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# ON A REGULARISATION OF A NONLINEAR DIFFERENTIAL EQUATION RELATED TO THE NON-HOMOGENEOUS AIRY EQUATION

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ABSTRACT. In this paper we study a nonlinear differential equation related to a non-homogeneous Airy equation. The linear equation has two families of solutions. We apply a procedure of resolution of points of indeterminacy to a system of first order differential equations equivalent to the nonlinear equation and study how the corresponding families of solutions are transformed.

KEYWORDS: the Airy equation, the Painlevé test, nonlinear differential equations, singularities

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## 1. INTRODUCTION

The non-homogeneous Airy equation is given by

$$\frac{d^2y(z)}{dz^2} = zy(z) + c, \tag{1.1}$$

where $c \in \mathbb{C}$ is a constant. In [13] some observations on the distribution of zeros of solutions of the non-homogeneous Airy equation were presented. The existence of a principal family of solutions, with simple zeros, and particular solutions, characterized by a double zero at a given position of the complex plane, was shown. In addition, a recursion describing the distribution of the zeros was introduced. These results generalise previous results on the distribution of zeros of solutions of the corresponding homogeneous equation [14, 15]. As shown in

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[13], a simple extension to the non-homogeneous case (by adding a constant term to the homogeneous equation) considerably changes the distribution of the zeros.

Equation (1.1) possesses entire solutions with order of growth equal to 3/2. Since this number is not an integer, all solutions of (1.1) possess an infinite number of zeros [12]. These zeros are movable, in the sense that if the initial conditions change, the positions of the zeros change.

Entire solutions of a linear second order non-homogeneous differential equation with entire coefficients may have double zeros. Indeed, for any $p \in \mathbb{C}$, equation (1.1) possesses always a solution with a double zero in $p$. More precisely (see [13]), given $p \in \mathbb{C}$, equation (1.1) possesses just one solution $\tau_d(z,p)$, proportional to $c$, with a double zero in $z = p$. This solution is defined by the series

(1.2)
$$\tau_d(z,p) = c\sum_{n=2} e_n(z-p)^n, \quad e_2 = \frac{1}{2},\ e_3 = 0,\ e_4 = \frac{p}{24},\ e_{n+2} = \frac{pe_n + e_{n-1}}{(n+1)(n+2)}.$$

The set of functions $\tau_d(z,p)$ then represents a family of particular solutions of equation (1.1). Besides the family $\tau_d(z,p)$, equation (1.1) possesses a *principal family* of solutions. Indeed, following [13] one has the following statement. Given $q \in \mathbb{C}$, equation (1.1) possesses a solution $\tau(z,q,\alpha)$ with a simple zero in $z = q$. This solution is defined by the series

(1.3)
$$\tau(z,q,\alpha) = \sum_{n=1} f_n(z-q)^n, \quad f_1 = \alpha,\ f_2 = \frac{c}{2},\ f_3 = \frac{q\alpha}{6},\ f_{n+2} = \frac{qf_n + f_{n-1}}{(n+1)(n+2)}.$$

Here $q$ and $\alpha$ are arbitrary parameters and so the series (1.3), that converges everywhere in the complex plane, can be considered as the general solution of equation (1.1).

## 2. Main results

The main objective of this paper is to study the nonlinear differential equation obtained from the non-homogeneous Airy equation using the logarithmic derivative of $y(z)$, i.e., by taking

$$u(z) = \frac{y'(z)}{y(z)}.$$

The function $u$ solves the following differential equation

$$\frac{d^2u(z)}{dz^2} + 3u(z)\frac{du(z)}{dz} + u(z)^3 = 1 + zu(z). \tag{2.1}$$

Equation (2.1) is a coupled Riccati equation in disguise, that is, it can be written as $w = u' + u^2 - z$, where $w' + uw = 0$. Therefore, it possesses a one parameter family of Riccati solutions when $w = 0$. Clearly, any entire solution $y$ of (1.1) gives a meromorphic solution $u$ of (2.1). There is the so-called Painlevé property, which demands that all movable singularities of the equation be poles. The Painlevé test is a useful criterion of integrability. Inserting a formal Laurent series into the equation, after determining the leading order of a possible solution, one can recursively compute the coefficients of the series. If there is no obstruction in computing the coefficients and a sufficient number of such formal Laurent series solutions exist, the equation is said to pass the test (see e.g. [1, 4]). Applying the Painlevé test to equation (2.1) it is not difficult to see [13] that the dominant balances give singularities of the type $c_0(z-p)^{-1}$, but, seeking the resonances, one finds that there are two families of solutions. One family is characterized by $c_0 = 1$ and with a resonance polynomial given by $(r-1)(r+1)$. The other one is characterized by $c_0 = 2$ and with a resonance polynomial given by $(r+1)(r+2)$. The resonance $r = -1$ corresponds to the arbitrariness of the position of the pole for $u(z)$, i.e. the arbitrariness of the position of the zero $z = p$ for $y(z)$. The coefficients of the Laurent expansion of the logarithmic derivative of $y(z)$ are explicitly connected with the zeros of $y(z)$. In particular, in the first family, $c_0 = 1$ implies that the zero in $z = p$ of $y(z)$ is simple, whereas the resonance $r = 1$ indicates that the constant term in the Laurent expansion of $u(z)$ is the other arbitrary constant describing the solutions of the second order equation (2.1). This is the principal family of solutions [1]. In the second family, $c_0 = 2$ implies that the zero in $z = p$ of $y(z)$ is double, whereas the resonance $r = -2$ is negative: this indicates that the second family is a particular solution of equation (2.1) [1].

As mentioned in [13], the non-homogeneous Airy equation has several applications in mathematical physics [8]. For example, it is related to the second member of the Burger's hierarchy [7]:

$$\psi_t + (\psi_{xx} + \psi^3 + 3\psi\psi_x)_x = 0. \tag{2.2}$$

Under the self-similarity transformation

$$\psi = \frac{1}{\eta(3t)^{1/3}} f(z), \quad z \doteq \frac{x}{\eta(3t)^{1/3}} - \frac{b}{\eta^3}, \quad \eta^3 = a \tag{2.3}$$

equation (2.2) becomes

$$\frac{d^3 f}{dz^3} + 3\left(\frac{df}{dz}\right)^2 + 3f\frac{d^2 f}{dz^2} + 3f^2\frac{df}{dz} = (az+b)\frac{df}{dz} + af. \tag{2.4}$$

Integrating once one gets

$$\frac{d^2 f}{dz^2} + 3\frac{df}{dz}f + f^3 = k + (az+b)f(z), \tag{2.5}$$

where $k$ is the integration constant. Some of the solutions of (2.5) have been considered in [7] for the description of liquids with gas bubbles. Equation (2.1) is a particular case of (2.5) with $k = a = 1$, $b = 0$. In general, in case $a = k$ we can scale solutions of (2.5) by taking $f(z) = a^{1/3}u(a^{1/3}z + b/a^{2/3})$ and get equation (2.1).

Solutions of a given non-linear second-order ordinary differential equation in general have infinitely many singularities in the complex plane, the location of which, apart from a finite number of fixed singularities of the equation, depends on the initial data of the equation. These singularities are therefore called movable. The method of blowing up points of indeterminacy of certain systems of two ordinary differential equations was applied to obtain information about the singularity structure of the solutions of the corresponding nonlinear differential equations in [2, 6].

Equation (2.1) is similar to the Painlevé-Ince equation

$$u'' + 3uu' + u^3 = 0 \tag{2.6}$$

considered in [2]. When the usual Painlevé analysis is applied, the Painlevé-Ince equation also possesses both positive and negative resonances. This equation can also be solved explicitely. The general solution is given by $u = 1/(z-a)+1/(z-b)$ with $a$ and $b$ constants [1]. All solutions are rational functions and, therefore, equation (2.6) possesses the Painlevé property. In [2] three equivalent to equation (2.6) systems were studied. It was shown that for all of them there is an infinite sequence of blow-ups for one of the base points and another one that terminates, which further gives a Laurent expansion of the solution around a movable pole. The aim of this paper is to study equation (2.1) in a similar way.

A blow-up is a construction originating in algebraic geometry to de-singularise an algebraic curve [10]. It can be adapted to the setting of differential equations where it serves to regularise a system of equations at points of indeterminacy of the equations. Okamoto studied all six Painlevé equations in their Hamiltonian form from a geometric point of view in [9], where he introduced the notion of a space of initial conditions, obtained by blowing up the phase space at a finite sequence of points. Points of indeterminacy also known as base points of the system are the points where the vector field is ill-defined. Geometrically the blow-up procedure separates lines through base points according to their slopes and, hence, adds a projective line, which is called an exceptional divisor. The blow-up at a point $(x, y) = (a, b)$, where $a = a(z)$ and $b = b(z)$ can in general be rational functions in $z$, is defined by the following construction. One introduces new coordinate charts, $x = a + u = a + UV$ and $y = b + uv = b + V$ and re-writes the system in new coordinates $(u, v)$ and $(U, V)$. The exceptional line then corresponds to $u = 0$ or $V = 0$. For more information and examples of application to the Painlevé equations see [5].

Let us consider the following system equivalent to equation (2.1):

$$u' = v - 3/2u^2, \quad v' = -(u^3 - 1 - zu). \tag{2.7}$$

Following the procedure described in [2], we extend the system to study it on $\mathbb{P}^1 \times \mathbb{P}^1$ and find points of indeterminacy of the vector field. Let us rename $u = q$, $v = p$ not to confuse the notation. Since we have a polynomial vector field, there are no points of indeterminacy in the coordinate chart $(q,\, p)$ (where both the numerator and the denominator on the right hand sides of equations vanish). In the chart $(Q, p) = (1/q,\, p)$ the system is

$$Q' = \frac{3 - 2pQ^2}{2}, \quad p' = \frac{Q^3 + zQ^2 - 1}{Q^3}.$$

We see that there are no points of indeterminancy of the vector field. In the chart $(q,\, P) = (q,\, 1/p)$ we have

$$q' = \frac{2 - 3Pq^2}{2P}, \quad P' = -P^2(1 + zq - q^3).$$

Here we also see that there are no points of indeterminancy of the vector field. In the chart $(Q,\, P) = (1/q,\, 1/p)$ the system is

$$Q' = \frac{3P - 2Q^2}{2P}, \quad P' = -\frac{P^2(Q^3 + zQ^2 - 1)}{Q^3}.$$

We see that when $Q = P = 0$ we have a point of indeterminacy.

To regularise the system (2.7), we need to resolve the base point at $(P,\, Q) = (0,\, 0)$. If the regularisation is successful (i.e., if the base point can be resolved in a finite number of blow-ups), then one can fully describe singularities of solutions of the differential equation [6, 2]. However, in some differential equations it might happen that one or more cascades of base points do not terminate after a reasonable (small) finite number of steps, in such case we call the cascade infinite. The calculations which we did for the case of system (2.7) show that the cascade splits into one finite and one infinite cascade. In particular, in the first step we see that by resolving the first point of indeterminacy $(Q,\, P) = (0,\, 0)$, that is, by taking $Q = u_1$, $P = u_1 v_1$, we can find the second point of indeterminacy $(u_1,\, v_1) = (0,\, 0)$. The corresponding system is

$$u_1' = \frac{3}{2} - \frac{u_1}{v_1}, \quad v_1' = 1 - \frac{3v_1}{2u_1} - zv_1^2 + \frac{v_1^2}{u_1^2} - u_1 v_1^2.$$

Then the cascade splits. We have new coordinates $u_1 = u_2$, $v_1 = u_2 v_2$ and new points of indeterminacy $(u_2,\, v_2) = (0,\, 1)$ and $(u_2,\, v_2) = (0,\, 2)$. The corresponding system is

$$u_2' = \frac{3}{2} - \frac{1}{v_2}, \quad v_2' = \frac{2 + v_2(v_2 - 3 - u^2(z + u_2)v_2)}{u_2}.$$

The first cascade is then infinite in the sense that it does not resolve in a reasonable number of steps. For the second one the system becomes regular after one more blow up. By taking $u_2 = u_3$, $v_2 = 2 + u_3 v_3$ one finds the system for $u_3$ and $v_3$ which has no more points of indeterminacy of the vector field. So one of the cascades of points of indeterminacy is resolved. We have the following statement.

**Theorem 2.1.** *The transformation*

$$u = \frac{1}{u_3}, \quad v = \frac{1}{u_3^2(2 + u_3 v_3)}.$$

*transforms system (2.7) into*

$$u_3' = \frac{4+3u_3v_3}{2(2+u_3v_3)}, \quad v_3' = -\frac{f(u_3, v_3)}{2(2+u_3v_3)}, \tag{2.8}$$

*where*

$$f(u_3, v_3) = 16(z+u_3)+24u_3(z+u_3)v_3+3(4u_3^2(z+u_3)-1)v_3^2+2u_3(u_3^2(z+u_3)-1)v_3^3,$$

*which has no more points of indeterminacy of the vector field.*

**Theorem 2.2.** *The principal family (1.3) corresponds to the following expansion for the function u:*

$$u(z) = \frac{1}{z-q} + \frac{c}{2\alpha} + \frac{(4q\alpha^2 - 3c^2)(z-q)}{12\alpha^2} + O((z-q)^2).$$

*The function v then has expansion*

$$v(z) = \frac{1}{2(z-q)^2} + \frac{3c}{2\alpha(z-q)} + \frac{32q\alpha^2 - 15c^2}{24\alpha^2} + O((z-q)).$$

*Expansions of the functions $u_3$ and $v_3$ are, respectively,*

$$u_3(z) = (z-q) - \frac{c(z-q)^2}{2\alpha} + \left(\frac{c^2}{2\alpha^2} - \frac{q}{3}\right)(z-q)^3 + O((z-q)^4),$$

$$v_3(z) = -\frac{4c}{\alpha} - \frac{4(q\alpha^2 - 3c^2)(z-q)}{\alpha^2} + O((z-q)^2).$$

On the other hand, if we use system (2.8) and search for its solutions in the form of the Taylor series with conditions $u_3(q) = 0$, $v_3(q) = -4c/\alpha$, then we recover this solution. Moreover, from the general theorems of regular systems we can find the radius of convergence of such series, so regularising the system is very important in undertanding the behaviour of solutions.

**Theorem 2.3.** *The second family (1.2) corresponds to*

$$u(z) = \frac{2}{z-p} + \frac{1}{6}p(z-p) + O((z-p)^2),$$

$$v(z) = \frac{4}{(z-p)^2} + \frac{7p}{6} + O((z-p)).$$

*We further find that*

$$u_3(z) = \frac{(z-p)}{2} - \frac{1}{24}p(z-p)^3 + O((z-p)^3),$$

$$v_3(z) = -\frac{2}{z-p} - \frac{5}{12}p(z-p) + O((z-p)^2),$$

*so $v_3$ has a pole at p.*

To reproduce this expansion from system (2.8) we take one further transformation $v_3 = 1/V_3$, so

$$V_3(z) = -\frac{z-p}{2} + \frac{5}{48}p(z-p)^3 + O((z-p)^4),$$

and from the system for $u_3$ and $V_3$ we can reproduce this expansion by searching for a solution with $u_3(p) = 0$, $v_3(p) = 0$ and $u_3'(p) = 1/2$, which also gives $v_3'(p) = -1/2$.

However, the system for $u_3$ and $V_3$ has further points of indeterminacy. This is refleced in the fact that we had an infinite cascade for the original system and we cannot regularise the system completely. We would also like to remark that we can find other equivalent systems to equation (2.1) and the cascades of points of indeterminacy will be different but still infinite. The computations are essentially the same but cumbersome so we omit them.

Since equation (2.1) has meromorphic solutions, it would be interesting to understand how to regularise the system completely and thus construct a space of initial conditions in the sense of Okamoto [9] for it. As it is well known, for the Painlevé equations, the method of blowing up the space of dependent variables at points of indeterminacy leads to the space of initial conditions. In certain proofs for the Painlevé property of the Painlevé equations (e.g. [3] and [11]) a main part is played by a system of equations in transformed coordinates where the system becomes regular at points where the original variables tend to infinity. As mentioned before, from the relation to the non-homogeneous Airy equation, equation (2.1) has meromorphic solutions. However, the regularisation by blow-ups, in comparison with the Painlevé equations, does not work properly in the sense that one of the cascades of points of indeterminacy seems to be infinite and so a similar formal proof of the Painlevé property using the regularising system is not possible. It would be interesting to understand further the question of connection between regularisation of the system of equations and the Painlevé property of the related nonlinear differential equation. This deserves further study. Other open problems include understanding the distribution of poles of nonlinear equations. In this case the dynamics of the zeros of the non-homogeneous Airy equation (1.1) may help understand the behavior of poles of the related nonlinear equation (2.1).

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