mathrm{d}(x, S^k) = 0\big\},$$
$$\operatorname*{Limsup}_{k\to\infty} \; S^k = \big\{x \in X \;:\; \liminf_{k\to\infty} \; \mathrm{d}(x, S^k) = 0\big\}.$$

An equivalent way to define these limits of sets is to say that $\operatorname{Liminf}_k S^k$ is the set of limits of sequences $(x^k)_k$ with $x^k \in S^k$, for any k, while the upper limit $\operatorname{Limsup}_k S^k$ is the set of cluster points of such sequences $(x^k)_k$. From the definitions, it is clear that the sets $\operatorname{Liminf}_k S^k$ and $\operatorname{Limsup}_k S^k$ are closed set. Thus, these lower and upper limits can be alternatively expressed as:

$$x \in \operatorname*{Liminf}_{k\to\infty} S^k \iff \exists (x^k)_k \subset \mathbb{R}^p \text{ such that } x = \lim_{k\to\infty} x^k \text{ and } x^k \in S^k, \; \forall k,$$
$$x \in \operatorname*{Limsup}_{k\to\infty} S^k \iff \left\{ \begin{array}{c} \exists (S^{k_t})_k \text{ subsequence of } (S^k)_k \text{ and } \exists (x^{k_t})_k \subset \mathbb{R}^p \\ \text{such that } x = \lim_{m\to\infty} x^{k_t} \text{ and } x^{k_t} \in S^{k_t}, \; \forall t. \end{array} \right.$$

Definition 2.2.4 ([74], Mosco convergence)**.** Let S be a subsets $\mathbb{R}^p$ and $(S^k)_k$ be a sequence of subsets of $\mathbb{R}^p$. We say that the sequence $(S^k)_k$ converges to a subset S in the sense of Mosco, if both of the following inclusions hold,

$$\operatorname*{Limsup}_{k\to\infty} \; S^k \subset S \qquad \text{and} \qquad S \supset \operatorname*{Liminf}_{k\to\infty} \; S^k.$$

For sake of convenience, the Mosco convergence of the sequence of sets S^k to a set S will be denoted by $S^k \xrightarrow[k\to\infty]{\text{Mosco}} S$. Moreover, for a sequence of sets $(S^k)_k$ and a sequence of set-valued maps $T^k : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$, we define the Mosco convergence of sequence of $\left(T^k\right)_k$ with respect to the sequence $\left(S^k\right)_k$ as follows:

$$\operatorname*{Limsup}_{S^k \ni x^k \to x} T^k(x^k) := \left\{ y \in \mathbb{R}^p : \begin{array}{c} \exists (S^{k_t})_t, \text{ a selection } (x^{k_t})_t \text{ with } x^{k_t} \in S^{k_t}, \\ \text{and a sequence } (y^{k_t})_t \text{ with } y^{k_t} \in T^{k_t}(x^{k_t}), \\ \text{such that } x^{k_t} \longrightarrow x, \; y^{k_t} \longrightarrow y \end{array} \right\}.$$

Let us end this section by recalling from [2] the following equivalent reformulation of Mosco convergence adapted to the case of subsets with non-empty interior. It will play an important role in the proofs of some results provided in the next chapters.

Proposition 2.2.5 ([2], Mosco convergence property)**.** Let $(S^k)_k$ be a sequence of convex subsets of $\mathbb{R}^p$ such that $\operatorname{int}(S^k) \neq \emptyset$, for any k. Let $S \subset \mathbb{R}^p$ be such that $\operatorname{int}(S) \neq \emptyset$. Then the following are equivalent.

i) The sequence $(S^k)_k$ Mosco-converges to S;

ii) $\mathrm{Limsup}_k\, S^k \subset S$, $\mathrm{int}(S) \subset \mathrm{Liminf}\,\mathrm{int}(S^k)$ and S is convex.

2.3 Quasi-convex optimization

2.3.1 Convexity and sub-level sets

Let us recall some basic notions of convexity.

Definition 2.3.1 (Set convexity)**.** A set K is convex if the line segment between any two points in K lies in K, that means for any $x_1,\ x_2 \in K$, for all $t \in [0,1]$,

$$tx_1 + (1-t)x_2 \in K.$$

Definition 2.3.2 (Convexity of functions)**.** A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be

i) *convex* if $\mathrm{dom}\, f$ is a convex set and for any $x = (x_1, x_2) \in \mathrm{dom}\, f$, for all $t \in [0,1]$ one has

$$f\big(tx_1 + (1-t)x_2\big) \leq tf(x_1) + (1-t)f(x_2);$$

ii) *strictly convex* if the previous inequality holds strictly, i.e.

$$f\big(tx_1 + (1-t)x_2\big) < tf(x_1) + (1-t)f(x_2);$$

for any $x = (x_1, x_2) \in \mathrm{dom}\, f$, $x_1 \neq x_2$ and for all $t \in]0,1[$.

iii) *σ-strongly convex* if $f - \frac{\sigma}{2}\|\cdot\|_2^2$ is convex, i.e.

$$f\big(tx_1 + (1-t)x_2\big) \leq tf(x_1) + (1-t)f(x_2) - \tfrac{\sigma}{2}t(1-t)\|x_1 - x_2\|^2.$$

for any $x = (x_1, x_2) \in \mathrm{dom}\, f$, $x_1 \neq x_2$ and for all $t \in]0,1[$.

The epigraph of function f is the set of points $epi(f) = \big\{(x,y) : x \in \mathrm{dom}\, f,\ y \geq f(x)\big\}$. It is well known that a function is convex if and only if its epigraph is convex.

For a proper function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\} \neq \emptyset$, let us now recall the concepts of sub-level set and strict sub-level sets.

Definition 2.3.3 ([7], Sub-level set)**.** For any $\alpha \in \mathbb{R}$, the sub-level set $S_\alpha(f)$ and the strict sub-level set $S^<_\alpha(f)$ associated with f and α can be defined as

$$S_\alpha(f) = \big\{x \in X : f(x) \leq \alpha\big\} \quad \text{and} \quad S^<_\alpha(f) = \big\{x \in X : f(x) < \alpha\big\}.$$

A classical assumption in economics (see, e.g. [5, 67]) is to assume that the cost functions θ_i are quasi-convex. The different classical notions of quasi-convexity are provided in the following definitions.

Definition 2.3.4 (Quasi-convexity)**.** A function f is said to be

i) *quasi-convex* on a convex subset $K \subset \mathbb{R}^p$ if for any $x,\ y \in K$ and any $t \in [0,1]$,

$$f\big(tx + (1-t)y\big) \leq \max\big\{f(x), f(y)\big\};$$

ii) *semi-strictly quasi-convex* on a convex subset $K \subset \mathbb{R}^p$ if f is quasi-convex on K and for any $x,\ y \in K$,

$$f(x) < f(y) \implies f(z) < f(y),\ \forall z \in [x, y[.$$

iii) *strictly quasi-convex* on a convex subset $K \subset \operatorname{dom} f$ if for any $x,\ y \in K$, $x \neq y$ and any $t \in]0, 1[$,

$$f(tx + (1-t)y) < \max\{f(x), f(y)\}.$$

It is clear that the strict quasi-convexity implies the semi-strict quasi-convexity, which induces, by definition, the quasi-convexity.

Roughly speaking a semi-strict quasi-convex function is a quasi-convex function such that its graph doesn't contain any full dimensional flat part. The concept of quasi-convex function is geometrical by nature. An equivalent and useful characterization of quasi-convexity is that the function f is quasi-convex on $\operatorname{dom} f$ if and only if its sub-level set $S_\alpha(f)$ is convex, for any $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f \text{ is quasi-convex on } \operatorname{dom} f &\iff \forall \alpha \in \mathbb{R},\ S_\alpha(f) \text{ is a convex subset} \\ &\iff \forall \alpha \in \mathbb{R},\ S_\alpha^<(f) \text{ is a convex subset.} \end{aligned}$$

Note that (see, e.g. [7]), a lower semi-continuous semi-strictly quasi-convex function f satisfies the following property

$$\forall \alpha > \inf_X f, \quad \operatorname{cl}\left(S_\alpha^<(f)\right) = S_\alpha(f).$$

Definition 2.3.5 (Pseudo-convexity)**.** Let now X be an open set, not necessarily convex, in $\mathbb{R}^p$, and f be differentiable and denote by ∇f its gradient. Then, f is called

$$\textit{pseudo-convex on } X \text{ iff} \quad \forall x, y \in X,\ \nabla f(x)(y-x) \geq 0 \implies f(y) \geq f(x).$$

It is well known that the concept of pseudo-convex is inspired by the fact that for this type of function any local minimum is a global minimum. Nevertheless pseudo-convexity is also clearly a too strong hypothesis for many applications.

A common concavity property in economic models, often used to investigate the uniqueness of solutions, is recalled as follows.

Definition 2.3.6 ([86], Diagonally strictly concavity)**.** A function $f(x, r)$ will be called *diagonally strictly concave* for $x \in \mathbb{R}$ and fixed $r \geq 0$ if for every $x^0,\ x^1 \in \mathbb{R}$ one has

$$(x^1 - x^0)' g(x^0, r) + (x^0 - x^1)' g(x^1, r) > 0,$$

where $g(x, r)$ is the pseudo-gradient of $f(x, r)$.

2.3.2 Normal Operator

In the following, we shall deal with the concept of normal cone and normal operator.

Definition 2.3.7 ([7], Normal cone)**.** For any convex subset $A \subset \mathbb{R}^p$, $N_A(x)$ stands for the *normal cone* to A at point x that is,

$$N_A(x) = \left\{x^* \in \mathbb{R}^p : \langle x^*, u - x\rangle \leq 0,\ \forall u \in A\right\}.$$

Similarly, one can define the so-called *normal operator* and *strict normal operator* of the function f at a point x by

$$N_f(x) = \{x^* \in \mathbb{R}^N : \langle x^*, y - x\rangle \le 0, \quad \forall y \in S_f(x)\},$$
$$N_f^<(x) = \{x^* \in \mathbb{R}^N : \langle x^*, y - x\rangle \le 0, \quad \forall y \in S_f^<(x)\},$$

for every $x \in \operatorname{dom} f$, while we set $N_f(x) = N_f^<(x) = \emptyset$ for $x \notin \operatorname{dom} f$. Equivalently, $x^* \in N_f(x)$ $\big($resp. $x^* \in N_f^<(x)\big)$ if and only if the following implication holds respectively

$$\langle x^*, y - x\rangle > 0 \implies f(y) > f(x); \qquad (\langle x^*, y - x\rangle > 0 \implies f(y) \ge f(x)).$$

As shown in Crouzeix [32], under mild assumptions, the normal operator is quasi-monotone while the adjusted normal operator is cone-upper semi-continuous. Rebounding on this work, Aussel-Hadjisavvas defined in [20] the concept of *adjusted normal operator*, which satisfies both of these properties (cone-upper semi-continuity and quasi-monotonicity). It is actually based on the so-called *adjusted sub-level sets*.

Definition 2.3.8 ([7], Adjusted sub-level set)**.** Let f be a real-valued function defined on $\mathbb{R}^p$ and $x \in \operatorname{dom} f$. The *adjusted sub-level set* $S_f^a(x)$ of f at x is defined by

$$S_f^a(x) = \begin{cases} S_f(x) \cap \overline{\mathcal{B}}\big(S_f^<(x), \rho_x\big) & \text{if } x \notin \operatorname{argmin} f, \\ S_f(x) & \text{otherwise,} \end{cases}$$

where, for any $x \in \operatorname{dom} f \setminus \operatorname{argmin} f$, ρ_x stands for the positive real number $\rho_x = \mathrm{d}\left(x, S_f^<(x)\right)$.

Clearly x is always an element of $S_f^a(x)$. If $x \in \operatorname{dom} f \setminus \operatorname{argmin} f$ is such that $\rho_x = 0$, then $S_f^a(x) = S_f^a(x) \cap \overline{S_f^<(x)}$; if, moreover, f is lower semi-continuous on $\operatorname{dom} f$, then $S_f^a(x) = \overline{S_f^<(x)}$.

The *adjusted normal operator* is simply the normal cone at x to the adjusted sub-level set.

Definition 2.3.9 ([7], Adjusted normal operator)**.** The *adjusted normal operator* of the real-valued function f defined on $\mathbb{R}^p$ is the set-valued map $N_f^a : \operatorname{dom} f \rightrightarrows \mathbb{R}^p$ defined for any $x \in \operatorname{dom} f$ as the normal cone to the adjusted sub-level set $S_f^a(x)$ at x, that is,

$$N_f^a(x) = \{x^* \in \mathbb{R}^p : \langle x^*, y - x\rangle \le 0, \quad \forall y \in S_f^a(x)\}.$$

Note that since $S_f^<(x) \subseteq S_f^a(x) \subseteq S_f(x)$, one immediately has $N(x) \subseteq N_f^a(x) \subseteq N^<(x)$. From the above definition, it is clear that the adjusted normal operator of f at x is simply the polar cone to the set $\left(S_f^a(x) - \{x\}\right)$. Let us recall that the notion of dual and polar cone are given as follows:

i) A *dual cone* C^* of a subset $C \in \mathbb{R}^p$ is the set

$$C^* = \{y \in \mathbb{R}^p : \langle y, x\rangle \ge 0, \forall x \in C\};$$

ii) A *polar set* S° of a set $S \in \mathbb{R}^p$ is defined as

$$S^\circ = \{y \in \mathbb{R}^p : \langle y, x\rangle \le 1, \ \forall x \in S\};$$

iii) A *polar cone* C° of a subset $C \in \mathbb{R}^p$ is the set

$$C^o = \{y \in \mathbb{R}^p : \langle y, x\rangle \le 0, \forall x \in C\}.$$

Obviously, the polar cone is equal to the negative of the dual cone, i.e. $C^\circ = -C^*$. For a closed convex cone C in $\mathbb{R}^p$, the polar cone and polar set of C are equivalent.

Quasi-monotonicity

Definition 2.3.10 (Monotonicity)**.** A set-valued map $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is said to be

i) *monotone* if for every (x, x^*), $(y, y^*) \in \operatorname{gr} T$, the following implication holds

$$\langle y^* - x^*, y - x\rangle \geq 0;$$

ii) *quasi-monotone* if for every (x, x^*), $(y, y^*) \in \operatorname{gr} T$, the following implication holds

$$\langle x^*, y - x\rangle > 0 \implies \langle y^*, y - x\rangle \geq 0.$$

iii) *properly quasi-monotone* on non-empty subset K of X if for all $x_1, \dots, x_n \in K$, and all $x \in \operatorname{conv}(x_1, \dots, x_n)$, there exists $i \in \{1, \dots, n\}$ such that

$$\langle x_i^*, x_i - x\rangle \geq 0, \quad \forall x_i^* \in T(x_i);$$

iv) *cyclically quasi-monotone* if for every $(x_i, x_i^*) \in \operatorname{gr} T$, $i \in \{1, \dots n\}$, the following implication holds

$$\forall i \in \{1, \dots n-1\},\ \langle x_i^*, x_{i+1} - x_i\rangle > 0 \implies \langle x_n^*, x_{n+1} - x_n\rangle \leq 0,$$

where $x_{n+1} = x_1$.

Clearly, any monotone map is quasi-monotone, every cyclically quasi-monotone or properly quasi-monotone operator is also quasi-monotone.

Variational Inequality

Besides the applications in mechanics, variational inequalities (VI) are perfect concepts to express optimality conditions for optimization problems. Therefore, it has been studied extensively since the last century. Particularly, the classical Stampacchia variational inequalities is perhaps the most well-known one.

Definition 2.3.11 (VI)**.** Let K be a non-empty subset of $\mathbb{R}^p$ and $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ be a set-valued maps. A variational inequality (VI) is defined as follows.

$$\text{VI}(T, K) \qquad \text{Find } x \in K \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall y \in K.$$

In classical context, that is whenever f is differentiable, $T(x)$ stands for ∇f and the considered variational inequality is $\text{VI}(\nabla f, K)$ (see, e.g. [16]). In this work, the set-valued map $T(x)$ is usually defined as an adjusted normal operator $N_f^a(x)$ associated to function f at point x.

Quasi-variational Inequality

Quasi-variational inequality (QVI) corresponds to a variational inequality for which the constraint set K depends on the considered point x.

Definition 2.3.12 (QVI)**.** Let C be a non-empty subset of $\mathbb{R}^p$, $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ and $K : C \rightrightarrows C$ be two set-valued maps. A quasi-variational inequality is defined as follows.

$$\text{QVI}(T, K) \qquad \text{Find } x \in C \text{ such that } x \in K(x) \text{ and } \exists x^* \in T(x)$$
$$\text{with } \langle x^*, y - x \rangle \geq 0,\ \forall y \in K(x).$$

2.4 Typical situations for non-cooperation

As explained at the beginning of this chapter, the common thread of this thesis is the non-cooperative interactions between a set of players. So let us now describe more in detail the different kinds of interactions which will be considered in the forthcoming chapters.

2.4.1 Nash game

Nash equilibrium problem

The Nash equilibrium problem (NEP) is a non-cooperative game in which each player's objective function depends on the other players' strategies. Namely, assume that there are n players and each player i controls variables $x_i \in \mathbb{R}^{N_i}$. In fact, x_i is a strategy of the player i denoted based on the vector $x = \{x_1, \ldots, x_n\}$ with $N = N_1 + N_2 + \cdots + N_n$ and thus $x \in \mathbb{R}^N$. The term strategy can be understood in various ways, mainly amount of production, consumption, buying, etc. To give an example, it can be electricity (in energy market), water (eco-park), goods (exchange) or materials (producing). Notation x_{-i} is also used to describe all players decision variables except the one of the player i so, with a commonly accepted abuse of notation, one can write $x = (x_i, x_{-i})$.

Definition 2.4.1 (NEP)**.** Given a finite number n of players and for each of them a cost function f_i of player i and strategy set K_i, a standard Nash equilibrium problem NEP(f, K) consists of finding a vector $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \subset \mathbb{R}^N$ such that, for any $i = 1 \ldots, n$, $\bar{x}_i$ solve the following problem

$$(P_i) \quad \min_{x_i} \ f_i(x_i, \bar{x}_{-i}),$$
$$\text{s.t.} \quad x_i \in K_i.$$

In the above notation NEP(f, K), it is understood that f and K stands respectively for $f(x) = (f_1(x), \ldots, f_n(x))$ and $K = \prod_{i=1}^n K_i$.

The aim of the player i, given the rival's strategies $\bar{x}_{-i}$, is to choose a strategy x_i that solves the optimization problem. Actually, objective function $f_i(x_i, \bar{x}_{-i})$ often denotes the loss that the player has to suffer when rivals take their decision.

The concept of Nash equilibrium is based on the following paradigm: a vector is a Nash equibrium if none of the players has advantage to unilaterally deviate from this vector. For NEP, the strategy of player i belongs to a strategy set $x_i \in X_i$ which is a fixed set. This implies, in the case of a goods shared by the different players, that each player chooses his strategy (amount of goods) independently of the strategies of the other players or, in other words, that the amount of available goods is assumed to be unlimited.

The analysis of Nash equilibrium is the method to examine the existence, uniqueness, qualitative/quantitative stability, etc. For example, [8, 9, 50, 88] are results about Nash equilibrium existence, or as in [76, 61, 95] are stability analyses.

There are many methods for examining Nash equilibrium problems, one of which, as mentioned, is the variational inequality. Instead of directly dealing with NEP, one can reformulate into VI and apply VI's results. For example, in [7], D. Aussel had results proving the equilibrium equivalence of VI and NEP in a quasi-convex optimization approach. Before that, B. Morgan also showed the close relationship between these two types of problems, and at the same time, proved the existence of Nash equilibria (see [73]). Some other results, presented over the past few years, can be mentioned [46, 84, 68, 70]. That is why throughout the research content, when studying non-cooperative problems, VI stated as a mandatory concept.

Generalized Nash equilibrium problem

The concept of Nash equilibrium problem has been generalized in the following form:

Definition 2.4.2 (GNEP)**.** Given a finite number n of players and for each of them a cost function f_i of player i, a set $C_i \subset \mathbb{R}^p$ and a strategy set-valued map $K_i : C_{-i} \rightrightarrows C_i$, a generalized Nash equilibrium problem GNEP(f, K) consists of finding a vector $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in C \subset \mathbb{R}^N$ such that, for any $i = 1, \dots, n$, $\bar{x}_i$ solves the following problem

$$(P_i) \quad \min_{x_i} \; f_i(x_i, \bar{x}_{-i}), \qquad \text{s.t. } \; x_i \in K_i(x_{-i}).$$

Again in the above notation GNEP(f, K), it is understood that f, C and K stands respectively for $f(x) = (f_1(x), \dots, f_n(x))$, $C = \prod_{i=1}^n C_i$ and $K(x) = \prod_{i=1}^n K_i(x)$.

Thus compared to NEP, in GNEP, the strategy set of player i depends on the decision variables of other players. It is thus a more realistic model since, in the case of a good shared by the different players, this model allows to take into account constraints expressing the limited availability of this goods on the market. Whenever the strategy set of each player becomes a constant set, that is,

$$\text{for any i, } \; K_i(x_{-i}) \text{ is a fixed set} := K_i$$

then the non-cooperative GNEP reduces to a NEP.

Similarly to Nash problems, GNEP has also received much attention because of its practical significance. Nevertheless, being more complicated than NEP, its popularity compared to NEP is quite limited and GNEP problems are less analysed in the literature. Few works on the existence of equilibrium are proved: [44, 13, 12, 34]. Note that in the same vein, this thesis also attempts to determine the existence and uniqueness of the weighted generalized Nash game. It will be demonstrated in the next chapter.

Just like the relationship between NEP and VI, here, GNEP and QVI also have a connection. Both of these problems have constraints that are set-valued maps. In fact, the constraint set varies, making this problem more difficult to solve than classical variational inequalities, both from the theoretical and numerical aspects. In particular, in the literature and as for GNEP, there are only a few existence results for quasi-variational inequalities. For instance, under some upper semi-continuity assumptions on T, or some hypotheses that the set of fixed points of K is closed, or even assumptions on upper sign-continuous quasi-monotone map T (see, e.g. [14]). Recently, a generalization to product maps (see, e.g. [11, 21]) has been established to explore assumptions for each component set-valued map T_i. Some remarkable works representing the link between QVI and GNEP can be found in [77] with [78, Erratum], [17] with [18, Addendum], or [26].

Without confusion, in order to emphasize the number of participants in the game, it is convenient to boldly denote by NEP_n and GNEP_n the standard/generalized n-player Nash games and their equilibria NE_n, GNE_n, respectively.

Bi-level optimization

Bi-level optimization is a particular sort of optimization in which one problem is embedded (nested) within another. The so-called *lower level* optimization problem is driven by a variable (called the lower level variable) and parametrized by the variable of the *upper level* problem. It corresponds to a situation where a player has a leading position on a market with two players but want to solve his optimization while taking into account the reaction of the other player. He then elaborates a reaction model of the other player (the lower-level problem) nested in his own optimization problem as a constraint. This leads to the following classic form of bi-level optimization:

$$\begin{aligned} &\min_{x,y} f(x,y) \\ \text{s.t.}\quad & x \in X, \quad y \in S(x), \end{aligned}$$

$$\text{where } y \in S(x) \text{ solves} \qquad \begin{aligned} &\min_{y} g(x,y) \\ \text{s.t.}\quad & y \in Y. \end{aligned}$$

In the formulation, x, f and X (resp. y, g and Y) represent the upper-level (resp. lower-level) variable, objective function and strategy constraint set. Note that this formulation is actually the *optimistic* version of the bi-level problem.

Bi-level problems are often applied to study real-world problems, such as transportation, economics, decision science, business, engineering, environmental economics etc. See for example the recent volume [40] dedicated to bi-level optimization.

What if we combine bi-level optimization problem and Nash game?
To answer, we do have a new class of problems that is more interesting but also much more complex. These problems are collectively known as multi-leader-follower games (MLF in short). Indeed, dealing with such problems requires a rational approach because of its complexity. They can be single-leader-multi-follower (SLMF), multi-leader-single-follower (MLSF) or multi-leader-multi-follower (MLMF) games. The third type, understandably, is the most complicated case among the three games. The other two forms introduced right after, are more common and have lots of practical meanings.

Single-leader-multi-follower game

Let leader decision variable x be an element of X which is a subset of $\mathbb{R}^p$ while each of the followers controls a variable $y_i \in \mathbb{R}^{N_i}$. Let $f : \mathbb{R}^p \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be the objective function of the leader (with $N = \sum_{i=1}^{n} N_i$), and $\theta_i : \mathbb{R}^p \times \mathbb{R}^N$ be the objective functions of the followers. Then a SLMF in its optimistic form is defined by the following bi-level model,

$$\begin{aligned} &\min_x \min_y \; f(x,y), \\ \text{s.t.} \quad &\begin{cases} x \in X, \\ y \in \text{GNEP}(x), \end{cases} \end{aligned}$$

where GNEP(x) is the solution set-valued map of generalized Nash equilibrium problem parametrized by x and defined by

$$\forall i \in \{1,\dots,n\}, \; (P_i^{x,y_{-i}}) \qquad \begin{aligned} &\min_{y_i} \; \theta_i(x, y_i, y_{-i}), \\ \text{s.t.} \quad & y_i \in K_i(x, y_{-i}), \end{aligned}$$

where $K_i(\cdot,\cdot)$ is a constraint map of player i parametrized by the leader decision x and the decisions y_{-i} of the other followers.

Multi-leader-single-follower game

With the same setting, but swapping the role leader/follower, then a MLSF is defined by the following bi-level model (multi-optimistic version),

$$\forall i \in \{1,\dots,n\}, \qquad (P_i) \qquad \begin{aligned} &\min_{y_i} \min_x \; \theta_i(x, y_i, y_{-i}), \\ \text{s.t.} \quad &\begin{cases} y_i \in Y_i(y_{-i}), \\ x \in Sol(y), \end{cases} \end{aligned}$$

where $Sol(\cdot)$ is the solution set of the follower parametrized by the leaders' decision $y = (y_i)_{i=1,\dots,n}$ in the optimization problem defined by

$$\begin{aligned} &\min_x \; f(x,y), \\ \text{s.t.} \quad & x \in K(y), \end{aligned}$$

where $K(y)$ is the constraint map of the follower parametrized by the leaders' decision y.

2.4.2 Radner problem

The Radner equilibrium problem is centrered on sequential trading exchange for allocating resources (goods) under uncertainty. This type of problem differs from the famous Nash equilibrium problem, which is not evolved in timing process, see the famous book of Mas Colell-Whinston-Green [67]. However, the technique to treat the two different types of problems is similar in manipulating the quasi-variational inequality to model and analyse with the constraint set depending on the variable.

Let us consider an I-player non-cooperative game in which there are 2 time points $t = 0$ and $t = 1$. Let us call i, l and s respectively the player i, the commodity l and the state of nature s of the market (or market state), which belong respectively to sets $\mathcal{I} = \{1,\dots,i,\dots,I\}$, $\mathcal{L} = \{1,\dots,i,\dots,L\}$ and $\mathcal{S} = \{1,\dots,i,\dots,S\}$, with $I, \; L, \; S \in \mathbb{N}$.

Let $(q, r, z, y) = \Big(q, r, \{z_1, \ldots, z_i, \ldots, z_I\}, \{y_1, \ldots, y_i, \ldots, y_I\}\Big)$ be a vector in $\mathbb{R}^r$. At $t = 1$, the following notation are introduced.

a) $q = \{q^1, \ldots, q^s, \ldots\}$ is the expected price at any state s of a special commodity 1 playing a role as money for trading, $q^s > 0$ for any s.

b) $z_i = \{z_i^1, \ldots, z_i^s, \ldots\}$ is the number of units of the special commodity 1 for all market states s that each player i decides to trade. For each state, if $z_i^s > 0$, player i will buy some units of commodity 1, or will sell if $z_i^s < 0$. Moreover, z is a kind of contract to be signed at $t = 0$ and be activated at $t = 1$.

c) $r = \{r^{11}, \ldots, r^{ls}, \ldots\}$ is the price of all commodities for all market states.

d) $y_i = \{y_i^{11}, \ldots, y_i^{ls}, \ldots\}$ is the number of units of all commodities for all market states that player i plans to buy.

Definition 2.4.3. A case of sequential trading exchange is to find a vector $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ to solve the following optimization problem

1) for each player i:

$$u_i(\bar{y}_i, \bar{y}_{-i}) = \max_{(z_i, y_i)} u_i(y_i, \bar{y}_{-i})$$

$$\text{subjects to} \quad \begin{cases} \text{(BC) budget constraint,} \\ \text{(CC) consumption constraint;} \end{cases}$$

2) for any market state s and commodity l:

$$\text{satisfies} \quad \begin{cases} \text{(TC) contract constraint,} \\ \text{(DC) demand constraint.} \end{cases}$$

The four constraints sets are defined as follows:

(BC) For each state s, player i can buy or sell, depending on the sign of z_i^s, the special commodity playing the role of an exchange money. Take a sum of all cases of market states for all s, the sign of $\sum_s (q^s z_i^s)$ must be non-positive that implies the player cannot always buy only a special commodity 1 but at least store some budget if selling it in some states. This ensures a portfolio that the player will use to acquire goods.

(CC) Each player i is not allowed to buy more than how much endowment provides. For any state of nature, the cost consumption for all commodities that the player wants to buy has to be less than the fund of initial endowment coupling with the payment in signed contract at $t = 0$.

Without providing complex formula for now, let us consider an example, simply for 2 commodities $\mathcal{M}$ (special) and $\mathcal{N}$, in order to have first in mind a general idea. Suppose that the initial endowment is 15 (shared 6 for $\mathcal{M}$ and 9 for $\mathcal{N}$) with prices 3€ and 5€ . Now, the player made a contract at $t = 0$ to buy commodity $\mathcal{M}$ at $t = 1$ with 4 units. Then, he can spend at time 1 at most $(3€ \times 6_{\mathcal{M}} + 5€ \times 9_{\mathcal{N}}) + 3€ \times 4_{\mathcal{M}} = 75€$. In the case that the contract is to sell 4 units of $\mathcal{M}$ at $t = 1$, he now cannot spend more than $(3€ \times 6_{\mathcal{M}} + 5€ \times 9_{\mathcal{N}}) - 3€ \times 4_{\mathcal{M}} = 51€$.

To further express, the initial fund is fixed at 63€. In the former case, the player has the right to spend up to 63€ plus the amount of 12€ he made in the contract previously. The latter case is little different, since he decided to sell commodity 1, the total fund at $t = 1$ contains which was sold. As a result, there is opportunity to buy all other commodities (in this case, the commodity 2) and the rest units of the first commodity. In fact, it is not reasonable to buy again what one has just sold. Hence, cost consumption $\leq 3€ \times (6_{\mathcal{M}} - 4_{\mathcal{M}}) + 5€ \times 9_{\mathcal{N}} = 51€$.
Basically, for each i, the maximization is an individual optimization in which the portfolio and consumption plan (z_i, y_i) solve the problem. This guarantees that every player is optimizing by (z_i, y_i) under the two constraints (BC) and (CC).

(TC) All contracts must be executed. For each state s, the sum of all contracts among players is equal to 0, thus $\sum_i z_i^s = 0$. Since this is a trading exchange, all players react altogether, any contract of each player i will be fulfilled by the ones of others. The aim is to make the market balanced and stable by keeping asset markets clear.

(DC) For each state and each commodity, the sum of all players' buying cannot be higher than the endowment of the market, therefore $\sum_i y_i^{ls} \leq \sum_i e_i^{ls}$. This is obviously understandable since people cannot buy what is not on the market. If the equality holds, it can be expressed that the products clear in any market state.

The vector $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ which solves the sequential trading exchange, is called a Radner equilibrium. Thus the sequential trading exchange is often called Radner equilibrium problem (REP). A recent result proving the existence of a Radner equilibrium can be stated now. Thus, it is convenient to quote [16, Assumption 1, Assumption 2 and Theorem 1] to give beforehand an idea of this topic.

Assumption 2.4.4 ([16], Assumption 1)**.** Let us consider the following set of assumptions.

i) For all $i \in \mathcal{I}$, $e_i > 0$, that is, $e_i^{ls} > 0$ for all $l \in \mathcal{L}$ and $s \in \mathcal{S}$;

ii) For all $i \in \mathcal{I}$, u_i is strictly increasing in the good $1s$, for every $s \in \mathcal{S}$, i.e.,
$\forall \hat{y}_i,\ \hat{\hat{y}}_i \in \mathbb{R}_+^{LS}\ :\ \hat{y}_i \geq \hat{\hat{y}}_i$ with $\hat{y}_i^{1s} > \hat{\hat{y}}_i^{1s} \implies u_i(\hat{y}_i) > u_i(\hat{\hat{y}}_i)$;

iii) For all $i \in \mathcal{I}$, u_i is locally non-satiated for all $s \in \mathcal{S}$:
$\forall y_i \in \mathbb{R}_+^{LS}$ and $\varepsilon > 0$, $\exists \hat{y}_i = (y_i^1, \ldots, \hat{y}_i^s, \ldots, y_i^S) \in \mathbb{R}_+^{LS}$ and $\|y_i - \hat{y}_i\| < \varepsilon$ such that $u_i(\hat{y}_i) > u_i(y_i)$.

Assumption 2.4.5 ([16], Assumption 2)**.** Additionally, consider the following alternative assumptions.

a) For all $i \in \mathcal{I}$, u_i is concave and continuously differentiable on $\mathbb{R}_+^{LS}$;

b) For all $i \in \mathcal{I}$, u_i is quasi-concave and continuous on $\mathbb{R}_+^{LS}$.

Theorem 2.4.6 ([16], Theorem 1)**.** Let items (*ii*), (*iii*) of Assumption 2.4.4, and alternatively one of (*a*) and (*b*) of Assumption 2.4.4 be satisfied. If $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a solution to the following quasi-variational inequality:

$$\begin{array}{l}\text{Find } (\bar{q},\bar{r},\bar{z},\bar{y}) \in \Delta \times B(\bar{q},\bar{r}) \text{ such that there exists } \bar{y}^* \in T_{\mathcal{U}}(\bar{y})\\ \text{satisfying, for any } (q,r,z,y) \in \Delta \times B(\bar{q},\bar{r}),\\ \Big\langle \Big(\sum_{i\in\mathcal{I}} \bar{z}_i, \sum_{i\in\mathcal{I}} (\bar{y}_i - e_i)\Big), (q,r) - (\bar{q},\bar{r}) \Big\rangle_{S+LS} + \big\langle \bar{y}^*, y - \bar{y} \big\rangle_{LSI} \leq 0.\end{array} \tag{2.4.1}$$

where $T_{\mathcal{U}}(\bar{y}) = \big\{\nabla \mathcal{U}(y)\big\} := \Big\{\big(\nabla u_1(y_1), \ldots, \nabla u_I(y_I)\big)\Big\}$ under hypothesis (a) and $T_{\mathcal{U}}(\bar{y}) = -N^a_{-\mathcal{U}}(\bar{y}) \setminus \{0\} := \Big(- N^a_{-u_1}(y_1) \setminus \{0\}, \ldots, -N^a_{-u_I}(y_I) \setminus \{0\}\Big)$ under hypothesis (b), then $(\bar{q},\bar{r},\bar{z},\bar{y})$ is a Radner equilibrium vector for the sequential trading.

This is the foundation for developing a new form of REP in this research. The concrete meaning of the theorem, through a formulation of QVI, is that if the QVI (2.4.1) admits a solution, this solution will also be a Radner equilibrium. More precisely, given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and two set-valued maps $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : P \rightrightarrows \mathbb{R}^n$ with P being a non-empty subset of $\mathbb{R}^m$, let us consider the special quasi-variational inequality QVI(G,f,K,P).

$$\begin{array}{c}\text{Find a couple } (\bar{p},\bar{x}) \in P \times K(\bar{p}) \text{ such that } \exists \bar{x}^* \in G(\bar{x}) \text{ with}\\ \big\langle f(\bar{x}), p - \bar{p} \big\rangle + \langle \bar{x}^*, x - \bar{x} \rangle \geq 0, \ \forall (p,x) \in P \times K(\bar{p}).\end{array} \tag{2.4.2}$$

This special structure is inspired by the recent work of Donato and co-authors [42, 41]. Problem QVI(G,f,K,P) can be equivalently reformulated as a classical quasi-variational inequality QVI(A,D) where $D : P \times \text{conv}\big(K(P)\big) \rightrightarrows P \times \text{conv}\big(K(P)\big)$ defined by $D(p,x) = P \times K(p)$ and $A : P \times \text{conv}\big(K(P)\big) \rightrightarrows \mathbb{R}^m \times \mathbb{R}^n$ defined by $A(p,x) = \Big\{\big(f(x), x^*\big) : x^* \in G(x)\Big\} = \big\{f(x)\big\} \times G(x)$. This means that problem (2.4.2) is equivalent to

$$\begin{array}{c}\text{Find a couple } (\bar{p},\bar{x}) \in D(\bar{p},\bar{x}) \text{ such that } \exists \big(f(\bar{x}), \bar{x}^*\big) \in A(\bar{p},\bar{x}) \text{ with}\\ \Big\langle \big(f(\bar{x}), \bar{x}^*\big), (p,x) - (\bar{p},\bar{x}) \Big\rangle \geq 0, \ \forall (p,x) \in D(\bar{p},\bar{x}).\end{array} \tag{2.4.3}$$

In Chapter 5, this link between the Radner equilibrium problem and the quasi-variational inequality formulations described above will play a key role for proving the existence of Radner equilibrium for the sequential trading exchange model.

Chapter 3
Player Position

As was briefly introduced in Chapter 1, we will explore here the topic of player position. This work aims to rebound on a paper of von Stengel [92] in which the author analyses the possible interactions between two players (GNEP, Multi-leader-follower game) and determines the most beneficial one for one of the players. In this chapter, we consider an interaction between a group of n players and a new player. A new selection process is the Nash game is also proposed.

3.1 Problem setting

The general setting of this chapter is as follows: while n players are interacting in a non-cooperative way, thus playing a generalized Nash game, an $(n+1)^{\text{th}}$ player wants to enter the game. Nevertheless, in this asymmetric situation this new player faces several possibilities to inter-operate with the group of n players. The game can be analysed in strategic form as a leadership game where players commit to a strategy without knowing what the other players will choose. Then, we have a sequential game with a leader-follower approach.

3.1.1 Multi-leader-follower models in two-period game context

Our aim is here to consider the asymmetric situation where a player, called here player $n+1$, plans to start a non-cooperative interaction with a group of n players already playing a generalized Nash game between them.

Thus there are three different possible interactions: of course the more "natural" is that player $n+1$ is inserted into the generalized Nash game with the other n players. This new $(n+1)$-player generalized Nash game would generate a Nash payoff for all $n+1$ players including player $n+1$'s. But the player $n+1$ can also consider to have hierarchical interactions with the group of n players. Indeed, he can have a position of *sole leader* in SLMF or a position of *single follower* in a MLSF. In the first case, it means that he will "play first" and that the group of n players will play a GNEP between them but this Nash game will be therefore parametrized (in the objective functions and/or in the constraint sets) by the decision of player $n+1$. This conducts to a result that player $n+1$ will gain a *leader payoff* while others' will be *follower payoffs*. In the second case, the group of n players will play a GNEP game between them but in which the optimization problem of a member among n players will be actually connected to a bi-level problem. In particular, each member will maximize his payoff at the upper level constrained by a lower-level optimization problem of player $n+1$. Player $n+1$ thus receives a *follower payoff* comparing to *leader payoffs* of others, and his optimization problems is parametrized by n players' decision.

Now the decision to follow one of these three possible interaction models will be made by a "two-step process". The group of n players and player $n+1$ will simultaneously declare if they want to play as leader (period 1) or as follower (period 2). There are thus four possible combinations that lead to the three above described interaction models:

New player $n+1$ / **The n players**	Single Leader (Plays in period 1)	Single Follower (Plays in period 2)
Multiple Leaders (Play in period 1)	GNEP_{n+1}	MLSF_{n+1}
Multiple Followers (Play in period 2)	SLMF_{n+1}	GNEP_{n+1}

Table 3.1: Description for situations of $(n+1)$-player games.

Clearly it is indifferent to have both choosing "leader" or both choosing "follower" and these two situations will lead to the same GNEP_{n+1} game.

Let us now describe the game more in detail. So let us consider a market in which q commodities can be bought and let β be a positive real number vector of $\mathbb{R}^q$ representing a maximum initial endowment or a maximum exchange volume of the commodities on the market. Let us assume that for any $i = 1, \ldots, n+1$, $C_i \subset \mathbb{R}^q$ denotes the constraint strategy set of player i while the real valued function θ_i, defined on $\mathbb{R}^q \times \mathbb{R}^{nq}$ stands for his payoff function. Thus the vector $x_i \in C_i$ is the strategies vector of player i, while x_{-i} is the vector composed of the strategy vector of all players but excluding i. The notation $C(\beta)$ will be used here to describe the common constraint set parametrized by the initial endowment β and shared by all the players. The following different models can be introduced:

a) **Single-leader-multi-follower game $\mathbf{SLMF_{n+1}(\boldsymbol{\beta})}$**:
A single-leader-multi-follower game between $n+1$ players (after the arrival of player $n+1$) is defined as

$$\left(\overline{P}(\beta)\right) \qquad \max_{x_{n+1}} \max_{x_1,\ldots,x_n} \; \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \quad \text{s.t.} \begin{cases} x_{n+1} \in C_{n+1}, \\ (x_1, \ldots, x_n) \in Eq(\beta - x_{n+1}), \end{cases}$$

where $Eq(\beta - x_{n+1})$ is the set of generalized Nash equilibria of the n-player $\mathrm{GNEP}_n(\beta - x_{n+1})$ defined by, for

$$\forall i = 1, \ldots, n, \; \left(\overline{P}_i(\beta - x_{n+1})\right) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i}) \quad \text{s.t.} \begin{cases} x_i \in C_i, \\ (x_1, \ldots, x_{n+1}) \in C(\beta). \end{cases}$$

Note that this problem will be well-posed only if for each possible value of x_{n+1} the equilibrium problem $\mathrm{GNEP}_n(\beta - x_{n+1})$ admits at least an equilibrium. It will be the case if, for example, the subsets C_i, $i = 1, \ldots, n$ are non-empty, convex compact, the functions θ_i are quasi-concave with regard to x_i and continuous on $\mathbb{R}^{(n+1)q}$ and, for any $x_{n+1} \in C_{n+1}$, the subset $C(\beta - x_{n+1})$ is non-empty convex and compact (see eg. [55]). In the case of several possible equilibria then a choice must be done by the leader (player $n+1$): optimistic, pessimistic or selection approach and the "$\max_{x_{n+1}}$" formulation adapted accordingly (see e.g. [39, 28]).

b) **Generalized Nash equilibrium problem GNEP$_{n+1}(\beta)$**:
A generalized Nash game between $n+1$ players (after the arrival of player $n+1$) is defined as

$$\forall i = 1, \dots, n+1, \; \big(\widetilde{P}_i(\beta)\big) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i}) \quad \text{s.t.} \begin{cases} x_i \in C_i, \\ (x_1, \dots, x_{n+1}) \in C(\beta). \end{cases}$$

c) **Multi-leader-single-follower game MLSF$_{n+1}(\beta)$**:
A multi-leader-single-follower game between $n+1$ players (after the arrival of player $n+1$) is defined as

$$\forall i = 1, \dots, n, \; \big(\widehat{P}_i(\beta)\big) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i}) \quad \text{s.t.} \begin{cases} x_i \in C_i, \\ x_{n+1} \text{ is the unique solution of the} \\ \text{optimization problem } \big(\widehat{P}(\beta, x_1, \dots, x_n)\big), \end{cases}$$

where

$$\big(\widehat{P}(\beta, x_1, \dots, x_n)\big) \qquad \max_{x_{n+1}} \; \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \quad \text{s.t.} \begin{cases} x_{n+1} \in C_{n+1}, \\ (x_1, \dots, x_{n+1}) \in C(\beta). \end{cases}$$

Regarding well-posedness, problem MLSF$_{n+1}(\beta)$ is well-defined if, for any $(x_1, \dots, x_n) \in \prod_{i=1}^{n} C_i$, the lower level problem $\big(\widehat{P}(\beta, x_1, \dots, x_n)\big)$ admits a unique solution. It will be the case if, for example, function θ_{n+1} is strictly quasi-concave upper semi-continuous with regard to variable x_{n+1} and the subsets C_{n+1} and $C(\beta)$ are non-empty convex and compact; or θ_{n+1} can be even strongly concave to drop the compactness of constraint set. This uniqueness assumption avoids to deal with the intrinsic ambiguity of multi-leader-follower games coming from the fact that the different leaders can consider different optimum of the lower level problem.

The interested reader can refer to [30, 27, 44, 69] for more information about Nash games and multi-leader-follower games.

All along the work we will make the following assumptions 3.1.1 and 3.1.2.

Assumption 3.1.1 (*Well-posedness assumption*)**.** For the considered maximum exchange volume β, each of the three problems SLMF$_{n+1}(\beta)$, GNEP$_{n+1}(\beta)$ and MLSF$_{n+1}(\beta)$ are assumed to be well-posed, that is

a) for each possible value of x_{n+1} the equilibrium problem GNEP$_n(\beta - x_{n+1})$ admits at least an equilibrium;

b) $\mathrm{GNEP}_{n+1}(\beta)$ admits at least a generalized Nash equilibrium;

c) for any $(x_1, \dots, x_n) \in \prod_{i=1}^{n} C_i$, the lower level problem $\big(\widehat{P}(\beta, x_1, \dots, x_n)\big)$ admits a unique solution.

Sufficient conditions for the well-posedness of $\mathrm{SLMF}_{n+1}(\beta)$ and $\mathrm{MLSF}_{n+1}(\beta)$ have been given above. Note that the uniqueness assumption *c)* is important for the well-posedness of model $\mathrm{MLSF}_{n+1}(\beta)$ since it allows to avoid a classical ambiguity of these models. Indeed, without this uniqueness hypothesis, one can face the unacceptable situation where the different leaders consider different solutions of the lower problem (see e.g. [58, 59]). For instance, in $\mathrm{SLMF}_{n+1}(\beta)$, suppose that $\mathrm{GNEP}_n(\beta - x_{n+1})$ admits several equilibria. Then, each time to solve $\big(\overline{P}(\beta)\big)$, one can make a selection to obtain a constraint among elements of $Eq(\beta - x_{n+1})$. However, things are not as easy in the case of solving the upper level of $\mathrm{MLSF}_{n+1}(\beta)$, which is $\big(\widehat{P}_i(\beta)\big)_{i=1,\dots,n}$. To be explicit, for $i \in \{1, \dots, n\}$, each player i needs to consider their own constraint relating to x_{n+1}. If Assumption 3.1.1 (c) doesn't hold, each player i can consider distinct values of x_{n+1} thus leads to what called "unacceptable".

Assumption 3.1.2 (*Uniqueness assumption*)**.** For the considered maximum exchange volume β, each of the three problems $\mathrm{SLMF}_{n+1}(\beta)$, $\mathrm{GNEP}_{n+1}(\beta)$ and $\mathrm{MLSF}_{n+1}(\beta)$ admits at most a solution $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1})$.

Knowing that it is quite difficult to ensure the uniqueness of equilibrium of a GNEP, the *uniqueness assumption* appears to be quite restrictive. Nevertheless, as it will be seen in the forthcoming Section 3.2, for the case of "bounded strategies", this uniqueness assumption will be fulfilled under mild assumption thanks to the concept of weighted generalized Nash equilibrium.

Thus now the question which we want to address is, taking into account some information that player $n+1$ has collected from the former GNEP_n game that the group of n players were having and not knowing what will be the choice of this group, what is the *more advantageous* choice for player $n+1$? Our aim in the forthcoming section is to define these notions and to provide some sufficient conditions under which such *better* strategy is possible.

3.1.2 Decision concepts

In endogenous time problems, the order of playing determines the difference in the player's benefit. An endogenous game is a game in which a leader and a follower arise spontaneously as a consequence of each player attempting to maximize their payoffs. A consideration of the ability to choose a specific moment to devise a strategy is called a decision. Both leader and follower may prefer to adopt sequential roles rather than engage in simultaneous competition in case that the sequential competition may bring them a higher payoff. There is no conflict over who moves first. In the introduction, player $n+1$ has the right to make a move in period 1 or 2 pro-actively and autonomously, just like the group of n players. Such these games, as mentioned, are said to exhibit endogenous timing behaviour because the roles of first and second movers are formed naturally from attempting to optimize their benefit, rather than being assigned exogenously by an existing rule.

Before entering into further details, under Assumption 3.1.1 and Assumption 3.1.2, let us introduce some notations concerning the payoffs of player $n+1$:

P^L_{n+1} : optimal payoff of player $n+1$ in role of a leader in $\text{SLMF}_{n+1}(\beta)$ that is P^L_{n+1} is optimal value of problem $\big(\overline{P}(\beta)\big)$;

P^G_{n+1} : optimal payoff of player $n+1$ in role of a player in $\text{GNEP}_{n+1}(\beta)$ that is $P^G_{n+1} = \theta_{n+1}(\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ where $(\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ is the unique equilibrium of $\text{GNEP}_{n+1}(\beta)$;

P^F_{n+1} : optimal payoff of player $n+1$ in role of a follower in $\text{MLSF}_{n+1}(\beta)$ that is $P^F_{n+1} = \theta_{n+1} = (\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ where $\bar{x}_{-(n+1)}$ is the unique equilibrium of $\text{MLSF}_{n+1}(\beta)$‘ and $\bar{x}_{n+1}$ is the associated solution of $\big(\widehat{P}(\beta, \bar{x}_{-(n+1)})\big)$;

P^m_{n+1} : lowest payoff of player i between the three payoffs P^L_{n+1}, P^G_{n+1}, P^F_{n+1}, therefore $P^m_{n+1} = \min_{\kappa \in \{L,G,F\}} P^\kappa_{n+1}$;

P^M_{n+1} : highest payoff of player i between the three payoffs P^L_{n+1}, P^G_{n+1}, P^F_i and thus $P^M_{n+1} = \max_{\kappa \in \{L,G,F\}} P^\kappa_{n+1}$.

Notice that, P^L_{n+1}, P^G_{n+1} and P^F_{n+1} represent for 3 different payoffs coming from distinct games. They are not related to each other and depend on the period that player $n+1$ takes. Based on these payoffs, let us now describe the meaning of *favourable strategies.*

Definition 3.1.3. Let us assume that player $n+1$ took a decision (either to play period 1 or to play period 2) which generates a payment P_{n+1}. This decision is said to be

i) a *safe strategy* for player $n+1$ if $P^m_{n+1} < P_{n+1}$;

ii) an *optimal strategy* for player $n+1$ if $P^m_{m+1} < P_{n+1} = P^M_{n+1}$;

iii) a *neutral strategy* for player $n+1$ if $P_{n+1} = P^m_{n+1} \leq P^M_{n+1}$.

Let us illustrate the three above definitions. A safe strategy is a strategy such that when applied, allows the player to avoid getting the lowest payoff and keeps the chance to gain the maximum without having to know about the others' decision. The term "lowest" here has to be understood in the sense of *lowest among three optimal payoffs in the three types of game.* The player's payoff can reach the maximum or not. Nevertheless, even though the best situation which is maximum can not occur, player $n+1$ is at least able to avoid the worst situation and to "feel safe". The fear that one player would get the lowest payoff reflects risk-aversion mentality of players. The lowest payoff is not necessarily a bad result, but the player could be "unsatisfied" if it is the smallest of the possible outcomes the player could reach. Therefore, the so-called "safe", which is based on a risk scale, comes from optimizing what a player can achieve.

Suppose now that player $n+1$ possesses a strategy which guarantees the maximum payoff, that is $P_{n+1} = P^M_{n+1}$. Two cases can occur: if $P_{n+1} > P^m_{n+1}$, then the decision will be an *optimal strategy*; otherwise $P_{n+1} = P^m_{n+1}$ which implies that $P^m_{n+1} = P^M_{n+1}$ and then we have $P^L_{n+1} = P^G_{n+1} = P^F_{n+1}$ and the strategy is a *neutral one.*

Remark 3.1.4. An optimal strategy is a safe strategy, but the reverse is not true. Although a neutral strategy is not the least, it is neither an optimal nor a safe strategy.

Let us assume, as a first step, that the three different payoffs P^L_{n+1}, P^F_{n+1} and P^G_{n+1} are known. Then proposition below describes some sufficient conditions for decision of player $n+1$ to be safe or optimal strategy.

Proposition 3.1.5. Assume Assumption 3.1.1 and Assumption 3.1.2. Then

i) If $P^L_{n+1} > P^F_{n+1}$ and $P^m_{n+1} \neq P^G_{n+1}$, the safe strategy for the player is to play in period 1. In addition, if $P^L_{n+1} = P^G_{n+1}$ then this safe strategy becomes optimal strategy.

ii) If $P^F_{n+1} > P^L_{n+1}$ and $P^m_{n+1} \neq P^G_{n+1}$, the safe strategy for the player is to play in period 2. In addition, if $P^F_{n+1} = P^G_{n+1}$ then this safe strategy becomes optimal strategy.

Proof. Let us preliminarily list the possible strategies and payoff of player $n+1$. Player $n+1$ has to decide if entering the game in period 1 or period 2, without knowing the strategy of the other n players. Let us, then, distinguish the different cases:

a) player $n+1$ decides to enter the game in period 1. Then, the following two situations can occur:

- a_1) the other n players enter the game in period 1, then a $\text{GNEP}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^G_{n+1};
- a_2) the other n players enter the game in period 2, then a $\text{SLMF}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^L_{n+1}.

b) player $n+1$ decides to enter the game in period 2. Then, the following two situations can occur:

- b_1) the other n players enter the game in period 1, then a $\text{MLSF}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^F_{n+1};
- b_2) the other n players enter the game in period 2, then a $\text{GNEP}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^G_{n+1}.

Now in case (i), since $\min_{k\in L,G,F} P^k_{n+1} \neq P^G_{n+1}$ and $P^L_{n+1} > P^F_{n+1}$ then

$$\min_{k\in L,G,F} P^k_{n+1} = P^F_{n+1}$$

and the only safe strategy for player $n+1$ is to play period 1. This strategy clearly becomes optimal if additionally $P^L_{n+1} = P^G_{n+1}$.

Symmetrically, in case (ii), since $\min_{k\in L,G,F} P^k_{n+1} \neq P^G_{n+1}$ and $P^F_{n+1} > P^L_{n+1}$ and

$$\min_{k\in L,G,F} P^k_{n+1} = P^L_{n+1}.$$

Playing period 2 is thus the only safe strategy for player $n+1$ which becomes optimal if $P^F_{n+1} = P^G_{n+1}$. □

The claim of Proposition 3.1.5 can be illustrated by the following strategic tables.

Group of n players \ Player $n+1$	Period 1		Period 2
Period 1	P^G_{n+1}	$>$	P^F_{n+1}
Period 2	P^L_{n+1}	$\geq$	P^G_{n+1}

Table 3.2: Playing in period 1 provides safe/optimal strategy to player $n + 1$.

Group of n players \ Player $n+1$	Period 1		Period 2
Period 1	P^G_{n+1}	$\leq$	P^F_{n+1}
Period 2	P^L_{n+1}	$<$	P^G_{n+1}

Table 3.3: Playing in period 2 provides safe/optimal strategy to player $n + 1$.

Group of n players \ Player $n+1$	Period 1		Period 2
Period 1	P^G_{n+1}	$\geq$	P^F_{n+1}
Period 2	P^L_{n+1}	$\leq$	P^G_{n+1}
Additionally	$P^L_{n+1} > P^F_{n+1}$		

Table 3.4: Another case gaining safe/optimal strategy in period 1.

Group of n players \ Player $n+1$	Period 1		Period 2
Period 1	P^G_{n+1}	$\geq$	P^F_{n+1}
Period 2	P^L_{n+1}	$\leq$	P^G_{n+1}
Additionally	$P^L_{n+1} < P^F_{n+1}$		

Table 3.5: Another case gaining safe/optimal strategy in period 2.

Let us now introduce two last senses/understandings of *favourable strategies* for player $n + 1$.

Definition 3.1.6. Let us assume that player $n + 1$ took a decision (either to play period 1 or to play period 2) which generates a payment P_{n+1}. This decision is said to be

i) *most beneficial* for player $n + 1$ if $P_{n+1} = P^M_{n+1}$;

ii) *lowest beneficial* for player $n + 1$ if $P_{n+1} = P^m_{n+1}$.

Remark 3.1.7. If a decision of player $n + 1$ is at the same time the most and the lowest beneficial for player $n + 1$, then any decision of this player is neutral.

As mentioned in Remark 3.1.4, if a player has a strategy which is optimal, safe or neutral, he can decide in which period (1 or 2) to play to optimize his payoff. Nevertheless, there are some cases in which we cannot provide a good enough advice for the player, since those cases can not guarantee avoiding the lowest payoff. Our aim is then to give as much information as possible to the player. If there is no knowledge about the best strategy to adopt, he can at least know which type of game he should take part in. If player $n + 1$ knows which period the other n players will enter in the game (in period 1 or 2), he can adopt a suitable strategy that maximizes his profit or at least avoids the worst payoff by choosing the appropriate game to play. The following proposition states the condition under which playing a SLMF, MLSF or GNEP game is the most or lowest beneficial game for player $n + 1$. In this vein, as a direct consequence of Definition 3.1.3, Definition 3.1.6 and Remark 3.1.7, one easily obtains the following conclusions.

Proposition 3.1.8. Assume Assumption 3.1.1 and Assumption 3.1.2. Then, *i)* If $P^G_{n+1} < P^F_{n+1} \leq P^L_{n+1}$ or $P^G_{n+1} = P^F_{n+1} < P^L_{n+1}$, then there exists at least a most beneficial game for player $n+1$ that is $\text{SLMF}_{n+1}(\beta)$ and at least a lowest beneficial game namely $\text{GNEP}_{n+1}(\beta)$.

ii) If $P^G_{n+1} < P^L_{n+1} \leq P^F_{n+1}$ or $P^G_{n+1} = P^L_{n+1} < P^F_{n+1}$, then there exists at least a most beneficial game for player $n+1$ that is $\text{MLSF}_{n+1}(\beta)$ and at least a lowest beneficial game namely $\text{GNEP}_{n+1}(\beta)$.

iii) If $P^G_{n+1} = P^F_{n+1} = P^L_{n+1}$, then any decision of player $n+1$ is neutral.

3.2 A selection process for bounded strategy models

One of the difficulty of the above presented models is that they required, for the well-posedness of models $\text{SLMF}_{n+1}(\beta)$, $\text{GNEP}_{n+1}(\beta)$ and $\text{MLSF}_{n+1}(\beta)$ that the generalized Nash equilibrium problem considered in these models admits a unique equilibrium (see assumption 3.1.2). This hypothesis is known to be quite difficult to guarantee. We thus propose here an adaptation of the formulation for which this uniqueness hypothesis will be automatically satisfied. It consists of a selection process on the different possible generalized Nash equilibria. This selection of generalized Nash equilibrium will be based on specific forms of the constraints sets C_i and $C(\beta)$ which corresponds to the case of bounded strategy sets. Namely, for the rest of the content we will assume as follows.

- For any $i = 1, \dots, n+1$, the constraint strategy set of player i is given by
$$C_i := \prod_{l=1}^{q} \left[0, \overline{X}_{i,l}\right]$$
where the consumption upper bound $\overline{X}_{i,l}$ of player i for commodity l is such that $0 < \overline{X}_{i,l} < \beta_l$. Note that the right inequality expresses here that no player can act in a monopolistic way for commodity l.
- The common constraint set $C(\beta)$ is given by
$$C(\beta) := \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{(n+1)q} : \sum_{i=1}^{n+1} x_{i,l} \leq \beta_l, \ \forall l = 1, \dots, q\right\},$$
where $\beta = (\beta_l)_l$, $x_{i,l}$ stands for the vector of consumption of commodity l by player i, and the constraint expresses the fact that the total consumption of commodity l cannot exceed the total available amount of this commodity. As a result, an equilibrium of any $(n+1)$-player game will yield for each player i, an optimal vector for q commodities.

3.2.1 Weighted Nash equilibrium

Now the selection process between the different possible generalized Nash equilibria will be based on a "weight vector" $(w_1, \dots, w_{n+1})$ of the players and the associated new concept of *weighted Nash equilibrium.*

Definition 3.2.1 (Weighted constraint)**.** Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \dots, q$,

$\sum_{i=1}^{n+1}\overline{X}_{i,l} \neq \beta_l$. Let $\mathcal{W} = \{w_1, \ldots, w_{n+1}\}$ be a family of weights of the players satisfying

$$\text{for any } l \quad \begin{cases} \text{for any } i, \quad w_{i,l} \in \left[0, \dfrac{\overline{X}_{i,l}}{\left|\sum_{i=1}^{n+1}\overline{X}_{i,l} - \beta_l\right|}\right[\\ \text{and } \sum_{i=1}^{n} w_{i,l} = 1. \end{cases} \tag{3.2.1}$$

Then, for any *pre-booking vector* $\delta \in [0, \beta_l]^q$, the *weighted consumption bounds* of player $i = 1, \ldots, n+1$ for commodity l is defined as follows:

- For a generalized Nash game between players $\{1, \ldots, n\}$, for any $l = 1, \ldots, q$,

$$\begin{aligned} \overline{X}^w_{i,l}(n, \delta_l) &= \overline{X}_{i,l} - w_{i,l} \max\left\{0, \left[\sum_{j=1}^{n}\overline{X}_{j,l} - (\beta_l - \delta_l)\right]\right\} \\ &= \begin{cases} \overline{X}_{i,l} & \text{if } \sum_{j=1}^{n}\overline{X}_{j,l} \leq \beta_l - \delta_l, \\ \overline{X}_{i,l} - w_{i,l}\left[\sum_{j=1}^{n}\overline{X}_{j,l} - (\beta_l - \delta_l)\right] & \text{otherwise;} \end{cases} \end{aligned}$$

- For a generalized Nash game between players $\{1, \ldots, n+1\}$, for any $l = 1, \ldots, q$,

$$\begin{aligned} \overline{X}^w_{i,l}(n+1, \delta_l) &= \overline{X}_{i,l} - w_{i,l} \max\left\{0, \left[\sum_{j=1}^{n+1}\overline{X}_{j,l} - (\beta_l - \delta_l)\right]\right\} \\ &= \begin{cases} \overline{X}_{i,l} & \text{if } \sum_{j=1}^{n+1}\overline{X}_{j,l} \leq \beta_l - \delta_l, \\ \overline{X}_{i,l} - w_{i,l}\left[\sum_{j=1}^{n+1}\overline{X}_{j,l} - (\beta_l - \delta_l)\right] & \text{otherwise.} \end{cases} \end{aligned}$$

The weights $w_i = (w_{i,l})_l$ could represent the vector of relative size/power of the players on a market of commodity l, in this case it is a negotiated coefficient, while the pre-booking vector $\delta = (\delta_l)_l$ stands for a part of the maximum exchange volume $\beta = (\beta_l)_l$ which has already been bought, for example, by player $n+1$ in the $\text{SLMF}_{n+1}(\beta)$.

It can appear to be quite surprising that the weights $w_{i,l}$ are assumed to satisfy the equality $\sum_{i=1}^{n} w_{i,l} = 1$ and not equality $\sum_{i=1}^{n+1} w_{i,l} = 1$. This specific choice of the weights is motivated by the fact that the weights are chosen before knowing what kind of game will be faced by the group of n players and player $n+1$ (SLMF, MLSF or GNEP) and the fact that condition $\sum_{i=1}^{n} w_{i,l} = 1$ is needed to ensure that SLMF will be well-posed. This will be explicitely proved in item (i) of the forthcoming Proposition 3.2.4.

Example 3.2.2. *For a game with* 4 *players, each of them need to buy a number of books* $\{x_1, x_2, x_3, x_4\}$. *The maximum endowments for each player are* $\overline{X}_1 = 3$, $\overline{X}_2 = 5$, $\overline{X}_3 = 6$ *and* $\overline{X}_4 = 4$, *respectively. If* $\beta = 20$, *thus four players can buy as much as they want, since* $\sum_{i=1}^{4}\overline{X}_i \leq \beta$. *For some reasons, the available books now reduce to* $\beta = 15$. *Suppose that the last player* 4 *has chance to buy first and* $x_4 = 4$, *then the three other players need to play in a Nash game such that* $x_1 + x_2 + x_3 \leq \beta - x_4 = 11$. *Since the weighted coefficients of three players* $\sum_{i=1}^{3} w_i = 1$, *all new weighted endowments* $\overline{X}^w_i$ *of thee players will adapt to be fit the new volume* $\beta - x_4$.

Replacing, in a generalized Nash game, the consumption bounds $\overline{X}_{i,l}$ by the weighted consumption bounds, leads to the concept of *weighted Nash equilibrium*.

Definition 3.2.3 (Weighted generalized Nash equilibrium problem)**.** Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \dots, q$, $\sum_{i=1}^{n+1} \overline{X}_{i,l} \neq \beta_l$. Let $\mathcal{W} = \{w_1, \dots, w_{n+1}\}$ be a family of weights of the players satisfying condition (3.2.1).

Then, for $p = n$ or $p = n+1$, and for any pre-booking $\delta \in [0, \beta_l]^q$, the *weighted generalized Nash equilibrium problem* $\mathrm{GNEP}_p^w(\beta - \delta)$ consists of:

$$\begin{aligned} &\text{Find } \bar{x} = (\bar{x}_1, \dots, \bar{x}_p) \in \mathbb{R}^{qp} \text{ such that } \forall i = 1, \dots, p,\ \bar{x}_i \text{ is a solution} \\ &\text{of the problem of player } i \\ &\quad \big(P_i^w(\bar{x}_{-i})\big) \qquad \max_{x_i}\ \theta_i(x_i, \bar{x}_{-i}) \\ &\qquad\qquad\qquad \text{s.t.}\quad x_{i,l} \in \big[0, \overline{X}_{i,l}^w(p, \beta_l - \delta_l)\big], \quad \forall l = 1, \dots, q. \end{aligned}$$

The equilibria $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$ are called weighted generalized Nash equilibria and their set will be denoted by $\mathrm{GNE}_p^w(\beta - \delta)$.

It follows immediately from the definitions 3.2.1 and 3.2.3 that if, for any $l = 1, \dots, q$, $\sum_{j=1}^{p} \overline{X}_{j,l} \leq \beta_l - \delta_l$, then for any $i = 1, \dots, p$ (with $p = n$ or $n+1$), one has $\overline{X}_{i,l}^w(p, \delta) = \overline{X}_{i,l}$ and thus $\mathrm{GNE}_p^w(\beta - \delta) = \mathrm{GNE}_p(\beta - \delta)$. This situation corresponds to the case where the maximum exchange volume is higher than the maximum cumulative consumption of the players.

A natural question arising is the link between the set of generalized Nash equilibria of $\mathrm{GNE}_p(\beta - \delta)$ and the set of weighted Nash equilibria $\mathrm{GNE}_p^w(\beta)$. As shown in the forthcoming proposition, any weighted generalized Nash equilibrium is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta - \delta)$ thus bringing to the fore that replacing the consumption bounds by the weighted consumption bounds leads to a selection process on the Nash equilibria.

Proposition 3.2.4. Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \dots, l$, $\sum_{i=1}^{n+1} \overline{X}_{i,l} \neq \beta_l$. Let $\mathcal{W} = \{w_1, \dots, w_{n+1}\}$ be a family of weights of the players satisfying condition (3.2.1).

Then, for $p = n$ or $p = n+1$ and any pre-booking $\delta \in [0, \beta_l]^q$, one has

i) Let $l \in \{1, \dots, q\}$. If, for any $i = 1, \dots, p$, $x_{i,l} \in \big[0, \overline{X}_{i,l}^w(p, \beta_l - \delta_l)\big]$ then $\sum_{i=1}^{p} x_{i,l} \leq \beta_l - \delta_l$;

ii) Any weighted generalized Nash equilibrium is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta - \delta)$, that is

$$\mathrm{GNE}_p^w(\beta - \delta) \subseteq \mathrm{GNE}_p(\beta - \delta).$$

Proof. Let us first observe that, as an immediate consequence of the definition of weighted consumption bounds, one has, for any $i = 1, \dots, p$ and any $l = 1, \dots, q$, $\overline{X}_{i,l}^w(p, \delta) \leq \overline{X}_{i,l}$.

To prove *i)*, let us first consider the case $p = n$. If $\sum_{j=1}^{n} \overline{X}_{j,l} \leq \beta_l - \delta_l$ then, for any i, $\overline{X}_{i,l}^w(p, \beta_l - \delta_l) = \overline{X}_{i,l}$ and the desired inequality is trivially fulfilled. So let us assume that $\sum_{j=1}^{n} \overline{X}_{j,l} > \beta_l - \delta_l$ and, for any $i = 1, \dots, n$, $x_{i,l} \in \big[0, \overline{X}_{i,l}^w(n, \beta_l - \delta_l)\big]$. Then one can deduce that

$$\begin{array}{rcl}\sum_{i=1}^{n} x_{i,l} & \leq & \sum_{i=1}^{n} \overline{X}_{i,l}^{w}(n, \beta_l - \delta_l) \\ & = & \sum_{i=1}^{n} \overline{X}_{i,l} - \sum_{i=1}^{n} w_{i,l}\Big(\sum_{k=1}^{n} \overline{X}_{k,l} - (\beta_l - \delta_l)\Big) \\ & = & \sum_{i=1}^{n} \overline{X}_{i,l} - \sum_{k=1}^{n} \overline{X}_{k,l} + (\beta_l - \delta_l) = \beta_l - \delta_l.\end{array}$$

Now in the case $p = n+1$, since the case $\sum_{j=1}^{n+1} \overline{X}_{j,l} \leq \beta_l - \delta_l$ is as immediate as above, let us assume that $\sum_{j=1}^{n+1} \overline{X}_{j,l} > \beta_l - \delta_l$ and, for any $i = 1, \ldots, n+1$, $x_{i,l} \in \big[0, \overline{X}_{i,l}^{w}(n+1, \beta_l - \delta_l)\big]$. Then, similarly,

$$\begin{array}{rcl}\sum_{i=1}^{n+1} x_{i,l} & \leq & \sum_{i=1}^{n+1} \overline{X}_{i,l}^{w}(n+1, \beta_l - \delta_l) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - \sum_{i=1}^{n+1} w_{i,l}\Big(\sum_{k=1}^{n+1} \overline{X}_{k,l} - (\beta_l - \delta_l)\Big) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - (1 + w_{n+1})\Big(\sum_{k=1}^{n+1} \overline{X}_{k,l} - (\beta_l - \delta_l)\Big) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - \sum_{k=1}^{n+1} \overline{X}_{k,l} + (\beta_l - \delta_l) - w_{n+1}\Big(\sum_{k=1}^{n+1} \overline{X}_{k,l} - (\beta_l - \delta_l)\Big) \\ & \leq & \beta_l - \delta_l.\end{array}$$

Now $ii)$ is a direct consequence of (i). Indeed, according to (i), $\mathrm{GNEP}_p(\beta - \delta)$ can be simplified in

> Find $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p) \in \mathbb{R}^{qp}$ such that, for any $i = 1, \ldots, p$, $\bar{x}_i$ is a solution of the problem of player i
>
> $$\begin{array}{ll}\max_{x_i} & \theta_i(x_i, \bar{x}_{-i}) \\ \text{s.t.} & x_{i,l} \in \big[0, \overline{X}_{i,l}\big], \quad l = 1 \ldots, q.\end{array}$$

and thus any weighted generalized Nash equilibrium of $\mathrm{GNEP}_p^w(\beta - \delta)$ is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta - \delta)$. □

Let us first observe that, as a consequence of item (i), the fundamental inequality "$\sum_{i=1}^{p} x_{i,l} \leq \beta_l - \delta_l$" stating that the total consumption of a commodity l cannot exceed the maximum exchange volume $\beta_l - \delta_l$ of this commodity can be dropped from the GNEP of the three models $\mathrm{SLMF}_{n+1}(\beta)$, $\mathrm{GNEP}_{n+1}(\beta)$ and $\mathrm{MLSF}_{n+1}(\beta)$ as soon as one considers "weighted formulation".

Thus according to Proposition 3.2.4 (ii), the weighted Nash equilibria can be interpreted as a selection process of the generalized Nash equilibria of $\mathrm{GNEP}_n(\beta)$. Besides the fact that the formulation of $\mathrm{GNEP}_n^w(\beta)$ is simpler than the one of $\mathrm{GNEP}_n(\beta)$, one can wonder what is the real improvement considering this selection process. There are two main reasons:

- First as it will be shown in the forthcoming Proposition 3.2.5, it can be proved that, given a family $\mathcal{W} = \{w_1, \ldots, w_{n+1}\}$ of weights of the players satisfying conditions (3.2.1) and under mild assumptions, there exists a unique weighted generalized Nash equilibrium. This uniqueness property will be of course of main importance when considering, in Section 3.2.2 and afterwards, weighted version of problems $\mathrm{SLMF}_{n+1}(\beta)$, GNEP$_{n+1}(\beta)$ and MLSF$_{n+1}(\beta)$. Indeed the "GNEP part" of Assumption 3.1.2-*b*) will be automatically satisfied while no "optimistic" or "pessimistic" formulations will be needed in the weighted version of SLMF$_{n+1}(\beta)$ and MLSF$_{n+1}(\beta)$.

- The second main reason to consider weighted Nash equilibrium is that it will drastically simplify the structure of the three models SLMF$_{n+1}(\beta)$, GNEP$_{n+1}(\beta)$ and MLSF$_{n+1}(\beta)$ (see Remark 3.2.6 and Proposition 3.2.8).

Proposition 3.2.5. Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \ldots, q$, $\sum_{i=1}^{n+1} \overline{X}_{i,l} \neq \beta_l$. Let $\mathcal{W} = \{w_1, \ldots, w_{n+1}\}$ be a family of weights of the players satisfying conditions (3.2.1).

Let $p = n$ or $p = n+1$ and $\delta \in [0, \beta_l]^q$ be a pre-booking vector. Assume that, for any $i = 1, \ldots, p$, the function $\theta_i(\cdot, x_{-i})$ is continuous in both variables x_i and x_{-i} and for any $x_{-i} \in \prod_{\substack{k=1\\k\neq i}}^{p} \prod_{l=1}^{q} \left[0, \overline{X}_{k,l}\right]$, the function $\theta_i(\cdot, x_{-i})$ is diagonally strictly concave.

Then,

i) GNEP$_p^w(\beta - \delta)$ admits a unique weighted Nash equilibrium $\bar{x}^w = \left(\bar{x}_1^w, \ldots, \bar{x}_p^w\right)$;

ii) If $q = 1$ and $\operatorname*{argmax}_{x_i \in \mathbb{R}_+^q} \theta_i(x_i, x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$) independently of the value of x_{-i}, then

$$\forall i = 1, \ldots, p, \quad \bar{x}_i^w = \begin{cases} \overline{X}_i^w(p, \beta - \delta) & \text{if } \overline{X}_i^w(p, \beta - \delta) < x_i^\star, \\ x_i^\star & \text{otherwise.} \end{cases}$$

Note that in case *ii)* (that is with $q = 1$), the notation $\overline{X}_i^w(p, \beta - \delta)$ is a shortcut for $\overline{X}_{i,1}^w(p, \beta - \delta)$.

Remark 3.2.6. The proof of Proposition 3.2.5 is a direct consequence of the following important observation: the use of the selection process through weighted Nash equilibrium allows to transform the bounded strategy generalized Nash game GNEP$_p(\beta)$ into the (classical) Nash game GNEP$_p^w(\beta)$. Indeed one can easily observe that in Definition 3.2.3 the constraint set of problem $\left(P_i^w(\bar{x}_{-i})\right)$ does not depend on the values of the other player strategies. It is thus an important advantage of the proposed selection process.

Proof. Taking into account Remark 3.2.6, one simply has to prove the existence and uniqueness of the classical Nash game GNEP$_p^w(\beta)$. For each $i = 1, \ldots, p$, the constraints set $\prod_{l=1}^q \left[0, \overline{X}_{i,l}^w\right]$ of the corresponding optimization problem $P_i^w(x_{-i})$ is non-empty, convex and compact in $\mathbb{R}_+^*$. On the other hand, the function $(x_i, x_{-i}) \longmapsto \theta_i(x_i, x_{-i})$ is continuous in both x_i and x_{-i} and strictly concave in x_i. Thus the existence of a weighted Nash equilibrium can be derived from [86].

Now let $q = 1$ and $\bar{x}$ be an equilibrium of GNEP$_p^w(\beta)$. Then for every $i = 1, \ldots, p$ $\bar{x}_i \in \operatorname*{argmax}_{[0, \overline{X}_i^w(p,\beta-\delta)]} \theta_i(x_i, \bar{x}_{-i})$. If $x_i^\star \in \left[0, \overline{X}_i^w(p, \beta - \delta)\right]$, then one clearly has $\bar{x}_i = x_i^\star$. Otherwise $x_i^\star > \overline{X}_i^w(p, \beta_l - \delta_l)$, then we have $\operatorname*{argmax}_{[0, \overline{X}_i^w]} \theta_i(x_i, \bar{x}_{-i}) = \overline{X}_i^w(p, \beta - \delta)$ since the function $\theta_i(\cdot, x_{-i})$ is strictly increasing on $\left[0, \overline{X}_i^w(p, \beta - \delta)\right]$. $\square$

Example 3.2.7. *In a game where there are* 3 *players sharing a single-commodity market, let us consider a generalized Nash equilibrium problem between* 2 *players with exchange volume* $\beta = 25$*, then* $\mathrm{GNEP}_2(25)$ *defined as follows.*

$$\begin{array}{ccc} \max\limits_{x_1}\ 60x_1 - 2x_1^2 & & \max\limits_{x_2}\ 60x_2 - 1.5x_2^2 \\ s.t. \begin{cases} x_1 \in [0,12], \\ x_1 + \bar{x}_2 \le 25, \end{cases} & \textit{and} & s.t. \begin{cases} x_2 \in [0,18], \\ x_2 + \bar{x}_1 \le 25. \end{cases} \end{array}$$

We are searching for a vector $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$ *such that* $\bar{x}_1$ *and* $\bar{x}_2$ *are solutions of the* $\mathrm{GNEP}_2(25)$*. The equilibrium for this problem is*

$$\begin{aligned} \mathrm{GNE}_2(25) &= \big\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+ \ : \ x_1 + x_2 = 25,\ x_1 \le 12,\ x_2 \le 18\big\} \\ &= \big\{(7,18),(8,17),(9,16),(10,15),(11,14),(12,13),...\big\}. \end{aligned}$$

It is clear to see that $\mathrm{GNE}_2(25)$ *is a non-empty set and not unique. However, for each weighted coefficient* $w = (w_1, w_2) = (w_1, 1-w_1) \in [0, 2/5] \times [0, 3/5]$*, there is a unique weighted equilibrium*

$$\begin{aligned} \mathrm{GNE}^w_2(25) = \big\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+ \ : \ & x_1 + x_2 = 25, \\ & x_1 \le 12 - w_1 \max\{0, 12+18-25\}, \\ & x_2 \le 18 - w_2 \max\{0, 12+18-25\}\big\} \\ = \big\{(12 - 5w_1&, 18 - 5w_2)\big\} \\ = \big\{(12 - 5w_1&, 13 + 5w_1)\big\}. \end{aligned}$$

Then, it can be easily observed that a weighted Nash equilibrium is a selection of the classical Nash equilibrium corresponding to the coefficient a, that yields the uniqueness of $\mathrm{GNEP}^w_2(25)$ *coming from original* $\mathrm{GNEP}_2(25)$*.*

Let us now consider the same setting except for a change to the value of $\beta = 32$*. The new* $\mathrm{GNEP}_2(32)$ *has a unique equilibrium and both classical and weighted* GNEP*s admit the same equilibrium,* $\mathrm{GNE}_2(32) = \mathrm{GNE}^w_2(32) = (12, 18)$*. Under some appropriate assumptions on the value of* β *and constraint sets,* GNEP *can achieve the equilibrium uniqueness, while for* GNEP^w *it is surely guaranteed.*

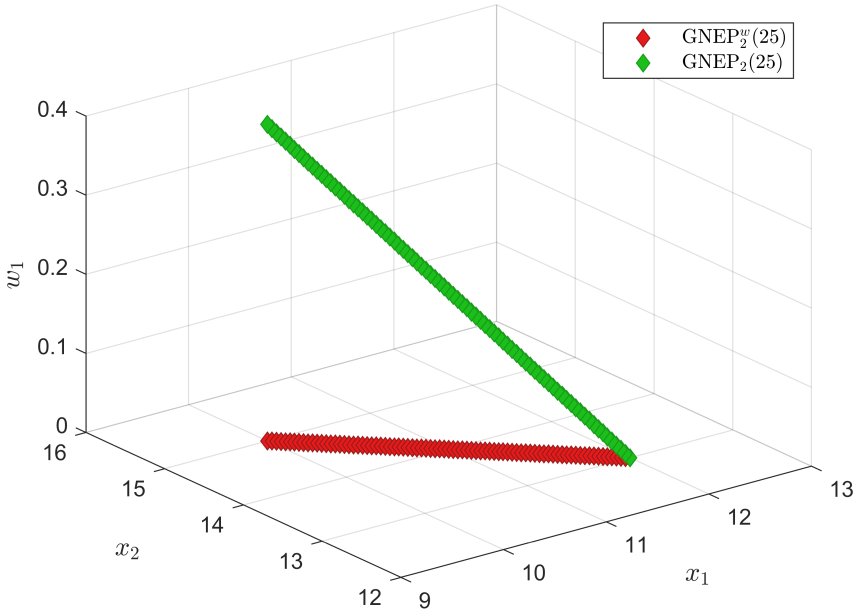

Figure 3.1: An illustration for classical and weighted Nash equilibria.

In Figure 3.1, a comparison between the unique equilibrium of the $\mathrm{GNEP}_2^w(25)$ *and the equilibria in the* $\mathrm{GNEP}_2(25)$ *is depicted.*

3.2.2 Weighted multi-leader-follower exchange models

Let us now come back to the three initial models $\mathrm{SLMF}_{n+1}(\beta)$, $\mathrm{GNEP}_{n+1}(\beta)$ and $\mathrm{MLSF}_{n+1}(\beta)$. By taking into account Proposition 3.2.4 (i) and Proposition 3.2.5, we can now replace them with $\mathrm{SLMF}^w_{n+1}(\beta)$, $\mathrm{GNEP}^w_{n+1}(\beta)$ and $\mathrm{MLSF}^w_{n+1}(\beta)$, by considering weighted Nash equilibrium instead of generalized Nash equilibrium. The three new formulations are defined as follows:

a) **The weighted single-leader-multi-follower game $\mathbf{SLMF}^{w}_{n+1}(\boldsymbol{\beta})$**:
An $(n+1)$-player single-leader-multi-follower game after the arrival of player $n+1$ is defined as

$$\left(\overline{P}^w(\beta)\right) \qquad \max_{x_{n+1}} \; \theta_{n+1}(x_{n+1}, x_{-(n+1)})$$
$$\text{s.t.} \; \begin{cases} 0 \le x_{n+1,l} \le \overline{X}_{n+1,l}, & l = 1, \ldots, q, \\ (x_1^w, \ldots, x_n^w) = Eq^w(\beta - x_{n+1}), \end{cases}$$

where $Eq^w(\beta - x_{n+1})$ is the unique weighted Nash equilibrium of $\mathrm{GNEP}^w_n(\beta - x_{n+1})$, defined by

$$\forall i = 1, \ldots, n, \; \left(\overline{P}^w_i(\beta - x_{n+1})\right) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i})$$
$$\text{s.t.} \; x_{i,l} \in \left[0, \overline{X}^w_{i,l}(n, x_{n+1,l})\right], \quad l = 1, \ldots, q, .$$

The uniqueness assumption (Assumption 3.1.2) for $\mathrm{SLMF}^w_{n+1}(\beta)$ can be satisfied by combining a strict quasi-convexity hypothesis of function $\theta_{n+1}(\cdot, x_{-(n+1)})$ with Proposition 3.2.5. The unique solution of $\mathrm{SLMF}^w_{n+1}(\beta)$ will be denoted by $(\bar{x}^L_{n+1}, \bar{x}^F_1, \ldots, \bar{x}^F_n)$ while the payoff of player $n+1$ will thus be $P^L_{n+1}(\beta)$.

b) **The weighted generalized Nash equilibrium problem $\mathbf{GNEP}^{\boldsymbol{w}}_{\boldsymbol{n+1}}(\boldsymbol{\beta})$:**
An $(n+1)$-player generalized Nash game after the arrival of player $n+1$ is defined as

$$\forall i = 1, \ldots, n+1, \ \big(\widetilde{P}_i^w(\beta)\big) \qquad \begin{array}{rl} \max\limits_{x_i} & \theta_i(x_i, x_{-i}) \\ \text{s.t.} & x_{i,l} \in \big[0, \overline{X}^w_{i,l}(n+1, 0)\big], \quad l = 1, \ldots, q. \end{array}$$

The uniqueness assumption (Assumption 3.1.2) for $\mathrm{GNEP}^w_{n+1}(\beta)$ can be satisfied simply through Proposition 3.2.5. The unique solution of $\mathrm{GNEP}^w_{n+1}(\beta)$ will be denoted by $(\bar{x}^G_1, \ldots, \bar{x}^G_n, \bar{x}^G_{n+1})$ while the payoff of player $n+1$ will thus be $P^G_{n+1}(\beta)$.

c) **The weighted multi-leader-single-follower game $\mathbf{MLSF}^{\boldsymbol{w}}_{\boldsymbol{n+1}}(\boldsymbol{\beta})$:**
An $(n+1)$-player multi-leader-single-follower game after the arrival of player $n+1$ is defined as

$$\forall i = 1, \ldots, n, \ \big(\widehat{P}_i^w(\beta)\big) \qquad \begin{array}{rl} \max\limits_{x_i} & \theta_i(x_i, x_{-i}) \\ \text{s.t.} & \begin{cases} 0 \le x_{i,l} \le \overline{X}^w_{i,l}(n, 0), \quad l = 1, \ldots, q, \\ x_{n+1} \text{ solves } \big(\widehat{P}(\beta - \sum\limits_{j=1}^n x_j)\big), \end{cases} \end{array}$$

where

$$\big(\widehat{P}(\beta - \sum_{j=1}^n x_j)\big) \qquad \begin{array}{rl} \max\limits_{x_{n+1}} & \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \\ \text{s.t.} & \begin{cases} 0 \le x_{n+1,l} \le \overline{X}_{n+1,l}, \quad l = 1, \ldots, q, \\ \sum\limits_{i=1}^{n+1} x_{i,l} \le \beta_l, \quad l = 1, \ldots, q. \end{cases} \end{array}$$

The uniqueness of *best response* x_{n+1} of player $n+1$ can be obtained thanks to a strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot, x_{-(n+1)})$ while the uniqueness of the equilibrium of the upper level generalized Nash game can be inferred from Proposition 3.2.5. The unique solution of $\mathrm{MLSF}^w_{n+1}(\beta)$ will be denoted by $(\bar{x}^L_1, \ldots, \bar{x}^L_n, \bar{x}^F_{n+1})$ while the payoff of player $n+1$ will thus be $P^F_{n+1}(\beta)$.

Note that, besides the use of weighted consumption bounds, at the upper level, the equation $\sum_{i=1}^{n+1} x_i \le \beta$ is maintained in the lower level because Proposition 3.2.4 (*i*) cannot be used here.

Let us now end this section by providing explicit formulae for the solutions of the three models that player $n+1$ can face, that are SLMF^w_{n+1}, GNEP^w_{n+1} and MLSF^w_{n+1}.

Proposition 3.2.8. Consider that $n+1$ players are interacting on a market with one commodity ($q = 1$) and a maximum exchange volume $\beta \in \mathbb{R}$ such that $\sum_{i=1}^{n+1} \overline{X}_i \neq \beta$. Let $\mathcal{W} = \{w_1, \ldots, w_{n+1}\}$ be a family of weights of the players satisfying conditions (3.2.1). Assume that, for any $i = 1, \ldots, n+1$, the function $\theta_i(\cdot, x_{-i})$ is continuous in both variables x_i and x_{-i}. Assume moreover that for any $x_{-i} \in \prod_{\substack{k=1 \\ k \neq i}}^{n+1} [0, \overline{X}_k]$, the function $\theta_i(\cdot, x_{-i})$ is diagonally strictly concave and $\operatorname{argmax}_{x_i \in \mathbb{R}^q_+} \theta_i(x_i, x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$).

Then, the following assertions hold:

i) The game $\mathrm{SLMF}^w_{n+1}(\beta)$ admits a unique solution $\bar{x}_l = (\bar{x}^L_{n+1}, \bar{x}^F_1, \ldots, \bar{x}^F_n)$ where the leader-type solution of player $n+1$ is given by

$$\bar{x}^L_{n+1} = \min\left\{x^\star_{n+1}, \overline{X}_{n+1}\right\},$$

coupled with the follower-type solutions in the lower-level $\text{GNEP}^w_n(\beta - x^L_{n+1})$,

$$\bar{x}^F_i = \min\left\{x^\star_i, \overline{X}^w_{n+1}(n, \bar{x}^L_{n+1})\right\}, \quad \forall i = 1, \dots, n.$$

ii) The $\text{GNEP}^w_{n+1}(\beta)$ admits a unique solution $\tilde{x}_l = (\bar{x}^G_1, \dots, \bar{x}^G_{n+1})$ where the solution of each player i is given by

$$\bar{x}^G_i = \min\left\{x^\star_i, \overline{X}^w_i(n+1, 0)\right\}, \quad \forall i = 1, \dots, n+1.$$

iii) The game $\text{MLSF}^w_{n+1}(\beta)$ admits a unique solution $\hat{x}_l = (\bar{x}^L_1, \dots, \bar{x}^L_n, \bar{x}^F_{n+1})$ where the leader-type solutions of the upper-level $\text{GNEP}^w_n(\beta)$,

$$\bar{x}^L_i = \min\left\{x^\star_i, \overline{X}^w_i(n, 0)\right\}, \quad \forall i = 1, \dots, n,$$

coupled the follower-type solution of player $n+1$ is given by

$$\bar{x}^F_{n+1} = \min\left\{x^\star_{n+1}, \overline{X}_{n+1}, \beta - \sum_{i=1}^n \bar{x}^L_i\right\}.$$

Proof. (*i*) (SLMF^w_{n+1}): Let us first observe that for solving the upper-level optimization problem, the equilibrium in the lower-level $\text{GNEP}^w_n(\beta - x_{n+1})$ has to be inferred. Since the objective function of each player i in the Nash game at the lower level is continuous and strictly concave, according to Proposition 3.2.5 (*ii*), a unique weighted Nash equilibrium can be obtained for the $\text{GNEP}^w_n(\beta - x_{n+1})$, that is

$$\begin{aligned} x^F_l &= \{x_1, \dots, x_n\} \\ &= \Big\{ \min\left\{x^\star_1, \overline{X}^w_{n+1}(n, x_{n+1})\right\}, \dots, \min\left\{x^\star_n, \overline{X}^w_{n+1}(n, x_{n+1})\right\} \Big\}. \end{aligned}$$

Thus, the single leader problem $(\overline{P}(\beta))$ turns out to express as

$$\big(\overline{P}(\beta)\big) \qquad \max_{x_{n+1}} \ \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \quad \text{s.t.} \begin{cases} 0 \le x_{n+1} \le \overline{X}_{n+1}, \\ (x_1, \dots, x_n) = x^F_l. \end{cases}$$

Thanks to concavity of function θ_{n+1}, this problem admits a unique solution

$$\bar{x}^L_{n+1} = \min\left\{x^\star_{n+1}, \overline{X}_{n+1}\right\}.$$

and

$$\bar{x}^F_i = \min\left\{x^\star_i, \overline{X}^w_{n+1}(n, \bar{x}^L_{n+1})\right\}, \quad \forall i = 1, \dots, n.$$

(*ii*) (GNEP^w_{n+1}): The results follows directly from Proposition 3.2.5 (*ii*) for $p = (n+1)$ and the $(n+1)$-player game $\text{GNEP}^w_{n+1}(\beta)$.

(*iii*) (MLSF^w_{n+1}): Let us consider the lower-level optimization problem. Player $n+1$ optimizes the objective function taking as already settled the optimal decision of the upper-level problem. Looking at the constraint set, we have $0 \le x_{n+1} \le \min\left\{\overline{X}_{n+1}, \beta - \sum_{i=1}^n x^L_i\right\}$. Then, since the objective function is strictly concave and the constraint set is boxed, we can conclude that the parametrized optimal solution of player $n+1$ is

$$\begin{aligned} x^F_{n+1} &= \min\left\{x^\star_{n+1}, \min\left\{\overline{X}_{n+1}, \beta - \sum_{i=1}^n x^L_i\right\}\right\} \\ &= \min\left\{x^\star_{n+1}, \overline{X}_{n+1}, \beta - \sum_{i=1}^n x^L_i\right\}. \end{aligned}$$

And thus the leader's problems $\big(\widehat{P}_i^w(\beta)\big)_{i=1,\dots,n}$ or namely $\text{GNEP}_n^w(\beta)$ turns out to be expressed as

$$\forall i=1,\dots,n,\ \big(\widehat{P}_i(\beta)\big) \qquad \max_{x_i}\ \theta_i(x_i,x_i), \\ \text{s.t.} \begin{cases} 0 \le x_i \le \overline{X}_i^w(n,0), \\ x_{n+1} = x_{n+1}^F. \end{cases}$$

Then by applying Proposition 3.2.5 (*ii*) with $p=n$, we get a unique weighted Nash equilibrium

$$\begin{aligned} \bar{x}_l^L &= \{\bar{x}_1^L,\dots,\bar{x}_n^L\} \\ &= \Big\{\min\big\{x_1^\star,\overline{X}_1^w(n,0)\big\},\dots,\min\big\{x_n^\star,\overline{X}_n^w(n,0)\big\}\Big\}. \end{aligned}$$

And thus,

$$\bar{x}_{n+1}^F = \min\Big\{x_{n+1}^\star,\overline{X}_{n+1},\beta-\sum_{i=1}^n \bar{x}_i^L\Big\}.$$

Finally, the $\text{MLSF}_{n+1}^w(\beta)$ admits a unique equilibrium $\hat{x}_l = (\bar{x}_1^L,\dots,\bar{x}_n^L,\bar{x}_{n+1}^F)$ and the proof is complete. □

3.3 Two-period games: a multi-leader-follower approach

This section treats only with a single-commodity market, that is with one commodity ($q=1$). Then, the notations are simplified by eliminating the index of commodity l. To prepare for the next section, let us recall some adaptive notions from previous sections, the three different models that raise from this two-period game context:

- if both players $\{1,\dots,n\}$ and player $n+1$ decide to play the same period (1 or 2), then they interact through a generalized Nash game, GNEP_{n+1};
- if player $n+1$ decides to play period 1 while the group $\{1,\dots,n\}$ opts for period 2, then a single-leader-multi-follower game will be played;
- third, if the group $\{1,\dots,n\}$ decides to play period 1 while player $n+1$ plays period 2 then it will be a multi-leader-single-follower game.

In each of these three cases, a generalized Nash game -possibly parametrized- will be considered, either with n or $n+1$ players. As explained in Section 3.2, we propose to consider a selection process for the resulting generalized Nash equilibrium. This selection process is based on the concept of *weighted Nash equilibrium* which we recall below:

Definition 3.3.1. Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that, $\sum_{i=1}^{n+1}\overline{X}_i \neq \beta$. Let $\mathcal{W}=\{w_1,\dots,w_{n+1}\}$ be a family of weights of the players satisfying the following conditions

$$\begin{cases} \text{for any } i, \quad w_i \in \left[0, \dfrac{\overline{X}_i}{\left|\sum_{i=1}^{n+1}\overline{X}_i-\beta\right|}\right[\\ \text{and } \sum_{i=1}^n w_i = 1. \end{cases} \tag{3.3.1}$$

Then, for $p = n$ or $p = n+1$ and any pre-booking vector $\delta \in [0, \beta]$, the *weighted generalized Nash equilibrium problem* $\mathrm{GNEP}_p^w(\beta - \delta)$ consists in:

Find $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p) \in \mathbb{R}^p$ such that $\forall i = 1, \ldots, p,\ \bar{x}_i$ is a solution of the problem of player i

$$\left(P_i^w(\bar{x}_{-i})\right) \qquad \max_{x_i}\ \theta_i(x_i, \bar{x}_{-i}) \quad \text{s.t.}\ \ x_i \in \left[0, \overline{X}_i^w(p, \beta - \delta)\right],$$

where the weighted consumption bounds $\overline{X}_i^w$ are defined as follows

- For a generalized Nash game between players $\{1, \ldots, n\}$,

$$\begin{aligned} \overline{X}_i^w(n, \delta) &= \overline{X}_i - w_i \max\left\{0, \left[\sum_{j=1}^{n} \overline{X}_j - (\beta - \delta)\right]\right\} \\ &= \begin{cases} \overline{X}_i & \text{if } \sum_{j=1}^{n} \overline{X}_j \leq \beta - \delta, \\ \overline{X}_i - w_i\left[\sum_{j=1}^{n} \overline{X}_j - (\beta - \delta)\right] & \text{otherwise;} \end{cases} \end{aligned}$$

- For a generalized Nash game between players $\{1, \ldots, n+1\}$,

$$\begin{aligned} \overline{X}_i^w(n+1, \delta) &= \overline{X}_i - w_i \max\left\{0, \left[\sum_{j=1}^{n+1} \overline{X}_j - (\beta - \delta)\right]\right\} \\ &= \begin{cases} \overline{X}_i & \text{if } \sum_{j=1}^{n+1} \overline{X}_j \leq \beta - \delta, \\ \overline{X}_i - w_i\left[\sum_{j=1}^{n+1} \overline{X}_j - (\beta - \delta)\right] & \text{otherwise.} \end{cases} \end{aligned}$$

The equilibria $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ are called weighted generalized Nash equilibria and their set will be denoted by $\mathrm{GNE}_p^w(\beta - \delta)$.

Then according to propositions 3.2.5 and 3.2.8, player $n+1$ will thus face one of the three following models $\mathrm{SLMF}_{n+1}^w(\beta)$, $\mathrm{GNEP}_{n+1}^w(\beta)$ and $\mathrm{MLSF}_{n+1}^w(\beta)$:

a) **The weighted single-leader-multi-follower game $\mathbf{SLMF}_{n+1}^{w}(\boldsymbol{\beta})$:**
An $(n+1)$-player single-leader-multi-follower game is defined as

$$\left(\overline{P}^w(\beta)\right) \qquad \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \quad \text{s.t.}\ \begin{cases} 0 \leq x_{n+1} \leq \overline{X}_{n+1}, \\ (x_1^w, \ldots x_n^w) = Eq^w(\beta - x_{n+1}), \end{cases}$$

where $Eq^w(\beta - x_{n+1})$ is the unique weighted Nash equilibrium of $\mathrm{GNEP}_n^w(\beta - x_{n+1})$, defined by

$$\forall i = 1, \ldots, n,\ \left(\overline{P}_i^w(\beta - x_{n+1})\right) \qquad \max_{x_i}\ \theta_i(x_i, x_{-i}) \quad \text{s.t.}\ \ x_i \in \left[0, \overline{X}_i^w(n, x_{n+1})\right].$$

b) **The weighted generalized Nash equilibrium problem $\mathbf{GNEP}_{n+1}^{w}(\boldsymbol{\beta_l})$:**
An $(n+1)$-player generalized Nash game is defined as

$$\forall i = 1, \ldots, n+1,\ \left(\widetilde{P}_i^w(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i, x_{-i}) \quad \text{s.t.}\ \ x_i \in \left[0, \overline{X}_i^w(n+1, 0)\right].$$

c) **The weighted multi-leader-single-follower game $\mathbf{MLSF}_{n+1}^{w}(\boldsymbol{\beta_l})$:**

An $(n+1)$-player multi-leader-single-follower game is defined as

$$\forall i = 1,\dots,n,\ \left(\widehat{P}_i^w(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i, x_{-i}) \quad \text{s.t.} \begin{cases} 0 \le x_i \le \overline{X}_i^w(n,0), \\ x_{n+1} \text{ solves } \left(\widehat{P}(\beta - \sum_{j=1}^{n} x_j)\right), \end{cases}$$

where

$$\left(\widehat{P}^w(\beta - \sum_{j=1}^{n} x_j)\right) \qquad \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1}, x_{-(n+1)}) \quad \text{s.t.} \begin{cases} 0 \le x_{n+1} \le \overline{X}_{n+1}, \\ \sum_{i=1}^{n+1} x_i \le \beta. \end{cases}$$

Let us adopt the well-posedness and uniqueness assumptions to weighted models as the following assumptions 3.3.2 and 3.3.3.

Assumption 3.3.2 (Well-posedness)**.** For the considered maximum exchange volume $\beta \in \mathbb{R}_+^*$, each of the three problems $\text{SLMF}_{n+1}^w(\beta)$, $\text{GNEP}_{n+1}^w(\beta)$ and $\text{MLSF}_{n+1}^w(\beta)$ are assumed to be well-posed, that is

a) for each possible value of x_{n+1} the equilibrium problem $\text{GNEP}_n^w(\beta - x_{n+1})$ admits at least an equilibrium;

b) $\text{GNEP}_{n+1}^w(\beta)$ admits at least a generalized Nash equilibrium;

c) for any $(x_1,\dots,x_n) \in \prod_{i=1}^{n}[0, \overline{X}_i]$, the lower level problem $\left(\widehat{P}^w(\beta, x_1, \dots, x_n)\right)$ admits a unique solution.

Assumption 3.3.3 (Uniqueness)**.** For the considered maximum exchange volume $\beta \in \mathbb{R}_+^*$, each of the three problems $\text{SLMF}_{n+1}^w(\beta)$, $\text{GNEP}_{n+1}^w(\beta)$ and $\text{MLSF}_{n+1}^w(\beta)$ admits at most a solution $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1})$.

The uniqueness in Assumption 3.3.3 for $\text{SLMF}_{n+1}^w(\beta)$ can be satisfied by combining a strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot, x_{-(n+1)})$ with Proposition 3.2.5. The unique solution of $\text{SLMF}_{n+1}^w(\beta)$ will be denoted by $(\bar{x}_{n+1}^L, \bar{x}_1^F, \dots, \bar{x}_n^F)$ while the payoff of player $n+1$ will thus be $P_{n+1}^L(\beta)$.

Similarly, for $\text{GNEP}_{n+1}^w(\beta)$, the uniqueness can be satisfied simply through Proposition 3.2.5. It will be denoted by $(\bar{x}_1^G, \dots, \bar{x}_n^G, \bar{x}_{n+1}^G,)$ the unique solution of $\text{GNEP}_{n+1}^w(\beta)$, while the payoff of player $n+1$ will thus be $P_{n+1}^G(\beta)$.

Finally, the uniqueness of *best response* x_{n+1} of player $n+1$ in game $\text{MLSF}_{n+1}^w(\beta)$ can be obtained thanks to a strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot, x_{-(n+1)})$ while the uniqueness of the equilibrium of the upper level generalized Nash game can be inferred from Proposition 3.2.5. Hence, $\text{MLSF}_{n+1}^w(\beta)$ admits a unique solution which is denoted by $(\bar{x}_1^L, \dots, \bar{x}_n^L, \bar{x}_{n+1}^F,)$, and the payoff of player $n+1$ is $P_{n+1}^F(\beta)$.

3.4 Strategic decision of the two-period game

Rebounding on the preliminary analysis done in sections 3.1 and 3.2, in particular, propositions 3.2.8, 3.1.5 and 3.1.8, our aim in this section is to analyse the “favourable” strategies for player $n+1$, only basing our decision making on the given constants of the problem. See definitions 3.1.3 and 3.1.6 for the classification/terminology of player $(n+1)$'s strategies.

Theorem 3.4.1. Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that, $\sum_{i=1}^{n+1} \overline{X}_i \neq \beta$. Let $\mathcal{W} = \{w_1, \ldots, w_{n+1}\}$ be a family of weights of the players satisfying conditions (3.3.1). Assume that, for any $i = 1, \ldots, n+1$,

- the function $\theta_i(\cdot, x_{-i})$ is continuous in both variables x_i and x_{-i};
- for any $x_{-i} \in \prod_{\substack{k=1 \\ k \neq i}}^{n+1} [0, \overline{X}_k]$, the function $\theta_i(\cdot, x_{-i})$ is strictly concave and $\underset{x_i \in \mathbb{R}_+}{\operatorname{argmax}}\, \theta_i(x_i, x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$).

Then, by setting $\chi^F = \beta - \sum_{i=1}^{n} \min\left\{x_i^\star, \overline{X}_i^w(n,0)\right\}$, the following assertions hold:

i) If $\chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\} \leq \overline{X}_{n+1}^w(n+1,0)$, then an optimal strategy for player $n+1$ is to play in period 1.

ii) If $\chi^F < \overline{X}_{n+1}^w(n+1,0) < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a safe strategy for player $n+1$ is to play in period 1.

iii) If $\overline{X}_{n+1}^w(n+1,0) < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\} \leq \chi^F$, then two most beneficial strategies for player $n+1$ are to be in $\mathrm{SLMF}_{n+1}^w(\beta)$ or to be in $\mathrm{MLSF}_{n+1}^w(\beta)$.

iv) If $\overline{X}_{n+1}^w(n+1,0) = \chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a most beneficial strategy for player $n+1$ is to be in $\mathrm{SLMF}_{n+1}^w(\beta)$.

v) If $\overline{X}_{n+1}^w(n+1,0) < \chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a most beneficial strategy for player $n+1$ is to be in $\mathrm{SLMF}_{n+1}^w(\beta)$ and a least beneficial strategy is to be in $\mathrm{GNEP}_{n+1}^w(\beta)$.

vi) Otherwise, any decision of player $n+1$ (that is to play period 1 or period 2) is a neutral strategy.

Regarding cases (*iii*)-(*v*), the conclusion is less precise since one cannot advise an optimal or safe or neutral strategy for player $n+1$. It could be understood in the sense that player $n+1$ would need some additional information from the group of players $\{1, \ldots, n\}$ to be able to elaborate a more favourable strategy. For example in case (*v*), player $n+1$ could choose the most favourable by playing period 1 if he knows that the group will avoid to play a GNEP_{n+1}^w game.

As quoted above, the proof of Theorem 3.4.1 is an essential consequence of propositions 3.2.8, 3.1.5 and 3.1.8.

Proof. From Proposition 3.2.8, one obtains that

$$\begin{aligned}
\bar{x}_{n+1}^L &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}, \\
\bar{x}_{n+1}^G &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}^w(n+1,0)\right\}, \\
\bar{x}_{n+1}^F &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}, \chi^F\right\}.
\end{aligned}$$

From Definition 3.3.1, one always has $\bar{x}_{n+1}^G \leq \bar{x}_{n+1}^L$. Let us know consider the different possible inequalities between the data and deduce, when possible, the favourable strategy for player $n+1$.

i) From the left inequality one can immediately deduce that $\bar{x}^F_{n+1} = \chi^F < \bar{x}^L_{n+1} \le x^\star_{n+1}$. Since $\theta_i(\cdot, x_{-i})$ is strictly concave and $\operatorname{argmax}_{x_i \in \mathbb{R}_+} \theta_{n+1}(x_{n+1}, x_{-i}) = \{\bar{x}^\star_{n+1}\}$, $\theta_{n+1}(\cdot, x_{-i})$ is increasing on $[0, x^\star_{n+1}]$ and thus $P^F_{n+1} < P^L_{n+1}$. On the other hand from the right side inequality one can easily deduce that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1}$. Thus $P^F_{n+1} < P^L_{n+1} = P^G_{n+1}$. and the conclusion follows from Proposition 3.1.5 (*i*);

ii) In this case, the two strict inequalities show us $\bar{x}^F_{n+1} < \bar{x}^G_{n+1} < \bar{x}^L_{n+1} \le x^\star_{n+1}$ and then, using that $\theta_{n+1}(\cdot, x_{-i})$ is increasing on $[0, x^\star_{n+1}]$, one deduces that $P^F_{n+1} < P^G_{n+1} < P^L_{n+1}$ and, according to Proposition 3.1.5 (*i*) that a safe strategy for the player $n+1$ is to play in period 1;

iii) From $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} \le \chi^F$ one has $\bar{x}^F_{n+1} = \bar{x}^L_{n+1}$. Moreover, $\overline{X}^w_{n+1}(n+1, 0)$ is strictly less than $\min\{x^\star_{n+1}, \overline{X}_{n+1}\}$ and therefore $\bar{x}^G_{n+1} < \bar{x}^F_{n+1} = \bar{x}^L_{n+1} \le x^\star_{n+1}$. By the same argument as in the previous case, $P^G_{n+1} < P^L_{n+1} = P^F_{n+1}$ and thus, combining (*i*) and (*ii*) of Proposition 3.1.8, one gets that there exists two most beneficial strategies for player $n+1$ which are $\text{SLMF}^w_{n+1}(\beta)$ as a leader or $\text{MLSF}^w_{n+1}(\beta)$ as a follower;

(iv) Here, the expression $\overline{X}^w_{n+1}(n+1, 0) = \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$ directly implies that $\bar{x}^G_{n+1} = \bar{x}^F_{n+1} < \bar{x}^L_{n+1} \le x^\star_{n+1}$ and thus again by the same arguments that $P^G_{n+1} = P^F_{n+1} < P^L_{n+1}$. The conclusion follows from Proposition 3.1.8 (*i*);

v) Similarly to case (*iv*), the condition $\overline{X}^w_{n+1}(n+1, 0) < \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$ implies that $\bar{x}^G_{n+1} < \bar{x}^F_{n+1} < \bar{x}^L_{n+1} \le x^\star_{n+1}$ and again, using the increasing property of function $\theta_{n+1}(\cdot, x_{-i})$, one has $P^G_{n+1} < P^F_{n+1} < P^L_{n+1}$. Conclusion then follows Proposition 3.1.8 (*i*);

vi) If none of the previous case occurs, it means that the relations between the constants is given by one of these three sub-cases:

a) $\overline{X}^w_{n+1}(n+1, 0) = \min\left\{x^\star_{n+1}, \overline{X}_{n+1}\right\} \le \chi^F$;
b) $\chi^F = \min\left\{x^\star_{n+1}, \overline{X}_{n+1}\right\} < \overline{X}^w_{n+1}(n+1, 0)$;
c) $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} \le \min\left\{\chi^F, \overline{X}^w_{n+1}(n+1, 0)\right\}$.

With these three sub-cases, playing period 1 or period 2 leads to the same payoff for player $n+1$ since one can show that $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1}$.
In sub-case (*a*) one immediately obtains that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \bar{x}^F_{n+1} = \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$.
Now in the sub-case (*b*), the equality shows that $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$.
But since one always has $\overline{X}^w_{n+1}(n+1, 0) \le \overline{X}_{n+1}$ it can be deduced that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \bar{x}^F_{n+1} = x^\star_{n+1} < \overline{X}^w_{n+1}(n+1, 0)$ and thus that $\bar{x}^G_{n+1} = x^\star_{n+1}$.
Finally in the last sub-case (*c*), when $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} = x^\star_{n+1}$, the desired double equality directly holds since $x^\star_{n+1} = \bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1}$. Otherwise, $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} = \overline{X}_{n+1}$ and, combining with the condition (*c*), they imply $\overline{X}^w_{n+1}(n+1, 0) = \overline{X}_{n+1} \le x^\star_{n+1} \le \chi^F$, and therefore $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \overline{X}_{n+1}$.
For these three sub-cases, no matter the period which the player $n+1$ will choose, his payoff will be the same for the three games that is $P^L_{n+1} = P^G_{n+1} = P^F_{n+1}$. Any decision of this player is a neutral strategy. □

Let us examine more in details cases (*iii*), (*iv*) and (*v*). In all these cases a (one of the) best/most beneficial strategy for the player $n+1$ is to be a leader. In other words, he can prioritize to play in period 1 in order to maximize his payoff. But there is still a risk, which can occur, there will be a possibility for him to be in the GNEP^w_{n+1}. This Nash game can bring the worst payoff among three types (leader, follower, Nash player) and drives him to have no idea for the playing decision. And the same situation happens even if player $n+1$ decides to play in MLSF^w_{n+1}, there is a bad chance to be in GNEP^w_{n+1} too. However, at least, he knows precisely the circumstance that he has to face, which games should be avoided and which games should be played although he doesn't know the strategy of the group of the others.

Remark 3.4.2. It is interesting to emphasize that if $\chi^F < \overline{X}^w_{n+1}(n+1, 0)$, then only case (i), (ii) and (iv) can occur which means that, by playing period 1, player $(n+1)$'s strategy will be optimal, safe or neutral.

Provided "not knowledge" on which period the n players will play, there is no safe or optimal choice outside of period 1 for the player $n+1$. In the corollaries below, a strategy will appear safe and optimal in period 2, if and only if the player $n+1$ has the full knowledge of which exact period the opponent group will play inevitably. These cases appear because items (iii)-(v) in Theorem 3.4.1 gain extra information.

Corollary 3.4.3. Let us use the same hypotheses as in Theorem 3.4.1 and assume player $n+1$ knows that the group of n players would like to play in period 1. Then the following assertions hold.

i) If $\chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\} \leq \overline{X}^w_{n+1}(n+1, 0)$, then an optimal strategy for player $n+1$ is to play in period 1.

ii) If $\chi^F < \overline{X}^w_{n+1}(n+1, 0) < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then a safe strategy for player $n+1$ is to play in period 1.

iii) If $\overline{X}^w_{n+1}(n+1, 0) < \min\{x^\star_{n+1}, \overline{X}_{n+1}\} \leq \chi^F$, then an optimal strategy for player $n+1$ is to play in period 2.

iv) If $\overline{X}^w_{n+1}(n+1, 0) = \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then any decision of player $n+1$ is a neutral strategy.

v) If $\overline{X}^w_{n+1}(n+1, 0) < \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then a safe strategy for player $n+1$ is to play in period 2.

vi) Otherwise, any decision of player $n+1$ is a neutral strategy.

Corollary 3.4.4. Let us use the same hypotheses as in Theorem 3.4.1 and assume player $n+1$ knows that the group of n players would like to play in period 2. Then the following assertions hold.

i) If $\chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\} \leq \overline{X}^w_{n+1}(n+1, 0)$, then any decision of player $n+1$ is an optimal solution.

ii) If $\chi^F < \overline{X}^w_{n+1}(n+1, 0) < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1, and safe strategy is to play in period 2.

iii) If $\overline{X}^w_{n+1}(n+1, 0) < \min\{x^\star_{n+1}, \overline{X}_{n+1}\} \leq \chi^F$, then an optimal strategy for player $n+1$ is to play in period 1.

iv) If $\overline{X}^w_{n+1}(n+1,0) = \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1.

v) If $\overline{X}^w_{n+1}(n+1,0) < \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1.

vi) Otherwise, any decision of player $n+1$ is a neutral strategy.

Corollary 3.4.3 (respectively Corollary 3.4.4) can be proved by adapting the proof of Theorem 3.4.1 to the fact that player $n+1$ cannot have P^L_{n+1} (respectively P^M_{n+1}) as payoff.

Remark 3.4.5. Assume that player $n+1$ knows certainly that the group of n players would like to play in period 1 (respectively 2), then he never can be a leader (resp. follower) because player $n+1$ loses the chance to be in SLMF^w_{n+1} (resp. MLSF^w_{n+1}). Now, the neutral strategy is just considered among the two left possible games between GNEP^w_{n+1} and MLSF^w_{n+1} (resp. between GNEP^w_{n+1} and SLMF^w_{n+1}).

3.5 Data estimation in a specific model

As already discussed in the previous sections, player $n+1$ takes his decision to play period 1 or 2 from an assumed knowledge of collected data $\left(\beta, \{\overline{X}_i\}_{i=1,\ldots,n}, \{x^\star_i\}_{i=1,\ldots,n}\right)$ of $\text{GNEP}^w_n(\beta)$. The most important one is the global maximum $x^\star_i$ of the payoff function $\theta_i(\cdot, x_{-i})$ which is assumed to be independent of x_{-i}, for any $i = 1, \ldots, n$. This value can be quite tricky to determine for player $n+1$. So all along this section let us assume that the objective function of each player is given by the following quadratic form:

$$\theta_i(x_i, x_{-i}) = \alpha x_i - c_i x_i^2. \tag{3.5.1}$$

In the context of electricity market, the constants can be interpreted as follows: let us assume that players = producers all use a certain rough material to produce electricity (coal, oil,...) and that x_i stands for the amount of rough material (in tons) that player i uses to produce his quantity of electricity. If the total amount of available rough product (in tons) is limited by $\beta > 0$ and the price of electricity is $\alpha \geq 0$ (in euro/tons of rough material, thus assuming that all the producers' plans have the same "efficiency") then function θ_i represents the revenue αx_i player i gets minus the cost of production $c_i x_i^2$.

Given this specific formula of the payoff functions θ_i, one clearly has, for any $i = 1, \ldots, n+1$, $x^\star_i = \alpha/(2c_i)$. But, from the perspective of player $n+1$, each C_i, $i = 1, \ldots, n$ is, a priori, only known by player i. That means player $n+1$ is just able to know c_{n+1}, not $\{c_i\}_{i=1,\ldots,n}$.

Let us assume that before entering into game with group of n players, player $n+1$ can observe a certain number of iterations of the GNEP game between the n players. Our aim in forthcoming subsection is to describe one context in which player $n+1$ can deduce the family of cost coefficients c_i from the observation of some GNEP_n between n players and how this knowledge can be used.

3.5.1 Observation phase

Let us assume now that player $n+1$ has observed a finite number $k \in \{1, \ldots, K\}$ of iterations of $\text{GNEP}_n^{w,k}$ played between the n players, each one with a different value of the price α^k and of the maximal exchange volume β^k. From this observation, player $n+1$ will deduce the values of cost coefficients $\{c_i\}_{i=1,\ldots,n}$ of the group of n players.

By using the same arguments about weighted generalized Nash game, let us introduce here a minor modification of $\text{GNEP}_n^w(\alpha, \beta)$ to get some results in the observation phase. Let us recall that the man idea of the weighted notion is to put some weights on the constraint sets of the optimization problems. In Definition 3.3.1, we assume $\sum_{i=1}^{n} w_i = 1$ in condition (3.3.1). These parameters appear to guarantee that the sum of bounds of all players cannot exceed the current market volume. But during this observation phase, only n players are involved into $\text{GNEP}_n^w(\alpha, \beta)$.

So let us consider that n players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that, $\sum_{i=1}^{n} \overline{X}_i \neq \beta$. Let $\mathcal{W} = \{w_1, \ldots, w_n\}$ be a family of weights of the players satisfying condition

$$\begin{cases} \text{for any } i, \quad w_i \in \left[0, \dfrac{\overline{X}_i}{\left|\sum_{i=1}^{n+1} \overline{X}_i - \beta\right|}\right[\\ \text{and } \sum_{i=1}^{n} w_i = 1. \end{cases} \tag{3.5.2}$$

Thus, given the values (α, β), the corresponding weighted generalized Nash equilibrium problem $\text{GNEP}_n^w(\alpha, \beta)$ can be defined here as

Find $\bar{x}^w = (\bar{x}_1^w, \ldots, \bar{x}_n^w) \in \mathbb{R}^n$ such that $\forall i = 1, \ldots, n$, $\bar{x}_i^w$ is a solution of the player's i problem

$$\left(P_i^w(\bar{x}_{-i})\right) \qquad \max_{x_i} \ \alpha x_i - c_i x_i^2$$
$$\text{s.t.} \quad x_i^w \in [0, \overline{X}_i^w(n, 0)],$$

where $\overline{X}_i^w(n,0) = \overline{X}_i - w_i \max\left\{0, \sum_{j=1}^{n} \overline{X}_j - \beta\right\}$.

Let us recall from Proposition 3.2.4 (i) that the inequality $x_i + \sum_{j \neq i}^{n} \bar{x}_j \leq \beta$ is implicitly considered. Let us also observe that due to the positiveness of the cost coefficients c_i, the well-posedness assumption (Assumption 3.3.2) and the uniqueness assumption (Assumption 3.3.3) are automatically fulfilled.

Now, in this context, let us specify the characterization of the solution of $\text{GNEP}_n^w(\alpha, \beta)$ in the following lemma with θ_i as just defined in (3.5.1). This Lemma directly follows Proposition 3.2.5.

Lemma 3.5.1. Consider that n players are interacting on a market with a price $\alpha \in \mathbb{R}_+$, a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that, $\sum_{i=1}^{n} \overline{X}_i \neq \beta$. Assume condition (3.5.2) is satisfied.
Then,

i) $\text{GNEP}_n^w(\alpha, \beta)$ admits a unique weighted Nash equilibrium $\bar{x}^w = (\bar{x}_1^w, \ldots, \bar{x}_n^w)$;

ii) For any $i = 1, \dots, n$, one has

$$\bar{x}_i^w = \begin{cases} \overline{X}_i^w(n,0) & \text{if } \overline{X}_i^w(n,0) < \frac{\alpha}{2c_i}, \\ \frac{\alpha}{2c_i} & \text{otherwise.} \end{cases}$$

So let us go back now to the observation phase and assume that player $n+1$ has observed a finite sequence of n-player generalized Nash game $\text{GNEP}_n^{w,k}(\alpha^k, \beta^k)$, with $k = 1, \dots, K$, where the optimization problem of player i at iteration k is defined as

$$\forall i = 1, \dots, n, \ \left(P_i^w(\alpha^k, \beta^k)\right) \qquad \max_{x_i} \ \alpha^k x_i - c_i x_i^2, \\ \text{s.t.} \quad 0 \le x_i \le \overline{X}_i^{w,k}(n, 0, \beta^k).$$

where $\overline{X}_i^{w,k}(n, 0, \beta^k)$ is the weighted bound with regarding to β^k at the iteration k.

By "observing the $\text{GNEP}_n^{w,k}(\alpha^k, \beta^k)$" we mean that player $n+1$ has accessed to the knowledge of the corresponding equilibrium $\text{GNE}_n^{w,k} = (\bar{x}_1^{w,k}, \dots, \bar{x}_n^{w,k})_k$. Let's consider the following assumption.

Assumption 3.5.2. For any $i = 1, \dots, n$, there exists $k(i) \in \{1, \dots, K\}$ such that $x_i^{w,k(i)} < \overline{X}_i^w(n, 0, \beta^{k(i)})$.

The interpretation is that, for a finite number K, there are some cycles/iterations of Nash games which n players took part in. Player $n+1$ inspects and collects data during the process. In fact, each time one cycle ends, this player has more information about the strategy value that each player i in the group opted for. By selecting a subset of cycles $k(i)$ corresponding player $i = 1, \dots, n$, such that the value of their strategies is strictly less than their maximal consumption bounds, one can obtain the cost coefficients of n players in this specific circumstance. As a direct consequence of Lemma 3.5.1, the forthcoming corollary allows then to deduce the values $\{c_i\}_{i=1,\dots,n}$.

Corollary 3.5.3. Consider that n players are interacting on a market with finite families of price $\{\alpha^k\}_{k=1,\dots,K} \in \mathbb{R}_+$ and maximum exchange volume $\{\beta^k\}_{k=1,\dots,K} \in \mathbb{R}_+^*$ such that, for any k, $\sum_{i=1}^{n} \overline{X}_i \neq \beta^k$.

Assume Assumption 3.5.2 hold. Then,

$$\text{for each } i = 1, \dots, n, \quad c_i = \frac{\alpha^{k(i)}}{2\bar{x}_i^{w,k(i)}}.$$

Assumption 3.5.2 can be consequence of specific structures of the finite sequence $(\alpha^k, \beta^k)_k$ or obtained after a "sufficiently large" number of observations. From that, player $n+1$ picks up the public outputs $\{\bar{x}_i^{w,k(i)}\}_{k(i)}$ for each player i, one by one, among the n players and detects their cost coefficients. The explanation is quite simple, since the optimal value is equal to either $\frac{\alpha}{2c_i^{k(i)}}$ or the individual maximal consumption $\overline{X}_i^{w,k(i)}$, if $\bar{x}_i^{w,k(i)} \neq \overline{X}_i^{w,k(i)}$, surely the optimal solution is the one relating to cost coefficient $c_i^{k(i)}$. With this achievement, player $n+1$ can decide how to act in the two-period game by using the results of Theorem 3.4.1.

3.5.2 Sensitivity analysis: a numerical illustration

In order to illustrate the above results and particularly their sensitivity to the exogeneous parameters α and β, we develop here some numerical simulations. The aim is here to show how the situation of player $n+1$ changes when these parameter evolve or, in other words, how the status of the "favourable strategy" (optimal, safe, neutral, most beneficial...) is sensible to the changes of parameters α and β.

The tests have been conducted by using MATLAB (version R2020b). For each pair of values (α, β), there will be a point in two-dimensional plane $O_{\alpha\beta}$, where the point will be coloured to represent the corresponding strategic property of player $n+1$. For instance,

- a green point ■ (OPTIMAL 1, resp. 2) represents a value (α, β) such that decision of player $n+1$ is an optimal strategy when he plays in period 1 (resp. 2);
- red point ■ (M.SLMF) implies the situation at which the most beneficial strategy for player $n+1$ is to be in SLMF game.

All possibilities and corresponding colours are described in Table 3.6.

■ OPTIMAL 1	■ M.SLMF, M.MLSF
■ SAFE 1	■ M.SLMF
■ NEUTRAL	■ M.SLMF, L.GNEP
■ OPTIMAL 2	■ OPTIMAL 1 or OPTIMAL 2
■ SAFE 2	■ OPTIMAL 1 or SAFE 2
*Abbreviations M. and L. stand for *most beneficial* and *lowest beneficial* strategies respectively.	

Table 3.6: Properties of strategic decition for player $n+1$.

The "status/colour" of the favourable strategy has been determined thanks to Theorem 3.4.1 and Corollaries 3.4.3 and 3.4.4.

The examples are built for three players which means that the role of player $n+1$ will be played by the third player ($n+1=3$). We will consider the four scenarios below described by the following family of four input data $\mathcal{I} = \{\mathcal{I}_k\}_{k=1,\ldots,4}$ as follows:

$\mathcal{I}_1$) $\beta \in]20, 60]$, $\overline{X} = (45, 35, 20)$, $c = (3, 5, 5/2)$ and $w = (9/16, 7/16, 1/10)$;
$\mathcal{I}_2$) $\beta \in]20, 60]$, $\overline{X} = (45, 35, 20)$, $c = (2, 2, 4)$ and $w = (9/16, 7/16, 1/10)$;
$\mathcal{I}_3$) $\beta \in]20, 90]$, $\overline{X} = (45, 35, 20)$, $c = (3, 5, 5/2)$ and $w = (9/16, 7/16, 1/10)$;
$\mathcal{I}_4$) $\beta \in]8, 60]$, $\overline{X} = (18, 24, 8)$, $c = (3, 5, 1)$ and $w = (3/7, 4/7, 1/10)$.

while the value of α will vary, for the four cases, in the interval $]0, 120]$, β is the maximum exchange volume of the market, $c = (c_1, c_2, c_3)$ is the vector of cost coefficients and $\overline{X} = (\overline{X}_1, \overline{X}_2, \overline{X}_3)$ stands for the vector of consumption bounds, and weight vector $w = (w_1, w_2, w_3)$, which is used when $\beta > \min\limits_{i=1,2,3} \overline{X}_i$, represents rate cut in bargain.

We take $\mathcal{I}_2$ as a benchmark and compare the other three cases with it. Input set $\mathcal{I}_1$ differs from $\mathcal{I}_2$ only by a change of c, whereas with $\mathcal{I}_3$ the difference is in the range of β. Lastly, in $\mathcal{I}_4$, there is a modification in c, and in $\overline{X}$ which entails the change of β and w as a consequence.

Simulation 3.5.4. The results of Theorem 3.4.1 are illustrated in figures 3.2-3.6 for input $\mathcal{I}_1$ and in figures 3.2, 3.6 for input $\mathcal{I}_2$.

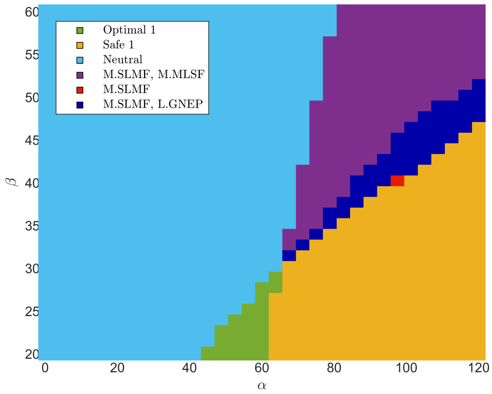

Figure 3.2: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_1$.

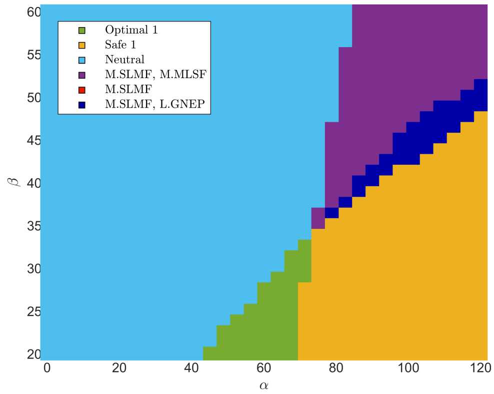

Figure 3.3: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_1$ with replacing $w_3 = 0.08$.

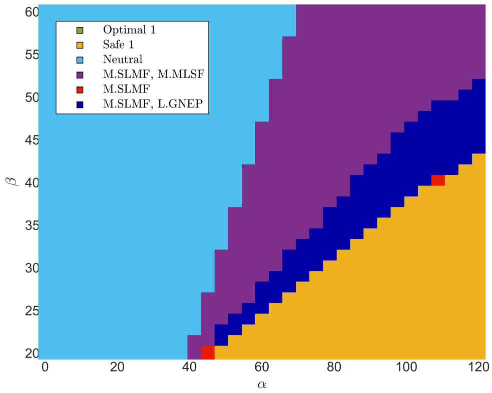

Figure 3.4: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_1$ with replacing $w_3 = 0.15$.

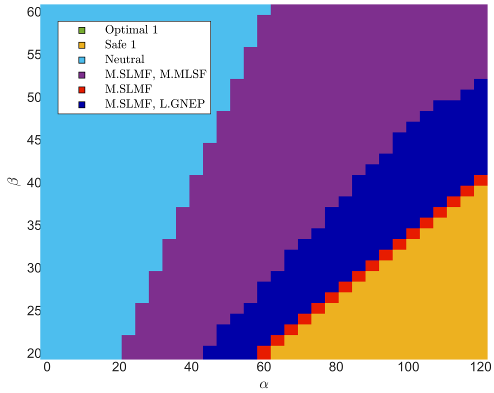

Figure 3.5: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_1$ with replacing $w_3 = 0.2$.

Let us first compare cases depicted in figures 3.2-3.5 respectively by changing coefficients α and β. In Figure 3.2, it is a situation containing all possibility of Theorem 3.4.1, in other words, all colours appear. The figures 3.3, 3.4 and 3.5 show different states at which w_3 varies. It is easy to see that, as the value of w_i increases, the overall graph moves down. The partitions green ■, yellow ■ and cyan ■ become narrower and give place to indigo ■, blue ■ and red ■. This implies that, if one player (not only player $n+1$) suffers more disadvantage because the w_i value is too large, the opportunity to achieve safe strategies is gradually reduced.

As soon as the price α decreases (resp. cost coefficient increases), the frequency of appearing of the cyan colour ■ is higher, meaning that the strategy will be neutral when all kinds of payoffs are the same. If α is too high (or c_{n+1} is too low), it infers that the value of $x^\star_{n+1}$ will be extremely high such that it cannot be a payoff of player $n+1$ since the payoff mainly depends on $\overline{X}_{n+1}$, $\overline{X}^w_{n+1}(n+1, 0)$ and χ^F. As a result, all the other colours will replace a part of cyan ■.

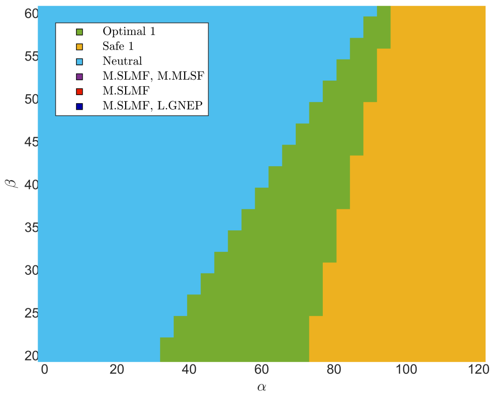

Figure 3.6: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_2$.

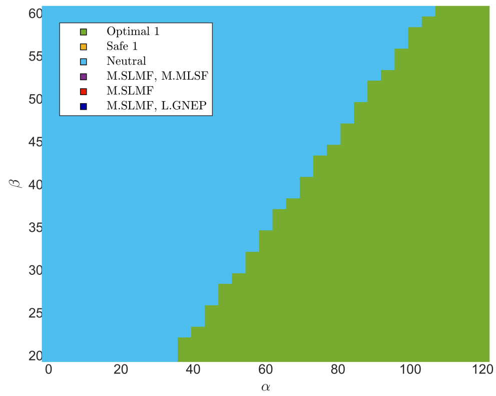

Figure 3.7: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_2$ with replacing $c_3 = 7$.

With the specific strictly concave quadratic objective function that we are considering in these simulations, it is clear that the solution $\bar{x}_{n+1}$ of player $n+1$'s problem belongs to the interval $]0, x^\star_{n+1}]$. But since $\overline{X}^w_{n+1}(n+1,0) \leq \overline{X}_{n+1}$, condition $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} \leq \overline{X}^w_{n+1}(n+1,0)$ will not happen unless either $x^\star_{n+1} \leq \overline{X}^w_{n+1}(n+1,0) \leq \overline{X}_{n+1}$ or $\overline{X}^w_{n+1}(n+1,0) = \overline{X}_{n+1} \leq x^\star_{n+1}$ (with $w_{n+1} = 0$). The case $w_{n+1} = 0$, is clearly a rare situation where player $n+1$ does not make any sacrifice for his production reserves comparing to other n players. Hence, the former case will be more likely to happen that means with a high value of c_{n+1} such that $x^\star_{n+1} \leq \overline{X}_{n+1}$, then a part of inequality conditions in Theorem 3.4.1 (i) is formed for optimal strategy, and the green colour ■ has more chance to appear. Figures 3.6 and 3.7 reflect this observation, since in the first one c_3 equals to 4 but rises to 7 in the second. It results that the green area of Figure 3.7 is greater than the one the other, and the yellow part (safe strategy) is empowered to become the green part (optimal strategy).

Remark 3.5.5. A rather special case to note is the red colour ■ matching with item (iv) in Theorem 3.4.1. Indeed, one can wonder why these points are so sparsely spread on the figures. Actually the case corresponds to the condition $\overline{X}^w_{n+1}(n+1,0) = \chi^F < \min\{x^\star_{n+1}, \overline{X}_{n+1}\}$ and thus involves an equality which is, from a simulation point of view, difficult to exactly reach. Reducing the mesh of simulation would possibly generate more such red points.

Simulation 3.5.6. The results in figures 3.8 and 3.9 correspond to input data $\mathcal{I}_3$ (expanding value of β) and $\mathcal{I}_4$ (reducing values of $\overline{X}$).

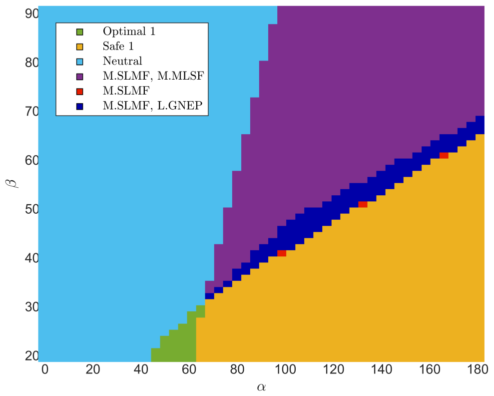

Figure 3.8: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_3$.

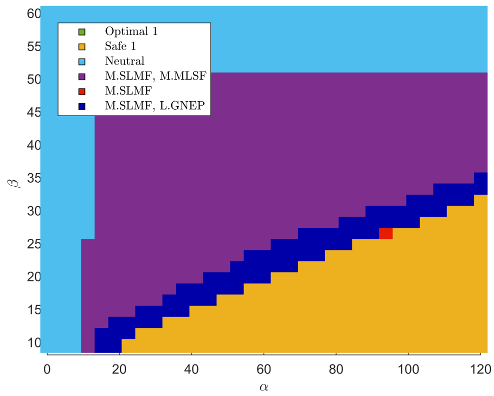

Figure 3.9: Illustrated result of Theorem 3.4.1 for input $\mathcal{I}_4$.

Comparing figures 3.2 and 3.8, the general shape is more or less the same, except that it appears clearly in Figure 3.8, for a fixed α_t, by increasing value of β, when it is exceeded over a certain threshold β_t then, the strategy will become neutral. For instance, at $\alpha_t = 80$, for all $\beta \geq \beta_t \approx 65$, the strategies of player $n+1$ are surely neutral. This threshold also exists in Figure 3.9 but with $\beta_t \approx 50$ since the consumption bounds are not the same. Indeed, this threshold β_t actually corresponds to $\beta_t = \sum_{i=1}^{n+1} \overline{X}_i$. This comes from the fact that when $\sum_{i=1}^{n+1} \overline{X}_i < \beta$ then $\overline{X}^w(n+1, 0) = \overline{X}_{n+1}$and thus $\bar{x}^L_{n+1} = \bar{x}^F_{n+1} = \bar{x}^G_{n+1}$ leading to a neutral situation where any strategy of player $n+1$ will give him the same payoff.

Now take a deeper look on Figure 3.9 which will reveal the effect of changing vector $\overline{X}$ in the input $\mathcal{I}_4$, namely shrinking consumption bounds from the input $\mathcal{I}_1$. Let us analyse the "flat part behaviour" in this figure. Assume that $\beta < \beta_t \approx 50$ is fixed and α is increasing. Since $x^\star_{n+1} = \alpha/(2c_{n+1})$ then for α greater than a certain threshold (for example, $\alpha > \alpha_t \approx 15$ in Figure 3.9) then $\min\{x^\star_{n+1}, \overline{X}_{n+1}\} = \overline{X}_{n+1}$ and thus the value of α has no longer influence and the case (iii) of Theorem 3.4.1 always occurs.

The next two simulations are built to illustrate corollaries 3.4.3 and 3.4.4 which are extensions of Theorem 3.4.1 in the case where player $n+1$ has information on the period chosen by the group of n players.

Simulation 3.5.7 (for Corollary 3.4.3)**.** The results in figures 3.10 and 3.11 describe cases $\mathcal{I}_1$ and $\mathcal{I}_2$ when the group of n players plays in period 1.

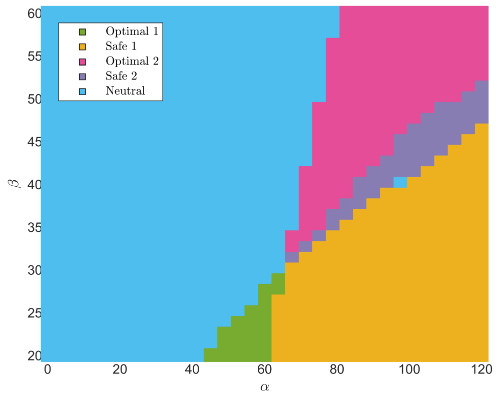

Figure 3.10: Illustrated result of Corollary 3.4.3 for input $\mathcal{I}_1$.

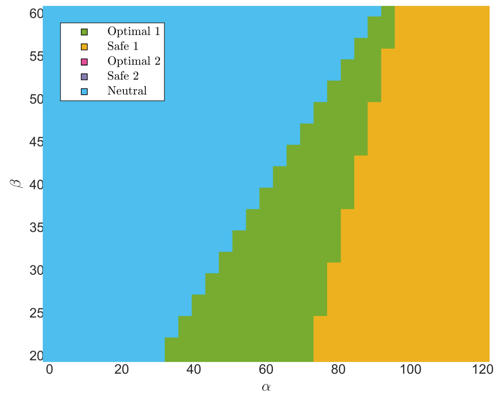

Figure 3.11: Illustrated result of Corollary 3.4.3 for input $\mathcal{I}_2$.

Obviously, for player $n+1$, with this additional information, stronger conclusions of the decision making (than for example “most beneficial”) can be reached. Specifically, player $n+1$ will possibly achieve Optimal 2, Safe 2. This leads to a remarkable fact. If the group of n players claims to play in period 1, then depending on the input data, strategic decisions of player $n+1$ will be very diversified containing various choices such as Optimal 1, Safe 1, Optimal 2, Safe 2 and Neutral.

When comparing figures 3.10, 3.11 with figures 3.2, 3.6, one can observe that colours ■, ■ and ■ are substituted for ■, ■ and ■ respectively.

Simulation 3.5.8 (for Corollary 3.4.4)**.** The results in figures 3.12 and 3.13 describe cases $\mathcal{I}_1$ and $\mathcal{I}_2$ when the group of n players plays in period 2.

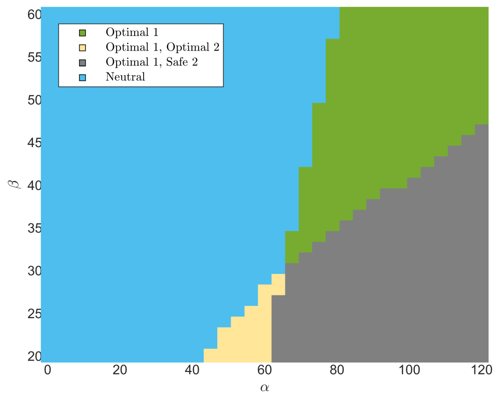

Figure 3.12: Illustrated result of Corollary 3.4.4 for input $\mathcal{I}_1$.

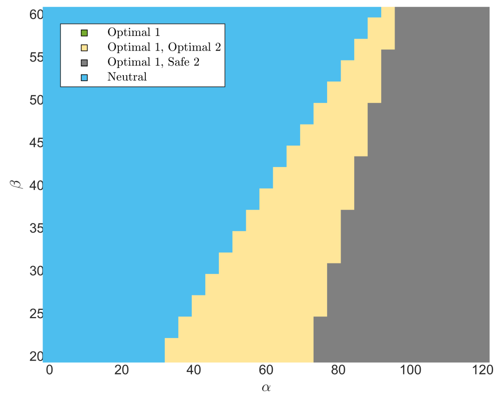

Figure 3.13: Illustrated result of Corollary 3.4.4 for input $\mathcal{I}_2$.

A noticeable comment in the two figures 3.12 and 3.13 that is, if not neutral, the strategies for player $n+1$ are at least safe. Here a new category of decision making appears in these two figures, a strategy which can be considered as “more effective” than any previously mentioned one: yellow case ■ (Optimal 1 and Optimal 2). Alike neutral strategy, the player can play in any period without concerning. While the neutral strategy provides the same payoff for each case (leader, follower or player of a Nash game), in this case player $n+1$ can avoid the lowest payoff $P^F_{n+1} = \min_{\kappa\in\{L,G,F\}} P^\kappa_{n+1}$ in period 2. In this situation, choices of player $n+1$ seem very positive. The strategies are most likely Optimal and Safe (■, ■, ■) or Neutral ■ in the worst case.

These last figures clearly confirm an evident fact: more information player $n+1$ has, better and less risky will be his decision making.

3.6 Conclusion

To conclude, a game, in which n players are interacting in a generalized Nash game with an additional player $n+1$ waiting for entry, is analysed. This work generalizes the approach which has been done by Bernhard von Stengel [92] in the case of a duopoly game. The new player can face several possibilities to inter-operate with the group of n players. In particular, the author has evaluated the gap of the new player's payoff between two possible models: on the one hand a non-cooperative model in which this player is one of the players of a Nash game (one-level game) and on the other hand a bi-level game in which this player plays the role of a common follower or common leader. Then, the three different kinds of games, SLMF_{n+1}, GNEP_{n+1} and MLSF_{n+1} are taken into account in order to estimate the $(n+1)$-player equilibria. The new concepts of optimal, safe and neutral strategy has been introduce for palyer $n+1$ by knowing exact period to move, or most/lowest beneficial strategy in the case that a clear decision cannot be made but provide a better information.

The next attention has been focused on defining the set of practicable equilibria in case of GNEP, and the new notion of weighted Nash equilibrium has been introduced. The weighted Nash equilibria can be interpreted as a selection process of the generalized Nash equilibria. It is then proved that, given a family of weights of the players satisfying some conditions (see (3.2.1)) and under mild assumptions, there exists a unique weighted generalized Nash equilibrium. This uniqueness property has been used when considering a weighted version of problems SLMF^w_{n+1}, GNEP^w_{n+1} and MLSF^w_{n+1}, to avoid "optimistic" or "pessimistic" formulations. Accordingly, a comprehensive consequence for equilibria of three kinds of games is established. This former setting, finally, is the basis for the latter step that is devoted to the study of a particular case where the utility function is a concave quadratic function and the constraint set is defined by inequalities. In this context a complete decision making policy is developed.

Basing on the assumptions of strictly concave utility functions and compact constraint sets, there is an elaboration of the less risky strategy for player $n+1$ with the knowledge of the model's constant data but not knowing the decision of the group of n players to play in period 1 or 2. It has stated all possibilities of entering games for player $n+1$ by utilizing the uniqueness of weighted and the decisions concept, thus being the decision making policy.

Numerical simulations has been examined bringing to the fore the sensitivity of the "favourable strategy" to the value of the price α and the maximal exchange volume β of the market. Even though, the treatment is used for a specific model, observing the behaviour of outcome reveals several exciting remarks.

One possible extension of this framework would be to weaken the initial hypothesis that whenever the additional players join the market, the group of n players has a common strategy/decision on the period they want to play. The situation would be then much more complicated since one can face a multi-leader-multi-follower game with a lot of possible combinations of leaders/followers groups.

Chapter 4

Qualitative Stability

In this section, the author considers Nash equilibrium problem perturbed by external parameters. The aim is then to estimate the stability of the solution set in the qualitative sense. Similar analysis has been conducted for example by Ait Mansour-Aussel (see, e.g. [1, 2]) for Variational Inequality. The analysis is done here through different approaches to examine qualitative stability. Then, a comparison with other existing results is made in order to reveal the difference between the old and new methods. At the same time, different sets of assumptions are, from here, provided to apply for distinguishing cases. In particular some stability results are obtained only using assumptions (quasi-monotonicity and locally upper-sign continuity) of the components of the game. Application to Single-leader-multi-follower game is also considered.

4.1 Parametrized Nash equilibrium problem

Let us first introduce the parametrized Nash equilibrium (parametrized NEP) for which we develop here a qualitative stability analysis.

Let us consider that n players are interacting in a non-cooperative way, each of them controlling a variable x_i chosen in a strategy set C_i of $\mathbb{R}^{N_i}$. As usually the notation x_{-i} stands for the vector of the strategies of the other players than player i and the abuse of notation $x = (x_i, x_{-i})$ will be used. Each player i aims to minimize his cost/loss function $\theta_i : \mathbb{R}^{N_i} \times \mathbb{R}^{N-N_i} \longrightarrow \mathbb{R}$ where $N = \sum_{i=1}^{n} N_i$. Now consider that some exogenous parameters affect the non-cooperative game between the n players: $\lambda \in \mathcal{L} \subset \mathbb{R}^L$ represents a perturbation in the objective functions $\theta_i(\lambda, \cdot)$ while $\mu \in \mathcal{M} \subset \mathbb{R}^M$ describes the one affecting the constraint sets $K_i(\mu)$. Thus the cost functions θ_i are now defined from $\mathcal{L} \times \mathbb{R}^{N_i} \times \mathbb{R}^{N-N_i}$ to $\mathbb{R}$ while the set K_i becomes a set-valued map $K_i : \mathcal{M} \rightrightarrows \mathbb{R}^{N_i}$.

For any $\lambda \in \mathcal{L}$, $\mu \in \mathcal{M}$, the *parametrized Nash equilibrium problem* is to find $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that $\forall i \in \{1, \ldots, n\}$, $\bar{x}_i$ solves

$$\big(P_i(\lambda, \mu, \bar{x}_{-i})\big) \qquad \begin{array}{rl} \min\limits_{x_i} & \theta_i(\lambda, x_i, \bar{x}_{-i}), \\ \text{s.t.} & x_i \in K_i(\mu). \end{array} \tag{4.1.1}$$

Our aim in this work is to study the regularity properties (semi-continuity, closedness) of the solution map $\mathrm{NEP} : \mathcal{L} \times \mathcal{M} \longrightarrow 2^Y$ of the parametrized NEP defined by

$$\mathrm{NEP}(\lambda, \mu) = \Big\{ \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \ : \ \forall i \in \{1, \ldots, n\},\ \bar{x}_i \text{ solves } \big(P_i(\lambda, \mu, \bar{x}_{-i})\big) \Big\}.$$

Nevertheless the continuity analysis will be here conducted in a very general setting since the cost function θ_i will be only assumed to be quasi-convex in the player's variable x_i.

4.2 Stability result by variational inequality approach

In this section and both forthcoming sections (4.3 and 4.4), three different approaches will be developed to prove the closedness of the solution map NEP of the parametrized Nash equilibrium problem (4.1.1). The approach proposed in this section is based on the reformulation of the Nash problem into a specific variational inequality and on the use of a qualitative stability result established in [2] for quasi-monotone Stampacchia variational inequality (that is a Stampacchia variational inequality defined by a quasi-monotone operator). As explained latter on, the resulting closedness result for the NEP map is called "product-type", in opposition with the "component-wise results" Theorem 4.3.1.

It is well known (see, e.g. [45, Proposition 1.4.2]) that if each cost function θ_i is continuously differentiable and convex in the player's variable x_i and the constraint sets K_i are non-empty closed and convex then, for any value $(\lambda, \mu) \in \mathcal{L} \times \mathcal{M}$, the parametrized Nash equilibrium problem (4.1.1) is equivalent to the following Stampacchia variational inequality

$$\text{find } \bar{x} \in K(\mu) \text{ such that } \quad \big\langle \nabla F(\lambda, \bar{x}), y - \bar{x} \big\rangle \geq 0, \quad \forall y \in K(\mu),$$

where $F : \mathcal{L} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is defined by $F(\lambda, x) = \big(F_1(\lambda, x), \ldots, F_n(\lambda, x)\big)$ with, for any $i = 1, \ldots, n$, $F_i(\lambda, x) = \nabla_{x_i} \theta_i(\lambda, x_i, x_{-i})$. Nevertheless our aim is here to establish some qualitative stability results for quasi-convex (and thus possibly non convex) and possibly non differentiable cost functions. Thus the above reformulation of the parametrized Nash equilibrium problem in terms of Stampacchia variational inequality cannot be used as it is and an extension to the general setting of quasi-convex continuous cost functions is needed. It is also well-known that out of the convex case, such variational inequality based on the gradients corresponds to the first order necessary conditions which are, in general, not sufficient optimality condition. This is essentially due to the fact that the good properties/behaviour of quasi-convex functions are on their sub-level sets while gradient and the generalization the sub-differentials are based on the epigraph of the functions (see [7] for more details). Following this observation, in [20], the author created a first order tool, called the *adjusted normal operator* N^a which are really adapted to quasi-convex optmization. Sufficient optimality conditions can be found in [20] and [31] while calculus rules have been developed in [25].

Using the developments on the adjusted sub-level set and adjusted normal operator ([7]), the link between generalized Nash Equilibrium problem with quasi-convex cost functions and their associated quasi-variational inequality has been studied in [17]. In the particular case of our parametrized Nash equilibrium problem one can deduce from [17, theorems 3.1 and 4.1] (see also [18, Addendum, Theorem 1]) the following equivalence result which is the keystone of the first approach developed in this section.

Proposition 4.2.1. Assume that each cost function θ_i is continuous in variable $x = (x_i, x_{-i})$ and semi-strictly quasi-convex in the player's variable x_i. Assume moreover that the constraint sets $K_i(\mu)$ are non-empty closed and convex. Then, for any value $(\lambda,\mu) \in \mathcal{L} \times \mathcal{M}$, the parametrized Nash equilibrium problem (4.1.1) is equivalent to the Stampacchia variational inequality $S\big(N^a_\theta(\lambda,\cdot),K(\mu)\big)$

$$\text{find } \bar{x} \in K(\mu) \text{ and } \bar{x}^* \in N^a_\theta(\lambda,\bar{x}) \text{ such that } \langle \bar{x}^*, y-\bar{x}\rangle \geq 0,\ \forall y \in K(\mu),$$

where $N^a_\theta : \mathcal{L} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is defined by $N^a_\theta(\lambda,x) = \big(T_1(\lambda,x),\dots,T_n(\lambda,x)\big)$ with, for any $i = 1,\dots,n$,

$$T_i(\lambda,x) = \begin{cases} N^a_f(x) \cap \overline{\mathcal{B}}_i(0,1) & \text{if } x_i \in \operatorname{argmin}_{\mathbb{R}^{N_i}} \theta_i(\lambda,\cdot,x_{-i}), \\ \operatorname{conv}\big(N^a_{\theta_i(\lambda,\cdot,x_{-i})}(x_i) \cap S_i(0,1)\big) & \text{otherwise.} \end{cases}$$

The proof follows essentially the lines of [17, theorems 3.1 and 4.1] and is given here for sake of completeness.

Proof. So let us assume that for a given couple $(\lambda,\mu) \in \mathcal{L}\times\mathcal{M}$, and $\bar{x}$ is a Nash equilibrium of the parametrized problem (4.1.1). It means that for any $i \in \{1,\dots,n\}$, $\bar{x}_i \in K_i(\mu)$ and $\theta_i(\lambda,\bar{x}_i,\bar{x}_{-i}) \leq \theta_i(\lambda,x_i,\bar{x}_{-i})$, for any $x_i \in K_i(\mu)$. Let us now fix $i \in \{1,\dots,n\}$. If $\bar{x}_i \in \arg\min_{\mathbb{R}^{N_i}} \theta_i(\lambda,\cdot,\bar{x}_{-i})$ then set $x^*_i = 0$. Otherwise the level set $S_{\theta_i(\lambda,\cdot,\bar{x}_{-i})}(\bar{x}_i)$ is closed convex with a non-empty interior

$$\operatorname{int} S_{\theta_i(\lambda,\cdot,\bar{x}_{-i})}(\bar{x}_i) = \{u_i \ : \ \theta_i(\lambda,u_i,\bar{x}_{-i}) < \theta_i(\lambda,\bar{x}_i,\bar{x}_{-i})\}.$$

Since the constraint set $K(\mu)$ is convex and does not intersect $\operatorname{int} S_{\theta_i(\lambda,\cdot,\bar{x}_{-i})}(\bar{x}_i)$, by a classical Hahn-Banach theorem, there exists $x^*_i \in T_i(\lambda,\bar{x})$ such that $\langle x^*_i, x_i-\bar{x}_i\rangle \geq 0$, for any $x_i \in K_i(\mu)$. Defining then $\bar{x}^* = (\bar{x}^*_1,\dots,\bar{x}^*_n)$, one can conclude that $\bar{x}$ is a solution of the Stampacchia variational inequality $S\big(N^a_\theta(\lambda,\cdot),K(\mu)\big)$. Conversely assume that $\bar{x} \in S(N^a_\theta(\lambda,\cdot),K(\mu))$, that is there exists $x^* \in N^a_\theta(\lambda,\bar{x})$ such that

$$\langle x^*, x-\bar{x}\rangle \geq 0, \text{ for any } x \in K(\mu). \tag{4.2.1}$$

Let us fix $i \in \{1,\dots,n\}$. If $\bar{x}_i \in \operatorname{argmin}_{\mathbb{R}^{N_i}} \theta_i(\lambda,\cdot,\bar{x}_{-i})$ then clearly $\bar{x}_i \in \operatorname{argmin}_{K(\mu)} \theta_i(\lambda,\cdot,\bar{x}_{-i})$. Otherwise $x^*_i \in \operatorname{conv}\big(N^a_{\theta_i(\lambda,\cdot,x_{-i})}(x_i) \cap S_i(0,1)\big)$ and thus, by [17, Lemma 3.1], there exists $\lambda > 0$ such that $\tilde{x}^*_i = \lambda.x^*_i \in N^a_{\theta_i(\lambda,\cdot,\bar{x}_{-i})}(\bar{x}_i) \setminus \{0\}$. Thus for any $x_i \in K(\mu)$, the vector $x = (\bar{x}_1,\dots,\bar{x}_{i-1},x_i,\bar{x}_{i+1},\dots,\bar{x}_n)$ is an element of $K(\mu)$ and therefore, together with (4.2.1), $\tilde{x}^*_i \in S\big(N^a_{\theta_i(\lambda,\cdot,\bar{x}_{-i})},K_i(\mu)\big)$. By [31, Proposition 3.2], $\bar{x}_i \in \arg\min_{K(\mu)} \theta_i(\lambda,\cdot,\bar{x}_{-i})$. And the proof is complete since the conclusion holds for any $i \in \{1,\dots,n\}$. □

We are now in position to prove our first stability result for the solution map NEP of the parametrized Nash equilibrium problem (4.1.1).

Theorem 4.2.2. Let us suppose that for any $(\lambda,\mu) \in \mathcal{L} \times \mathcal{M}$, $\mathrm{NEP}(\lambda,\mu)$ is non-empty and

i) for all μ, $K(\mu)$ is non-empty closed convex with non-empty interior;

ii) for all i, the cost function θ_i is continuous in variable $x = (x_i,x_{-i})$ and semi-strictly quasi-convex in the player's variable x_i;

iii) for all (λ,μ), the map $N^a_\theta(\lambda,\cdot)$ is quasi-monotone and locally upper sign-continuous on $K(\mu)$;

iv) for all $(\lambda^k, \mu^k) \xrightarrow{k\to\infty} (\lambda, \mu)$, $u^k \xrightarrow{k\to\infty} u$, and $v^k \xrightarrow{k\to\infty} v$ with $u^k \in \operatorname{int} K(\mu^k)$, $u \in \operatorname{int} K(\mu)$, $v^k \in K(\mu^k)$, $v \in K(\mu)$,

$$\sup_{u^* \in N^a_\theta(\lambda,u)} \langle u^*, v-u\rangle \le \liminf_{k\to\infty} \sup_{u^{k,*} \in N^a_\theta(\lambda^k,u^k)} \langle u^{k,*}, v^k - u^k\rangle;$$

v) for any $\mu^k \xrightarrow{k\to\infty} \mu$, $K(\mu^k) \xrightarrow[k\to\infty]{\text{Mosco}} K(\mu)$.

vi) for any $i \in \{1,\dots,n\}$, $\operatorname{argmin}_{\mathbb{R}^{N_i}} \theta_i(\lambda,\cdot,x_{-i}) \cap K_i(\mathcal{M}) = \emptyset$.

Then the solution map NEP of parametrized Nash equilibrium problem (4.1.1) is closed on $\mathcal{L} \times \mathcal{M}$.

Proof. Let us denote by S the set-valued map $S : \mathcal{L} \times \mathcal{M} \rightrightarrows \mathbb{R}^N$ defined, according to assumption (vi), as

$$S(\lambda,\mu) = S\big(N^a_\theta(\lambda,\cdot), K(\mu)\big) = S^*\big(N^a_\theta(\lambda,\cdot), K(\mu)\big).$$

Taking into account assumptions (i), (iii), (iv) and (v), all the hypotheses of Theorem 4.2 in [2] are fulfilled and one can conclude to the closedness of the map S^* on $\mathcal{L} \times \mathcal{M}$. Now the conclusion follows directly from Proposition 4.2.1. □

It is here important to notice that the scope of Theorem 4.2.2 is quite limited. While hypotheses (i), (ii), (v) are quite natural and not so restrictive, and assumption (iv) and (vi) are a technical requirement, assumption (iii) clearly restricts the applicability of this stability result. Indeed according to [20], one immediately gets from assumption (ii) that, for any $i = 1,\dots,n$ and for any $x_{-i} \in \prod_{j=1, j\neq i}^{n} K_j(\mu)$, the map $N^a_{\theta_i}(\lambda,\cdot,x_{-i})$ is quasi-monotone and locally upper sign-continuous on $K(\mu)$. But, as illustrated by Example 4.2.3 (respectively Example 4.2.4), the product $T = \prod_{i=1}^{n} T_i$ of quasi-monotone operators (respectively locally upper sign-continuous operators) is in general not quasi-monotone (respectively not locally upper sign-continuous).

And thus, this clearly motivates us to obtain the closedness of the solution map NEP under alternative assumptions, typically under component-wise assumptions. This will be reached in the forthcoming Theorem 4.3.1.

Example 4.2.3. *Let* $C_1 = [-2,2]$, $C_2 = [-2,2]$, $\mathcal{L}_1 = [1/2, 1]$, $\mathcal{L}_2 = [0,2]$ *and* $C = C_1 \times C_2$, $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. *For any* $x_2 \in C_2$ *and for any* $\lambda_1 \in \mathcal{L}_1$, *let* $T_1(\lambda_1,\cdot,x_2) : C_1 \rightrightarrows \mathbb{R}$ *be defined by* $T_1(\lambda_1,x_1,x_2) = \{x_1^2 + \lambda_1\}$. *For* $x_1 \in C_1$, $\lambda_2 \in \mathcal{L}_2$, *let* $T_2(\lambda_2,x_1,\cdot) : C_2 \rightrightarrows \mathbb{R}$ *be defined by* $T_2(\lambda_2,x_1,x_2) = \{x_1^2 + 1 + 2\lambda_2\}$. *For any* $(\lambda_1,\lambda_2,x_1,x_2) \in \mathcal{L} \times C$, *the maps* T_1 *and* T_2 *are clearly quasi-monotone respectively on* C_1 *and* C_2 *since they are respectively derivatives of the quasi-convex functions* $x_1 \longmapsto x_1^3/3 + \lambda_1 x_1$ *and* $x_2 \longmapsto x_2^3/3 + (1+2\lambda_2)x_2$. *But the product operator* $T : \mathcal{L}_1 \times \mathcal{L}_2 \times C_1 \times C_2 \rightrightarrows \mathbb{R}^2$ *defined by* $T(\lambda,x) = \{x_1^2 + \lambda_1\} \times \{x_1^2 + 1 + 2\lambda_2\}$ *is not quasi-monotone on* $C_1 \times C_2$. *Indeed, if one considers the points* $x = (x_1,x_2) = (0, 1/2)$ *and* $y = (y_1,y_2) = (-2,1)$ *then, for any* λ_1, λ_2, $x^* = (x_1^2 + \lambda_1, x_2^2 + 1 + 2\lambda_2) \in T(\lambda,x)$ *and*

$y^* = (y_1^2 + \lambda_1, y_2^2 + 1 + 2\lambda_2) \in T(\lambda, y)$. *Then one has*

$$\langle x^*, y - x\rangle = \left\langle \begin{pmatrix} x_1^2 + \lambda_1 \\ x_2^2 + 1 + 2\lambda_2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \lambda_1 \\ 5/4 + 2\lambda_2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle$$
$$= -2\lambda_1 + \lambda_2 + \frac{5}{8} \in [-3/8, 5/8],$$

thus being positive for some λ *(take e.g.* $\lambda = (1, 2)$*) while*

$$\langle y^*, y - x\rangle = \left\langle \begin{pmatrix} y_1^2 + \lambda_1 \\ y_2^2 + 1 + 2\lambda_2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 4 + \lambda_1 \\ 2 + 2\lambda_2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle$$
$$= -2\lambda_1 + \lambda_2 - 7 < 0.$$

for any $\lambda \in \mathcal{L}$. *This example is a "parametrized version" of [7, Example 1].*

Similarly, the example below, extracted from [11], shows an extremely simple case for which locally upper-sign continuity is not preserved by product.

Example 4.2.4. *Let* $C_1 = [-1, 1]$, $C_2 = [-1, 1]$, $\mathcal{L}_1 = [1/2, 1]$, $\mathcal{L}_2 = [0, 2]$ *and* $C = C_1 \times C_2$, $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. *For any* $x_2 \in C_2$ *and for any* $\lambda_1 \in \mathcal{L}_1$, *let* $T_1(\lambda_1, \cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ *be defined by* $T_1(\lambda_1, x_1, x_2) = \{-1\}$. *For* $x_1 \in C_1$, $\lambda_2 \in \mathcal{L}_2$, *let* $T_2(\lambda_2, x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ *be defined by* $T_2(\lambda_2, x_1, x_2) = \begin{cases} \{1\} & \text{if } x_2 < 0 \\ \{1/2\} & \text{if } x_2 = 0 \\ \{1\} & \text{if } x_2 > 0 \end{cases}$. *Then, each component operator is upper-sign continuous and locally upper sign-continuous respectively on* C_1 *and* C_2 *but the product operator* $T : \mathcal{L} \times C \rightrightarrows \mathbb{R}^2$ *given by* $T(\lambda, x) = T_1(\lambda, x) \times T_2(\lambda, x)$ *is not even locally upper-sign continuous on* C *(see [11, Example 2] for the proof).*

4.3 Stability result by direct approach

Our aim in the section is to obtain a so-called *"componentwise-type"* stability result for the solution map NEP, that is a closedness result in which the needed assumptions are made directly on the *"component operators"* T_i, avoiding thus the product-type assumption (*iii*) of Theorem 4.2.2.

Theorem 4.3.1. Let us suppose that for any $(\lambda, \mu) \in \mathcal{L} \times \mathcal{M}$, NEP$(\lambda, \mu)$ is non-empty and

i) for all μ and any i, $K_i(\mu)$ is convex with non-empty interior;

ii) for all i, the cost function θ_i is continuous in variable $x = (x_i, x_{-i})$ and semi-strictly quasi-convex in the player's variable x_i;

iii) for all i, $(\lambda^k, \mu^k) \xrightarrow{k\to\infty} (\lambda, \mu)$, $u_i^k \xrightarrow{k\to\infty} u_i$, and $v_i^k \xrightarrow{k\to\infty} v_i$ with $u_i^k \in \operatorname{int} K_i(\mu^k)$, $u_i \in \operatorname{int} K_i(\mu)$, $v_i^k \in K_i(\mu^k)$, $v_i \in K_i(\mu)$,
$$\sup_{u_i^* \in T_i(\lambda, u_i, v_{-i})} \langle u_i^*, v_i - u_i\rangle \le \liminf_{k\to\infty} \sup_{u_i^{k,*} \in T_i(\lambda^k, u_i^k, v_{-i}^k)} \langle u_i^{k,*}, v_i^k - u_i^k\rangle;$$

iv) for all i, for any $\mu^k \xrightarrow{k\to\infty} \mu$, $K_i(\mu^k) \xrightarrow[k\to\infty]{\text{Mosco}} K_i(\mu)$.

Then the solution map NEP of parametrized Nash equilibrium problem (4.1.1) is closed on $\mathcal{L} \times \mathcal{M}$.

Proof. Let $(\lambda^k, \mu^k)_k$ and $(x^k)_k = \big((x_1^k)_k, \ldots, (x_n^k)_k\big)$ be respectively sequences of $\mathcal{L}\times\mathcal{M}$ and $K(\mu^k) = \prod_{i=1}^{n} K_i(\mu^k)$ such that the sequence $(\lambda^k, \mu^k)_k$ converges to $(\hat{\lambda}, \hat{\mu})$ and the sequence $(x^k)_k$ converges to $x = (x^1, \ldots, x^n)$ with, for any $k \in \mathbb{N}$, $x^k \in \mathrm{NEP}(\lambda^k, \mu^k)$.

For any i, let V_{x_i} be a convex neighbourhood of x_i and $\varphi^i_{x_i}(\hat{\lambda}, \cdot, x_{-i}) : K_i(\hat{\mu}) \cap V_{x_i} \rightrightarrows \mathbb{R}^{N_i}$ be a set-valued map, upper sign-continuous at x_i and such that, for any $\hat{v}_i \in K_i(\hat{\mu}) \cap V_{x_i}$, $\varphi^i_{x_i}(\hat{\lambda}, \hat{v}_i, x_{-i})$ is a non-empty compact subset of $T_i(\hat{\lambda}, \cdot, \hat{v}_{-i})$.

Due to the Mosco convergence of $\big(K_i(\mu^k)\big)_k$ and since $x_i^k \in K_i(\mu^k)$, for any i, we immediately obtain that $x_i \in K_i(\hat{\mu})$. Let y_i be an arbitrary point of $[\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}]\backslash\{x_i\}$. Since V_{x_i} and $K_i(\hat{\mu})$ are convex, the segment $[y_i, x_i[$ is included in $\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}$. Let $z_{i,t} = ty_i + (1-t)x_i$ with $t \in]0,1]$ be an element of $[y_i, x_i[$.

Since, for any k, x^k is an element of $\mathrm{NEP}(\lambda^k, \mu^k)$, we can deduce that, for any i, x_i^k is element of $\operatorname{argmin}_{K_i(\mu^k)} g_i(\lambda^k, \cdot, x_{-i}^k)$. Let us fix $i \in \{1, \ldots, n\}$. Now according to [17, Theorem 4.1], there exists $y_i^{k,*} \in T_i(\lambda^k, x_i^k, x_{-i}^k)$ such that

$$\langle x_i^{k,*}, z_i^k - x_i^k \rangle \geq 0, \quad \forall z_i^k \in K_i(\mu^k). \tag{4.3.1}$$

Now combining hypothesis (iv) with Proposition 3.2, $(i) \Longrightarrow (ii)$ in [2], for any i one can find a sequence $(z_i^k)_k$ converging to $z_{i,t}$ such that $z_i^k \in \operatorname{int} K_i(\mu^k)$, $z_i^k \neq x_i^k$. But since $x_i^{k,*} \neq 0$ and z_i^k is an element of $\operatorname{int} K_i(\mu^k)$, we can assume, without loss of generality, that the inequality in (4.3.1) is strict.

Now since the cost function θ_i are quasi-convex with respect to the player's variable x_i, the map T_i is quasi-monotone with regard to x_i and therefore, for any k

$$\langle z_i^{k,*}, z_i^k - x_i^k \rangle \geq 0, \quad \forall z_i^{k,*} \in T_i(\lambda^k, z_i^k, x_i^k).$$

from which we can deduce, by coupling with hypothesis (iii), that

$$\sup_{z_{i,t}^* \in T_i(\hat{\lambda}, z_{i,t}, x_{-i})} \langle z_{i,t}^*, x_i - z_{i,t} \rangle \leq \liminf_{k\to\infty} \sup_{z_i^{k,*} \in T_i(\lambda^k, z_i^k, x_{-i}^k)} \langle z_i^{k,*}, x_i^k - z_i^k \rangle \leq 0.$$

Therefore we can devive that, any $z_{i,t}^* \in \varphi^i_{x_i}(\hat{\lambda}, z_{i,t}, x_{-i})$,

$$\langle z_{i,t}^*, x_i - z_{i,t} \rangle \leq 0. \tag{4.3.2}$$

and thus, for any $t \in]0,1]$,

$$0 = t\langle z_{i,t}^*, y_i - z_{i,t} \rangle + (1-t)\langle z_{i,t}^*, x_i - z_{i,t} \rangle \leq t\langle z_{i,t}^*, y_i - z_{i,t} \rangle.$$

which yields

$$\inf_{z_{i,t}^* \in \varphi^i_{x_i}(\hat{\lambda}, z_{i,t}, x_{-i})} \langle z_{i,t}^*, y_i - x_i \rangle \geq 0, \quad \forall t \in]0,1[.$$

Therefore according to the upper sign-continuity of $\varphi^i_{x_i}(\hat{\lambda}, \cdot, x_{-i})$ on $K_i(\hat{\mu}) \cap V_{x_i}$ and to the compactness of $\varphi^i_{x_i}(\hat{\lambda}, z_{i,t}, x_{-i})$, one has

$$\max_{x_i^* \in \varphi^i_{x_i}(\hat{\lambda}, x_i, x_{-i})} \langle x_i^*, y_i - x_i \rangle \geq 0,$$

which means that for any $y_i \in [\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}]\backslash\{0\}$ there exists $x_i^* \in T_i(\hat{\lambda}, x_i, x_{-i})$ such that $\langle x_i^*, y_i - x_i \rangle \geq 0$. The latter still holds for any $y_i \in \operatorname{int} K_i(\hat{\mu})$ since in this case, $K_i(\hat{\mu})$

being convex, the point $x_i + \frac{\rho}{\|y_i - x_i\|}(y_i - x_i)$ is an element of $[\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}] \setminus \{x_i\}$ for ρ sufficiently small. This can be summarized as

$$\inf_{y_i \in [\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}]} \sup_{x_i^* \in \varphi^i_{x_i}(\hat{\lambda}, x_i, x_{-i})} \langle x_i^*, y_i - x_i \rangle \geq 0, \quad \forall i.$$

and thus, since the operator $\varphi^i_{x_i}(\hat{\lambda}, \cdot, x_{-i})$ is convex valued, by a Sion minimax theorem, we immediately deduce that there exists an element $x_i^* \in \varphi^i_{x_i}(\hat{\lambda}, x_i, x_{-i}) \subset T_i(\hat{\lambda}, x_i, x_{-i})$ such that,

$$\langle x_i^*, y_i - x_i \rangle \geq 0, \quad \forall y_i \in [\operatorname{int} K_i(\hat{\mu}) \cap V_{x_i}].$$

But $K_i(\hat{\lambda})$ being convex,the previous inequality still holds true for any $y_i \in K_i(\hat{\lambda})$. Using again [17, Theorem 4.1], the point x_i is an element of $\operatorname{argmin}_{K_i(\hat{\mu})} g_i(\hat{\lambda}, \cdot, x_{-i})$.

Combining all as once,

$$x \in \prod_{i=1}^{n} \operatorname*{argmin}_{K_i(\hat{\mu})} \; g_i(\hat{\lambda}, \cdot, x_{-i}) \equiv \mathrm{NEP}(\hat{\lambda}, \hat{\mu}).$$

Since the latter is true for any $(\hat{\lambda}, \hat{\mu}) \in \mathcal{L} \times \mathcal{M}$, the set-valued map NEP is closed on $\mathcal{L} \times \mathcal{M}$. □

Theorem 4.3.1 allows to obtain the same closedness conclusion as in Theorem 4.2.2 but without the restrictive assumption (*iii*) on the quasi-monotonicity and locally upper sign-continuity of the product map N^a_θ. Nevertheless Theorem 4.2.2 cannot be deduced from Theorem 4.3.1 because actually in the latter theorem the technical assumption (*iii*) is made on each of the component maps T_i while the same technical assumption (assumption (*iv*)) is assumed only for the product map N^a_θ in Theorem 4.2.2.

4.4 Alternative approach and comparison

In the above established Theorem 4.2.2 and Theorem 4.3.1, we proved the closedness of the solution map NEP of the parametrized Nash equilibrium problem 4.1.1 under the semi-strict quasi-convexity of the cost function of the different players. In this section, our aim is to explore the possibility of such a closedness result without assuming any convexity of the cost functions. This approach is based on the basic following observation: if one considers, for any i, the map $\mathrm{NEP}_i : \mathcal{L} \times \mathcal{M} \rightrightarrows \mathbb{R}^N$ defined, for any $(\lambda, \mu) \in \mathcal{L} \times \mathcal{M}$, by the set

$$\mathrm{NEP}_i(\lambda, \mu) = \Big\{ (x_i, x_{-i}) \in K(\mu) \;:\; x_i \in \operatorname*{argmin}_{y_i \in K_i(\mu)} \theta_i(\lambda, y_i, x_{-i}) \Big\}.$$

then the solution set $\mathrm{NEP}(\lambda, \mu)$ of the parametrized Nash equilibrium problem (4.1.1) can be equivalently defined as

$$\mathrm{NEP}(\lambda, \mu) = \bigcap_{i=1}^{n} \mathrm{NEP}_i(\lambda, \mu).$$

This approach has been used in [27, Theorem 3.1] to prove the existence of solutions for a Single-Leader-Multi-Follower problem (see defintion in Section 4.5).

Theorem 4.4.1. Let us suppose that for any $(\lambda, \mu) \in \mathcal{L} \times \mathcal{M}$, $\mathrm{NEP}(\lambda, \mu)$ is non-empty and

i) for all i, the set-valued map $K_i(\cdot)$ is lower semi-continuous on $\mathcal{M}$ with non-empty closed graph;

ii) for all i, the cost function θ_i be continuous with respect to the three variables λ, x_i and x_{-i};

Then the set-valued map NEP is closed on $\mathcal{L} \times \mathcal{M}$.

Proof. Taking into account (4.4), we immediately have that if, for any i, the map NEP_i is closed on $\mathcal{L} \times \mathcal{M}$ then so is the map NEP. So let us fix i and take sequences $(\lambda^k, \mu^k)_k \xrightarrow{k\to\infty} (\hat{\lambda}, \hat{\mu})$, $(x_i^k, x_{-i}^k)_k \xrightarrow{k\to\infty} (x_i, x_{-i})$ such that, for any k, $x^k = (x_i^k, x_{-i}^k) \in \mathrm{NEP}_i(\lambda^k, \mu^k)$.

Since $\mathrm{NEP}_i(\lambda, \mu) \subset K_i(\mu)$, we can deduce that the sequence $(x_i^k)_k$ is included into $K_i(\mu)$ and thus, thanks to the closedness of K_i that $x_i \in K_i(\mu)$. For any fixed x_{-i} and μ, take an arbitrary $\tilde{x}_i \in K_i(\mu)$. By lower semi-continuity of the map K_i, there exists a sequence $(\tilde{x}_i^k)_k$ converging to $\tilde{x}_i$ and such that, for any k, $\tilde{x}_i^k \in K_i(\mu^k)$.

Now since, for any k, $x^k \in \mathrm{NEP}_i(\lambda^k, \mu^k)$, we have in particular that $x_i^k \in \operatorname{argmin}_{K_i(\mu^k)} \theta_i(\lambda^k, \cdot, x_{-i}^k)$ and thus, in particular,

$$\theta_i(\lambda^k, x_i^k, x_{-i}^k) \le \theta_i(\lambda^k, \tilde{x}_i^k, x_{-i}^k).$$

Since the latter is true for any k, by continuity of θ_i, we conclude that

$$\theta_i(\lambda, x_i, x_{-i}) \le \theta_i(\lambda, \tilde{x}_i, x_{-i}).$$

The vector $\tilde{y}_i$ being arbitrary chosen in $K_i(\mu)$, we infer that $y_i \in \mathrm{NEP}_i(\lambda, \mu)$ and thus the map NEP_i is closed on $\mathcal{L} \times \mathcal{M}$. Then the proof is complete since this holds true for any i. □

In order to emphasize the complementarity of Theorem 4.3.1 and Theorem 4.4.1, let us provide here two examples of parametrized Nash equilibrium problems: for the first one, the closedness of the solution map NEP can be proved thanks to Theorem 4.3.1 but not using Theorem 4.4.1 while for the second example the reverse situation holds true.

Example 4.4.2. *Let us consider a parametrized Nash equilibrium problem of 2 players ($i \in \{1, 2\}$) controlling respectively the real variables x_1 and x_2 and for which the strategy maps are described by the set-valued maps $K_i : \mathbb{R}_+^* \rightrightarrows \mathbb{R}$ as*

$$K_1(\mu) = \{x_1 : 0 \le x_1 \le \mu_1\}, \quad K_2(\mu) = \{x_2 : 0 \le x_2 \le \mu_2\}.$$

and the cost functions $\theta_i : \mathbb{R}_+^* \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined as

$$\theta_1(\lambda, x_1, x_2) = h_1(\lambda, x_2) + \begin{cases} x_1^2 - 1 & \text{if } 1 \le x_1 \\ 0 & \text{if } 0 \le x_1 < 1 \\ x_1 & \text{otherwise} \end{cases},$$

$$\theta_2(\lambda, x_1, x_2) = h_2(\lambda, x_1) + \begin{cases} (x_2 - 1)^2 - 4 & \text{if } x_2 \le -1 \\ 0 & \text{if } -1 < x_2 \le 0 \\ -x_2 & \text{otherwise} \end{cases},$$

where h_i are non continuous functions, for any $i \in \{1, 2\}$.

Obviously, for any μ, $K_i(\mu)$ *is non-empty closed convex interval of* $\mathbb{R}$ *and for any* $\mu^k \xrightarrow{k\to\infty} \mu$, *the sequence of sets* $\big(K(\mu^k)\big)_k$ *Mosco converges to* $K(\mu)$. *It's clear that, the cost functions* θ_1 *and* θ_2 *are continuous quasi-convex respectively with respect to* x_1 *and* x_2. *And thus, since* $\arg\min \theta_i(\lambda, x)$ *is empty, for* $i \in \{1,2\}$,

$$\begin{aligned}
T_1(\lambda, x_1, x_2) &= N^a_{\theta_1(\lambda,\cdot,x_2)}(x_1) \setminus \{0\} = \big\{x_1^* \in \mathbb{R} \mid \langle x_1^*, u - x_1\rangle \le 0,\ \forall u \in S^a_{\theta_1(\lambda,\cdot,x_2)}(x_1)\big\} \setminus \{0\} \\
&= \big\{x_1^* \in \mathbb{R} \mid \langle x_1^*, \hat{u}\rangle \le 0,\ \forall \hat{u} \in S^a_{\theta_1(\lambda,\cdot,x_2)}(x_1) - \{x_1\}\big\} \setminus \{0\} \\
&= \big\{x_1^* \in \mathbb{R} \mid \langle x_1^*, \hat{u}\rangle \le 0,\ \forall \hat{u} \le 0\big\} \setminus \{0\} \\
&= \big(S^a_{\theta_1(\lambda,\cdot,x_2)}(x_1) - \{x_1\}\big)^\circ \setminus \{0\} = \mathbb{R}^*_+, \\
T_2(\lambda, x_1, x_2) &= N^a_{\theta_2(\lambda,x_1,\cdot)}(x_2) \setminus \{0\} = \big\{x_2^* \in \mathbb{R} \mid \langle x_2^*, v - x_2\rangle \le 0,\ \forall v \in S^a_{\theta_2(\lambda,x_1,\cdot)}(x_2)\big\} \setminus \{0\} \\
&= \big\{x_2^* \in \mathbb{R} \mid \langle x_2^*, \hat{v}\rangle \le 0,\ \forall \hat{v} \in S^a_{\theta_2(\lambda,x_1,\cdot)}(x_2) - \{x_2\}\big\} \setminus \{0\} \\
&= \big\{x_2^* \in \mathbb{R} \mid \langle x_2^*, \hat{v}\rangle \le 0,\ \forall \hat{v} \ge 0\big\} \setminus \{0\} \\
&= \big(S^a_{\theta_2(\lambda,x_1,\cdot)}(x_2) - \{x_2\}\big)^\circ \setminus \{0\} = \mathbb{R}^*_-.
\end{aligned}$$

Now let us verify that assumption (iii) *in Theorem 4.3.1 holds true for the first variable* x_1. *Suppose that there are sequences* $(\lambda^k, \mu^k) \xrightarrow{k\to\infty} (\lambda, \mu)$, $(\bar{x}_1^k, \bar{x}_2^k) \xrightarrow{k\to\infty} (\bar{x}_1, \bar{x}_2)$ *and* $(x_1^k, x_2^k) \xrightarrow{k\to\infty} (x_1, x_2)$. *Then we claim that,*

$$\sup_{x_1^* \in T_1(\lambda, x_1, \bar{x}_2)} \langle x_1^*, \bar{x}_1 - x_1\rangle \le \liminf_{k\to\infty} \sup_{x_1^{k,*} \in T_1(\lambda^k, x_1^k, \bar{x}_2^k)} \langle x_1^{k,*}, \bar{x}_1^k - x_1^k\rangle. \tag{4.4.1}$$

First, if $\bar{x}_1 < x_1$ *then* $\langle x_1^*, \bar{x}_1 - x_1\rangle \le 0$, *for any* $x_1^* \in T_1(\lambda, x_1, \bar{x}_2)$ *and thus* $\sup_{x_1^* \in T_1(\lambda,x_1,\bar{x}_2)} \langle x_1^*, \bar{x}_1 - x_1\rangle = 0$. *Due to the convergence of* $(\bar{x}_1^k)_k$ *and* $(x_1^k)_k$ *to* $\bar{x}_1$ *and* x_1 *respectively, for* k *large enough, we have* $\bar{x}_1^k - x_1^k \le 0$, *thus* $\sup_{x_1^{k,*} \in T_1(\lambda^k, x_1^k, \bar{x}_2^k)} \langle x_1^{k,*}, \bar{x}_1^k - x_1^k\rangle = 0$ *implying that inequality (4.4.1) is satisfied. Now in the second case, that is whenever* $x_1 \le \bar{x}_1$, *leads to* $\infty \le \infty$ *for inequality (4.4.1) and the claim is proved.*

By the same calculus and arguments, assumption (iii) *is fulfilled for variable* x_2. *Hence, thanks to Theorem 4.3.1, the set-valued solution map* NEP *is closed on* $\mathcal{L} \times \mathcal{M}$. *Nevertheless, this closedness cannot be obtained using Theorem 4.4.1 since the cost functions are not continuous for all variables* λ, x_1, x_2 *and set-valued maps* K_i *is not lower semi-continuous.*

Example 4.4.3. *Let us consider a parametrized Nash equilibrium problem of* 2 *players* ($i \in \{1,2\}$) *controlling respectively the real variables* x_1 *and* x_2 *and for which the strategy maps are described by the set-valued maps* $K_i : (\mathbb{R}^*_+)^2 \rightrightarrows \mathbb{R}$ *as*

$$K_1(\mu_1, \mu_2) = \big\{x_1 : x_1 \in [a^3, a^3 + 20] \ \ \textit{with} \ \ a \in [-\mu_1, \mu_1]\big\}, \tag{4.4.2}$$

$$K_2(\mu_1, \mu_2) = \big\{x_2 : x_2 \in [{}^1\!/_4\, b^2, {}^1\!/_4\, b^2 + 3] \ \ \textit{with} \ \ b \in [-\mu_2, \mu_2]\big\}. \tag{4.4.3}$$

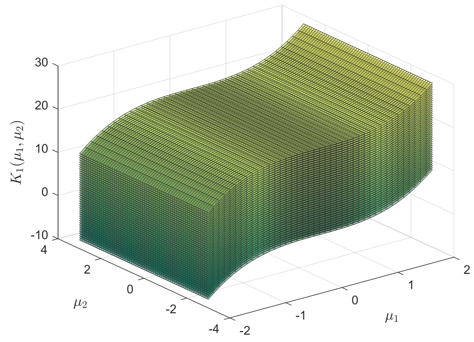

Figure 4.1: Set-valued map $K_1(\mu_1, \mu_2)$ with $\mu_1 = 2$.

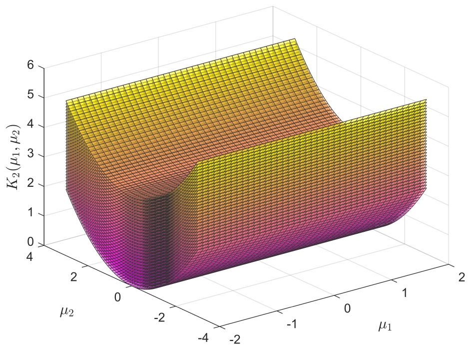

Figure 4.2: Set-valued map $K_2(\mu_1, \mu_2)$ with $\mu_2 = 3$.

It is not difficult to show that both set-valued maps K_1 and K_2 are lower semi-continuous with non-empty and closed graph. Let us suppose that the cost functions $\theta_i : (\mathbb{R}_+^*)^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined as

$$\theta_1(\lambda, x_1, x_2) = x_1^3 + x_2^2 + x_1 x_2 + \lambda_1 + 3, \tag{4.4.4}$$

$$\theta_2(\lambda, x_1, x_2) = {}^1\!/_2\, x_1^2 + 2x_2^2 + 2x_1^2 x_2 + \lambda_2. \tag{4.4.5}$$

Clearly, both cost functions $\theta_{i=1,2}$ are continuous on $\mathbb{R}_+^* \times \mathbb{R}^2$ and therefore, according to Theorem 4.4.1, the solution map NEP *is a closed set-valued map. Nevertheless, even if the cost functions θ_i are very regular $\big($actually $\mathcal{C}^\infty(\mathbb{R}^3)\big)$, it is also clearly not quasi-convex. Therefore, one cannot use Theorem 4.3.1 to prove the closedness of the solution map for this example.*

Let us end this section by comparing Theorem 4.4.1 with similar results of the literature. First a very similar closedness result has been proved, in infinite dimensional setting, in [64] also assuming continuity of the objective functions over all the variables. Nevertheless, in Theorem 4.4.1 we only assume the lower semi-continuity of the constraint map while in [64, Theorem 4.1] the lower semi-continuity and closedness of the constraint map were required.

Other results with the same assumptions (closedness, lower semi-continuity) on constraint maps but a stronger assumption (pseudo-continuity) on cost functions are investigated in [72, Theorem 1] and [73, Theorem 1]. Another similar case is [73, Theorem 2] in which constraint maps are not parametrized while a technical assumption coupling with the upper pseudo-continuity for all the variables is assumed for the objective functions.

Finally, in [66, Theorem 3.1], the author extended the previously quoted works by only assuming upper semi-continuity to the objective functions (plus technical assumptions) with regard to all variables. Nonetheless, they do not consider perturbation on the constraint sets.

4.5 Application to SLMF game

In mathematical economics, a classical situation in which a Nash game is parametrized corresponds to the so-called *single-leader-multi-follower* model (SLMF in short). It corresponds to a hierarchical exchange model in which a group of n players, called the *followers* interacts in a non-cooperative way through a Nash equilibrium problem but a NEP parametrized by an external data which corresponds to the decision variable of an additional player called the *leader*. This kind of model has been proved to be extremely useful for many applications, for example in energy management (see, e.g. [80, 81, 87]) or transport design [93]. Our aim in this last section is to use the above qualitative stability results to deduce properties of SLMF models. For a general presentation of SLMF models, the interested reader can consult the recent chapter [30].

This SLMF game is well-posed only if, for any decision of the leader, NEP(x) admits at least a solution. Our aim here is to derive some regularity properties of the *marginal map* of the leader's problem.

Definition 4.5.1. The (optimistic) marginal map of the SLMF problem is $L : X \longrightarrow \mathbb{R}$ defined by

$$\forall x \in X, \quad L(x) := \min_{y \in \mathrm{NEP}(x)} f(x, y).$$

Using the above analysis of the qualitative stability of parametrized Nash games, we obtain, as a consequence of Theorem 4.3.1, the upper semi-continuity of the marginal function L.

Proposition 4.5.2. For $i \in \{1, \dots, n\}$ and for each $x \in X$, let us suppose that X is closed, NEP(x) is non-empty and

i) for all i, for all x, $K_i(x)$ is convex with non-empty interior and $K_i(X) = \bigcup_{x \in X} K_i(x)$ is compact in $\mathbb{R}^{N_i}$;

ii) for all i, $g_i(x, \cdot, y_{-i})$ be continuous quasi-convex in $y_i \in K_i(x)$ with all x, y_{-i};

iii) for all i, $(x^k) \xrightarrow{k\to\infty} x$, $u_i^k \xrightarrow{k\to\infty} u_i$, and $v_i^k \xrightarrow{k\to\infty} v_i$ with $u_i^k \in \operatorname{int} K_i(x^k)$, $u_i \in \operatorname{int} K_i(x)$, $v_i^k \in K_i(x^k)$, $v_i \in K_i(x)$,

$$\sup_{u_i^* \in T_i(x, u_i, v_{-i})} \langle u_i^*, v_i - u_i \rangle \le \liminf_{k\to\infty} \sup_{u_i^{k,*} \in T_i(x^k, u_i^k, v_{-i}^k)} \langle u_i^{k,*}, v_i^k - u_i^k \rangle;$$

iv) for all i, for any $x^k \xrightarrow{k\to\infty} x$, $K_i(x^k) \xrightarrow[k\to\infty]{\text{Mosco}} K_i(x)$.

Then L is upper semi-continuous.

Before proving this proposition let us state, as an immediate consequence of Theorem 4.3.1 and [6, Proposition 1.4.7] and thanks to the compactness of the set $K(\mathcal{M}) = \prod_{i=1}^{n} K_i(\mathcal{M})$, the following corollary.

Corollary 4.5.3. Let us suppose that for any $(\lambda, \mu) \in \mathcal{L} \times \mathcal{M}$, NEP($\lambda, \mu$) is non-empty and

i) for all μ and any i, $K_i(\mu)$ is convex with non-empty interior and $K_i(\mathcal{M}) = \bigcup_{\mu \in \mathcal{M}} K_i(\mu)$ is compact in $\mathbb{R}^{N_i}$;

ii) for all i, the cost function θ_i is continuous in variable $x = (x_i, x_{-i})$ and semi-strictly quasi-convex in the player's variable x_i;

iii) for all i, $(\lambda^k, \mu^k) \xrightarrow{k\to\infty} (\lambda, \mu)$, $u_i^k \xrightarrow{k\to\infty} u_i$, and $v_i^k \xrightarrow{k\to\infty} v_i$ with $u_i^k \in \operatorname{int} K_i(\mu^k)$, $u_i \in \operatorname{int} K_i(\mu)$, $v_i^k \in K_i(\mu^k)$, $v_i \in K_i(\mu)$,

$$\sup_{u_i^* \in T_i(\lambda, u_i, v_{-i})} \langle u_i^*, v_i - u_i \rangle \leq \liminf_{k\to\infty} \sup_{u_i^{k,*} \in T_i(\lambda^k, u_i^k, v_{-i}^k)} \langle u_i^{k,*}, v_i^k - u_i^k \rangle;$$

iv) for all i, for any $\mu^k \xrightarrow{k\to\infty} \mu$, $K_i(\mu^k) \xrightarrow[k\to\infty]{\text{Mosco}} K_i(\mu)$.

Then solution map NEP of the parametrized Nash equilibrium problem (4.1.1) is upper semi-continuous on $\mathcal{L} \times \mathcal{M}$.

Proof of Proposition 4.5.2. Using Corollary 4.5.3 with $\mathcal{L} = \mathcal{M} = X$ we conclude that the solution map NEP is upper semi-continuous on x. On the other hand, for any $x \in X$, $K(x) = \prod_{i=1}^{n} K_i(x)$ is a compact set of $\mathbb{R}^N$ and thus the values of the map NEP are compact in $\mathbb{R}^N$. Finally, since f is upper semi-continuous in (x, y) then, according to [6, Theorem 1.4.16], the marginal map L is upper semi-continuous on x. □

It is clear from the above proof that conclusion of Corollary 4.5.3 would hold true by using Theorem 4.2.2 or Theorem 4.4.1 and by adapting accordingly the set of assumptions.

4.6 Conclusion

Completing existing qualitative stability results for the closedness of the solution map of a parametrized Nash game, the author proposes in this work three different sets of hypotheses which are shown to be complementary. An application to multi-leader-follower games is also provided through a semi-continuity result for single-leader-multi-follower game. Other types of multi-leader-follower games could be considered with possible application but it is out of the scope of this work.

Chapter 5

Radner Existence

Being a non-cooperative game, the Radner problem describes a broad class of problems in which a two period time process occurs, say "now" and "tomorrow"; an equilibrium point at present is determined by a previously devised strategy. This is done by considering all possible real states of the market, that is, modelling, in a sense an uncertainty on the state of the world tomorrow. However, by predicting a finite set of possible scenarios, the players can prepare some strategies to react in the future (or the next period). In comparision, in a Nash game, the time is fixed according to each occurrence of the game, whereas the Radner problem describes the more general case from the very beginning. This obviously makes handling the Radner problem more complex in its approach. Recently, Radner equilibrium problems have attracted more attention (see, e.g. [16, 71]). The present chapter proposes a contribution in this vein.

5.1 Sequential trading exchange under uncertainty

In general a REP considers a timeline with a finite sequence of future time periods with uncertain future realizations. Nevertheless, for a classical REP, one can just consider two time periods: $t = 0$ and $t = 1$ representing in real case studies a spot and future decision, respectively. In $t = 0$, the agents have perfect knowledge of the market, whereas in $t = 1$, uncertainty plays a key role and different uncertain situations can occur. Let us denote with $l \in \mathcal{L} := \{1, \dots, L\}$, the commodities, $s \in \mathcal{S} := \{1, \dots, S\}$ each state of the world and with $i \in \mathcal{I} := \{1, \dots, I\}$ each consumer. The numbers $\{L,\ S,\ I\}$ belong to $\mathbb{N}$ and are greater than 1.

Let us then denote with "commodity 1", a reference commodity that plays a special role in the market as an intermediary exchanging goods (for example gold, silver or wheat).

Being public in the market, a common prediction/anticipation $q = (q^1, \dots, q^s, \dots, q^S) \in \mathbb{R}^S_+$ is known by all consumers, where q^s stands for the expected price of the commodity 1 in case state s occurs at $t = 1$. Also declared in market, there is a vector $e_i = (e_i^{11}, \dots, e_i^{1S}, \dots, e_i^{ls}, \dots, e_i^{L1}, \dots, e_i^{LS}) \in \mathbb{R}^{LS}_{+,*}$ of initial endowment where e^{ls} describes the endowment of commodity l that each consumer i will receive at $t = 1$ if the state of the world is s. An example consists of the wheat quantity harvested by consumers, depending on the weather conditions.

When $t = 0$, consumers sign contracts to buy or sell commodities to be consumed at time $t = 1$. The real consumption will occur in $t = 1$ and it will depend on the state s that will be realized. Then, for consumer i, z_i^s stands for the number of units of commodity 1 that will be traded at time 1 if state s occurs: if $z_i^s > 0$, then i will receive, at time 1, this amount of commodity 1, while $z_i^s < 0$ means that i, at time 1, will deliver this amount of the commodity 1. The vector $z_i := (z_i^1, \dots, z_i^S) \in \mathbb{R}^S$ represents thus the vector of contracts that consumer i signs. Since the commission on these contracts is made at time 0, the associated unit price/value of commodity 1 is the anticipated one, that is q. More precisely, if $z_i^s > 0$ then consumer i promises to buy a quantity z_i^s of commodity 1 at time 1 if state s occurs and he will thus pay, at time 0, $q^s \cdot z_i^s$. If $z_i^s < 0$ then consumer i promises to sell a quantity z_i^s of commodity 1 at time 1 if state s occurs and will thus receive, at time 0, $q^s \cdot z_i^s$. For each consumer i, these incomes/payments will be made for any possible states of the world in such a way that the total cost at time 0 for consumer i is $\langle \bar{q}, z_i \rangle_S$, where $\bar{q}$ is the initial vector of price at $t = 0$.

At time 0, consumer i also decides about his consumption plans for time 1. The corresponding state-contingent commodity vector for consumer i, $y_i = (y_i^{11}, \ldots, y_i^{1S}, \ldots, y_i^{ls}, \ldots, y_i^{L1}, \ldots, y_i^{LS}) \in \mathbb{R}_+^{LS}$, describes for any commodity l the quantity y_i^{ls} that i plans to buy at time 1 if state s occurs. The common state-contingent prices of commodities at time 1 is given by $r = (r^{11}, \ldots, r^{1S}, \ldots, r^{ls}, \ldots, r^{L1}, \ldots, r^{LS}) \in \mathbb{R}_+^{LS}$. A simple example of such a contingent delivery is inspired from [33], fire insurance for a house. As raised in the contract at the date α, the insurance is paid independent of which unknown event is chosen by Nature at the date ω: fire or no fire. Delivery from the insurance company to an insured customer with corresponding money enables the customer to rebuild the house only if Nature chooses the event "fire" (not "flood" or any else state).

At the time $t = 1$, the uncertainty is resolved and the state of the world s becomes known, then the price vector r of commodities is determined (often by a spot market) and each agent i receives his endowment $e_i^s = (e_i^{1s}, \ldots, e_i^{Ls}) \in \mathbb{R}^L$. Each consumer i aims to maximize his utility function $u_i : \mathbb{R}_+^{LS} \longrightarrow \mathbb{R}$ and it is denoted by $u : \mathbb{R}_+^{LSI} \longrightarrow \mathbb{R}^I$ the vector valued function defined by

$$u(y) := \big(u_1(\cdot, y_{-1}), \ldots, u_I(\cdot, y_{-I})\big) \qquad \forall y = (y_1, \ldots, y_I) \in \mathbb{R}_+^{LSI}.$$

Moreover, consumer i faces budget constraints, one for each state. For the initial state, the budget constraint is $\langle \bar{q}, z_i \rangle_S \leq 0$, that is the payment cannot exceed the income in the future market. For all other states, the budget constraints are $\langle \bar{r}^s, y_i^s \rangle_L \leq \langle \bar{r}^s, e_i^s \rangle_L + \bar{r}^{1s} z_i^s$, for any $s \in \mathcal{S}$. The left-hand side is the cost of consumption y_i; the right-hand side is the value of her endowment plus the value in that state of her position in the future market z_i^s.

For the sets of contracts we set $C = \prod_{i \in \mathcal{I}} \widetilde{C} \subset \mathbb{R}^{SI}$ where $\widetilde{C} = \prod_{s \in \mathcal{S}} C^s \subset \mathbb{R}^S$ and $C^s := \Big[-\sum_{i \in \mathcal{I}} e_i^{1s}, \sum_{i \in \mathcal{I}} e_i^{1s}\Big] \subset \mathbb{R}$ while for the sets of state-contingent commodity vectors we define $\widetilde{K} := \prod_{s \in \mathcal{S}, l \in \mathcal{L}} K^s \subset \mathbb{R}_+^{LS}$ where $K^s := \big[0, E^s\big] \subset \mathbb{R}_+$ with $E^s = \sum_{l \in \mathcal{L}} \sum_{i \in \mathcal{I}} e_i^{ls} \in \mathbb{R}_{+,*}$ is the total endowment if state s occurs at time 1. Although $\widetilde{C}$ and $\widetilde{K}$ do not depend on i and hold for all consumers, we can still interpret it as dependent on each i as a generalization instead of a common constant constraint, that means $\widetilde{C}_i := \widetilde{C}$ and $\widetilde{K}_i := \widetilde{K}$ for any i. This setting might be technical but just a way to adapt using result of *net-lower-sign continuity* which will be introduced later on.

Let us now recall the concept of Radner equilibrium for a sequential trading exchange.

Definition 5.1.1 (Radner equilibrium problem)**.** The vector $(\bar{q}, \bar{r}, \bar{z}, \bar{y}) \in \mathbb{R}_+^S \times \mathbb{R}_+^{LS} \times C \times \mathbb{R}_+^{LSI}$ is a Radner equilibrium if

1. for any $i \in \mathcal{I}$:

$$u_i(\bar{y}_i, \bar{y}_{-i}) = \max_{(z_i,y_i)\,\in\,\widetilde{C}_i\times\mathbb{R}_+^{LS}} u_i(y_i, \bar{y}_{-i})$$
$$\text{s.t.} \quad \begin{cases} \langle \bar{q}, z_i \rangle_S \le 0 \\ \langle \bar{r}^s, y_i^s \rangle_L \le \langle \bar{r}^s, e_i^s \rangle_L + \bar{r}^{1s} z_i^s, \quad \forall s \in \mathcal{S} \end{cases} \tag{5.1.1}$$

2. for any $s \in \mathcal{S}$ and $l \in \mathcal{L}$:

$$\sum_{i\in\mathcal{I}} \bar{z}_i^s = 0, \quad \sum_{i\in\mathcal{I}} \bar{y}_i^{ls} \le \sum_{i\in\mathcal{I}} e_i^{ls}. \tag{5.1.2}$$

It is worth noticing that Definition 5.1.1 differs from Definition 1 in [16] since the utility function $u_i(\bar{y}_i, \bar{y}_{-i})$ depends on both the decision vector y_i of consumer i and the decision vectors y_{-i} of the other consumers except i. This framework arises in many concrete problems where the sequential game can be seen as a generalized Nash game in which the payoff of each player is influenced by the decisions of all market players.

An equivalent formulation of Radner equilibrium could be described in the two step process below:

a) For any i and (q, r), denote by $\Psi_i(q, r)$ the set of the solutions $\big(\bar{z}_i(q, r), \bar{y}_i(q, r)\big)$ of the maximization problem (5.1.1);

b) Now $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a Radner equilibrium of the considered sequential trading if $(\bar{q}, \bar{r}) \in \mathbb{R}_+^S \times \mathbb{R}_+^{LS}$ and $\big(\bar{z}_i(\bar{q}, \bar{r}), \bar{y}_i(\bar{q}, \bar{r})\big) \in \Psi_i(\bar{q}, \bar{r})$ for any i and

$$\sum_{i\in\mathcal{I}} \bar{z}_i^s(\bar{q}, \bar{r}) = 0 \quad \text{and} \quad \sum_{i\in\mathcal{I}} \bar{y}_i^s(\bar{q}, \bar{r}) \le \sum_{i\in\mathcal{I}} e_i^{ls}.$$

Let us observe that, since all the budget constraints are homogeneous of degree zero with respect to price, we can assume, without loss of generality that the prices belong to the following sets

$$q \in \Delta^S := \Big\{ q \in \mathbb{R}_+^S : \sum_{s\in\mathcal{S}} q^s = 1 \Big\},$$
$$r^s \in \Delta^L := \Big\{ r^s \in \mathbb{R}_+^L : \sum_{l\in\mathcal{L}} r^{ls} = 1 \Big\}, \text{ for any } s \in \mathcal{S} \text{ and } r \in \prod_{s\in\mathcal{S}} \Delta^L.$$

Then, let us define $\Delta = \Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L \subset \mathbb{R}_+^{S+LS}$. For any $(q, r) \in \Delta$, let us introduce the bounded set $B(q, r) := \prod_{i\in\mathcal{I}} B_i(q, r)$ where

$$B_i(q, r) := \Big\{ (z_i, y_i) \in X_i : \langle q, z_i \rangle_S \le 0,$$
$$\langle r^s, y_i^s \rangle_L \le \langle r^s, e_i^s \rangle_L + r^{1s} z_i^s, \ \forall s \in \mathcal{S} \Big\}.$$

Please note that for any $i \in \mathcal{I}$ and for any (q, r), the subset $B_i(q, r)$ is non-empty and convex since it is defined from affine continuous functions under non-empty domain.

5.2 Quasi-variational Inequality formulation

In the sequel, we will consider the following set of assumptions.

Assumption 5.2.1. For $l \in \mathcal{L}$, $s \in \mathcal{S}$, $i \in \mathcal{I}$, $y_i \in \mathbb{R}_+^{LS}$ and $y_{-i} \in \mathbb{R}_+^{LS(I-1)}$, let us assume that

i) for any i, $e_i \in \mathbb{R}_+^{LS}$, that is, $e_i^{ls} > 0$ for any l and s;

ii) for any i, for all y_{-i}, $u_i(\cdot, y_{-i})$ is strictly increasing in terms of the goods $1s$ for any s, which means

$$\forall \hat{y}_i,\ \tilde{y}_i \in \mathbb{R}_+^{LS} :\ \hat{y}_i \geq \tilde{y}_i,\ \hat{y}_i^{1s} > \tilde{y}_i^{1s} \Longrightarrow u_i(\hat{y}_i, y_{-i}) > u_i(\tilde{y}_i, y_{-i});$$

iii) for any i, for all y_{-i}, $u_i(\cdot, y_{-i})$ is locally non-satiated for any s and for all l, which means

$$\forall y_i \in \mathbb{R}_+^{LS},\ \varepsilon > 0,\ \exists\, \hat{y}_i = (\dots, y_i^{l1}, \dots, \hat{y}_i^{ls}, \dots, y_i^{lS}, \dots) \in \mathbb{R}_+^{LS}$$
$$\text{and } \|\hat{y}_i - y_i\| < \varepsilon :\ u_i(\hat{y}_i, y_{-i}) > u_i(y_i, y_{-i});$$

iv) for any i, for all y_{-i}, $u_i(\cdot, y_{-i})$ is continuous quasi-concave in y_i.

These assumptions are very classical in a wide range of models. From an economic point of view, in Assumption 5.2.1, the item (i) is called *survivability assumption*, that is, each agent i is endowed with each commodity l in each state s. This implies that at any point of time, for any kind of goods and any circumstance, the endowment is strict positive thus any player always has at least something (not nothing) to create something afterwards. In reality, the endowment can be non-positive but consequently the agent is forced to be out of the game. Here, in short term, we require this assumption to ensure the presence of all players. The item (ii) reflexes the core feature of non-decreasing utility function and business desire. The more agents have, the more satisfied they are. In the next item (iii), at any state of the world at $t = 1$, there is always an intention of buying commodity. Obviously, business plan will keep going and agents won't feel enough.

For the last one, item (iv) shows us a terminology *classical agent* who is expected as a utility maximizer with quasi-concave utility function. In fact, the agent never wants to stop finding out a way to reach the highest pay-off. Here, in mathematical viewpoint, if these utility functions are not differentiable or sub-differentiable, the gradient of them will be replaced by the adjusted normal operator N^a. However, quasi-concave function u_i for maximization problems can be treated as an opposite quasi-convex function $-u_i$ for minimization problems. Thus, instead of handling with u_i one can utilize $-u_i$ and the adjusted normal operator $N^a_{-u_i}$ is characterized. Namely, for each i, let us set $T_u(y) = \prod_{i \in \mathcal{I}} T_{u_i(\cdot, y_{-i})}(y_i)$ where $T_{u_i(\cdot, y_{-i})}(y_i) = -N^a_{-u_i(\cdot, y_{-i})}(y_i) \setminus \{0\}$.

Theorem 5.2.2. Let items (ii), (iii) and (iv) in Assumption 5.2.1 be satisfied. If $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a solution to the following quasi-variational inequality QVI(5.2.1):

$$\begin{aligned} &\text{Find } (\bar{q}, \bar{r}, \bar{z}, \bar{y}) \in \Delta \times B(\bar{q}, \bar{r}) \text{ such that there exists } \bar{y}^* \in T_u(\bar{y}) \\ &\text{satisfying, for any } (q, r, z, y) \in \Delta \times B(\bar{q}, \bar{r}), \\ &\Big\langle \big(\sum_{i \in \mathcal{I}} \bar{z}_i, \sum_{i \in \mathcal{I}} (\bar{y}_i - e_i)\big), (q, r) - (\bar{q}, \bar{r}) \Big\rangle_{S+LS} + \langle \bar{y}^*, y - \bar{y} \rangle_{LSI} \leq 0. \end{aligned} \tag{5.2.1}$$

Then $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a Radner equilibrium for the sequential trading exchange.

The proof of this theorem is based on the proof of Theorem 2.4.6 in Chapter 2 with an adaptation for Assumption 5.2.1. For the sake of conciseness we only report the main differences in the proof and a reader can refer to [16, Theorem 1] for the full proof. The proof consists of 9 steps whose conclusions are stated for each step. The main difference comes from the properties of the utility function u_i including strictly increasing monotonicity, local non-satiety and quasi-concavity in Assumption 5.2.1. Notice that, as previously pointed out, from item (ii)-(iv) of Assumption 5.2.1, function $u_i(\cdot)$ in this scheme depends not only on y_i but also on y_{-i}. Nevertheless, in the proof of Theorem 1 we just need to verify the properties of $u_i(\cdot, y_{-i})$ with respect to variable y_i by assuming y_{-i} as given.

Proof. One can firstly observe that, $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a solution of the QVI(5.2.1) if and only if the three conditions hold

(a) $\bar{q}$ is a solution of the variational inequality

$$\left\langle \sum_{i\in\mathcal{I}} \bar{z}_i, q - \bar{q} \right\rangle_S \leq 0, \quad \forall q \in \Delta^S. \tag{5.2.2}$$

(b) $\bar{r}$ is a solution of the variational inequality

$$\left\langle \sum_{i\in\mathcal{I}} (\bar{y}_i - e_i), r - \bar{r} \right\rangle_{LS} \leq 0, \quad \forall r \in \prod_{s\in\mathcal{S}} \Delta^L. \tag{5.2.3}$$

(c) For any $i \in \mathcal{I}$, $\bar{y}_i$ is a solution of the variational inequality

$$\left\langle T_{u_i(\cdot, y_{-i})}(\bar{y}_i), y_i - \bar{y}_i \right\rangle_{LS} \leq 0, \quad \forall (z_i, y_i) \in B_i(\bar{q}, \bar{r}). \tag{5.2.4}$$

Indeed, for any $i \in \mathcal{I}$, $(q, r) \in \Delta$, one has that $\big(q, \bar{r}, \bar{z}_i, \bar{y}_i\big)$ and $\big(\bar{q}, r, \bar{z}_i, \bar{y}_i\big)$ are elements of $\Delta \times B_i(\bar{q}, \bar{r})$. While for any $(z_i, y_i) \in B_i(\bar{q}, \bar{r})$, the element $\big(\bar{q}, \bar{r}, (\bar{z}_1, \ldots, z_i, \ldots, \bar{z}_I), (\bar{y}_1, \ldots, y_i, \ldots, \bar{y}_I)\big)$ also belongs to the set $\Delta \times B_i(\bar{q}, \bar{r})$.

Moreover, it is worth to note that the solution to the QVI(5.2.1) over a bounded domain becomes a solution to REP which is over an unbounded domain in Definition 5.1.1. Now, let us recall all assertions of the 9 steps as follows.

Step 1. For all $i \in \mathcal{I}$, $\bar{q}$, $\bar{r}$ and $\bar{y}_i$ are solutions to VI (5.2.2), (5.2.3) and (5.2.4), respectively. Therefore, since $B_i(\bar{q}, \bar{r})$ is a non-empty convex set, for any $i \in \mathcal{I}$, $\bar{y}_i$ is a solution to the maximization problem:

$$u_i(\bar{y}_i, \bar{y}_{-i}) = \max_{(z_i, y_i)\in B_i(\bar{q}, \bar{r})} u_i(y_i, \bar{y}_{-i}). \tag{5.2.5}$$

Step 2-Step 3 see proof of Theorem 1 in [16].

Step 4. For all $i \in \mathcal{I}$, $(\bar{z}_i, \bar{y}_i)$ is a solution to maximum problem (5.1.1).

$$u_i(\bar{y}_i, \bar{y}_{-i}) = \max_{(z_i, y_i) \in \widetilde{C}_i \times \mathbb{R}_+^{LS}} u_i(y_i, \bar{y}_{-i})$$

$$\text{s.t.} \begin{cases} \langle \bar{q}, z_i \rangle_S \leq 0 \\ \langle \bar{r}^s, y_i^s \rangle_L \leq \langle \bar{r}^s, e_i^s \rangle_L + \bar{r}^{1s} z_i^s, \quad \forall s \in \mathcal{S}. \end{cases}$$

Step 5-Step 8 The proof follows the same steps as in Theorem 1 in [16].

Step 9. Since from Step 2, Step 7 and Step 8, one has respectively $\sum_{i\in\mathcal{I}} \bar{z}_i^s \leq 0$, $\forall s \in \mathcal{S}$ and $\bar{q}^s > 0$, $\forall s \in \mathcal{S}$ and $\langle \bar{q}, \sum_{i\in\mathcal{I}} \bar{z}_i \rangle_S = 0$, thus it follows that $\sum_{i\in\mathcal{I}} \bar{z}_i^s = 0$ for all $s \in \mathcal{S}$. By combining with steps 3 and 4, for all $i \in \mathcal{I}$, we can conclude that $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a Radner equilibrium vector. □

Remark 5.2.3. Let us point out that QVI of type (5.2.1) is a particular case of a classical Stampacchia quasi-variational inequality problem:

$$\begin{gathered}\text{Find an element } x \in C \text{ such that } x \in K(x) \\ \text{and } \exists x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall y \in K(x).\end{gathered} \tag{5.2.6}$$

with $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows C$ being two set-valued maps with C being a non-empty subset of $\mathbb{R}^n$. Indeed, it can be easily observed that QVI(G, f, K, P) can be equivalently reformulated as QVI$(A, \tilde{K})$ where $\tilde{K} : P \times \operatorname{conv}\big(K(P)\big) \rightrightarrows P \times \operatorname{conv}\big(K(P)\big)$ is defined by $\tilde{K}(p, x) = P \times K(p)$ and $A : P \times \operatorname{conv}\big(K(P)\big) \rightrightarrows \mathbb{R}^m \times \mathbb{R}^n$ is defined by $A(p, x) = \Big\{\big(f(x), x^*\big) : x^* \in G(x)\Big\} = \big\{f(x)\big\} \times G(x)$. Problem (5.2.6) is then equivalent to

$$\begin{gathered}\text{Find a couple } (\bar{p}, \bar{x}) \in \tilde{K}(\bar{p}, \bar{x}) \text{ such that } \exists\big(f(\bar{x}), \bar{x}^*\big) \in A(\bar{p}, \bar{x}) \text{ with} \\ \Big\langle \big(f(\bar{x}), \bar{x}^*\big), (p, x) - (\bar{p}, \bar{x})\Big\rangle \geq 0,\ \forall (p, x) \in \tilde{K}(\bar{p}, \bar{x}).\end{gathered} \tag{5.2.7}$$

Notice that the constraint set is not fixed, making this problem more difficult to solve than classical variational inequalities, from both the theoretical and numerical points of view.

This remark is based on Subsection **??**, by using equivalence of quasi-variational inequality to obtain solution set of sequential trading exchange. Theorem 5.2.2 shows that any solution of QVI(5.2.1) is also a Radner equilibrium of the sequential trading exchange. Then, existence results for the solutions of QVI(5.2.1) can guarantee the existence of a solution for REP. In the next section we will provide the main result of this work for existence of Radner equilibria based on the property of net-lower-sign continuity of a set valued map.

5.3 Existence result for sequential trading exchange

Using the quasi-variational inequality reformulation of the sequential trading exchange with uncertainty, our aim in this section is to prove a first existence result of Radner equilibrium. But compared to [16, Theorem 6] the existence will be obtain for continuous quasi-concave utility function for each player i with respect to variables y_i and y_{-i}. Let us now recall a concept of net-lower-sign continuity.

Definition 5.3.1 ([11], Net-lower-sign continuity)**.** Let X and Y be sets of $\mathbb{R}^n$ and Y^* be a dual space of Y. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. Then, the pair (T, K) is net-lower-sign continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$ if and only if for every sequence $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ converging to (μ, λ), every $z \in \operatorname{cl} K(\mu, \lambda)$ and every selection $(z_n)_n$ of $(\operatorname{cl} K(\mu_n, \lambda_n))_n$

converging to z, the following condition holds:

$$\begin{cases} \text{If for every subsequence } (\mu_{n_k}, \lambda_{n_k})_k \text{ of } (\mu_n, \lambda_n)_n \text{ and every selection} \\ (y_{n_k})_k \text{ of } (K(\mu_{n_k}, \lambda_{n_k}))_k \text{ converging to } y \text{ one has that} \\ \qquad \limsup_k \sup_{y^*_{n_k} \in T(y_{n_k}, \lambda_{n_k})} \langle y^*_{n_k}, z_{n_k} - y_{n_k} \rangle \leq 0, \\ \text{then, } \sup_{y^* \in T(y,\lambda)} \langle y^*, z - y \rangle \leq 0. \end{cases} \tag{5.3.1}$$

We simply say that (T, K) is net-lower-sign continuous with respect to the pair (U, Λ) if it is so at each $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$.

Remark 5.3.2. From the above definition one clearly has that if the pair (T, K) is net-lower-sign continuous with respect to the pair (U, Λ) then, for any non-empty valued sub-map $\widetilde{K}$ of the map K, the pair $(T, \widetilde{K})$ is net-lower-sign continuous with respect to the pair (U, Λ).

Theorem 5.3.3. Suppose that Assumption 5.2.1 holds and let us define, for any $i \in \mathcal{I}$, the set-valued maps $A_i : \Delta^S \times \prod_{s \in \mathcal{S}} \Delta^L \times \widetilde{C} \times \widetilde{K} \rightrightarrows \mathbb{R}^S \times \mathbb{R}^{LS}_+$ and $D_i : \Delta^S \times \prod_{s \in \mathcal{S}} \Delta^L \times \widetilde{C} \times \widetilde{K} \rightrightarrows \Delta^S \times \prod_{s \in \mathcal{S}} \Delta^L$ by

$$A_i : (q, r, z, y) \longmapsto \left\{ (z^*_i, y^*_i) \in \widetilde{C}_i \times \widetilde{K}_i : y^*_i \in -T_{u_i(\cdot, y_{-i})}(y_i),\ z^*_i = 0 \right\},$$

and

$$D_i : (q, r, z, y) \longmapsto B_i(q, r).$$

Assume that, for any $i \in \mathcal{I}$, $(A_i, \operatorname{int} D_i)$ is net-lower-sign continuous with respect to the pair (X_i, X_{-i}). Then, the sequential trading exchange with uncertainty admits at least a Radner equilibrium.

The proof of this Theorem 5.3.3 will be inferred based on Theorem 5.2.2 and the one in [11, Theorem 2.6].

Before going further, let us state some remarks. The Assumption 5.2.1 is the most demanding requirement in commonly seen patterns. While the extra condition of the so-called *net-lower-sign continuity* in Theorem 5.3.3 is such a rather technical hypothesis. This is a quite unfamiliar concept and has only appeared in recent studies (e.g. see [11, 12]). At this moment, we temporarily accept the feasibility of this condition. In next section, this criteria will be observed to detect the sufficient condition in verifying the net-lower-sign continuity. In addition, it is necessary to notice that all required assumptions are expressed in component-type sense for each i. That means any mentioned property is applied for each component i, not in terms of product-type for all i.

The scheme of this theorem is in the same vein with [16, Theorem 7] which also treats REP. However, a big gap between two researching works is that, in [16] the author investigates different types of utility functions while here we focus only on quasi-concave function. The values of objective functions $u_i(y_i)$ for each i and properties of those functions in [16] are independent to variables y_{-i} of agent $-i$, but in our work utility functions $u_i(\cdot, y_{-i})$ are defined as depending on y_{-i}. In case of u_i is quasi-concave, assumptions used in [16] are local upper sign-continuity and quasi-monotonicity over the product map of normal operator coupling with dually lower semi-continuity on constraint sets. However, by using the concept of net-lower-sign continuity, we simplify the assumption on constraint set and normal operator in terms of component-type and tackle the problem with different approach to obtain the existence of REP.

Let us now recall the concept net-lower-sign continuity. This is known as a weaker assumption with respect to the settings in [2, Theorem 4.2. (*iii*),(*iv*)] and [15, Lemma 3.1. (*iv*),(*v*)] where require Mosco convergence, lower semi-continuity or dual lower semi-continuity. Basically, this hypothesis is about to link two set-valued maps, namely one is a constrained map while the other is a normal operator generated from objective function. The following definition corresponds to a particular case of [11, Prop. 3.2].
We are now in position to prove Theorem 5.3.3.

Proof of Theorem 5.3.3. Let us first define a set-valued map

$$\begin{array}{rccc} A: & \Delta^S \times \prod\limits_{s\in\mathcal{S}} \Delta^L \times \widetilde{C} \times \widetilde{K} & \rightrightarrows & \mathbb{R}^S \times \mathbb{R}^{LS}_+ \times \widetilde{C} \times \widetilde{K} \\ & (q,r,z,y) & \longmapsto & A(q,r,z,y) = A_{I+1}(q,r,z,y) \times \prod\limits_{i\in\mathcal{I}} A_i(q,r,z,y) \end{array}$$

where $A_{I+1} : (q,r,z,y)$ is a set-valued map such that

$$A_{I+1} : (q,r,z,y) \longmapsto \Big\{(q^*,r^*) \in \mathbb{R}^S \times \mathbb{R}^{LS}_+ : \\ (q^*,r^*) = -\big(\sum_{i\in\mathcal{I}} z_i, \sum_{i\in\mathcal{I}} (y_i - e_i)\big)\Big\}.$$

Similarly, define

$$\begin{array}{rccc} D: & \Delta^S \times \prod\limits_{s\in\mathcal{S}} \Delta^L \times \widetilde{C} \times \widetilde{K} & \rightrightarrows & \Delta^S \times \prod\limits_{s\in\mathcal{S}} \Delta^L \times \widetilde{C} \times \widetilde{K} \\ & (q,r,z,y) & \longmapsto & D(q,r,z,y) = D_{I+1}(q,r,z,y) \times \prod\limits_{i\in\mathcal{I}} D_i(q,r,z,y) \end{array}$$

where

$$D_{I+1} : (q,r,z,y) \longmapsto \Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$$

Setting $\mathcal{I}^+ := \mathcal{I} \cup \{I+1\}$, one has $A = A_{I+1} \times \prod_{i\in\mathcal{I}} A_i = \prod_{j\in\mathcal{I}^+} A_j$ and $D = D_{I+1} \times \prod_{i\in\mathcal{I}} D_i = \prod_{j\in\mathcal{I}^+} D_j$. Moreover, let us set X_{I+1} be a subset of $\Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$ and $X_{-(I+1)} = \prod_{i\in\mathcal{I}} X_i$.

According to Theorem 5.2.2, it is sufficient to prove the existence of a solution for the QVI(5.2.1). Let us assign, for $i \in \mathcal{I}$, $x_i = (z_i, y_i)$, $x_i^* = (z_i^*, y_i^*)$, and set $f(x) = p^* = (q^*, r^*)$, $P = \Delta$, $G(x) = \prod_{i\in\mathcal{I}} A_i(q,r,z,y)$ and $K(p) = B(q,r)$. Taking into account Remark 5.2.3, the QVI(G,f,K,P) is reformulated as follows.

$$\begin{array}{c} \text{Find } (\bar p, \bar x) \in D(\bar p, \bar x) \text{ such that } \exists \big(f(\bar x), \bar x^*\big) \in A(\bar p, \bar x) \text{ with} \\ \Big\langle \big(f(\bar x), \bar x^*\big), (p,x) - (\bar p, \bar x) \Big\rangle \geq 0, \ \forall (p,x) \in D(\bar p, \bar x). \end{array} \tag{5.3.2}$$

It is equivalent to find a vector $(\bar q, \bar r, \bar z, \bar y) \in \prod_{j\in\mathcal{I}^+} D_j(\bar q, \bar r, \bar z, \bar y)$ such that there exists $(\bar q^*, \bar r^*, \bar z^*, \bar y^*) \in \prod_{j\in\mathcal{I}^+} A_j(\bar q, \bar r, \bar z, \bar y)$ satisfying, for any $(q,r,z,y) \in \prod_{j\in\mathcal{I}^+} D_j(\bar q, \bar r, \bar z, \bar y)$,

$$\Big\langle (\bar q^*, \bar r^*), (q,r) - (\bar q, \bar r) \Big\rangle_{S+LS} + \Big\langle (\bar z^*, \bar y^*), (z,y) - (\bar z, \bar y) \Big\rangle_{SI+LSI} \leq 0. \tag{5.3.3}$$

Clearly quasi-variational inequalities QVI(5.3.3) are equivalent QVI(5.2.1) thus in particular any solution of QVI(5.3.3) is also a solution of QVI(5.2.1).

Now, we claim QVI(5.3.3) admits at least a solution.

a) Due to Assumption 5.2.1.(*i*), for any $i \in \mathcal{I}$, sets $X_i = \widetilde{C}_i \times \widetilde{K}_i$ are products of non-empty real compact segments. Moreover the set $X_{I+1} = \Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$ is a convex compact set defined by equality constraints. These all imply the non-emptiness, compactness and convexity of X_j for $j \in \mathcal{I}^+$.

b) We detect that the set-valued map D_j is closed and convex-valued with non-empty interior for any $j \in \mathcal{I}^+$, in view of the fact as follows. As described in previous argument, $\Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$ is a closed set, this implies $D_{I+1}(q,r,z,y)$ is a closed map. Next, $B_i(q,r)$ is a subset of non-empty closed set X_i. The setting of $B_i(q,r)$, for all $i \in \mathcal{I}$, reveals that it is a set defined from affine continuous functions under non-empty domain. Therefore, take any sequence $(q^k, r^k, z^k, y^k)_k$ converges to (q,r,z,y), thank to continuity of the affine functions $\lim_{k\to\infty} D_i(q^k, r^k, z^k, y^k) = D_i(q,r,z,y) = B_i(q,r)$. Together, one gets D_j is a closed map for any $j \in \mathcal{I}^+$ and so is its product D.

As we already know the set $\Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$ and for any $i \in \mathcal{I}$, the bounded set $B_i(q,r)$ are non-empty and convex. In the end, D_j is convex-valued with non-empty interior for any $j \in \mathcal{I}^+$.

c) Thanks to the continuity and quasi-concavity of utility function $u_i(\cdot, y_{-i})$, for $i \in \mathcal{I}$ (in other words, the quasi-convexity of $-u_i(\cdot, y_{-i})$) in Assumption 5.2.1.(*iv*), we claim the normal operator $-T_{u_i(\cdot,y_{-i})}(y_i)$ is quasi-monotone and locally upper-sign continuous. This inference is explained as follows. Since $-u_i$ is continuous, it is also lower semi-continuous. This property and the quasi-convexity fulfils conditions in [20, Proposition 3.5] and helps us to gain the cone upper semi-continuity of $-T_{u_i(\cdot,y_{-i})}(y_i)$. Then, the locally upper-sign continuity of it will be inferred. It is free to have the quasi-monotonicity of $-T_{u_i(\cdot,y_{-i})}(y_i)$ by referring [7, Proposition 5.13] for utility function $-u_i$. As a consequence, for any $i \in \mathcal{I}$, A_i is quasi-monotone and locally upper-sign continuous. This also holds for any $j \in \mathcal{I}^+$ since the map A_{I+1} is defined as an affine function.

d) Let us observe now, a couple of set-valued map $(A_{I+1}, \operatorname{int} D_{I+1})$. Suppose that sequences $(q^k, r^k)_k \subset X_{I+1}$ and $(z^k, y^k)_k \subset X_{-(I+1)}$ converge to (q,r) and (z,y) respectively. Also take a sequence $(\mu^k, \nu^k)_k \subset \operatorname{cl}\left(\Delta^S \times \Delta^{LS}\right)$ converging to (μ,ν). Since $\Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$ is a closed constant set, one can take any selection $(u^l, v^l)_l \subset D_{I+1}(q^l, r^l, z^l, y^l)$ converges to $(u,v) \in \Delta^S \times \prod_{s\in\mathcal{S}} \Delta^L$, where $(q^l, r^l)_l$, $(z^l, y^l)_l$ are sub-sequences of $(q^k, r^k)_k$, $(z^k, y^k)_k$ respectively. Now, let us investigate the following inequality.

$$\limsup_{l} \sup_{(u^{l,*}, v^{l,*}) \in A_{I+1}(q^l, r^l, z^l, y^l)} \Big\langle (u^{l,*}, v^{l,*}), (\mu^l, \nu^l) - (u^l, v^l) \Big\rangle \le 0$$

Just as stated previously, A_{I+1} is defined as an constant function. The inequality can be shown as

$$\limsup_{l} \sup_{(u^{l,*}, v^{l,*}) = -\left(\sum_{i\in\mathcal{I}} z_i^l, \sum_{i\in\mathcal{I}} (y_i^l - e_i)\right)} \Big\langle (u^{l,*}, v^{l,*}), (\mu^l, \nu^l) - (u^l, v^l) \Big\rangle \le 0$$

Since $-\big(\sum_{i\in\mathcal{I}} z_i, \sum_{i\in\mathcal{I}} (y_i - e_i)\big)$ is a constant function, it is continuous and thus

$$\lim_{l\to\infty} -\big(\sum_{i\in\mathcal{I}} z_i^l, \sum_{i\in\mathcal{I}} (y_i^l - e_i)\big) = -\big(\sum_{i\in\mathcal{I}} z_i, \sum_{i\in\mathcal{I}} (y_i - e_i)\big) \in A_{I+1}(q,r,z,y).$$

That being so, all sequences pass to limits, one can come up with

$$\sup_{(u^*, v^*) \in A_{I+1}(q,r,z,y)} \big\langle (u^*, v^*), (\mu,\nu) - (u,v) \big\rangle \le 0.$$

Thus, the condition (H) in Definition 5.3.1 holds. Then, the net-lower-sign continuity will satisfy for this couple regarding to the parameters pair $(X_{I+1}, X_{-(I+1)})$. Hence, from the assumption of the same property corresponding to A_i and D_i, for $i \in \mathcal{I}$, the net-lower-sign continuity of $(A_j, \operatorname{int} D_j)$ with regard to the pair (X_j, X_{-j}) is deduced, for any $j \in \mathcal{I}^+$.

By combining all above implications *a*)-*d*), it is well-founded to apply the result of [11, Theorem 2.6] to affirm that the QVI(5.3.3) has solution and the claim is proved.

Finally, the solution set of QVI(5.2.1) is non-empty. Thanks to Theorem 5.2.2, we terminate the proof by concluding that $(\bar{q}, \bar{r}, \bar{z}, \bar{y})$ is a Radner equilibrium of the sequential trading exchange with uncertainty. □

5.4 A possible case for verifying sufficient condition

In the existence result Theorem 5.3.3, the assumption on the net-lower-sign continuity of the pairs $(A_i, \operatorname{int} D_i)$ can appear to be quite technical and difficult to verify. Thus the aim of this section is to describe a case in which this technical assumption is fulfilled thanks a "separability structure" of the game.

Definition 5.4.1 (Sub-boundarily constant function). A function $f : \mathbb{R}^p \longrightarrow \mathbb{R}$ is said to be sub-boundarily constant on a subset C if, for every $x \in C$, one has that

$$f(y) < f(x) \Longrightarrow [y, x[\cap \operatorname{int} S_f^a(x) \neq \emptyset.$$

This concept of *sub-boundarily constant property* has been introduced in [12] and is fulfilled, for example, in the following cases:

- If f is radially continuous and quasi-convex then f is sub-boundarily constant on $\operatorname{int}(\operatorname{dom} f)$;
- If f is defined over $\mathbb{R}$, and if f is quasi-convex, then f is sub-boundarily constant on its domain.

Let us now define the following special "separability structure" of a game:

Definition 5.4.2 (Separability). For any $i \in \mathcal{I}$, given a function $\theta_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ (with $n = \sum_{i \in \mathcal{I}} n_i$) and a set-valued map $K_i : \mathbb{R}^{n-i} \rightrightarrows \mathbb{R}^{n_i}$.

i) θ_i is called *i-separable*, if there exists two core functions $\eta_i^i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$ and $\eta_{-i}^i : \mathbb{R}^{n-i} \longrightarrow \mathbb{R}$ such that

$$\theta_i(x_i, x_{-i}) = \eta_i^i(x_i) + \eta_{-i}^i(x_{-i}), \quad \forall x \in \mathbb{R}^n.$$

ii) $K_i(x_{-i})$ is said to satisfy *Separable Inequality Constrained* (SIC) property if for any $x_{-i} \in \mathbb{R}^{n-i}$, the set $K_i(x_{-i})$ is described by a finite set of inequalities, that is

$$K_i(x_{-i}) = \left\{ y_i \in \mathbb{R}^{n_i} \ : \ g_j^i(y_i) \leq h_j^i(x_{-i}),\ j \in \{1, \ldots, J_i\} \right\}$$

where, for any i, $J_i \in \mathbb{N}$ is the number of inequalities corresponding to each i and, for $j = 1, \ldots, J_i$, $g_j^i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$ and $h_j^i : \mathbb{R}^{n-i} \longrightarrow \mathbb{R}$.

This specific separability structure of a game, which is actually satisfied for many applications, will be a key tool in the case of sequential trading exchange model in order to obtain an existence result without any technical assumption as the net-lower-sign continuity of the pairs $(A_i, \operatorname{int} D_i)$, see the forthcoming Corollary 5.4.5.

The idea in Definition 5.4.2 is to use a technique to reformulate expressions of functions and set-valued maps into a specific structure which allows us to obtain an assertion in Proposition 5.4.4. In the sequel, we will consider an additional assumption for the utility function $-u_i$.

Thus in the sequel the sequential trading exchange model will be now assumed to satisfy such a separability structure

Assumption 5.4.3. For any $l \in \mathcal{L},\ s \in \mathcal{S},\ i \in \mathcal{I}$, let us assume that, for all y_{-i}, the function $-u_i(\cdot, y_{-i})$ is i-separable and sub-boundarily constant with the core function η_i^i being quasi-convex and lower semi-continuous.

Actually Assumption 5.4.3 will replace item (iv) of Assumption 5.2.1, that is the continuity and quasi-concavity assumption of the utility functions $u_i(\cdot, y_{-i})$. thus allowing, by combining the sub-boundarily constance and the lower-semi-continuity, to drop the continuity assumption.The usage of this assumption for sub-boundarily constance is about to reduce the continuity of the utility function for each consumer in REP.

Now, we come up with a central result of this section for establishing sufficient condition of n.l.s. continuity.

Proposition 5.4.4. For $l \in \mathcal{L},\ s \in \mathcal{S},\ i \in \mathcal{I}$, let assumptions (i), (iv) of Assumption 5.2.1 and Assumption 5.4.3 be satisfied. Let

$$\widetilde{A}_i : (q, r, z, y) \longmapsto \big\{(z_i^*, y_i^*) \in \widetilde{C}_i \times \widetilde{K}_i : y_i^* \in -\widetilde{T}_{u_i(\cdot, y_{-i})}(y_i),\ z_i^* = 0\big\}, \tag{5.4.1}$$

be a set-valued map with non-empty convex values, where the set-valued map $-\widetilde{T}_i : \widetilde{K}_i \rightrightarrows \widetilde{K}_i$ is described as

$$-\widetilde{T}_i = -\widetilde{T}_{u_i(\cdot, y_{-i})}(y_i) = \begin{cases} N^a_{-u_i(\cdot, y_{-i})}(y_i) \cap \overline{\mathcal{B}}_{n_i} & \text{if } y_i \in \operatorname{argmax} u_i(\cdot, y_{-i}), \\ \operatorname{conv}\big(N^a_{-u_i(\cdot, y_{-i})}(y_i) \cap \overline{\mathcal{S}}_{n_i}\big) & \text{otherwise,} \end{cases} \tag{5.4.2}$$

and $D_i : (q, r, z, y) \longmapsto B_i(q, r)$ be the set-valued map defined by

$$D_i(q, r, z, y) = \Big\{(z_i, y_i) \in X_i : \langle q, z_i \rangle_S \leq 0 \text{ and } \forall s \in \mathcal{S}, \\ \langle r^s, y_i^s \rangle_L \leq \langle r^s, e_i^s \rangle_L + r^{1s} z_i^s \Big\}.$$

Then, for any $i \in \mathcal{I}$, the couple $\big(\widetilde{A}_i, D_i\big)$ is net-lower-sign continuous w.r.t. the parameter pair (X_i, X_{-i}).

Proof. Let us first recall

$$\begin{aligned} D_i(q,r,z,y) &= \Big\{(z_i,y_i)\in X_i : \langle q, z_i\rangle_S \le 0 \text{ and } \forall s\in\mathcal{S},\\ &\qquad\qquad \langle r^s, y_i^s\rangle_L \le \langle r^s, e_i^s\rangle_L + r^{1s} z_i^s\Big\}\\ &= \Big\{(z_i,y_i)\in \mathbb{R}^S\times\mathbb{R}_+^{LS} : z_i\in \widetilde{C}_i = \prod_{s\in\mathcal{S}}\Big[-\sum_{i\in\mathcal{I}} e_i^{1s}, \sum_{i\in\mathcal{I}} e_i^{1s}\Big],\\ &\qquad\qquad y_i\in \widetilde{K}_i = \prod_{l\in\mathcal{L}, s\in\mathcal{S}}\Big[0, \sum_{l\in\mathcal{L}}\sum_{i\in\mathcal{I}} e_i^{ls}\Big],\\ &\qquad\qquad \langle q, z_i\rangle_S \le 0,\\ &\qquad\qquad \text{and } \forall s\in\mathcal{S},\ \langle r^s, y_i^s\rangle_L \le \langle r^s, e_i^s\rangle_L + r^{1s} z_i^s\Big\}\\ &= \Big\{(z_i,y_i)\in \mathbb{R}^S\times\mathbb{R}_+^{LS} \text{ such that (5.4.3) holds}\Big\} \end{aligned}$$

where

$$\begin{cases} (i): & z_i^s - \sum_{i\in\mathcal{I}} e_i^{1s} \le 0,\ \forall s\in\mathcal{S},\\ (ii): & -z_i^s - \sum_{i\in\mathcal{I}} e_i^{1s} \le 0,\ \forall s\in\mathcal{S},\\ (iii): & y_i^s - \prod_{l\in\mathcal{L}}\sum_{l\in\mathcal{L}}\sum_{i\in\mathcal{I}} e_i^{ls} \le 0,\ \forall s\in\mathcal{S},\\ (iv): & -y_i^s \le 0,\ \forall s\in\mathcal{S},\\ (v): & \langle q, z_i\rangle_S \le 0,\\ (vi): & \langle r^s, y_i^s\rangle_L - \langle r^s, e_i^s\rangle_L - r^{1s} z_i^s \le 0,\ \forall s\in\mathcal{S}, \end{cases} \tag{5.4.3}$$

to clarify the map D_i, for $i\in\mathcal{I}$. Let us also set the left hand side of inequalities in (5.4.3) be $g_j^i(x_i) = g_j^i(z_i,y_i)$ and the right hand side be $h_j^i(x_{-i}) = h_j^i(z_{-i},y_{-i}) = 0$ with $j\in\{1,\ldots,J_i = 5S+1\}$ in which, J_i is the sum of number of inequalities inferred from constraints (5.4.3.i)-(5.4.3.vi), respectively. It's clear to observe that, for any i, the map D_i is SIC with g_j^i being continuous and semi-strictly quasi-convex, and h_j^i being continuous.

a) We claim that, for fixed (q,r) and $k\in\mathbb{N}$, given any i, any $(z,y)\in\widetilde{C}\times\widetilde{K} = X$ and any sequence $(z_{-i}^k, y_{-i}^k)_k \subset X_{-i}$ converging to (z_{-i}, y_{-i}), the sequence of convex sets $(Z_i^k)_{k\in\mathbb{N}}$ given by $Z_i^k := D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k)$ Mosco-converges to the convex set $Z_i := D_i(q,r,z_i,y_i,z_{-i},y_{-i})$. The proof of this claim follows the one of [2, Proposition 3.3] and use the reformulation of Mosco convergence of [2, Proposition 3.2].

Regarding the Mosco convergence, the first part which will imply the closedness of the map D_i is a consequence of the lower semi-continuity of the function g_j^i and the continuity of the function h_j^i. Indeed, let $(Z_i^{k_t})_{t\in\mathbb{N}}$ be any sub-sequence of $(Z_i^k)_k$ and $(\hat{z}_i^t, \hat{y}_i^t)_t \subset \mathbb{R}^{n_i}$ be a sequence converging to $(\hat{z}_i, \hat{y}_i)$ with $(\hat{z}_i^t, \hat{y}_i^t)_t \subset (Z_i^{k_t})_t$. Since D_i is SIC, one has

$$g_j^i(\hat{z}_i^t, \hat{y}_i^t) \le h_j^i(z_{-i}^t, y_{-i}^t), \quad t\in\mathbb{N},\ \forall j = 1,\ldots,J_i.$$

Then, thanks to the continuity assumption on g_j^i and h_j^i one has, for each i and for all j,

$$g_j^i(\hat{z}_i,\hat{y}_i) \le \liminf_{t\to\infty} g_j^i(\hat{z}_i^t,\hat{y}_i^t) \le \liminf_{t\to\infty} h_j^i(z_{-i}^t, y_{-i}^t) = h_j^i(z_{-i},y_{-i}) = 0$$

which implies that $(\hat{z}_i,\hat{y}_i)$ is an element of Z_i.

To prove the second part of Mosco convergence, let us assume that, for fixed (q,r), the pair $(\hat{z}_i,\hat{y}_i)$ is an element of $\operatorname{int} Z_i$. We claim that, for k large enough, $(\hat{z}_i,\hat{y}_i)\in\operatorname{int} Z_i^k$.

Let us define $\mathcal{J}_i = \{j \in \{1,\dots,J_i\} : \inf_{\mathbb{R}^{n_i}} g_j^i = h_j^i(z_{-i},y_{-i})\}$ and $\mathcal{J}_i^c = \{1,\dots,J_i\} \setminus \mathcal{J}_i$. Thanks to the continuity and the semi-strictly quasi-convexity of the functions g_j^i, one has, for any $j \in \mathcal{J}_i^c$ and any $\alpha > \inf_{\mathbb{R}^{n_i}} g_j^i$,

$$\operatorname{int} S_\alpha(g_j^i) = \operatorname{int}\operatorname{cl}\left(S_\alpha^<(g_j^i)\right) = S_\alpha^<(g_j^i).$$

Therefore,

$$\begin{aligned}(\hat z_i,\hat y_i) &\in \operatorname{int}\bigcap_{j=1}^{J_i} S_{h_j^i(z_{-i},y_{-i})}(g_j^i)\\ &= \bigcap_{j=1}^{J_i}\operatorname{int} S_{h_j^i(z_{-i},y_{-i})}(g_j^i)\\ &= \left[\bigcap_{j\in\mathcal{J}_i}\operatorname{int}\operatorname*{argmin}_{\mathbb{R}^{n_i}} g_j^i\right]\bigcap\left[\bigcap_{j\in\mathcal{J}_i^c} S^<_{h_j^i(z_{-i},y_{-i})}(g_j^i)\right].\end{aligned}$$

One the one hand, for any $j \in \mathcal{J}_i^c$, $g_j^i(z_i,y_i) < h_j^i(z_{-i},y_{-i})$ and thus, for k large enough, $(\hat z_i,\hat y_i) \in S^<_{h_j^i(z_{-i}^k,y_{-i}^k)}(g_j^i)$. On the other hand, for any $j \in \mathcal{J}_i$ and any k, $\hat y_i \in \operatorname{int}\operatorname*{argmin}_{\mathbb{R}^{n_i}} g_j^i \subset \operatorname{int} S_{h_j^i(z_{-i}^k,y_{-i}^k)}(g_j^i)$. Combining both cases and denoting by $\mathcal{J}_i^k = \{j \in \mathcal{J}_i \ :\ h_j^i(z_{-i}^k,y_{-i}^k) = \inf_{\mathbb{R}^{n_i}} g_j^i\}$, one gets

$$\begin{aligned}(\hat z_i,\hat y_i) &\in \left[\bigcap_{j\in\mathcal{J}_i}\operatorname{int}\operatorname*{argmin}_{\mathbb{R}^{n_i}} g_j^i\right]\bigcap\left[\bigcap_{j\in\mathcal{J}_i^c\cup(\mathcal{J}_i\setminus\mathcal{J}_i^k)} S^<_{h_j^i(z_{-i}^k,y_{-i}^k)}(g_j^i)\right]\\ &= \operatorname{int} Z_i^k = \operatorname{int} D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k).\end{aligned}$$

Then, together with [2, Proposition 3.2], the claim is proved, completing the proof that $(Z_i^k)_k \xrightarrow[k\to\infty]{\text{Mosco}} Z_i$.

b) Next, according to Assumption 5.4.3, each of the functions $-u_i$ is i-separability and therefore we immediately have that, for any $(z,y) \in X$, one has $N^a_{-u_i(\cdot,y_{-i})}(y_i) = N^a_{\eta_i^i}(y_i)$. Again, let $(z_{-i}^k,y_{-i}^k)_k \subset X_{-i}$ converging to (z_{-i},y_{-i}) and let $(z_i,y_i) \in \operatorname{int} D_i(q,r,z_i,y_i,z_{-i},y_{-i})$. As we have shown above, there exists $k_0 \in \mathbb{N}$ large enough such that $(z_i,y_i) \in \operatorname{int} D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k)$ for all $k \geq k_0$. Thus, we can write

$$\begin{aligned}N^a_{-u_i(\cdot,y_{-i})}(y_i) &= N^a_{\eta_i^i}(y_i)\\ &\subseteq \operatorname*{Limsup}_{\operatorname{int} D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k)\ni(z_i^k,y_i^k)\to(z_i,y_i)} N^a_{\eta_i^i}(y_i^k)\\ &\subseteq \operatorname*{Limsup}_{\operatorname{int} D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k)\ni(z_i^k,y_i^k)\to(z_i,y_i)} N^a_{-u_i(\cdot,y_{-i}^k)}(y_i^k).\end{aligned}$$

Let us observe that the first inclusion is obtained thanks to the use of Lemma 3.8 of [12] with the sequence $(K_n)_k = \left(D_i(q,r,z_i,y_i,z_{-i}^k,y_{-i}^k)\right)_k$ of sets, which Mosco converges according to item *a*) above, and with the constant sequence of functions $(f_k)_k = (\eta_i^i)_k$.

c) From Assumption 5.2.1.(i), for any i, X_i is a non-empty, compact and convex subset of $\mathbb{R}^{n_i}$.

Coupling assertions *a*), *b*) and *c*), all assumptions of [12, Proposition 3.5] and one thus obtains that the pair $\left(-\widetilde{T}_i, D_i\right)$ is net-lower-sign continuous w.r.t. the pair (X_i,X_{-i}). Finally,taking into account the definition (5.4.1) of set-valued map $\widetilde{A}$, we can deduce the net-lower-sign continuity of $\left(\widetilde{A}, D_i\right)$ w.r.t. the pair (X_i,X_{-i}) and the proof is completed.

□

Finally, combining Proposition 5.4.4 with Remark 5.3.2, one immediately gets that under Assumption 5.2.1 and Assumption 5.4.3, for any $i \in \mathcal{I}$, the couple $\left(\widetilde{A}_i, \operatorname{int} D_i\right)$ is also net-lower-sign continuity w.r.t. the pair (X_i, X_{-i}). Thus the following corollary becomes a consequence of Theorem 5.3.3.

Corollary 5.4.5. Assume that items *(i)*, *(ii)* and *(iii)* of Assumption 5.2.1 and Assumption 5.4.3 hold true. Then, the sequential trading exchange with uncertainty admits at least a Radner equilibrium.

5.5 Conclusion

In this work, the Radner equilibrium problem is used to model a sequential trading exchange with non differentiable quasi-concave utility functions. By describing a real-state market, an existence result is obtained using the recent notion of net-lower-sign continuity. A particular case in which it is possible to verify sufficient conditions of the mentioned continuity property under some mild assumptions is also provided.

Chapter 6

Epilogue

6.1 Summary

In this essay, models of economic problems are presented from the perspective of game theory. Explaining the interaction between players in a non-cooperative way leads to an observation of the existence and stability of equilibrium. It has many implications for competitive situations in economics, energy management, etc.

The first part is also the most important one on which this thesis focuses in Chapter 3. It simulates a phenomenon where a new player (company, agent, consumer, producer, etc.) wants to enter the play. There are lots of complex circumstances that may occur for this player to get into the game. Here, the problem is described as an endogenous timing game in which the player must decide when to enter the game and maximize his payoff. At the end of the chapter, the results are pretty positive, showing that in all cases, players will have at least a little information to be more proactive when making decisions. In essence, all that the single player has is the shared information of the market. But it's worth adding that the more information this player knows from his opponent group, the clearer and simpler his choices will be.

The biggest challenge of this theme is dealing with three types of games at the same time and comparing them with each other. Missing any game result or failing to determine a solution for each game will not advance to the next steps. Processing GNEP_n for n players and linking it to two sort of multi-leader-follower games and GNEP_{n+1} led to the concept of weighted GNEP. The idea is quite manageable when creating a selection process for GNEP. Since then, the weighted GNEP solution uniqueness result has been used to establish the three types of games mentioned above. The decision making policy was developed to give an accurate judgement about the optimal strategy that new players can use.

In the second part of the thesis, a mathematical analysis investigating the qualitative properties of the Nash games is tackled. The problem presented here is a parametrized form of Nash equilibrium problem in which the perturbations affect objective function and constraint set. The author would like to know whether the solution set of the problem is stable when changing perturbed parameters. By various approaches and different settings of assumptions, the closedness and semi-continuity of the solution set are demonstrated. In each case, the assumptions are mild and oriented for each component i of objective function and constraint set. The reason is that, as discussed in Chapter 4, the properties of quasi-monotonicity and locally upper-sign continuity of component normal operators are not preserved for their product. Therefore, it requires some additional hypotheses for the problem that should be better to avoid. Not stopping there, from parametrized Nash game, it is easy to convert to SLMF game considering that each perturbation of Nash game in the lower level is equivalent to a decision variable of the top leader. And thus, concerning the reformulation of margin function, a result of semi-continuity for solution set is also proved. It can be added that this result somehow connects with the first topic when learning about Nash game and a single-leader-multi-follower game.

The last part of the work series addresses a problem of recent interest, Radner equilibrium problem. Compared to Nash equilibrium problem, it is the same essence when dealing with non-cooperative games. However, the most significant difference is that REP takes care of different time points in the market state. It is more or less related to time, like in the first topic about two-period games. In a classical REP, time is considered at two points ($t = 0, 1$) with a finite-state variable s occurring in the market. REP is also formulated by quasi-variational inequality, and recent literature shows the feasibility of this approach.

As shown previously, an existence result of Radner equilibrium for sequential trading exchange is concluded. Under the mild assumptions, the work provided sufficient conditions to verify net-lower-sign continuity. One link between this and the result of the qualitative analysis topic is to assume requirements for each component i. Besides, a relationship of it and the topic of player position is about the imperfect knowledge for the outcome of opponents. Knowing more about the state of market volatility will give players more options to prepare their strategy.

6.2 Prospect

As it has gone through all the dissertation details, this section will give some more preliminary ideas for upcoming projects out of the framework.

In Chapter 3, we looked at an extension of the two-period game for multi-player. This idea is a generalization to the symmetric duopoly game of [92]. Essentially, this expansion removed the symmetry property and increased the number of players. The critical difficulty of this idea can be expressed in the following three points.

- Give the optimal strategy for the $(n+1)^{\text{th}}$ player without knowing how the remaining group of n players will react.
- The Nash equilibrium problem does not always have a unique solution, so the theory of weighted Nash game is proposed to solve this issue.
- Reducing costs by not solving problems but by giving the "safest advice" to consider since Nash game and also multi-leader-follower game are known challenging to solve.

Based on the results obtained from this study, different cases can be studied in the future. For instance, the uniqueness of an equilibrium to the Nash game can be not required, but it must be limited to a selection of possible equilibria. The author can also investigate other cases where the objective function is not necessarily quasi-concave or with changing the constraint set.

For the second topic, we have already seen an investigation of the conditions required to achieve the qualitative properties of a solution set. As presented, quasi-monotonicity, locally upper-sign continuity and Mosco convergence are the assumptions on which this work focuses. However, this study can be considered in future with weaker conditions but still ensures the possibility of verification. The idea can also be analysed for the case of quantitative analysis, as in [1], which has shown that the assumptions about monotonicity or pseudo-monotonicity become too strong and do not really adapt for "less classical" problems. The stability of variational inequality in the qualitative sense can be observed as Hölder-type inequalities under strong quasi-monotonicity, known as the weakest assumption at the moment. Several algorithms can be considered for evaluating perturbation in numerical computations.

Last but not least, the sequential trading exchange problem is an appealing topic in the economic field. It describes a "real world" market that is subject to change in time based on the participants' decisions. In the narrow scope of this essay, the author has shown sufficient conditions for the Radner problem to admit a solution. This is an important point, but it should be added that it is only the first step to consider the problem under a particular case. The next target would be to consider a similar aspect to the more general case or to use the idea in [16] but to reformulate and deal directly with a complete quasi-variational inequality. If this is successful, some qualitative properties will also be considered for the solution set of the Radner problem.

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Titre: Une analyse des équilibres pour les problèmes de Nash, les problèmes de Radner et les jeux multi-leader-suiveur.

Mots clés: Jeux de Nash, Problèmes de Radner, Jeux multi-leader-suiveur, Inégalité quasi-variationnelle, Existence de solutions, Marché de l'électricité.

Résumé: Le premier sujet de cette thèse est consacré à la généralisation au cas de $n+1$ joueurs du travail de B. von Stengel. L'auteur définit un jeu à $(n+1)$-joueurs où un groupe de n joueurs interagit selon un GNEP et un nouveau joueur $n+1$ veut entrer dans le jeu. Le problème est abordé grace à l'introduction du nouveau concept d'équilibre de Nash pondéré. Une stratégie de décision pour le joueur $n+1$ est discutée.

La deuxième partie est consacrée à l'étude de la stabilité qualitative des problèmes de Nash paramétrés par des perturbations sur les fonctions objectives des différents joueurs et leurs ensembles de stratégies.

Le dernier travail traite des conditions suffisantes pour l'existence d'un équilibre de Radner pour un problème d'échange séquentiel à deux périodes dans lequel les fonctions d'utilité des joueurs sont quasi-concaves. Des conditions suffisantes pour vérifier le concept récent de continuité net-lower-sign sont également présentées sous certaines propriétés de séparabilité.

Title: An analysis of equilibria for Nash problems, Radner problems and Multi-leader-follower games.

Keywords: Nash games, Radner problems, Multi-leader-follower games, Quasi-variational inequality, Solution existence, Electricity market.

Abstract: The first topic of this thesis is devoted to the generalisation of the work by B. von Stengel to the case of $n+1$ players. The author defines an $(n+1)$-player game where a group of n players interacts in an existing GNEP and a new player $n+1$ wants to enter the game. The problem is tackled with the introduction of the new concept of weighted Nash equilibrium. A decision making strategy for the $n+1$ player to maximise his profit is discussed.

The second part is devoted to a study of qualitative stability of parametrised Nash problems in which parameters represent perturbations on the different players' objective functions and their strategy sets.

The last work deals with sufficient conditions for the existence of a Radner equilibrium for a sequential trading exchange problem with two periods in which the players' utility functions are quasi-concave. Sufficient conditions to verify the recent concept of net-lower-sign continuity are also presented under some separability properties.

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Titolo: Un'analisi degli equilibri per problemi di Nash, problemi di Radner e giochi Multi-leader-follower.

Parole chiave: Giochi di Nash, problemi di Radner, giochi multi-leader-follower, disuguaglianza quasi variazionale, esistenza di soluzioni, mercato elettrico.

Riassunto: Il primo argomento di questa tesi è dedicato alla generalizzazione del lavoro di B. von Stengel al caso dei giocatori $n + 1$. L'autore definisce un gioco $(n + 1)$-player in cui un gruppo di n giocatori interagisce in un GNEP esistente e un nuovo giocatore $n + 1$ vuole entrare nel gioco. Il problema viene affrontato con l'introduzione del nuovo concetto di equilibrio di Nash ponderato. Viene discussa una strategia decisionale per il giocatore $n + 1$ per massimizzare il suo profitto.

La seconda parte è dedicata allo studio della stabilità qualitativa di problemi di Nash parametrizzati in cui i parametri rappresentano perturbazioni sulle funzioni obiettivo dei diversi attori e sui loro insiemi di strategie.

L'ultimo lavoro tratta delle condizioni sufficienti per l'esistenza di un equilibrio di Radner per un problema di scambio commerciale sequenziale con due periodi in cui le funzioni di utilità dei giocatori sono quasi-concave. Sotto alcune proprietà di separabilità vengono presentate anche condizioni sufficienti per verificare il recente concetto di continuità di segno inferiore.

Title: An analysis of equilibria for Nash problems, Radner problems and Multi-leader-follower games.

Keywords: Nash games, Radner problems, Multi-leader-follower games, Quasi-variational inequality, Solution existence, Electricity market.

Abstract: The first topic of this thesis is devoted to the generalisation of the work by B. von Stengel to the case of $n + 1$ players. The author defines an $(n + 1)$-player game where a group of n players interacts in an existing GNEP and a new player $n + 1$ wants to enter the game. The problem is tackled with the introduction of the new concept of weighted Nash equilibrium. A decision making strategy for the $n + 1$ player to maximise his profit is discussed.

The second part is devoted to a study of qualitative stability of parametrised Nash problems in which parameters represent perturbations on the different players' objective functions and their strategy sets.

The last work deals with sufficient conditions for the existence of a Radner equilibrium for a sequential trading exchange problem with two periods in which the players' utility functions are quasi-concave. Sufficient conditions to verify the recent concept of net-lower-sign continuity are also presented under some separability properties.

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