Grassmann embeddings of polar Grassmannians

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Abstract

In this paper we compute the dimension of the Grassmann embeddings of the polar Grassmannians associated to a possibly degenerate Hermitian, alternating or quadratic form with possibly non-maximal Witt index. Moreover, in the characteristic 2 case, when the form is quadratic and non-degenerate with bilinearization of minimal Witt index, we define a generalization of the so-called Weyl embedding (see [4]) and prove that the Grassmann embedding is a quotient of this generalized 'Weyl-like' embedding. We also estimate the dimension of the latter.

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1. Introduction

Let $V := V(N, \mathbb{F})$ be a vector space of dimension $1 < N < \infty$ over a field $\mathbb{F}$, equipped with a (possibly degenerate) sesquilinear or quadratic form η such that V is spanned by the set of the vectors that are singular for η. Let $R := \mathrm{Rad}(\eta)$ be the radical of η and let n be the *reduced Witt index* of η, namely the Witt index of the non-degenerate form induced by η on V/R. The numbers $d := \dim(R)$ and $n + d$ are the (*singular*) *defect* and the *Witt index* of η, respectively. With respect to η, the space V admits a direct sum decomposition

$$V = (\bigoplus_{i=1}^{n} V_i) \oplus V_0 \oplus R \qquad (1)$$

where $V_1, V_2, \ldots, V_n$ are mutually orthogonal hyperbolic 2-spaces and V_0 is an $(N - 2n - d)$-dimensional totally anisotropic subspace orthogonal to each of $V_1, V_2, \ldots, V_n$. In order to avoid trivial cases we assume $n > 1$. We call the number $d_0 := \dim(V_0)$ the *anisotropic defect* of η and we denote it $\mathrm{def}_0(\eta)$, while $\mathrm{def}(\eta)$ stands for d.

For $1 \leq k < N$ denote by $\mathcal{G}_k$ the k-Grassmannian of V, that is the point-line geometry having as points the subspaces of V of dimension k and as lines the sets of the form $\ell_{X,Y} := \{Z : X < Z < Y, \dim(Z) = k\}$, where X and Y are two subspaces of V with $\dim(X) = k - 1$, $\dim(Y) = k + 1$ and $X < Y$. Incidence is containment.

Let $e_k : \mathcal{G}_k \to \mathrm{PG}(\bigwedge^k V)$ be the Plücker (or Grassmann) embedding of $\mathcal{G}_k$, mapping the point $\langle v_1, \ldots, v_k\rangle$ of $\mathcal{G}_k$ to the projective point $\langle v_1 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(\bigwedge^k V)$. The dimension of an

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embedding is defined as the vector dimension of the space spanned by its image. It is well known that $\dim(e_k) = \binom{N}{k}$.

With η a form of reduced Witt index $n > 1$ and singular defect d, for $k = 1, \ldots, n + d$ the *polar k-Grassmannian* associated to η is the point-line geometry having as points the totally η–singular k-dimensional subspaces of V. Lines are defined as follows:

1. For $k < n + d$, the lines are the lines $\ell_{X,Y}$ of $\mathcal{G}_k$ with Y totally η-singular.
2. For $k = n + d$, if η is sesquilinear then the lines are the sets as follows

$$\ell_X := \{Y : X \leq Y, \dim(Y) = n + d, \; Y \text{ totally } \eta\text{–singular}\} \text{ with } \dim(X) = n + d - 1$$

and X totally η-singular, while when η is a quadratic form they are the sets

$$\ell_X := \{Y : X \leq Y \leq X^{\perp}, \dim(Y) = n+d, \; Y \text{ totally } \eta\text{–singular}\} \text{ with } \dim(X) = n+d-1,$$

where X is totally η-singular and $X^{\perp}$ is the orthogonal of X w.r.t. the bilinearization of η.

Let now $\mathcal{P}_k$ be the polar k-Grassmannian defined by η. If $k = 1$ the geometry $\mathcal{P}_1$ is the polar space associated to η. When $d = 0$ (namely η is non-degenerate) and $k = n$, the polar Grassmannian $\mathcal{P}_n$ is usually called *dual polar space*.

If $k < n + d$ then $\mathcal{P}_k$ is a full-subgeometry of $\mathcal{G}_k$. In any case, all points of $\mathcal{P}_k$ are points of $\mathcal{G}_k$. So, we can consider the restriction $\varepsilon_k := e_k|_{\mathcal{P}_k}$ of the Plücker embedding e_k of $\mathcal{G}_k$ to $\mathcal{P}_k$. The map ε_k is an embedding of $\mathcal{P}_k$ called *Plücker (or Grassmann) embedding* of $\mathcal{P}_k$.

Note that the span $\langle \varepsilon_k(\mathcal{P}_k) \rangle$ of the ε_k-image $\varepsilon_k(\mathcal{P}_k)$ of (the point-set of) $\mathcal{P}_k$ might not coincide with $\mathrm{PG}(\bigwedge^k V)$, although the equality $\langle \varepsilon_k(\mathcal{P}_k) \rangle = \mathrm{PG}(\bigwedge^k V)$ holds in several cases, as we shall see in this paper. The *dimension* $\dim(\varepsilon_k)$ of ε_k is the dimension of the vector subspace of $\bigwedge^k V$ corresponding to $\langle \varepsilon_k(\mathcal{P}_k) \rangle$.

When $k < n + d$ the embedding ε_k is *projective*, namely it maps the lines of $\mathcal{P}_k$ surjectively onto lines of $\mathrm{PG}(\bigwedge^k V)$. When $k = n + d$ this is not true, except when $d = 0$ and η is an alternating form. Indeed when η is degenerate $\mathcal{P}_{n+d}$ also admits lines consisting of just one point. Accordingly, when $d > 0$ the geometry $\mathcal{P}_{n+d}$ does not admit any projective embedding. If η is non-degenerate but not alternating then ε_n may map the lines of $\mathcal{P}_n$ onto proper sublines of lines of $\mathrm{PG}(\bigwedge^k V)$ (in which case ε_n is *laxly projective*) or onto conics (when ε_n is a *veronesean embedding* as defined in [4], also [18]) or other curves or varieties, depending on d_0 and the type of η.

When η is sesquilinear but not bilinear we can always assume that it is Hermitian. If it is bilinear then it can be either alternating or symmetric. However, if η is symmetric we can always replace it with the quadratic form associated to it. Thus, henceforth, we shall consider only Hermitian, alternating and quadratic forms.

Definition 1.1. If η is a Hermitian form, then $\mathcal{P}_k$ is called *Hermitian k-Grassmannian.* We denote it by $\mathcal{H}_k(n, d_0, d; \mathbb{F})$, as to have a notation which keeps record of the reduced Witt index n and the anisotropic and singular defects d_0 and d of η.

When η is alternating we call $\mathcal{P}_k$ a *symplectic k-Grassmannian* and denote it $\mathcal{S}_k(n, d; \mathbb{F})$ (recall that if η is alternating then $d_0 = 0$, so we may omit to keep a record of d_0). Finally, when $\mathcal{P}_k$ is associated to a non-degenerate quadratic form, then it is called *orthogonal k-Grassmannian.* We denote it by $\mathcal{Q}_k(n, d_0, d; \mathbb{F})$.

We will also often use the following shortenings: $\mathcal{H}_k$ for $\mathcal{H}_k(n, d_0, d; \mathbb{F})$, $\mathcal{S}_k$ for $\mathcal{S}_k(n, d; \mathbb{F})$ and $\mathcal{Q}_k$ for $\mathcal{Q}_k(n, d_0, d; \mathbb{F})$.

In this paper we shall compute the dimensions of the Grassmann embeddings of $\mathcal{H}_k(n, d_0, d; \mathbb{F})$, $\mathcal{S}_k(n, d; \mathbb{F})$ and $\mathcal{Q}_k(n, d_0, d; \mathbb{F})$ for any k, with no hypotheses on either d or d_0. Note that, in general, the possible values that the anisotropic defect d_0 can take depend on the field $\mathbb{F}$. For instance if $\mathbb{F}$ is finite and η is Hermitian then $d_0 \leq 1$. If $\mathbb{F}$ is quadratically closed, then a quadratic form defined over $\mathbb{F}$ has anisotropic defect $d_0 \leq 1$ and if $\mathbb{F}$ is finite then $d_0 \leq 2$.

Remark 1.2. In the literature the word "defect" is sometimes given a meaning different from either of those stated above. Indeed a number of authors use it to denote the defect of the bilinearization of a non-degenerate quadratic form in characteristic 2. We shall consider this defect in Subsections 1.2, 5.2 and 7.2, denoting it by the symbol d_0'. Clearly, $d_0' \leq d_0$.

Remark 1.3. A number of authors (Tits [23, Chapter 8], for instance), when dealing with vectors (or subspaces) that are singular (totally singular) for a given sesquilinear form, prefer the word "isotropic" rather than "singular", keeping the latter only for pseudoquadratic forms. Other authors (e.g. Buekenhout and Cohen [2, Chapters 7–10]) use "singular" in any case. We have preferred to follow these latter ones.

1.1. A survey of known results

Before stating our main results, we provide a brief summary of what is currently known about the dimension of the Grassmann embedding of a polar k-Grassmannian. In this respect, only non-degenerate forms are considered in the literature. So, throughout this subsection we assume $d = 0$.

We consider $\mathcal{H}_k(n, d_0, 0; \mathbb{F})$ first. The dimension of the Grassmann embedding of $\mathcal{H}_k(n, d_0, 0; \mathbb{F})$ has been proved to be equal to $\binom{N}{k}$ for $d_0 = 0$ and k arbitrary by de Bruyn [15] (see also Blok and Cooperstein [1]) and for d_0 arbitrary and $k = 2$ by Cardinali and Pasini [6]. When $k = 1$ there is nothing to say: ε_1 is just the canonical embedding of the polar space $\mathcal{H}_1(n, d_0, 0; \mathbb{F})$ in $\mathrm{PG}(V)$. As far as we know, the case where $k > 2$ and $d_0 > 0$ has never been considered so far.

It is worth to spend a few words on $\mathcal{H}_n = \mathcal{H}_n(n, d_0, 0; \mathbb{F})$. When $d_0 = 0$ the Grassmann embedding ε_n of $\mathcal{H}_n$ is laxly projective: it maps the lines of $\mathcal{H}_n$ onto Baer sublines of $\mathrm{PG}(\bigwedge^n V)$; by replacing $\mathrm{PG}(\bigwedge^n V)$ with a suitable Baer subgeometry containing $\varepsilon_n(\mathcal{H}_n)$, the embedding ε_n is turned into a projective embedding (see e.g. [15] or [12]; also [8, Section 4]). This modification has no effect on the dimension of ε_n, which remains the same. On the other hand, when $d_0 > 0$ then ε_n maps the lines of $\mathcal{H}_n$ onto Hermitian hypersurfaces in $(d_0 + 1)$-dimensional subspaces of $\mathrm{PG}(\bigwedge^n V)$ (Hermitian plane curves when $d_0 = 1$). In this case $\mathcal{H}_n$ does not admit any projective embedding, as it follows from the classification of Moufang quadrangles (Tits and Weiss [24]).

As for $\mathcal{S}_k(n, 0; \mathbb{F})$, it is well known that its Grassmann embedding has dimension $\binom{N}{k} - \binom{N}{k-2}$, with the usual convention that $\binom{N}{-1} := 0$ in the case $k = 1$; see e.g. see De Bruyn [14] or Premet and Suprunenko [20] (also De Bruyn [13], Cooperstein [11]).

Let now ε_k be the Plücker embedding of $\mathcal{Q}_k(n, d_0, 0; \mathbb{F})$. The dimension of ε_k is known only for $d_0 \leq 1$ with the further restriction $k < n$ when $d_0 = 0$. Indeed, in [4] it is shown that

$$\dim(\varepsilon_k) = \begin{cases} \binom{N}{k} & \text{if char}(\mathbb{F}) \text{ is odd and } d_0 \leq 1. \\ \binom{N}{k} - \binom{N}{k-2} & \text{if char}(\mathbb{F}) \text{ is even and } d_0 \leq 1. \end{cases}$$

The embedding ε_n of $\mathcal{Q}_n = \mathcal{Q}_n(n, d_0, 0; \mathbb{F})$ also deserves a few comments. When $d_0 = 0$ the lines of $\mathcal{Q}_n$ are just pairs of points. This case does not look very interesting. Suppose $d_0 > 0$. Then ε_n maps the lines of $\mathcal{Q}_n$ onto non-singular quadrics in $(d_0 + 1)$-dimensional subspaces of $\mathrm{PG}(\bigwedge^n V)$ (conics for $d_0 = 1$ and elliptic ovoids for $d_0 = 2$). If $d_0 = 1$ then $\mathcal{Q}_n$ admits the so-called *spin embedding*, which is projective and 2^n-dimensional. Interesting relations exist between this embedding and ε_n (see [4], [5]; also Section 7.3 of this paper). Furthermore, still assuming $d_0 = 1$, if $\mathbb{F}$ is a perfect field of characteristic 2 then $\mathcal{Q}_n \cong \mathcal{S}_n = \mathcal{S}_n(n, 0; \mathbb{F})$. In this case the Grassmann embedding of $\mathcal{S}_n$ yields a projective embedding of $\mathcal{Q}_n$ which, as proved in [4], is a quotient of ε_n. A 2^n-dimensional projective embedding also exists for $\mathcal{Q}_n$ when $d_0 = 2$ (see e.g. Cooperstein and Shult [12, §2.2]). It is likely that some relation also exists between this embedding and ε_n (see Section 7.3). If $d_0 > 2$, then $\mathcal{Q}_n$ admits no projective embedding, as it follows from [24].

1.2. The main results of this paper

In Sections 3, 4 and 5 of this paper we shall compute the dimension of the Grassmann embedding of a polar k-Grassmannian for k not greater than the reduced Witt index n of the associated form but with no restrictions on the anisotropic and singular defects d_0 and d. As a by-product, we obtain anew the results mentioned in the previous subsection. Explicitly, we prove the following:

Theorem 1. *Let V be a vector space of finite dimension $N > 1$ over a field $\mathbb{F}$ and let n, d_0 and d be non-negative integers with $n > 1$, $0 \leq d, d_0$ and $2n + d_0 + d = N$. Let η be a Hermitian, alternating or quadratic form defined over V, with reduced Witt index n, anisotropic defect d_0 and singular defect d, provided that such a form exists. For $1 \leq k \leq n$ let $\mathcal{P}_k$ be the polar k-Grassmannian associated to η and ε_k its Grassmann embedding. Then:*

1. *If η is Hermitian then $\dim(\varepsilon_k) = \binom{N}{k}$, namely $\varepsilon_k(\mathcal{P}_k)$ spans $\mathrm{PG}(\bigwedge^k V)$.*
2. *If η is alternating then $\dim(\varepsilon_k) = \binom{N}{k} - \binom{N}{k-2}$.*
3. *Suppose η to be a quadratic form. Then $\dim(\varepsilon_k) = \binom{N}{k}$ if $\mathrm{char}(\mathbb{F}) \neq 2$ and $\dim(\varepsilon_k) = \binom{N}{k} - \binom{N}{k-2}$ if $\mathrm{char}(\mathbb{F}) = 2$. In other words, if either $\mathrm{char}(\mathbb{F}) \neq 2$ or $k = 1$ then $\varepsilon_k(\mathcal{P}_k)$ spans $\mathrm{PG}(\bigwedge^k V)$, otherwise $\langle \varepsilon_k(\mathcal{P}_k) \rangle$ coincides with the span of the image of the Grassmann embedding of the symplectic k-Grassmannian associated with the bilinearization of η.*

Remark 1.4. As noticed in Subsection 1.1, the dual polar space $\mathcal{Q}_n(n, 0, 0; \mathbb{F})$ is not considered in [4]. Part 3 of the above theorem includes this case too.

In Theorem 1 we have assumed $k \leq n$, but $n < k \leq n + d$ is also allowed by the definition of polar Grassmannian when $d > 0$. We consider this case in the next corollary, to be proved in Section 6.

Corollary 2. *With the notation of Theorem 1, let $n < k \leq n + d$.*

1. *If η is Hermitian or quadratic, but with $\mathrm{char}(\mathbb{F}) \neq 2$ in the latter case, then*

$$\dim(\langle \varepsilon_k(\mathcal{P}_k) \rangle) = \binom{N}{k} - \sum_{i=0}^{k-n-1} \binom{N-d}{k-i}\binom{d}{i},$$

with the usual convention that a binomial coefficient $\binom{m}{h}$ is 0 when $h > m$.

2. *If η is alternating or quadratic, with $\mathrm{char}(\mathbb{F}) = 2$ in the latter case, then*

$$\dim(\langle \varepsilon_k(\mathcal{P}_k) \rangle) = \binom{N}{k} - \binom{N}{k-2} - \sum_{i=0}^{k-n-1} \binom{N-d}{k-i}\binom{d}{i} + \sum_{i=0}^{k-n-1} \binom{N-d}{k-i-2}\binom{d}{i}.$$

In any case $\langle \varepsilon_k(\mathcal{P}_k) \rangle$ is a proper subspace of $\mathrm{PG}(\bigwedge^k V)$.

Remark 1.5. A linear system of $\binom{N}{k-2}$ equations is proposed in [19, Section 4] which, combined with the (non-linear) equations describing the Grassmann variety $e_k(\mathcal{G}_k)$, characterizes the variety $\mathbb{S}_k = \varepsilon_k(\mathcal{S}_k)$. It is asked in [19] whether those linear equations are linearly independent and if they characterize $\langle \mathbb{S}_k \rangle$. In general, the answer to either of these questions is negative. In view of Corollary 2, the answer is certainly negative when $k > n$. However, it is negative even if $k \leq n$ (in particular, when $d = 0$). For instance, let $d = 0$. Then if $k \leq 3$ those equations are independent, whence they indeed describe $\langle \mathbb{S}_k \rangle$; perhaps they are independent for any $k \leq n$ when $\mathrm{char}(\mathbb{F}) \neq 2$, but when $\mathrm{char}(F) = 2$ and $k > 3$ they are dependent, as one can see by a straightforward check. So, here is one more problem: find a linear system that describes $\langle \mathbb{S}_k \rangle$.

We now turn to the second problem studied in this paper. Let $\mathcal{Q}_k = \mathcal{Q}_k(n, d_0, 0; \mathbb{F})$ be a non-degenerate orthogonal k-Grassmannian with $k < n$. As $k < n$, the Grassmann embedding ε_k of $\mathcal{Q}_k$ is projective. Assume moreover that $\mathrm{char}(\mathbb{F}) = 2$. So $\dim(\varepsilon_k) = \binom{N}{k} - \binom{N}{k-2}$ by Theorem 1. As proved in [4], if $k > 1$ and $d_0 \leq 1$ then ε_k is a proper quotient of an $\binom{N}{k}$-dimensional projective embedding e_k^W of $\mathcal{Q}_k$, which in [4] is called *Weyl embedding* and lives in the Weyl module $W(\mu_k)$ for the Chevalley group $G = \mathrm{O}(N, \mathbb{F})$ (when $d_0 = 1$) or $G = \mathrm{O}^+(N, \mathbb{F})$ (when $d_0 = 0$), for a suitable weight μ_k of the root system of G (see e.g. [17] or [22] for these notions). Explicitly, μ_k is the k-th fundamental weight λ_k except when $k = n-1$ and $d_0 = 0$. In the latter case $\mu_{n-1} = \lambda_{n-1} + \lambda_n$. We refer to [4] (also [7]) for more information on e_k^W. We only recall here that the Weyl embedding e_k^W also exists when $\mathrm{char}(\mathbb{F}) \neq 2$ and when $k = 1$, however $e_k^W \cong \varepsilon_k$ for any $k < n$ when $\mathrm{char}(\mathbb{F}) \neq 2$ (as proved in [4]) and $e_1^W \cong \varepsilon_1$ whatever $\mathrm{char}(\mathbb{F})$ is. Note also that, in any case, $\dim(W(\mu_k)) = \binom{N}{k}$. So, $e_k^W(\mathcal{Q}_k)$ spans $\mathrm{PG}(W(\mu_k))$.

It is natural to ask whether an analogue of the Weyl embedding can be defined when $d_0 > 1$. In the last section of this paper we propose such a generalization, but only for orthogonal Grassmannians associated to absolutely non-degenerate quadratic forms.

We recall that a non-degenerate quadratic form $q : V \to \mathbb{F}$ is said to be *absolutely non-degenerate* if its natural extension $\overline{q} : \overline{V} \to \overline{\mathbb{F}}$ to $\overline{V} := V \otimes \overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$, is still non-degenerate. When $\mathrm{char}(\mathbb{F}) \neq 2$ this notion is devoid of interest: in this case all non-degenerate quadratic forms are absolutely non-degenerate. On the other hand, when $\mathrm{char}(\mathbb{F}) = 2$, let f_q be the bilinearization of q and $d_0' := \dim(\mathrm{Rad}(f_q))$. Then $\dim(\mathrm{Rad}(\overline{q})) = \max(0, d_0' - 1)$. So, q is absolutely non-degenerate if and only if $d_0' \leq 1$.

Let q be absolutely non-degenerate with anisotropic defect d_0. Then, as we shall prove in Section 7.2, the field $\mathbb{F}$ admits an algebraic extension $\widehat{\mathbb{F}}$ such that the extension $\widehat{q} : \widehat{V} \to \widehat{\mathbb{F}}$ of q to $\widehat{V} = V \otimes \widehat{\mathbb{F}}$ is non-degenerate with anisotropic defect $\mathrm{def}_0(\widehat{q}) = d_0'' \leq 1$, where $d_0'' = d_0'$ when $\mathrm{char}(\mathbb{F}) = 2$ while, if $\mathrm{char}(\mathbb{F}) \neq 2$, then $d_0'' = 0$ or 1 according to whether N is even or odd. In any case, keeping the hypothesis $k < n$, the k-Grassmannian $\widehat{\mathcal{Q}}_k$ associated to $\widehat{q}$ admits the Weyl embedding $\widehat{e}_k^W : \widehat{\mathcal{Q}}_k \to \mathrm{PG}(\widehat{V}_k^W)$, where $\widehat{V}_k^W$ is the appropriate Weyl module. Clearly, the orthogonal k-Grassmannian $\mathcal{Q}_k$ associated to q is a subgeometry of $\widehat{\mathcal{Q}}_k$, $\mathrm{PG}(\bigwedge^k V)$ is a subgeometry of $\mathrm{PG}(\bigwedge^k \widehat{V})$ and the Grassmann embedding $\widehat{\varepsilon}_k : \widehat{\mathcal{Q}}_k \to \mathrm{PG}(\bigwedge^k \widehat{V})$ of $\widehat{\mathcal{Q}}_k$ induces on $\mathcal{Q}_k$ its Grassmann embedding $\varepsilon_k : \mathcal{Q}_k \to \mathrm{PG}(\bigwedge^k V)$.

The following theorem will be proved in Section 7.2. In order to make its statement a bit shorter, we take the liberty of using the symbols $\langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k) \rangle$ and $\langle \varepsilon_k(\mathcal{Q}_k) \rangle$, which actually stand for subspaces of $\mathrm{PG}(\bigwedge^k \widehat{V})$ and $\mathrm{PG}(\bigwedge^k V)$ respectively, to denote the corresponding vector subspaces of $\bigwedge^k \widehat{V}$ and $\bigwedge^k V$.

Theorem 3. *Let q be absolutely non-degenerate, $k < n$ and let $\widehat{\mathcal{Q}}_k$, $\widehat{V}$, $\widehat{V}_k^W$, $\widehat{\varepsilon}_k$ and $\widehat{e}_k^W$ be as defined above. Then the Weyl module $\widehat{V}_k^W$, regarded as an $\mathbb{F}$-space, contains an $\mathbb{F}$-subspace V_k^W such that:*

1. *The Weyl embedding* $\widehat{e}_k^W$ *induces on* $\mathcal{Q}_k$ *a projective embedding* $e_k^W : \mathcal{Q}_k \to \mathrm{PG}(V_k^W)$*. Moreover* $e_k^W(\mathcal{Q}_k)$ *spans* V_k^W*.*

2. *The (essentially unique) morphism* $\widehat{\varphi} : \widehat{V}_k^W \to \langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k)\rangle$ *from the Weyl embedding* $\widehat{e}_k^W$ *to the Grassmann embedding* $\widehat{\varepsilon}_k$ *of* $\widehat{\mathcal{Q}}_k$ *maps* $\widehat{V}_k^W$ *onto* $\langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k)\rangle$ *and induces a morphism from* e_k^W *to the Grassmann embedding* ε_k *of* $\mathcal{Q}_k$*.*

3. *If* $\widehat{\varphi}$ *is an isomorphism (which is the case precisely when either* $\mathrm{char}(\mathbb{F}) \neq 2$ *or* $k = 1$*) then* φ *also is an isomorphism.*

We call e_k^W the *Weyl-like* embedding of $\mathcal{Q}_k$ (Subsection 7.2, Definition 7.10). Needless to say, when $d_0 \leq 1$ then e_k^W is just the Weyl embedding of $\mathcal{Q}_k$. Clearly, when φ is an isomorphism, the Weyl-like embedding $e_k^W \cong \varepsilon_k$ is $\binom{N}{k}$-dimensional. Otherwise, as we shall prove in Subsection 7.2,

$$\binom{N}{k} - \binom{N}{k-2} \;\leq\; \dim(e_k^W) \;\leq\; \binom{N}{k} + \binom{N}{k-2}(g-1)$$

where $g := |\widehat{\mathbb{F}} : \mathbb{F}|$. As we shall see in Subsection 7.2, we can always choose $\widehat{\mathbb{F}}$ in such a way that $g \leq \max(1, d_0 - d_0')$ (note that we are assuming $\mathrm{char}(\mathbb{F}) = 2$, otherwise φ is an isomorphism). However, even with $g \leq \max(1, d_0 - d_0')$, the above bounds are likely to be rather lax. We leave the task of improving them for a future work.

To finish, we mention an important problem which stands in the background of this paper: under which conditions the embeddings considered in this paper are universal? Of course, this question makes sense only if universality can be defined in a sensible way for the family of embeddings we consider, as when they are projective (but not only in that case). Apart from the trivial case of $k = 1$, where ε_1 is just the canonical embedding of the polar space $\mathcal{P} = \mathcal{P}_1$, which is indeed universal except when $\mathcal{P}$ is symplectic and $\mathrm{char}(\mathbb{F}) = 2$, sticking to non-degenerate cases, a clear answer was known only for $\mathcal{H}_k(n, 0, 0; \mathbb{F})$ and $\mathcal{S}_k(n, 0; \mathbb{F})$: the Grassmann embedding of $\mathcal{H}(n, 0, 0; \mathbb{F})$ is universal provided that $|\mathbb{F}| > 4$ and that of $\mathcal{S}_k(n, 0; \mathbb{F})$ is universal provided that $\mathrm{char}(\mathbb{F}) \neq 2$ (Blok and Cooperstein [1]). Partial answers for $\mathcal{Q}_k(n, d_0, 0; \mathbb{F})$ with $d_0 \leq 1$ are also known, which might suggest that e_k^W is universal when $k < n$ and ε_n is universal when $\mathrm{char}(\mathbb{F}) \neq 2$ (see e.g. [4, Theorem 1.5] for $1 < k \leq 3$, $k < n$ and [5, Theorem 5] for $k = n = 2$). In a recent paper [10], the authors have investigated the generating rank of polar Grassmannians; in particular, for $\mathcal{H}_k(n, d_0, 0; \mathbb{F})$ with $d_0 \geq 0$ and $k < n$ it is shown that the Grassmann embedding of $\mathcal{H}_k$ is universal; see [10, Corollary 2]. For $k = 2$, and $k = 3 < n$ for $d_0 \leq 1$, the Grassmann embedding of $\mathcal{Q}_k(n, d_0, 0, \mathbb{F})$ is universal; see [4]. Very little is known for $\mathcal{Q}_k(n, d_0, 0; \mathbb{F})$ with $d_0 > 1$ or $k > 2$. However, we are not going to further address this problem in the present paper.

Structure of the paper. In Section 2 we set some notation and prove some preliminary general results. Parts 1, 2 and 3 of Theorem 1 will be proved in Sections 3, 4 and 5 respectively. Finally, in Section 7 we propose a general definition of *liftings* of embeddings and use this notion to prove Theorem 3.

2. Preliminaries

Let $V := V(N, \mathbb{F})$ be a vector space of dimension N over a field $\mathbb{F}$. Let $E := (e_i)_{i=1}^N$ be a given basis of V. For any set $J = \{j_1, \ldots, j_h\}$ of indexes with $1 \leq j_i \leq N$ denote by V_J the subspace of V generated by $E_J := (e_{j_1}, \ldots, e_{j_h})$.

We shall write in brief $V_k := \bigwedge^k V$ and $V_{J,k} := \bigwedge^k V_J$. It is well known that a basis E_k of V_k is given by all vectors of the form $e_T := e_{t_1} \wedge e_{t_2} \wedge \cdots \wedge e_{t_k}$ with $t_1 < t_2 < \cdots < t_k$

where $T = \{t_1, \ldots, t_k\}$ varies among all k-subsets of $\{1, \ldots, N\}$. Consistently with the notation introduced above we shall write $E_{J,k}$ for the basis of $V_{J,k}$ induced by E_J.

Given a Hermitian, alternating, or quadratic form η defined over V, for any $J \subseteq \{1, 2, \ldots, N\}$ let η_J be the restriction of η to V_J. A k-Grassmannian associated to η_J will be denoted by $\mathcal{P}_{J,k}$. We shall write the image of $\mathcal{P}_k$ under its Grassmann embedding ε_k as:

$$\mathbb{P}_k := \{\varepsilon_k(X) \colon X \text{ is a point of } \mathcal{P}_k\} \subseteq \mathrm{PG}(V_k).$$

According to the notation introduced above we put

$$\mathbb{P}_{J,k} := \{\varepsilon_k(X) \colon X \text{ is a point of } \mathcal{P}_{J,k}\} \subseteq \mathrm{PG}(V_{J,k}).$$

In particular,

$$\begin{aligned}\mathbb{H}_{J,k} &:= \{\varepsilon_k(X) \colon X \text{ is a point of } \mathcal{H}_{J,k}\},\\ \mathbb{S}_{J,k} &:= \{\varepsilon_k(X) \colon X \text{ is a point of } \mathcal{S}_{J,k}\},\\ \mathbb{Q}_{J,k} &:= \{\varepsilon_k(X) \colon X \text{ is a point of } \mathcal{Q}_{J,k}\},\end{aligned}$$

for respectively the image of Hermitian, alternating and orthogonal Grassmannians. If $J = \{1, \ldots, N\}$ then $\mathbb{H}_{J,k}$, $\mathbb{S}_{J,k}$ and $\mathbb{Q}_{J,k}$ will just be denoted by $\mathbb{H}_k$, $\mathbb{S}_k$ and $\mathbb{Q}_k$ respectively. We have defined $\mathbb{P}_k$ and $\mathbb{P}_{J,k}$ as sets of points in $\mathrm{PG}(V_k)$ and $\mathrm{PG}(V_{J,k})$. In the sequel, with some abuse of notation, we shall often take the liberty to regard them also as sets of vectors of respectively V_k and $V_{J,k}$, implicitly replacing $\{\varepsilon_k(X) : X \text{ is a point of } \mathcal{P}_k\}$ with $\{v \in \varepsilon_k(X) : X \text{ is a point of } \mathcal{P}_k\}$ and similarly for $\mathbb{P}_{J,k}$. With these conventions, we can define

$$\mathbb{P}_{I,h} \wedge \mathbb{P}_{J,k} := \{\langle v \wedge w\rangle \colon \langle v\rangle \in \mathbb{P}_{I,h} \text{ and } \langle w\rangle \in \mathbb{P}_{J,k}\} \subseteq \mathrm{PG}(V_{I\cup J,h+k}).$$

We always regard $\langle \mathbb{P}_k\rangle$ and $\langle \mathbb{P}_{J,k}\rangle$ as subspaces of V_k and $V_{J,k}$ respectively (as we did in the Introduction when defining $\dim(\varepsilon_k)$).

2.1. Orthogonal decompositions

As above, for $I, J \subseteq \{1, 2, \ldots, n\}$ let η_I and η_J be the forms induced by η on V_I and V_J respectively. We put $d_I := \mathrm{def}(\eta_I)$, $d_J := \mathrm{def}(\eta_J)$ and we denote by n_I and n_J the reduced Witt index of η_I and η_J.

Lemma 2.1. *Suppose that $I \cap J = \emptyset$ and $V_I \perp V_J$ with respect to η (or its bilinearization if η is quadratic). Then, for $1 \leq h \leq n_I + d_I$ and $1 \leq k \leq n_J + d_J$ we have $\mathbb{P}_{I,h} \wedge \mathbb{P}_{J,k} \subseteq \mathbb{P}_{I\cup J,h+k}$ and $\langle \mathbb{P}_{I,h}\rangle \wedge \langle \mathbb{P}_{J,k}\rangle \subseteq \langle \mathbb{P}_{I\cup J,h+k}\rangle$.*

Proof. Take $\langle v\rangle \in \mathbb{P}_{I,h}$ and $\langle w\rangle \in \mathbb{P}_{J,k}$. Since V_I and V_J are orthogonal by hypothesis, the h-dimensional vector space $X_v := \varepsilon_h^{-1}(v)$ and the k-dimensional vector space $X_w := \varepsilon_k^{-1}(w)$ are mutually orthogonal. Hence the space $X_v + X_w$ is totally singular and it has dimension $h + k$ (recall that $V_I \cap V_J = \{0\}$ as $I \cap J = \emptyset$ by assumption). So, $\langle v \wedge w\rangle = \varepsilon_{h+k}(\langle X_v, X_w\rangle) \in \mathbb{P}_{I\cup J,h+k}$. The condition on the linear spans is now immediate. □

In the hypotheses of Lemma 2.1 the form $\eta_{I\cup J}$ induced by η on $V_{I\cup J}$ is the orthogonal sum of η_I and η_J. Accordingly, $d_I + d_J = \mathrm{def}(\eta_{I\cup J})$, namely $\mathrm{Rad}(\eta_{I\cup J}) = \mathrm{Rad}(\eta_I) \oplus \mathrm{Rad}(\eta_J)$. Moreover $n_I + n_J$ is the reduced Witt index of $\eta_{I\cup J}$. Clearly, $n_I + n_J \leq n$ but no relation can be stated between $d_I + d_J$ and d in general. Indeed, although we always have $V_{I\cup J} \subseteq \mathrm{Rad}(\eta_{I\cup J})^\perp$, in general $\mathrm{Rad}(\eta_{I\cup J})^\perp \subset V$.

2.2. Reduction to the non-degenerate case

As above, let η be a Hermitian, alternating, or quadratic form defined over V, with $\dim(V) = N$. Let $R := \mathrm{Rad}(\eta)$, $d = \dim(R) = \mathrm{def}(\eta)$ and let n be the reduced Witt index of η.

With $V_0, V_1, \dots, V_n$ as in decomposition (1), let $\overline{V} := V_0 \oplus V_1 \oplus \cdots \oplus V_n$ and let $\overline{\eta}$ be the form induced by η on $\overline{V}$. Note that $\dim(\overline{V}) = 2n + d_0 = N - d$, the form $\overline{\eta}$ is non-degenerate and it is isomorphic to the reduction of η, namely the form induced by η on V/R.

Given $k \le n$, for $1 \le j \le k$ the polar j-Grassmannian $\overline{\mathcal{P}}_j$ associated to $\overline{\eta}$ is a full subgeometry of $\mathcal{P}_j$. Its ε_j-image $\overline{\mathbb{P}}_j = \varepsilon_j(\overline{\mathcal{P}}_j)$ is contained in $\overline{V}_j := \bigwedge^j \overline{V}$, which is a subspace of V_j.

Lemma 2.2. *For $1 \le k \le n$ we have*

$$\langle \mathbb{P}_k \rangle = \bigoplus_{i=0}^{\min(d,k)} \langle \overline{\mathbb{P}}_{k-i} \rangle \wedge \bigwedge^i R$$

where $\bigwedge^0 R := \mathbb{F}$ (as usual) and $\langle \overline{\mathbb{P}}_0 \rangle := \bigwedge^0 \overline{V} = \mathbb{F}$ by convention (when $d \ge k$).

Proof. Every vector $x \in V$ splits as $\overline{x} + x^R$ for uniquely determined vectors $\overline{x} \in \overline{V}$ and $x^R \in R$. Moreover, x is η-singular if and only if $\overline{x}$ is $\overline{\eta}$-singular and $x \perp y$ where $y = \overline{y} + y^R$ if and only if $\overline{x} \perp \overline{y}$. It follows that every vector of $\mathbb{P}_k$ is a sum $\sum_{i=0}^{\min(d,k)} u_i \wedge v_i$ with $u_i \in \overline{\mathbb{P}}_{k-i}$ and v_i a pure power in $\bigwedge^i R$, for $i = 0, 1, \dots, \min(d,k)$. Conversely, every wedge product $u_i \wedge v_i$ as above belongs to $\mathbb{P}_k$. The conclusion follows from these remarks and the fact that, since $V = \overline{V} \oplus R$, we also have $\bigwedge^k V = \bigoplus_{i=0}^{\min(d,k)} \bigwedge^{k-i} \overline{V} \wedge \bigwedge^i R$. □

Corollary 2.3. *For $1 \le k \le n$ we have*

$$\dim(\langle \mathbb{P}_k \rangle) = \sum_{i=0}^{\min(d,k)} \dim(\langle \overline{\mathbb{P}}_{k-i} \rangle) \cdot \binom{d}{i}.$$

Proof. It follows directly from Lemma 2.2, recalling that $\dim(\bigwedge^i R) = \binom{d}{i}$ and $\dim(\langle X \wedge Y \rangle) = \dim(X) \cdot \dim(Y)$ for any two vector spaces X and Y with trivial intersection. □

We recall the following property of binomial coefficients.

Lemma 2.4. *We have:*

$$\sum_{i=0}^{\infty} \binom{N-d}{k-i} \binom{d}{i} = \binom{N}{k} \tag{2}$$

where, as usual, we put $\binom{N-d}{k-i} := 0$ if either $k - i < 0$ or $k - i > N - d$.

Proof. By the binomial theorem, the coefficient of x^k in $(x+1)^N$ is the right hand side of (2); however $(x+1)^N = (x+1)^{N-d}(x+1)^d$ and the coefficient of x^k in $(x+1)^{N-d}(x+1)^d$ is the left hand side of (2). □

Theorem 2.5. *Both of the following hold for $1 \le k \le n$.*

1. *If $\dim(\langle \overline{\mathbb{P}}_h \rangle) = \binom{N-d}{h}$ for every $h = 1, 2, \dots, n$ then $\dim(\langle \mathbb{P}_k \rangle) = \binom{N}{k}$.*
2. *If $\dim(\langle \overline{\mathbb{P}}_h \rangle) = \binom{N-d}{h} - \binom{N-d}{h-2}$ for every $h = 1, 2, \dots, n$, then $\dim(\langle \mathbb{P}_k \rangle) = \binom{N}{k} - \binom{N}{k-2}$.*

Proof. To prove Part 1, replace $\dim(\langle\overline{\mathbb{P}}_{k-i}\rangle)$ with $\binom{N-d}{k-i}$ in Corollary 2.3 and apply (2) of Lemma 2.4. Turning to Part 2, replace $\dim(\langle\overline{\mathbb{P}}_{k-i}\rangle)$ with $\binom{N-d}{k-i} - \binom{N-d}{k-i-2}$ in Corollary 2.3. We obtain

$$\dim(\langle\mathbb{P}_k\rangle) = \sum_{i=0}^{\min(d,k)} \left(\binom{N-d}{k-i} - \binom{N-d}{k-i-2}\right)\cdot\binom{d}{i} =$$
$$\sum_{i=0}^{\min(d,k)} \binom{N-d}{k-i}\binom{d}{i} - \sum_{i=}^{d} \binom{N-d}{k-2-i}\binom{d}{i}.$$

Hence $\dim(\langle\mathbb{P}_k\rangle) = \binom{N}{k} - \binom{N}{k-2}$ by Lemma 2.4. □

3. Hermitian *k*-Grassmannians

In this section we shall prove Part 1 of Theorem 1. In view of Part 1 of Theorem 2.5, it is sufficient to prove Part 1 of Theorem 1 in the non-degenerate case. Accordingly, throughout this section $h : V \times V \to \mathbb{F}$ is a non-degenerate σ-Hermitian form of Witt index n and anisotropic defect $\mathrm{def}_0(h) = d_0 = N - 2n$, where $N = \dim(V)$ and σ is an involutory automorphism of $\mathbb{F}$. Let $\mathbb{F}_0 := \mathrm{Fix}(\sigma)$ be the subfield of $\mathbb{F}$ consisting of the elements fixed by σ. It is always possible to choose a basis

$$E = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_N)$$

of V and $\kappa_{2n+1}, \dots, \kappa_N \in \mathbb{F}_0 \setminus \{0\}$ such that

$$h\left(\sum_{i=1}^{N} e_i x_i, \sum_{j=1}^{N} e_j y_j\right) = \sum_{i=1}^{n}(x_{2i-1}^{\sigma} y_{2i} + x_{2i}^{\sigma} y_{2i-1}) + \sum_{j=2n+1}^{N} \kappa_j x_j^{\sigma} y_j \tag{3}$$

where the form induced by h on $\langle e_{2n+1}, \dots, e_N\rangle$ is anisotropic in $\mathbb{F}$, that is

$$\sum_{j=2n+1}^{N} \kappa_j x_j^{\sigma+1} = 0 \Leftrightarrow x_{2n+1} = x_{2n+2} = \dots = x_N = 0,$$

see [3, §6]. Observe that for all $i \in \{1, \dots, n\}$ the vectors (e_{2i-1}, e_{2i}) form a hyperbolic pair for h.

For $1 \le k \le n$, let $\mathcal{H}_k$ be the Hermitian k-Grassmannian associated to h and $\mathbb{H}_k = \varepsilon_k(\mathcal{H}_k)$ be its ε_k-image in V_k. According to the conventions stated in Section 2, given $J \subseteq \{1, 2, \dots, N\}$ we denote by h_J the restriction of h to $V_J \times V_J$, by $\mathcal{H}_{J,k}$ the Hermitian k-Grassmannian associated to h_J (if k is not greater than the Witt index of h_J) and we put $\mathbb{H}_{J,k} = \varepsilon_k(\mathcal{H}_{J,k})$ $(\subseteq V_{J,k})$.

Lemma 3.1. *Suppose $n \ge 2$. Given $i, j \in \{1, 2, \dots, n\}$ with $i \ne j$, let $J = \{2i-1, 2i, 2j-1, 2j\}$. Then, $\langle\mathbb{H}_{J,2}\rangle = V_{J,2}$.*

Proof. Clearly, $\langle\mathbb{H}_{J,2}\rangle \subseteq V_{J,2}$. Recall that $V_{J,2} = \bigwedge^2\langle e_{2i-1}, e_{2i}, e_{2j-1}, e_{2j}\rangle$. By definition of h, the vectors $e_{2i-1} \wedge e_{2j-1}$, $e_{2i-1} \wedge e_{2j}$, $e_{2i} \wedge e_{2j-1}$ and $e_{2i} \wedge e_{2j}$ represent totally h-singular lines of $\mathrm{PG}(V_J)$; so, all of them are elements of $\langle\mathbb{H}_{J,2}\rangle$. In order to complete the proof we need to show that both $e_{2i-1} \wedge e_{2i}$ and $e_{2j-1} \wedge e_{2j}$ lie in the span of $\mathbb{H}_{J,2}$. To this purpose, take $\alpha, \beta \in \mathbb{F}^*$ such that $\alpha\beta^{-1} \notin \mathbb{F}_0$ and consider the four vectors

$$u_1^x = xe_{2i-1} + e_{2j-1}, \qquad u_2^x = -x^{-\sigma} e_{2i} + e_{2j}$$

with $x \in \{\alpha, \beta\}$. It is immediate to see that u_1^x and u_2^x are mutually orthogonal singular vectors. So, $\langle u_1^x \wedge u_2^x \rangle \in \mathbb{H}_{J,2}$ and

$$u_1^x \wedge u_2^x = -e_{2i-1} \wedge e_{2i} x^{1-\sigma} + e_{2j-1} \wedge e_{2j} + e_{2i-1} \wedge e_{2j} x + e_{2i} \wedge e_{2j-1} x^{-\sigma} \in \langle \mathbb{H}_{J,2} \rangle.$$

Consequently,

$$u_1^\beta \wedge u_2^\beta - u_1^\alpha \wedge u_2^\alpha = (\alpha^{1-\sigma} - \beta^{1-\sigma})(e_{2i-1} \wedge e_{2i}) + w \in \langle \mathbb{H}_{J,2} \rangle, \tag{4}$$

with $w = (\beta - \alpha) e_{2i-1} \wedge e_{2j} + (\beta^{-\sigma} - \alpha^{-\sigma}) e_{2i} \wedge e_{2j-1} \in \langle \mathbb{H}_{J,2} \rangle$. If $\beta^{1-\sigma} = \alpha^{1-\sigma}$, then $\alpha\beta^{-1} = (\alpha\beta^{-1})^\sigma \in \mathbb{F}_0$ — a contradiction. So by (4), $e_{2i-1} \wedge e_{2i} \in \langle \mathbb{H}_{J,2} \rangle$. An analogous argument shows that also $e_{2j-1} \wedge e_{2j} \in \langle \mathbb{H}_{J,2} \rangle$ and this completes the proof. □

The following is shown by De Bruyn [15, Corollary 1.2] (also Block and Cooperstein [1, Corollary 3.2]). For completeness's sake, we provide a proof here.

Lemma 3.2. *Suppose $d_0 = 0$. Then for all k we have $\langle \mathbb{H}_k \rangle = V_k$.*

Proof. Clearly, $\langle \mathbb{H}_k \rangle \subseteq V_k$. To prove the reverse containment, we proceed by induction on k. Recall from Section 2 that V_k is spanned by the set E_k consisting of all $(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k})$ where $\{j_1, \ldots, j_k\}$ varies among all k-subsets of $\{1, \ldots, 2n\}$ and $j_1 < j_2 < \cdots < j_k$.

If $k = 1$, it is well known that the polar space $\mathbb{H}_1$ generates V and there is nothing to prove. Suppose the assertion holds for all values up to k and consider $e := e_{j_0, j_1, \ldots, j_k} \in E_{k+1}$. We can assume without loss of generality $j_0 \leq 2$. If $j_1 > 2$, let $J = \{3, 4, \ldots, N\}$. By induction, $\mathbb{H}_{J,k}$ spans $V_{J,k}$. Since $\{1, 2\}$ and J are disjoint and $V_{\{1,2\}}$ is orthogonal to V_J, by Lemma 2.1 we have

$$e \in \langle e_{j_0} \rangle \wedge V_{J,k} = \langle e_{j_0} \rangle \wedge \langle \mathbb{H}_{J,k} \rangle \subseteq \langle \mathbb{H}_{\{1,2\},1} \wedge \mathbb{H}_{J,k} \rangle \subseteq \langle \mathbb{H}_{\{1,2\} \cup J, k+1} \rangle \subseteq \langle \mathbb{H}_{k+1} \rangle.$$

Suppose now $j_1 = 2$; consequently $j_0 = 1$ and $j_2 > 2$. Since, by hypothesis, $n \geq k + 1$ and the indexes $j_2, j_3, \ldots, j_k$ are at most $k-1 (\leq n-2)$, there is at least one subset of the form $\{2i-1, 2i\}$ with $i = 2, 3, \ldots, n$ which is disjoint from $\{j_2, \ldots, j_k\}$.

For simplicity of notation suppose $\{3, 4\}$ to be such that $\{3, 4\} \cap \{j_2, \ldots, j_k\} = \emptyset$. Let $J = \{5, 6, \ldots, 2n\}$. By induction, $\langle \mathbb{H}_{J,k-1} \rangle = V_{J,k-1}$. On the other hand, $e_1 \wedge e_2 \in V_{\{1,2,3,4\},2} = \langle \mathbb{H}_{\{1,2,3,4\},2} \rangle$ by Lemma 3.1. So, $e \in \langle \mathbb{H}_{\{1,2,3,4\},2} \rangle \wedge \langle \mathbb{H}_{J,k-1} \rangle \subseteq \langle \mathbb{H}_{k+1} \rangle$ by Lemma 2.1. The lemma follows. □

Theorem 3.3. *We have $\langle \mathbb{H}_k \rangle = V_k$ for all k, independently of the anisotropic defect d_0 of h.*

Proof. Clearly, $\langle \mathbb{H}_k \rangle \subseteq V_k$. To prove the reverse containment we proceed by induction on d_0. For $d_0 = 0$, the result is given by Lemma 3.2. Suppose $d_0 > 0$ and then argue by induction on k. For $k = 1$ there is nothing to prove. So assume $k > 1$. We want to prove that for all J with $|J| = k$ we have $e_J \in \langle \mathbb{H}_k \rangle$. Define $s := k - |J \cap \{1, 2, \ldots, n\}|$. If $s = 0$, then $e_J \in V_{\{1,\ldots 2n\},k}$. As $h_{\{1,\ldots,2n\}}$ is non-degenerate with anisotropic defect 0, the result follows from Lemma 3.2 applied to $\mathbb{H}_{\{1,2,\ldots,2n\},k} \subseteq \mathbb{H}_k$. Suppose $s > 0$ and $J = \{j_1, j_2, \ldots, j_{k-1}, j\}$ with $j_1 < j_2 < \cdots < j_{k-1} < j$ and $j > 2n$. We can assume without loss of generality $j = N$. Thus $e_J = e_{J \setminus \{N\}} \wedge e_N$.

Since $k - 1 < n$, there is at least one pair $X_i := \{2i-1, 2i\}$ with $X_i \cap J = \emptyset$ and $1 \leq i \leq n$. Since the trace $\mathrm{Tr} : \mathbb{F} \to \mathbb{F}_0$ is surjective, there exists $t \in \mathrm{Tr}^{-1}(-\kappa_N)$. Then, the vector $u = e_{2i-1} + t e_{2i} + e_N$ is singular and $\langle u \rangle$ belongs to $\mathbb{H}_{\{2i-1,2i,N\},1}$. On the other hand, also $\langle e_{2i-1} \rangle, \langle e_{2i} \rangle \in \mathbb{H}_{\{2i-1,2i,N\},1}$. So $e_N \in \langle \mathbb{H}_{\{2i-1,2i,N\},1} \rangle$.

Put $I_i := \{1, 2, \ldots, N-1\} \setminus X_i$. Clearly, $e_{J \setminus \{N\}} \in V_{I_i, k-1}$. The form h_{I_i} induced by h on V_{I_i} is non-degenerate with anisotropic defect $d_0 - 1$. Thus, by the inductive hypothesis (on k or d_0, as we like) referred to h_{I_i} we obtain that $e_{J \setminus \{N\}} \in \langle \mathbb{H}_{I_i, k-1} \rangle$. So $e_J \in \langle \mathbb{H}_{I_i, k-1} \rangle \wedge \langle \mathbb{H}_{\{2i-1,2i,N\},1} \rangle$. However $V_{2i-1,2i,N} \perp V_{I_i}$. Hence, by Lemma 2.1, $\langle \mathbb{H}_{I_i,k-1} \rangle \wedge \langle \mathbb{H}_{\{2i-1,2i,N\},1} \rangle \subseteq \langle \mathbb{H}_{I_i \cup \{2i-1,2i,N\},k} \rangle = \langle \mathbb{H}_k \rangle$. Therefore $e_J \in \langle \mathbb{H}_k \rangle$. The theorem follows. □

Corollary 3.4. $\dim(\langle \mathbb{H}_k \rangle) = \binom{N}{k}$.

Theorem 3.3 and Corollary 3.4 yield Part 1 of Theorem 1 in the non-degenerate case. As noticed at the beginning of this section, the complete statement of Part 1 of Theorem 1 follows by combining this partial result with Theorem 2.5.

4. Symplectic k-Grassmannians

As recalled in Subsection 1.1, Part 2 of Theorem 1 holds in the non-degenerate case. By Theorem 2.5, it holds in the general case as well: $\dim(\langle \mathbb{S}_k \rangle) = \binom{N}{k} - \binom{N}{k-2}$ provided that $k \leq n$, no matter how large the defect of the underlying alternating form can be.

Our goal in this section is to describe a generating set for $\langle \mathbb{S}_k \rangle$ for $1 \leq k \leq n$. This description will be crucial to prove the main result of Subsection 5.2.

Let $s \colon V \times V \to \mathbb{F}$ be an alternating bilinear form with Witt index n and singular defect $d = N - 2n$, where $N = \dim(V)$. It is always possible to choose a basis $E = (e_1, e_2, \ldots, e_N)$ of V such that

$$s(\sum_{i=1}^{N} x_i e_i, \sum_{i=1}^{N} y_i e_i) = \sum_{i=1}^{n} (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}),$$

see [3, §5]. The subspace $\langle e_{2n+1}, \ldots, e_N \rangle$ is the radical of s. In the sequel, it will be convenient to keep a record of the form s in our notation. Thus, we write $\mathbb{S}_k(s)$ instead of $\mathbb{S}_k$. A basis of $\langle \mathbb{S}_k(s) \rangle$ when s is non-degenerate (namely $d = 0$) is explicitly described by De Bruyn [14] for arbitrary fields (see also Premet and Suprunenko [20] for fields of odd characteristic). In this section we shall provide a generating set $E_k(s)$ for $\langle \mathbb{S}_k(s) \rangle$ in the general case.

We first introduce some notation. For $J = \{j_1, j_2, \ldots, j_\ell\} \subseteq \{1, 2, \ldots, N\}$ with $j_1 < j_2 < \cdots < j_\ell$ define

$$e_J := e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_\ell}$$

with $e_\emptyset = 1$ by convention when $J = \emptyset$.

For $A \subseteq \{1, 2, \ldots, n\}$, put $(2A - 1) := \{2i - 1 : i \in A\}$, $(2A) := \{2i : i \in A\}$ and define

$$e_A^+ := e_{(2A-1)} \quad \text{and} \quad e_A^- := e_{(2A)}.$$

In particular, $e_\emptyset^+ = e_\emptyset^- = 1$ when $A = \emptyset$. For $1 \leq i < j \leq n$ let

$$u_i := e_{2i-1} \wedge e_{2i} \ \text{ and } \ u_{i,j} := e_{2i-1} \wedge e_{2i} - e_{2j-1} \wedge e_{2j}.$$

Consider a set $C = \{C_1, \ldots, C_h\}$ where $C_1 := \{i_1, j_1\}$, $C_2 := \{i_2, j_2\}, \ldots, C_h = \{i_h, j_h\}$ are disjoint pairs of elements of $\{1, 2, \ldots, n\}$, with the further assumptions $i_1 < i_2 < \cdots < i_h$ and $i_r < j_r$ for $r = 1, 2, \ldots, h$. Let also $\overline{C} := \{i_1, \ldots, i_h, j_1, \ldots, j_h\}$; clearly $|\overline{C}| = 2h$. Given such a set C, let

$$u_C := u_{i_1,j_1} \wedge u_{i_2,j_2} \wedge \cdots \wedge u_{i_h,j_h} \ \text{ and } \ u_\emptyset := 1.$$

Setting 4.1. *In the sequel (A, B, C, D) always stands for a quadruple with*

$$A, B \subseteq \{1, \ldots, n\}, \ D \subseteq \{2n + 1, 2n + 2, \ldots, N\} \ \text{ and } \ C = \{C_1, \ldots, C_h\}$$

where $C_1 := \{i_1, j_1\}$, $C_2 := \{i_2, j_2\}, \ldots, C_h = \{i_h, j_h\}$ are disjoint pairs of elements of $\{1, 2, \ldots, n\}$ such that $1 \leq i_1 < i_2 < \cdots < i_h$, $i_r < j_r \leq n$. Moreover, with $\overline{C} := \{i_1, \ldots, i_h, j_1, \ldots, j_h\}$, we assume that $A, B, \overline{C}$ and D are pairwise disjoint with $|A \cup B \cup \overline{C} \cup D| = k$.

With (A,B,C,D) as in Setting 4.1, put $e_{A,B,C,D} := e_A^+ \wedge e_B^- \wedge u_C \wedge e_D$. Define

$$E_k(s) := \{e_{A,B,C,D} \colon (A,B,C,D) \text{ as in Setting 4.1}\}. \tag{5}$$

For short, denote by $e_{A,B,C}$ the factor $e_A^+ \wedge e_B^- \wedge u_C$ of $e_{A,B,C,D} = (e_A^+ \wedge e_B^- \wedge u_C) \wedge e_D$. Clearly, $e_{A,B,C} \in \bigwedge^{k-r} \overline{V}$, where $\overline{V} := \langle e_1, e_2, \dots, e_{2n}\rangle$ and $r = |D|$. The numbers $k-r = |A|+|B|+2|C| = |A|+|B|+|\overline{C}|$ and r will be called the *rank* and the *corank* of $e_{A,B,C}$, respectively.

Given $r \le \min(d,k)$, the set of vectors $e_{A,B,C}$ of corank r as defined above coincides with the set $E_{k-r}(\overline{s})$ defined as in (5), but with $k-r$ instead of k and s replaced by its restriction $\overline{s}$ to $\overline{V}\times\overline{V}$. The form $\overline{s}$ is non-degenerate and $E_{k-r}(\overline{s})$ is a standard generating set for $\langle \mathbb{S}_{k-r}(\overline{s})\rangle$ (see e.g. De Bruyn [14]; also [20] in odd characteristic). Accordingly, the set

$$E_{k,r}(s) := \{e_{A,B,C,D} \in E_k(s) : |D| = r\} = E_{k-r}(\overline{s}) \wedge \{e_D : D \subseteq \{2n+1,\dots,N\}, |D| = r\}$$

is a generating set for $\langle \mathbb{S}_{k-r}(\overline{s})\rangle \wedge \bigwedge^r R$, where $R := \mathrm{Rad}(s) = \langle e_{2n+1},\dots,e_N\rangle$. By this fact, Lemma 2.2 and the fact that $E_k(s)$ is the disjoint union of the sets $E_{k,0}(s), E_{k,1}(s), \dots, E_{k,m}(s)$ (where $m := \min(k,d)$), we immediately obtain the main result of this section:

Lemma 4.2. *If $k \le n$ then $E_k(s)$ is a generating set for $\langle \mathbb{S}_k(s)\rangle$.*

5. Orthogonal k-Grassmannians

In this section we shall prove Part 3 of Theorem 1. We firstly deal with the non-degenerate case. Having done that, a few words will be enough to fix the general case. We will treat separately the cases in which $\mathrm{char}(\mathbb{F})$ is odd or even.

5.1. The non-degenerate case in odd characteristic

Suppose $\mathrm{char}(\mathbb{F}) \neq 2$. Let $q : V \to \mathbb{F}$ be a non-degenerate quadratic form of Witt index n and anisotropic defect $\mathrm{def}_0(q) = d_0 = N - 2n$. It is always possible to choose a basis $E = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_N)$ of V and $\kappa_{2n+1}, \dots, \kappa_N \in \mathbb{F}$ such that

$$q(\sum x_i e_i) = \sum_{i=1}^{n} x_{2i-1}x_{2i} + \sum_{j=2n+1}^{N} \kappa_j x_j^2, \tag{6}$$

where each pair (e_{2i-1}, e_{2i}) for $i = 1,\dots,n$ is hyperbolic and the space $\langle e_{2n+1},\dots,e_N\rangle$ is anisotropic in $\mathbb{F}$, i.e.

$$\sum_{j=2n+1}^{N} \kappa_j x_j^2 = 0 \Leftrightarrow x_{2n+1} = x_{2n+2} = \cdots = x_N = 0;$$

see [3, §6]. As in Section 2, $\mathcal{Q}_k$ is the polar k-Grassmannian associated to q and $\mathbb{Q}_k = \varepsilon_k(\mathcal{Q}_k)$ is its image by the Grassmann embedding. Given a subset $J \subset \{1,2,\dots,N\}$, q_J is the form induced by q on V_J and, for a positive integer t not greater than the Witt index of q_J, $\mathcal{Q}_{J,t}$ is the t-Grassmannian associated to q_J and $\mathbb{Q}_{J,t} = \varepsilon_t(\mathcal{Q}_t)$. Let $b : V \times V \to \mathbb{F}$ be the bilinear form associated to q, i.e.

$$b\left(\sum_{i=1}^{N} e_i x_i, \sum_{j=1}^{N} e_j y_j\right) = \sum_{i=1}^{n}(x_{2i-1}y_{2i} + x_{2i}y_{2i-1}) + 2\sum_{j=2n+1}^{N} \kappa_j x_j y_j.$$

The content of the following Lemma has been proved in [4] for $k < n$. In order to keep our treatment as self-contained as possible (and to clarify that we do not take here the assumption $k < n$) we shall provide a new and more elementary proof.

Lemma 5.1. *Suppose $d_0 = 0$. Then $\langle \mathbb{Q}_k \rangle = V_k$ for all $k \leq n$.*

Proof. Clearly, $\langle \mathbb{Q}_k \rangle \subseteq V_k$. To prove $\langle \mathbb{Q}_k \rangle \supseteq V_k$ we argue by induction on k. When $k = 1$ it is well known that the polar space $\mathbb{Q}_1$ spans V. We now show that $\langle \mathbb{Q}_2 \rangle \supseteq V_2$. Indeed, we shall prove that for all i, j with $1 \leq i < j \leq n$ we have $e_i \wedge e_j \in \langle \mathbb{Q}_2 \rangle$. If $\{i, j\} \neq \{2x-1, 2x\}$ for some $x \in \{1, \dots, n\}$, then the vectors e_i and e_j are q-singular and mutually orthogonal. Hence $\langle e_i, e_j \rangle$ is a q-singular line and so $\langle e_i \wedge e_j \rangle \in \mathbb{Q}_2$. Suppose $\{i, j\} = \{2x-1, 2x\}$ for some $x \in \{1, \dots, n\}$, take $h \neq x$ and let

$$u_1 = e_{2x-1} - e_{2h-1}, \qquad u_2 = e_{2x} + e_{2h},$$
$$u_3 = e_{2x-1} - e_{2h}, \qquad u_4 = e_{2x} + e_{2h-1}.$$

By construction $q(u_1) = q(u_2) = q(u_3) = q(u_4) = 0$ and $b(u_1, u_2) = b(u_3, u_4) = 0$ hence $\langle u_1 \wedge u_2 \rangle, \langle u_3 \wedge u_4 \rangle \in \mathbb{Q}_2$. Furthermore,

$$u_1 \wedge u_2 + u_3 \wedge u_4 = 2(e_{2x-1} \wedge e_{2x}) + w + w'$$

with $w = e_{2x-1} \wedge e_{2h} - e_{2h-1} \wedge e_{2x} \in \langle \mathbb{Q}_2 \rangle$ and $w' = e_{2x-1} \wedge e_{2h-1} - e_{2h} \wedge e_{2x} \in \langle \mathbb{Q}_2 \rangle$. So, $e_{2x-1} \wedge e_{2x} = e_i \wedge e_j \in \langle \mathbb{Q}_2 \rangle$.

Take now $k \geq 2$ and suppose that $\langle \mathbb{Q}_k \rangle = V_k$; we claim $\langle \mathbb{Q}_{k+1} \rangle = V_{k+1}$. Let $J \subseteq \{1, \dots, 2n\}$ with $|J| = k+1$. We show that $e_J \in \langle \mathbb{Q}_{k+1} \rangle$. Since $k + 1 \leq n$, clearly $k < n$. Take $i \in \{1, \dots, n\}$ such that $X_i \cap J \neq \emptyset$ where $X_i = \{2i-1, 2i\}$. Let also $I_i := \{1, \dots, 2n\} \setminus X_i$. Put $t := |X_i \cap J|$; clearly $t \in \{1, 2\}$. By the inductive hypothesis on k, $e_{J \setminus X_i} \in \langle \mathbb{Q}_{I_i, k-t+1} \rangle$ and $e_{J \cap X_i} \in \langle \mathbb{Q}_{X_i, t} \rangle$. (Note that q_{I_i} is non-degenerate with anisotropic defect 0). So, by Lemma 2.1, being X_i disjoint from I_i and V_{X_i} orthogonal to V_{I_i}, we have $e_J \in \langle \mathbb{Q}_{X_i, t} \rangle \wedge \langle \mathbb{Q}_{I_i, k-t+1} \rangle \subseteq \langle \mathbb{Q}_{X_i \cup I_i, k+1} \rangle = \langle \mathbb{Q}_{k+1} \rangle$. □

Theorem 5.2. *We have $\langle \mathbb{Q}_k \rangle = V_k$ for any value of the anisotropic defect d_0 of q and any positive integer $k \leq n$.*

Proof. Clearly, $\langle \mathbb{Q}_k \rangle \subseteq V_k$. To prove $\langle \mathbb{Q}_k \rangle \supseteq V_k$ we argue by induction on d_0. The case $d_0 = 0$ is settled in Lemma 5.1. Suppose $d_0 > 0$.

Let $J \subseteq \{1, 2, \dots, N\}$ with $|J| = k$. If $N \notin J$, then $e_J \in V_{I,k}$ where $I := \{1, 2, \dots, N-1\}$. On the other hand, the form q_I induced by q on V_I is non-degenerate with anisotropic defect $d_0 - 1$. So $\langle \mathbb{Q}_{I,k} \rangle = V_{I,k}$ by the inductive hypothesis. Hence $e_J \in V_{I,k} = \langle \mathbb{Q}_{I,k} \rangle \subseteq \langle \mathbb{Q}_k \rangle$.

Suppose $N \in J$ and let $\kappa_N = q(e_N)$. So, $e_J = e_{J \setminus \{N\}} \wedge e_N$. Since $|J| \leq k$ and $N > 2n$, we have $|J \setminus \{N\}| \leq k - 1 \leq n - 1$. Hence there is necessarily at least one index $i \in \{1, \dots, n\}$ such that $\{2i-1, 2i\} \cap J = \emptyset$. For such a choice of i, let $v_1 := e_{2i-1} - \kappa_N e_{2i} + e_N$ and $v_2 := e_{2i-1} + \kappa_N e_{2i} - e_N$; so $\langle v_1 \rangle, \langle v_2 \rangle \in \mathbb{Q}_{\{2i-1, 2i, N\}, 1}$. Clearly, $v_1 - v_2 = -2\kappa_N e_{2i} + 2e_N$. So $e_N \in \langle \mathbb{Q}_{\{2i-1, 2i, N\}, 1} \rangle$. On the other hand, $e_{J \setminus \{N\}} \in V_{I_i, k-1}$, where $I_i := \{1, 2, \dots, N-1\} \setminus \{2i-1, 2i\}$. The form q_{I_i} is non-degenerate and has anisotropic defect $d_0 - 1$. Then $V_{I_i, k-1} = \langle \mathbb{Q}_{I_i, k-1} \rangle$ by induction on d_0 (or on k). Consequently, $e_{J \setminus \{N\}} \in \langle \mathbb{Q}_{I_i, k-1} \rangle$. It follows that

$$e_J = e_{J \setminus \{N\}} \wedge e_N \in \langle \mathbb{Q}_{I_i, k-1} \rangle \wedge \langle \mathbb{Q}_{\{2i-1, 2i, N\}, 1} \rangle.$$

However $\langle \mathbb{Q}_{I_i, k-1} \rangle \wedge \langle \mathbb{Q}_{\{2i-1, 2i, N\}, 1} \rangle \subseteq \langle \mathbb{Q}_{I_i \cup \{2i-1, 2i, N\}, k} \rangle$ by Lemma 2.1 and $\mathbb{Q}_{I_i \cup \{2i-1, 2i, N\}, k} = \mathbb{Q}_k$. Therefore $e_J \in \langle \mathbb{Q}_k \rangle$. This completes the proof. □

Corollary 5.3. $\dim(\langle \mathbb{Q}_k \rangle) = \binom{N}{k}$.

Remark 5.4. Theorem 5.2 includes Theorem 1.1 of [4] as the special case where $d = 1$, but the proof given in [4] is not as easy as our proof of Theorem 5.2. Note also that we have obtained Theorem 5.2 from Lemma 5.1 while in [4] the analogue of our Lemma 5.1 is obtained as a consequence of Theorem 1.1 of that paper.

5.2. The non-degenerate case in even characteristic

Let now $\mathrm{char}(\mathbb{F}) = 2$ and $q : V \to \mathbb{F}$ be a non-degenerate quadratic form of Witt index n and anisotropic defect $\mathrm{def}_0(q) = d_0 = N - 2n$ as in Equation (6). As $\mathrm{char}(\mathbb{F}) = 2$, the anisotropic part of the expression of q can assume a more complex form than (6). Actually, it is always possible to determine a basis

$$E = (e_1, e_2, \ldots, e_{2n}, e_{2n+1}, \ldots, e_{2n+2m}, e_{2n+2m+1}, \ldots, e_N)$$

of V such that q can be written as

$$q\left(\sum_{i=1}^{N} x_i e_i\right) = q_0\left(\sum_{i=1}^{2n} x_i e_i\right) + q_1\left(\sum_{i=2n+1}^{2n+2m} x_i e_i\right) + q_2\left(\sum_{i=2n+2m+1}^{N} x_i e_i\right), \qquad (7)$$

where

$$q_0\left(\sum_{i=1}^{2n} x_i e_i\right) = \sum_{i=1}^{n} x_{2i-1}x_{2i},$$

$$q_1\left(\sum_{i=2n+1}^{2n+2m} x_i e_i\right) = \sum_{i=1}^{m}(x_{2n+2i-1}x_{2n+2i} + \lambda_i x_{2n+2i-1}^2 + \mu_i x_{2n+2i}^2),$$

$$q_2\left(\sum_{i=2n+2m+1}^{N} x_i e_i\right) = \sum_{j=2n+2m+1}^{N} \kappa_j x_j^2,$$

and $\lambda_i, \mu_i, \kappa_j \in \mathbb{F}$ are such that the form

$$q_{1,2}(\sum_{i=2n+1}^{N} x_i e_i) := q_1(\sum_{i=2n+1}^{m} x_i e_i) + q_2(\sum_{i=2n+2m}^{N} x_i e_i)$$

defined on $V_{\{2n+1,\ldots,N\}}$ is totally anisotropic; see, for instance, [16, Proposition 7.31] and also [3]. In particular, for any $2n + 2m + 1 \le i, j \le N$ each of the ratios κ_i/κ_j must be a non-square in $\mathbb{F}$ and the equations $\lambda_i t^2 + t + \mu_i = 0$ must admit no solution in $\mathbb{F}$. We will say that q has parameters $[n, m, d_0']$ where n is its Witt index and $d_0' := N - 2n - 2m$. Clearly, $d_0 = 2m + d_0'$.

Denote by $f_q \colon V \times V \to \mathbb{F}$, $f_q(x, y) := q(x + y) + q(x) + q(y)$ the alternating bilinear form polarizing q. Explicitly, by Equation (7), we have

$$f_q(\sum_{i=1}^{N} x_i e_i, \sum_{i=1}^{N} y_i e_i) = \sum_{i=1}^{n+m}(x_{2i-1}y_{2i} + x_{2i}y_{2i-1}).$$

Note that the Witt index of f_q is $n + m + d_0'$ and $d_0' = \dim(\mathrm{Rad}(f_q))$. So, f_q is non-degenerate if and only if $d_0' = 0$. Also, $n + m$ is the reduced Witt index of f_0.

Any totally singular k-space for q is necessarily totally singular for f_q, but the converse does not hold. This gives $\mathcal{Q}_k \subset \mathcal{S}_k(f_q)$, where by $\mathcal{S}_k(f_q)$ we mean the symplectic k-Grassmannian related to the form f_q. Consequently, $\langle \mathbb{Q}_k \rangle \le \langle \mathbb{S}_k(f_q) \rangle$, where $\mathbb{S}_k(f_q)$ is the image of $\mathcal{S}_k(f_q)$ under the Grassmann embedding. In the remainder of this section we shall prove that actually $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$.

We shall stick to the notation of Sections 2 and 4. In particular, $E_k(f_q)$ is the generating set for $\langle \mathbb{S}_k \rangle$ defined in (5). We add the following to the notation of Section 4. Given $X \subseteq \{1, 2 \ldots, n\}$ we write $[X]$ for $\{2x - 1, 2x \colon x \in X\}$.

The statement of the next lemma is proved in [4, Proposition 4.1(2)] for $k < n$. We give an easier proof here, which works also for the case $k = n$.

Lemma 5.5. *Let $d_0 = 0$. Then $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$ for any $1 \leq k \leq n$.*

Proof. As noticed above, $\langle \mathbb{Q}_k \rangle \leq \langle \mathbb{S}_k(f_q) \rangle$. We shall show that $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$.

Let $n = k = 2$ and suppose first that $\mathbb{F} = \mathbb{F}_2$. A direct computation shows that the 6 vectors representing the lines of $\mathcal{Q}_1 = Q^+(3, 2)$ span $\langle \mathbb{S}_2(f_q) \rangle$, which has dimension 5, and we are done. As linear independence is preserved taking field extensions, the 5 vectors forming a basis of $\langle \mathbb{Q}_2 \rangle$ are linearly independent over any algebraic extension of $\mathbb{F}$. So, $\langle \mathbb{Q}_2 \rangle = \langle \mathbb{S}_2(f_q) \rangle$ for $n = 2$.

We now show that the equality $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$ holds for any n and any $2 \leq k \leq n$. The generating set $E_k(f_q)$ for $\langle \mathbb{S}_k(f_q) \rangle$ is formed by the vectors $e_{A,B,C,\emptyset}$. Observe first that for any $1 \leq k \leq n$, all vectors of the form $e_{A,B,\emptyset,\emptyset}$ with $|A \cup B| = k$ represent totally singular spaces for the quadratic form q; so $e_{A,B,\emptyset,\emptyset} \in \langle \mathbb{Q}_k \rangle$. Take now $e_{A,B,C,\emptyset} \in E_k(f_q)$ with $t := |A \cup B|$, $0 \leq t \leq k$. We have

$$e_{A,B,C,\emptyset} = e_{A,B,\emptyset,\emptyset} \wedge e_{\emptyset,\emptyset,C,\emptyset}$$

with $e_{A,B,\emptyset,\emptyset}$ and $e_{\emptyset,\emptyset,C,\emptyset}$ as in Setting 4.1 but with k replaced by t and $k - t$ respectively (note that $k - t$ is even). Then $e_{A,B,\emptyset,\emptyset} \in \langle \mathbb{Q}_{(2A-1)\cup(2B),t} \rangle \leq \langle \mathbb{Q}_{[A\cup B],t} \rangle$ and $e_{\emptyset,\emptyset,C,\emptyset} \in \langle \mathbb{S}_{[\overline{C}],k-t} \rangle$.

The vectors of the form $e_{\emptyset,\emptyset,C,\emptyset}$ are in $\langle \mathbb{Q}_{[\overline{C}],k-t} \rangle$ since, by Setting 4.1, $e_{\emptyset,\emptyset,C,\emptyset} = u_C = u_{i_1,j_1} \wedge u_{i_2,j_2} \wedge \cdots \wedge u_{i_{k/2},j_{(k-t)/2}}$ and each of the u_{i_x,j_x} is in $\langle \mathbb{S}_{\{2i_x-1,2i_x,2j_x-1,2j_x\},2} \rangle$. Indeed, for each x we have $\mathcal{Q}_{\{2i_x-1,2i_x,2j_x-1,2j_x\},1} \cong Q^+(3, \mathbb{F})$, so, by what has been shown for $n = k = 2$, we have $\langle \mathbb{Q}_{\{2i_x-1,2i_x,2j_x-1,2j_x\},2} \rangle = \langle \mathbb{S}_{\{2i_x-1,2i_x,2j_x-1,2j_x\},2} \rangle$ and

$$u_C \in \bigwedge_{x=1,\ldots,(k-t)/2} \langle \mathbb{Q}_{\{2i_x-1,2i_x,2j_x-1,2j_x\},2} \rangle = \langle \mathbb{Q}_{[\overline{C}],k-t} \rangle.$$

Hence $e_{A,B,C,\emptyset} \in \langle \mathbb{Q}_{[A\cup B],t} \rangle \wedge \langle \mathbb{Q}_{[\overline{C}],k-t} \rangle$. However, by Lemma 2.1,

$$\langle \mathbb{Q}_{[A\cup B],t} \rangle \wedge \langle \mathbb{Q}_{[\overline{C}],k-t} \rangle \subseteq \langle \mathbb{Q}_{[A\cup B]\cup\overline{C},k-t} \rangle \subseteq \langle \mathbb{Q}_k \rangle.$$

Therefore $e_{A,B,C,\emptyset} \in \langle \mathbb{Q}_k \rangle$. It follows that $E_k(f_q) \subseteq \langle \mathbb{Q}_k \rangle$. Consequently $\langle \mathbb{S}_k(f_q) \rangle \leq \langle \mathbb{Q}_k \rangle$; this gives the thesis. □

Lemma 5.6. *Let $d_0 = 2m$, namely $d'_0 = 0$. Then $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$.*

Proof. If $m = 0$ then the thesis holds by Lemma 5.5. Suppose $m \geq 1$. We will show that any vector of $E_k(f_q)$ is contained in $\langle \mathbb{Q}_k \rangle$. We argue by induction on k. If $k = 1$, it is well known that $\langle \mathbb{Q}_1 \rangle = V = \langle \mathbb{S}_1(f_q) \rangle$ and we are done. Suppose now that the lemma holds for all values up to $k - 1$. Let $(A, B, C, \emptyset)$ be as in Setting 4.1 with $|A \cup B \cup \overline{C}| = k - 1$. Then, by the inductive hypothesis, $e_{A,B,C} := e_{A,B,C,\emptyset}$ belongs to $\langle \mathbb{Q}_{k-1} \rangle$. Take $i \notin A \cup B \cup \overline{C}$.

If $i \leq n$, since $q(e_{2i-1}) = q(e_{2i}) = 0$, we have that $e_{A\cup\{i\},B,C}$ and $e_{A,B\cup\{i\},C}$ are in $\langle \mathbb{Q}_k \rangle$. Indeed $e_{A\cup\{i\},B,C} = e_{A,B,C} \wedge e_{2i-1}$ and $e_{A,B,C} \in \langle \mathbb{Q}_{\{1,2,\ldots,2i-2,2i+1,\ldots,2n\},k-1} \rangle$, since $i \notin A \cup B$. So, $e_{A\cup\{i\},B,C} = e_{A,B,C} \wedge e_{2i-1} \in \langle \mathbb{Q}_{[A\cup B\cup\overline{C}],k-1} \rangle \wedge \langle \mathbb{Q}_{\{2i-1,2i\},1} \rangle$. However

$$\langle \mathbb{Q}_{[A\cup B\cup\overline{C}],k-1} \rangle \wedge \langle \mathbb{Q}_{\{2i-1,2i\},1} \rangle \subseteq \langle \mathbb{Q}_{[A\cup B\cup\overline{C}]\cup\{2i-1,2i\},k-1+1} \rangle \subseteq \langle \mathbb{Q}_k \rangle$$

by Lemma 2.1. Hence $e_{A\cup\{i\},B,C} \in \langle \mathbb{Q}_k \rangle$. The proof that $e_{A,B\cup\{i\},C} \in \langle \mathbb{Q}_k \rangle$ is entirely analogous.

Suppose $i > n$. Since $|A \cup B \cup \overline{C}| = k - 1 \leq n - 1$, there exists j with $1 \leq j \leq n$ and $j \notin A \cup B \cup \overline{C}$. Let $v_1 = e_{2j-1} + \lambda_{i-n} e_{2j} + e_{2i-1}$, where $\lambda_{i-n} = q(e_{2i-1})$. By construction, $\langle v_1 \rangle \in \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1}$; since $q(e_{2j-1}) = q(e_{2j}) = 0$, also $\langle e_{2j-1} \rangle, \langle e_{2j} \rangle \in \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1}$, thus forcing $e_{2i-1} \in \langle \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1} \rangle$. Thus

$$e_{A\cup\{i\},B,C} = e_{A,B,C} \wedge e_{2i-1} \in \langle \mathbb{Q}_{[A\cup B\cup\overline{C}],k-1} \rangle \wedge \langle \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1} \rangle.$$

By Lemma 2.1, $\langle \mathbb{Q}_{[A\cup B\cup \overline{C}],k-1}\rangle \wedge \langle \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1}\rangle \subseteq \langle \mathbb{Q}_k\rangle$. Therefore $e_{A\cup\{i\},B,C} \in \langle \mathbb{Q}_k\rangle$. Likewise, let $v_2 = e_{2j-1} + \mu_{i-n}e_{2j} + e_{2i}$, where $\mu_{i-n} = q(e_{2i})$. The same argument as above shows that

$$e_{A,B\cup\{i\},C} = e_{A,B,C} \wedge e_{2i} \in \langle \mathbb{Q}_{[A\cup B\cup \overline{C}],k-1}\rangle \wedge \langle \mathbb{Q}_{\{2j-1,2j,2i-1,2i\},1}\rangle \subseteq \langle \mathbb{Q}_k\rangle.$$

Take now $(A, B, C, \emptyset)$ as in Setting 4.1 with $|A \cup B \cup \overline{C}| = k - 2$. By the inductive hypothesis, $e_{A,B,C} \in \langle \mathbb{Q}_{k-2}\rangle$. We claim that $e_{A,B,C\cup\{\{i,h\}\}} \in \langle \mathbb{Q}_k\rangle$ for any pair $\{i, h\} \subseteq \{1, 2, \ldots, n + m\}$ such that $\{i, h\} \cap (A \cup B \cup \overline{C}) = \emptyset$.

Since $k - 2 \leq n - 2$, there exist at least two distinct indexes x, y with $1 \leq x < y \leq n$ and $x, y \notin A \cup B \cup \overline{C}$. Also, there exists at least one index less or equal to n which does not belong to $A \cup B \cup \overline{C} \cup \{i, h\}$ because $|(A \cup B \cup \overline{C} \cup \{i, h\}) \cap \{1, 2, \ldots, n\}| \leq k - 1 \leq n - 1$. Assume x is that index. We now distinguish two cases, according as y coincides or not with h. Note that in the case $k < n$ it is always possible to find x and y distinct such that $\{i, h\} \cap \{x, y\} = \emptyset$ while if $k = n$ then we may have to take $y = h$.

1. Suppose $y \neq h$. Let $u_1 = e_{2x-1} + \alpha e_{2x} + e_{2h-1} + e_{2i-1}$ and $u_2 = e_{2y-1} + \beta e_{2y} + e_{2h} + e_{2i}$ with $\alpha = q(e_{2h-1} + e_{2i-1})$ and $\beta = q(e_{2h} + e_{2i})$ and $J := \{2x-1, 2x, 2y-1, 2y, 2i-1, 2i, 2h-1, 2h\}$. By construction, $f_q(u_1, u_2) = 0$; so $\langle u_1 \wedge u_2\rangle \in \mathbb{Q}_{J,2}$. On the other hand,

   $$u_1 \wedge u_2 = (e_{2h-1} \wedge e_{2h} + e_{2i-1} \wedge e_{2i}) + w$$

   where

   $$w = (e_{2x-1} + \alpha e_{2x}) \wedge (e_{2y-1} + \beta e_{2y}) + (e_{2x-1} + \alpha e_{2x}) \wedge e_{2h} + (e_{2y-1} + \beta e_{2y}) \wedge e_{2h-1}+ \\ + (e_{2y-1} + \beta e_{2y} + e_{2h}) \wedge e_{2i-1} + (e_{2x-1} + \alpha e_{2x} + e_{2h-1}) \wedge e_{2i}$$

   is a linear combination of vectors of the form $e_{A,B,\emptyset}$. So $w \in \langle \mathbb{Q}_{J,2}\rangle$ and, consequently, $u_{h,i} = e_{2h-1} \wedge e_{2h} + e_{2i-1} \wedge e_{2i} \in \langle \mathbb{Q}_{J,2}\rangle$. Since $J \cap (A \cup B \cup \overline{C}) = \emptyset$ we have

   $$e_{A,B,C\cup\{\{i,h\}\}} = e_{A,B,C} \wedge u_{h,i} \in \langle \mathbb{Q}_{\{1,2,\ldots,N\}\setminus J,k-2}\rangle \wedge \langle \mathbb{Q}_{J,2}\rangle \subseteq \langle \mathbb{Q}_k\rangle.$$

2. Suppose $y = h$. Let $u_1 = \alpha e_{2x-1} + \beta e_{2x} + e_{2h-1} + e_{2i-1}$ and $u_2 = e_{2x} + \alpha' e_{2h} + \beta' e_{2h-1} + e_{2i}$ with $\alpha, \alpha', \beta, \beta'$ such that $q(u_1) = q(u_2) = 0$. This yields $\alpha\beta = q(e_{2i-1})$ and $\alpha'\beta' = q(e_{2i})$. Note $q(e_{2i-1}) \neq 0 \neq q(e_{2i})$. We also want $f_q(u_1, u_2) = \alpha + \alpha' + 1 = 0$. Take $\alpha' = \alpha + 1$. Let now $J := \{2x - 1, 2x, 2h - 1, 2h, 2i - 1, 2i\}$. Then, $\langle u_1 \wedge u_2\rangle \in \mathbb{Q}_{J,2}$ and

   $$u_1 \wedge u_2 = \alpha(e_{2x-1} \wedge e_{2x}) + \alpha'(e_{2h-1} \wedge e_{2h}) + (e_{2i-1} \wedge e_{2i}) + w$$

   where

   $$w = \alpha e_{2x-1} \wedge (\alpha' e_{2h} + \beta' e_{2h-1} + e_{2i}) + \beta e_{2x} \wedge (\alpha' e_{2h} + \beta' e_{2h-1} + e_{2i})+ \\ e_{2h-1} \wedge (e_{2x} + e_{2i}) + e_{2i-1} \wedge (e_{2x} + e_{2h} + \beta' e_{2h-1}) \in \langle \mathbb{Q}_{J,2}\rangle,$$

   since w is a linear combination of vectors of the form $e_{A,B,\emptyset}$. On the other hand,

   $$u_1 \wedge u_2 - w = \alpha(e_{2x-1} \wedge e_{2x}) + (1 + \alpha)(e_{2h-1} \wedge e_{2h}) + (e_{2i-1} \wedge e_{2i}) = \\ \alpha(\underbrace{e_{2x-1} \wedge e_{2x} + e_{2h-1} \wedge e_{2h}}_{v_1}) + (\underbrace{e_{2h-1} \wedge e_{2h} + e_{2i-1} \wedge e_{2i}}_{v_2}).$$

Since $1 \leq x, y \leq n$ and $h = y$, we have that $v_1 \in \langle \mathbb{S}_{\{2x-1,2x,2y-1,2y\},2}(f_q) \rangle$; however the quadratic form $q' = q_{\{2x-1,2x,2y-1,2y\}}$ is non-singular with anisotropic defect 0. So, by Lemma 5.5,

$$\langle \mathbb{S}_{\{2x-1,2x,2y-1,2y\},2}(f_{q'}) \rangle = \langle \mathbb{Q}_{\{2x-1,2x,2y-1,2y\},2} \rangle \subseteq \langle \mathbb{Q}_{J,2} \rangle.$$

It follows that $v_2 = u_1 \wedge u_2 - v_1 - w \in \langle \mathbb{Q}_{J,2} \rangle$. By Lemma 2.1 we now have

$$e_{A,B,C\cup\{\{i,h\}\}} = e_{A,B,C} \wedge v_2 \in \langle \mathbb{Q}_{[A\cup B\cup \overline{C}],k-2} \rangle \wedge \langle \mathbb{Q}_{J,2} \rangle \subseteq \langle \mathbb{Q}_k \rangle.$$

This completes the proof. □

Theorem 5.7. *We have $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$ for any choice of the parameters $[n, m, d'_0]$ of q and any $k \leq n$.*

Proof. When $d'_0 = 0$ the statement holds true by Lemma 5.6. Also, for $k = 1$ it is well known that $\langle \mathbb{Q}_1 \rangle = V = \langle \mathbb{S}_1(f_q) \rangle$. In this case there is nothing to prove.

Suppose $k > 1$, take $e_{A,B,C,D} \in E_k(f_q)$ and let $J = A \cup B \cup \overline{C} \cup D$. Clearly $|J| = k$. Let also $h = k - |J \cap \{1, \ldots, n, n+1, \ldots, n+m\}|$. We argue by induction on h. If $h = 0$, then $e_{A,B,C,D} \in \langle \mathbb{S}_{\{1,\ldots,2n+2m\},k}(f_q) \rangle = \langle \mathbb{Q}_{\{1,2,\ldots,2n+2m\},k} \rangle$ by Lemma 5.6, since the polar space $\mathcal{Q}_{\{1,2,\ldots,2n+2m\}}$ has $d'_0 = 0$ and we are done. Suppose $h > 1$; in particular $D \neq \emptyset$ and take $j \in D$. As $D \subseteq \{2n+2m+1, \ldots, N\}$ (see Setting 4.1), it must be $j > 2n+2m$. Moreover, there necessarily exists an index $1 \leq i \leq n$ such that $\{2i-1, 2i\} \cap J = \emptyset$. Let $u = e_{2i-1} + \kappa_j e_{2i} + e_j$ where $q(e_j) = \kappa_j$. Clearly $q(u) = 0$, so $\langle u \rangle \in \mathbb{Q}_{\{2i-1,2i,j\},1}$. Since $\langle e_{2i-1} \rangle, \langle e_{2i} \rangle \in \mathbb{Q}_{\{2i-1,2i,j\}}$ we have $e_j \in \langle \mathbb{Q}_{\{2i-1,2i,j\},1} \rangle$. Now let $D' = D \setminus \{j\}$ and $J' = A \cup B \cup \overline{C} \cup D'$.Then, $h' := (k-1) - |J' \cap \{1, 2, \ldots, n+m\}| = (k-1) - |J \cap \{1, 2, \ldots, n+m\}| = h - 1$. Moreover, if $I_{ij} := \{1, 2, \ldots, N\} \setminus \{2i-1, 2i, j\}$, the form $q_{I_{ij}}$ is non-degenerate with parameters $[n-1, m, d'_0-1]$. So $e_{A,B,C,D'} \in \langle \mathbb{Q}_{I_{ij},k-1} \rangle$ by the inductive hypothesis on h (but induction on k or on d'_0 would work as well). By Lemma 2.1,

$$e_{A,B,C,D} = e_{A,B,C,D'} \wedge e_j \in \langle \mathbb{Q}_{I_{ij},k-1} \rangle \wedge \langle \mathbb{Q}_{\{2i-1,2i,j\},1} \rangle \subseteq \langle \mathbb{Q}_k \rangle.$$

This completes the proof. □

Corollary 5.8. $\dim(\langle \mathbb{Q}_k \rangle) = \binom{N}{k} - \binom{N}{k-2}$.

Remark 5.9. Theorem 5.7 includes Theorem 1.2 of [4] as the special case where $d = 1$, but the proof given in [4] is far more elaborate than our proof of Theorem 5.7. Note also that we have obtained Theorem 5.7 from Lemmas 5.5 and 5.6 while in [4] the analogue of our Lemma 5.5 is obtained as a consequence of Theorem 1.2 of that paper.

5.3. End of the proof of Part 3 of Theorem 1

Let now q be degenerate with reduced Witt index n and let $k \leq n$. When $\mathrm{char}(\mathbb{F}) \neq 2$ the equality $\dim(\langle \mathbb{Q}_k \rangle) = \binom{N}{k}$ follows from Corollary 5.3 and Part 1 of Theorem 2.5. In this case $\langle \mathbb{Q}_k \rangle = V_k$.

Let $\mathrm{char}(\mathbb{F}) = 2$. Then $\dim(\langle \mathbb{Q}_k \rangle) = \binom{N}{k} - \binom{N}{k-2}$ by Corollary 5.8 and Part 2 of Theorem 2.5. Let f_q be the bilinearization of q. Then $\langle \mathbb{Q}_k \rangle \leq \langle \mathbb{S}_k(f_q) \rangle$. Moreover $\dim(\langle \mathbb{S}_k(f_q) \rangle) = \binom{N}{k} - \binom{N}{k-2}$ by Part 2 of Theorem 1, as already proved in Section 4. The equality $\langle \mathbb{Q}_k \rangle = \langle \mathbb{S}_k(f_q) \rangle$ follows.

The proof of Theorem 1 is complete.

6. The case $n < k \leq n + d$

We shall now prove Corollary 2. Let $n < k \leq n + d$. Note firstly that the formula of Corollary 2.3 also holds for $n < k \leq n + d$ provided that the summation index i is subject to the restriction $k - i \leq n$, namely $i \geq k - n$. Similarly for the statement of Lemma 2.2. Thus,

$$\left.\begin{array}{rcl} \langle \mathbb{P}_k \rangle & = & \displaystyle\bigoplus_{i=k-n}^{\min(d,k)} \langle \overline{\mathbb{P}}_{k-i} \rangle \wedge \bigwedge^i R, \\ \dim(\langle \mathbb{P}_k \rangle) & = & \displaystyle\sum_{i=k-n}^{\min(d,k)} \dim(\langle \overline{\mathbb{P}}_{k-i} \rangle) \cdot \binom{d}{i}. \end{array}\right\} \qquad (8)$$

The dimensions of the spaces $\langle \overline{\mathbb{P}}_{k-i} \rangle$ are known by Theorem 1 (recall that the form $\overline{\eta}$ induced by η on $\overline{V}$ is non-degenerate): explicitly, $\dim(\langle \overline{\mathbb{P}}_{k-i} \rangle)$ is equal to $\binom{N-d}{k-i}$ or $\binom{N-d}{k-i} - \binom{N-d}{k-i-2}$ according to the type of η and the characteristic of $\mathbb{F}$ (when η is quadratic). By putting these values in the second equation of (8) and using Lemma 2.4 we obtain the formulas of Corollary 2.

The last claim of Corollary 2, namely $\langle \mathbb{P}_k \rangle \subset V_k$, is clear. Indeed $\sum_{i=0}^{k-n-1} \binom{N-d}{k-i}\binom{d}{i} > 0$. So, $\dim(\langle \mathbb{P}_k \rangle) < \binom{N}{k} = \dim(V_k)$ in Case 1 of Corollary 2. In Case 2 we have $\langle \overline{\mathbb{P}}_{k-i} \rangle \subset \overline{V}_{k-i}$ for every admissible value of i. Consequently $\langle \mathbb{P}_k \rangle \subset V_k$ by the first equation of (8).

7. Generalizing the Weyl embedding

In this section we will show that it is possible to define a projective embedding $\widetilde{\varepsilon}_0$ of a subgeometry Γ_0 of a given geometry Γ (in general Γ_0 is defined over a subfield) starting from a given projective embedding ε_0 of Γ_0 and knowing that there exist two projective embeddings ε and $\widetilde{\varepsilon}$ of Γ such that ε induces ε_0 on Γ_0 and is a quotient of $\widetilde{\varepsilon}$. We will then apply this result to obtain a generalization of the Weyl embedding to orthogonal Grassmannians in even characteristic defined by non-degenerate quadratic forms with anisotropic defect $d_0 > 1$ but $d_0' \leq 1$.

In the Introduction of this paper we have used the word "embedding" in a somewhat loose way, avoiding a strict definition but thinking of an embedding $\varepsilon : \Gamma \to \Sigma$ of a point-line geometry Γ in a projective space Σ as an injective mapping from the point-set P of Γ to the point-set of Σ, neither requiring that ε is 'projective', namely it maps lines of Γ onto lines of Σ (although this requirement is satisfied in most of the cases we consider), nor that $\varepsilon(P)$ spans Σ, even when ε is indeed 'projective'. This free way of talking was the right one in that context, since we focused on the problem of determining the span of the set of points $\varepsilon(P)$ (namely $\varepsilon_k(\mathcal{P}_k)$) inside Σ (namely $\mathrm{PG}(\bigwedge^k V)$). However, in the sequel, sticking to that lax setting would cause some troubles. Thus, from now on, we shall adopt a sharper terminology. Following Shult [21], given a point-line geometry Γ where the lines are sets of points, we say that an injective mapping from the point-set P of Γ to the point-set of a projective geometry $\mathrm{PG}(V)$ is a *projective embedding* of Γ in $\mathrm{PG}(V)$ if:

(E1) for every line ℓ of Γ, the set $\{\varepsilon(p) : p \in \ell\}$ is a line of $\mathrm{PG}(V)$;

(E2) the set $\varepsilon(P)$ spans $\mathrm{PG}(V)$.

When talking about projective embeddings in the Introduction of this paper we assumed (E1) but not (E2), but now we also require (E2). According to (E2), the dimension $\dim(\varepsilon)$ of ε is just $\dim(V)$.

All embeddings to be considered henceforth are projective in the sense we have now fixed. In the sequel we shall also deal with morphisms of embeddings. We refer to Shult [21] for this notion.

7.1. Liftings of projective embeddings

Let Γ be a point-line geometry and consider two projective embeddings

$$\widetilde{\varepsilon} : \Gamma \to \mathrm{PG}(\widetilde{V}), \qquad \varepsilon : \Gamma \to \mathrm{PG}(V)$$

defined over the same field $\mathbb{F}$ such that ε is a quotient of $\widetilde{\varepsilon}$. Let also $\varphi : \widetilde{\varepsilon} \to \varepsilon$ be the morphism from $\widetilde{\varepsilon}$ to ε, i.e. $\varphi\colon \widetilde{V} \to V$ is a semilinear mapping such that $\varphi \circ \widetilde{\varepsilon} = \varepsilon$. Note that φ is uniquely determined up to scalars. In particular, $\mathrm{PG}(\widetilde{V}/K) \cong \mathrm{PG}(V)$ where $K := \ker(\varphi)$. Note also that φ induces a bijection from $\widetilde{\varepsilon}(\Gamma)$ to $\varepsilon(\Gamma)$ (in fact an isomorphism of point-line geometries). Equivalently, K contains no point $\widetilde{\varepsilon}(p) \in \widetilde{\varepsilon}(\Gamma)$ and $\langle \widetilde{\varepsilon}(p), \widetilde{\varepsilon}(q) \rangle \cap K = 0$ for any two points p and q of Γ.

Lemma 7.1. *For any point p of Γ and any vector $v \in \varepsilon(p)$ with $v \neq 0$, there is exactly one vector $\widetilde{v} \in \widetilde{\varepsilon}(p)$ such that $\varphi(\widetilde{v}) = v$.*

Proof. We warn the reader that in the following we will often regard the point $\varepsilon(p)$ as a 1-dimensional vector subspace.

Since φ is surjective, $\varphi^{-1}(\varepsilon(p)) \neq \emptyset$. Hence, since by hypothesis $v \in \varepsilon(p)$ (i.e. $\langle v \rangle = \varepsilon(p)$), there is $\widetilde{x} \in \widetilde{V}$ such that $\varphi^{-1}(v) = \widetilde{x} + K$ and $\widetilde{\varepsilon}(p) = \langle \widetilde{x} \rangle$. Hence $|(\widetilde{x} + K) \cap \widetilde{\varepsilon}(p)| \geq 1$. If $|(\widetilde{x} + K) \cap \widetilde{\varepsilon}(p)| \geq 2$ then $\widetilde{\varepsilon}(p) \cap K \neq 0$ and, since $\dim(\widetilde{\varepsilon}(p)) = 1$, we would have $\widetilde{\varepsilon}(p) \subseteq K$ — a contradiction. So, $|(\widetilde{x} + K) \cap \widetilde{\varepsilon}(p)| = 1$. Take $\widetilde{v} \in \widetilde{\varepsilon}(p)$. Then $\langle \varphi(\widetilde{v}) \rangle = \varepsilon(p) = \langle v \rangle$; so $\varphi(\widetilde{v}) = tv$ for some $t \neq 0$. Up to replacing $\widetilde{v}$ with $t^{-1}\widetilde{v}$, we can suppose $\varphi(\widetilde{v}) = v$, so $\widetilde{v} \in (\widetilde{x} + K) \cap \widetilde{\varepsilon}(p)$. Consequently, $\varphi^{-1}(v) \cap \widetilde{\varepsilon}(p) = \{\widetilde{v}\}$. □

Definition 7.2. With $\widetilde{v}$ as in Lemma 7.1, we call $\widetilde{v}$ the *lifting* of v to $\widetilde{V}$ through φ and write $\widetilde{v} = \varphi^{-1}(v)$.

Let now Γ_0 be a subgeometry of Γ defined over a subfield $\mathbb{F}_0$ of $\mathbb{F}$. Suppose that the vector space V admits a basis E such that the restriction of ε to Γ_0 is the natural field extension of a projective embedding $\varepsilon_0 : \Gamma_0 \to \mathrm{PG}(V_0)$ where V_0 is the span of E over $\mathbb{F}_0$.

In order to avoid unnecessary complications, we assume that φ is linear. This hypothesis suits our needs in Section 7.2. Moreover, it is not as restrictive as it can look (see below, Remark 7.7).

Let $\varphi^{-1}(\varepsilon_0(\Gamma_0))$ be the set of all liftings to $\widetilde{V}$ of vectors of V representing points of the form $\varepsilon_0(p)$ with $p \in \Gamma_0$ and let $\widetilde{V}_0$ be the span of $\varphi^{-1}(\varepsilon_0(\Gamma_0))$ over $\mathbb{F}_0$. By Lemma 7.1, every vector $v_0 \in \varepsilon_0(p)$ with $p \in \Gamma_0$ admits a unique lifting $\widetilde{v}_0 \in \widetilde{\varepsilon}(\Gamma)$. Furthermore, since φ is $\mathbb{F}$–linear (whence also $\mathbb{F}_0$–linear) it is immediate to see that the set of the liftings of the non-zero vectors of $\varepsilon_0(p)$ is the $\mathbb{F}_0$-span $\langle \widetilde{v}_0 \rangle_{\mathbb{F}_0}$ of the lifting $\widetilde{v}_0$ of any non-zero vector $v_0 \in \varepsilon_0(p)$. So, the following definition is well posed.

Definition 7.3. Let $\widetilde{\varepsilon}_0 : \Gamma_0 \to \mathrm{PG}(\widetilde{V}_0)$ be the map defined by the clause $\widetilde{\varepsilon}_0(p) := \langle \widetilde{v}_0 \rangle_{\mathbb{F}_0}$ for p a point of Γ_0, $\varepsilon_0(p) = \langle v_0 \rangle_{\mathbb{F}_0}$ and $\widetilde{v}_0 := \varphi^{-1}(v_0)$. We call $\widetilde{\varepsilon}_0$ the *lifting* of ε_0 to $\widetilde{V}_0$ through φ.

Theorem 7.4. *The lifting $\widetilde{\varepsilon}_0$ of ε_0 is a projective embedding of Γ_0 in $\mathrm{PG}(\widetilde{V}_0)$ and φ induces a linear morphism $\varphi_0 : \widetilde{\varepsilon}_0 \to \varepsilon_0$.*

Proof. We first show that the image of any three collinear points $r, s, t \in \Gamma_0$ is contained in a line of $\mathrm{PG}(\widetilde{V}_0)$. Let $r' \in \varepsilon_0(r)$, $s' \in \varepsilon_0(s)$, $t' \in \varepsilon_0(t)$, with $r', s', t' \neq 0$. Since ε_0 is a projective embedding, there exist $x, y \in \mathbb{F}_0$ such that $r' = xs' + yt'$. By Lemma 7.1 the vectors $\widetilde{r}'$, $\widetilde{s}'$, $\widetilde{t}'$ such that $\varphi(\widetilde{r}') = r'$, $\varphi(\widetilde{s}') = s'$, $\varphi(\widetilde{t}') = t'$ are uniquely determined. Clearly $\widetilde{\varepsilon}_0(r) = \langle \widetilde{r}' \rangle$, $\widetilde{\varepsilon}_0(s) = \langle \widetilde{s}' \rangle$, $\widetilde{\varepsilon}_0(t) = \langle \widetilde{t}' \rangle$. Since φ is linear we have also

$$\varphi(x\widetilde{s}' + y\widetilde{t}') = x\varphi(\widetilde{s}') + y\varphi(\widetilde{t}') = xs' + yt' = r'.$$

So, by Lemma 7.1 we have $x\widetilde{s}' + y\widetilde{t}' = \widetilde{r}'$ with $x, y \in \mathbb{F}_0$. Thus, $\widetilde{r}' \in \langle \widetilde{s}', \widetilde{t}' \rangle_{\mathbb{F}_0}$.

Since ε_0 is projective, for each value of $x, y \in \mathbb{F}_0$, the vector $xs' + yt'$ represents a point $r \in \Gamma_0$, so also $x\widetilde{s}' + y\widetilde{t}'$ represents a point of Γ_0. This proves that the image of a line of Γ_0 by means of $\widetilde{\varepsilon}_0$ is a (full) line in $\mathrm{PG}(\widetilde{V}_0)$.

Define φ_0 by $\varphi_0(\widetilde{v}_0) = v_0$ for any $\widetilde{v}_0 \in \widetilde{\varepsilon}_0(p)$ and $p \in \Gamma_0$ and extend the function by linearity to all $\widetilde{V}_0$. So, $\varphi_0(\widetilde{\varepsilon}_0(p)) = \langle v_0 \rangle_{\mathbb{F}_0}$ where $\varepsilon_0(p) = \langle v_0 \rangle_{\mathbb{F}_0}$ and $\varphi_0 : \widetilde{V}_0 \to V_0$ is a restricted truncation of the $\mathbb{F}$–linear map $\varphi : \widetilde{V} \to V$. In particular, φ_0 is $\mathbb{F}_0$–linear. This completes the proof. $\square$

Remark 7.5. Clearly, $\ker(\varphi_0) = \widetilde{V}_0 \cap \ker(\varphi)$. So, if φ is an isomorphism then φ_0 too is an isomorphism.

Remark 7.6. It follows from Theorem 7.4 that $\dim(\widetilde{V}_0) \geq \dim(V)$. On the other hand, it can happen that $\widetilde{V}_0$ is not contained in the $\mathbb{F}_0$-span of any basis of $\widetilde{V}$. If that is the case, then it might happen that $\dim(\widetilde{V}_0) > \dim(\widetilde{V})$.

Remark 7.7. We have assumed that φ is linear but what we have said remains valid if φ is semi-linear without being linear, with the following unique modification: if σ is the automorphism of $\mathbb{F}$ associated to φ, then we must define $\widetilde{V}_0$ as the $\mathbb{F}_0'$-span of $\varphi^{-1}(\varepsilon_0(\Gamma_0))$ with $\mathbb{F}_0' := \sigma^{-1}(\mathbb{F}_0)$.

7.2. Weyl-like embeddings

Consider an orthogonal k-Grassmannian $\mathcal{Q}_k$ defined by a non-degenerate quadratic form of Witt index n and anisotropic defect d_0. Suppose $k < n$ and $d_0 \leq 1$. Then, as explained in Section 1.2, the geometry $\mathcal{Q}_k$ affords both the Weyl embedding and the Grassmann embedding. These are both projective. If either $\mathrm{char}(\mathbb{F}) \neq 2$ or $k = 1$ then the Weyl embedding and the Grassmann embedding are essentially the same while if $\mathrm{char}(\mathbb{F}) = 2$ and $k > 1$ then the Grassmann embedding is a proper quotient of the Weyl embedding.

If $d_0 > 1$ then the Weyl embedding cannot be considered. Nevertheless we shall manage to define a generalization of the Weyl embedding for orthogonal Grassmannians defined by quadratic forms with defect greater than 1. To this aim, we need a couple of lemmas on algebraic extensions of the underlying field $\mathbb{F}$ of a quadratic form q.

Lemma 7.8. *Let $\mathrm{char}(\mathbb{F}) = 2$ and let $q : V \to \mathbb{F}$ be the generic non-degenerate quadratic form with parameters $[n, m, d_0']$ as given by (7). Then there exists a field extension $\widehat{\mathbb{F}}$ such that the extension $\widehat{q}$ of q to $\widehat{V} := V \otimes \widehat{\mathbb{F}}$ admits the following representation with respect to a suitable basis $\widehat{E}$ or $\widehat{V}$:*

$$\widehat{q}(x_1, \ldots, x_N) \;=\; \begin{cases} \displaystyle\sum_{i=1}^{n+m} x_{2i-1}x_{2i} & \text{if } 2n + 2m = N, \\ \displaystyle\sum_{i=1}^{n+m} x_{2i-1}x_{2i} + \kappa_{2n+2m+1}x_{2n+2m+1}^2 & \text{if } 2n + 2m < N. \end{cases} \tag{9}$$

Moreover, if we require that $\widehat{\mathbb{F}}$ has minimal degree over $\mathbb{F}$, then $\widehat{\mathbb{F}}$ is a uniquely determined algebraic extension of $\mathbb{F}$ of degree $|\widehat{\mathbb{F}} : \mathbb{F}| \leq 2m + 2d_0''$ where $d_0'' := \max(0, d_0' - 1)$.

Proof. If $2n+2m<N$, put $\kappa := \kappa_{2n+2m+1}$, for short. Consider an extension $\widehat{\mathbb{F}}$ containing the two roots of each equation $\lambda_i t^2 + t + \mu_i = 0$ for $i = 1,\dots,m$, say α_i and β_i, and also all elements $\delta_j = (\kappa_j/\kappa)^{1/2} \in \mathbb{F}$ for all $j = 2n+2m+2,\dots,N$. Define $\gamma_i := (\alpha_i+\beta_i)^{1/2}$. Since $\mathrm{char}(\mathbb{F}) = 2$, we have $\gamma_i \in \widehat{\mathbb{F}}$. Now let

$$\left.\begin{array}{rcl} \widehat{e}_{2n+2i-1} & = & e_{2n+ri-1}\alpha_i\gamma_i^{-1} + e_{2n+2i}\gamma_i^{-1} \\ \widehat{e}_{2n+2i} & = & e_{2n+2i-1}\beta_i\gamma_i^{-1} + e_{2n+2i}\gamma_i^{-1} \end{array}\right\} \text{ for } i = 1,2,\dots,m.$$

Then,

$$\widehat{q}(\widehat{e}_{2n+2i-1}) = \lambda_i\alpha_i^2\gamma_i^{-2} + \alpha_i\gamma_i^{-2} + \mu_i\gamma_i^{-2} = \gamma_i^{-2}(\lambda_i\alpha_i^2 + \alpha_i + \mu_i) = 0,$$
$$\widehat{q}(\widehat{e}_{2n+2i}) = \lambda_i\beta_i^2\gamma_i^{-2} + \beta_i\gamma_i^{-2} + \mu_i\gamma_i^{-2} = \gamma_i^{-2}(\lambda_i\beta_i^2 + \beta_i + \mu_i) = 0,$$
$$f_{\widehat{q}}(\widehat{e}_{2n+2i-1}, \widehat{e}_{2n+2i})\gamma_i^{-2}(\alpha_i+\beta_i) = 1.$$

When $2n+2m<N$, put also $\widehat{e}_{2m+2n+1} := e_{2m+2n+1}$ and

$$\widehat{e}_j := e_{j-1}\delta_{j-1}^{-1} + e_j\delta_j^{-1}, \quad \text{for } j = 2n+2m+2,\dots,N,$$

with $\delta_{j-1} := 1$ when $j = 2n+2m+2$. Clearly, $\widehat{q}(\widehat{e}_{2n+2m+1}) = \kappa$ and, for $j = 2n+2m+2,\dots,N$,

$$\widehat{q}(\widehat{e}_j) = \kappa_{j-1}\delta_{j-1}^2 + \kappa_j\delta_j^{-2} = \kappa+\kappa = 0, \quad \text{for } j = 2n+2m+2,\dots,N$$

$$f_{\widehat{q}}(\widehat{e}_i, \widehat{e}_j) = 0, \quad \text{for } i,j = 2n+2m+1,\dots,N.$$

(As for the latter equality, recall that $f_q(e_i,e_j) = 0$ for any $i,j = 2n+2m+1,\dots,N$). So, for $j \geq 2n+2m+1$ the vector $\widehat{e}_j$ is orthogonal to all vectors of $\widehat{V}$, namely it belongs to the radical of $\widehat{q}$. Let now

$$\widehat{E} := (e_1,\dots,e_{2n},\widehat{e}_{2n+1},\dots,\widehat{e}_{2n+2m},\widehat{e}_{2n+2m+1},\dots,\widehat{e}_N). \tag{10}$$

This is a basis of $\widehat{V}$. With respect to this basis, $\widehat{q}$ assumes the form (9).

The last claim of the Lemma remains to be proved. Suppose that $\widehat{\mathbb{F}}$ has minimal degree over $\mathbb{F}$. Then $\widehat{\mathbb{F}}$ is an algebraic extension of $\mathbb{F}$ obtained from $\mathbb{F}$ as a series of quadratic extensions by adding roots of the equations $\lambda_i t^2 + t + \mu_i = 0$ for $i = 1,2,\dots,m$ and $t^2 = \kappa_j/\kappa$ for $j = 2n+2m+2,\dots,N$. The degree of such an extension is at most $2m + 2d_0''$ with $d_0'' = d_0' - 1$ if $d_0' > 0$, otherwise $d_0'' = 0$.

Still assuming that $\widehat{\mathbb{F}}$ has minimal degree, let $\widehat{\mathbb{F}}'$ be another extension of $\mathbb{F}$ such that the extension $\widehat{q}'$ of q to $\widehat{V}' = V \otimes \widehat{\mathbb{F}}'$ admits the representation (9). If some of the polynomials $\lambda_i t^2 + t + \mu_i$ or $t^2 + \kappa_j/\kappa$ are irreducible in $\widehat{\mathbb{F}}'$, then $\widehat{V}'$ admits 2-subspaces X totally anisotropic for $\widehat{q}'$ and such that $X + X^\perp = \widehat{V}'$. However this situation does not fit with (9). Therefore all polynomials $\lambda_i t^2 + t + \mu_i$ for $i = 1,2,\dots,m$ are reducible in $\widehat{\mathbb{F}}'$ as well as all polynomials $t^2 + \kappa_j/\kappa$ for $j = 2n+2m+2,\dots,N$. It follows that $\widehat{\mathbb{F}}'$ contains $\widehat{\mathbb{F}}$ (modulo isomorphisms, of course). $\square$

Clearly, the form $\widehat{q}$ of Lemma 7.8 is non-degenerate if and only if $d_0' \leq 1$. If this is the case then $d_0' = \mathrm{def}_0(\widehat{q})$. On the other hand, if $d_0' > 1$ then the radical of q has dimension $d_0' - 1$ and is generated by the last $d_0' - 1$ vectors $\widehat{e}_{2m+2n+2},\dots,\widehat{e}_N$ of the basis $\widehat{E}$ defined in (10).

In Lemma 7.8 we have assumed that $\mathrm{char}(\mathbb{F}) = 2$. An analogue of Lemma 7.8 also holds for $\mathrm{char}(\mathbb{F}) \neq 2$. Its proof is similar to that of Lemma 7.8. We leave it to the reader.

Lemma 7.9. *Let* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let* $q : V \to \mathbb{F}$ *be the generic non-degenerate quadratic form as given by* (6)*. Then there exists a field extension* $\widehat{\mathbb{F}}$ *such that the extension* $\widehat{q}$ *of* q *to* $\widehat{V} := V \otimes \widehat{\mathbb{F}}$

admits the following representation with respect to a suitable basis $\widehat{E}$ of $\widehat{V}$:

$$\widehat{q}(x_1,\ldots,x_N) \;=\; \begin{cases} \sum_{i=1}^{N/2} x_{2i-1}x_{2i} & \text{if } N \text{ is even} \\ \sum_{i=1}^{(N-1)/2} x_{2i-1}x_{2i} + \kappa_N x_N^2 & \text{if } N \text{ is odd.} \end{cases} \tag{11}$$

Moreover, if we require that $\widehat{\mathbb{F}}$ has minimal degree over $\mathbb{F}$, then $\widehat{\mathbb{F}}$ is a uniquely determined algebraic extension of $\mathbb{F}$ of degree at most $g = 2d_0''$ where $d_0'' := \max(0, \mathrm{def}_0(q)-1)$.

The form $\widehat{q}$ of Lemma 7.9 is always non-degenerate with anisotropic defect equal to 0 or 1 according to whether N is even or odd.

We shall now prove Theorem 3 of Section 1.2. Let $\widehat{\mathbb{F}}$, $\widehat{V}$ and $\widehat{q}$ be as in Lemmas 7.8 or 7.9. In particular, q is non-degenerate. To fix ideas, suppose we have chosen $\widehat{\mathbb{F}}$ so that it has minimal degree over $\mathbb{F}$. Thus $\widehat{\mathbb{F}}$ is uniquely determined. Accordingly, $\widehat{V}$ and $\widehat{q}$ are uniquely determined as well. Let $\mathcal{Q}_k$ and $\widehat{\mathcal{Q}}_k$ be the orthogonal k-Grassmannians associated to q and $\widehat{q}$ respectively and

$$\varepsilon_k : \mathcal{Q}_k \to \langle \varepsilon_k(\mathcal{Q}_k)\rangle \subseteq \mathrm{PG}(V_k), \quad \widehat{\varepsilon}_k : \widehat{\mathcal{Q}}_k \to \langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k)\rangle \subseteq \mathrm{PG}(\widehat{V}_k)$$

be their Grassmann embeddings. So, $\mathcal{Q}_k$ is a subgeometry of $\widehat{\mathcal{Q}}_k$ and $\widehat{\varepsilon}_k$ induces ε_k on $\mathcal{Q}_k$. (Recall that $\mathrm{PG}(V_k)$ is a subgeometry of $\mathrm{PG}(\widehat{V}_k)$, non-full if and only if $\widehat{\mathbb{F}} \supset \mathbb{F}$).

Suppose $k < n$ and, when $\mathrm{char}(\mathbb{F}) = 2$, assume $d_0' \leq 1$. So, $\widehat{q}$ is non-degenerate with anisotropic defect $\mathrm{def}_0(\widehat{q}) \leq 1$. Consequently, the k-Grassmannian $\widehat{\mathcal{Q}}_k$ admits the Weyl embedding, say $\widehat{e}_k^W : \widehat{\mathcal{Q}}_k \to \mathrm{PG}(\widehat{V}_k^W)$, and $\widehat{\varepsilon}_k$ is a quotient of $\widehat{e}_k^W$, as explained in Section 1.2.

Let $\widehat{\varphi}$ be the (linear) morphism from $\widehat{e}_k^W$ to $\widehat{\varepsilon}_k$ and let e_k^W be the lifting of ε_k to $V_k^W := \langle \widehat{\varphi}^{-1}(\varepsilon_k(\mathcal{Q}_k))\rangle_{\mathbb{F}}$ through $\widehat{\varphi}$. We know by Theorem 7.4 that e_k^W is a projective embedding of $\mathcal{Q}_k$ in $\mathrm{PG}(V_k^W)$ and that $\widehat{\varphi}$ induces a (linear) morphism $\varphi : V_k^W \to \langle \varepsilon_k(\mathcal{Q}_k)\rangle$ from e_k^W to ε_k. As $\widehat{q}$ is uniquely determined, in view of the hypotheses made on $\widehat{\mathbb{F}}$, the embedding e_k^W is uniquely determined as well.

Definition 7.10. The embedding $e_k^W : \mathcal{Q}_k \to \mathrm{PG}(V_k^W)$ is the *Weyl-like embedding* of $\mathcal{Q}_k$.

As e_k^W is a projective embedding, $\dim(e_k^W) = \dim(V_k^W)$ by Property (E2) of projective embeddings. As noticed in Section 1.2, the morphism $\widehat{\varphi}$ is an isomorphism when either $\mathrm{char}(\mathbb{F}) \neq 2$ or $k = 1$. In this case $e_k^W \cong \varepsilon_k$ (Remark 7.5). If $d_0 = \mathrm{def}_0(q) \leq 1$ then $\widehat{q} = q$. In this case $\mathbb{F} = \widehat{\mathbb{F}}$ and $e_k^W = \widehat{e}_k^W$. Accordingly, $\dim(e_k^W) = \dim(V_k^W) = \binom{N}{k}$.

All parts of Theorem 3 have been proved. It remains to estimate $\dim(e_k^W)$ (namely $\dim(V_k^W)$) when $\mathrm{char}(\mathbb{F}) = 2$, $k > 1$ and $d_0 > 1$. Recall that we have assumed the degree $|\widehat{\mathbb{F}} : \mathbb{F}|$ to be minimal and $d_0' \leq 1$. Hence $|\widehat{\mathbb{F}} : \mathbb{F}| \leq 2m$, by Lemma 7.8.

Theorem 7.11. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$. *Assume the hypotheses $k < n$ and $d_0' \leq 1$, let $k > 1$ and $d_0 > 1$. Then, with $g := |\widehat{\mathbb{F}} : \mathbb{F}|$, we have $2 \leq g \leq 2m$ and*

$$\binom{N}{k} - \binom{N}{k-2} \leq \dim(e_k^W) \leq \binom{N}{k} + \binom{N}{k-2}(g-1). \tag{12}$$

Proof. By construction, φ is the restriction of $\widehat{\varphi}$ to V_k^W. Hence $\ker(\varphi) \subseteq \ker(\widehat{\varphi}) \cap V_k^W$. Moreover, $\ker(\varphi)$, regarded as an $\mathbb{F}$-space, has dimension $\dim_{\mathbb{F}}(\ker(\widehat{\varphi})) = g \cdot \dim_{\widehat{\mathbb{F}}}(\ker(\widehat{\varphi}))$. So,

$$\dim(\ker(\varphi)) \leq \dim_{\mathbb{F}}(\ker(\widehat{\varphi})) = g \cdot \dim_{\widehat{\mathbb{F}}}(\ker(\widehat{\varphi})). \tag{13}$$

However

$$\dim_{\widehat{\mathbb{F}}}(\ker(\widehat{\varphi})) = \binom{N}{k-2} \tag{14}$$

since $\dim(\widehat{V}_k^W) = \binom{N}{k}$ while $\widehat{\varphi}(\widehat{V}_k^K) = \langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k)\rangle$ and $\dim_{\widehat{\mathbb{F}}}(\langle \widehat{\varepsilon}_k(\widehat{\mathcal{Q}}_k)\rangle) = \binom{N}{k} - \binom{N}{k-2}$ (Theorem 1, Part 3). By combining (13) with (14) we get

$$\dim(\ker(\varphi)) \leq g \cdot \binom{N}{k-2}. \tag{15}$$

However φ maps V_k^W onto $\langle \varepsilon_k(\mathcal{Q}_k)\rangle$ and $\dim(\langle \varepsilon_k(\mathcal{Q}_k)\rangle) = \binom{N}{k} - \binom{N}{k-2}$ by Theorem 1, Part 3. By comparing these facts with (15) we obtain

$$\begin{aligned} \dim(V_k^W) &= \dim(\langle \varepsilon_k(\mathcal{Q}_k)\rangle) + \dim(\ker(\varphi)) \leq \\ &\leq \binom{N}{k} - \binom{N}{k-2} + g \cdot \binom{N}{k-2} = \binom{N}{k} + \binom{N}{k-2} \cdot (g-1) \end{aligned}$$

which yields the right hand inequality of (12). Similarly, from

$$\dim(V_k^W) = \dim(\langle \varepsilon_k(\mathcal{Q}_k)\rangle) + \dim(\ker(\varphi)) \geq \binom{N}{k} - \binom{N}{k-2}$$

we obtain the left part of (12). □

Remark 7.12. Since the Grassmann embedding ε_k is *transparent* [9] and ε_k is a quotient of e_k^W, the latter is also transparent, i.e. the e_k^W-preimage of every projective line contained in $e_k^W(\mathcal{Q}_k)$ is a line of the geometry $\mathcal{Q}_k$.

7.3. Comments on the case $k = n$

We have not considered the case $k = n$ in Section 1.2. When $d_0 = 0$ no Weyl embedding can be defined for $\mathcal{Q}_n$. Let $d_0 = 1$. Then $\mathcal{Q}_n$ admits two interesting Weyl embeddings. One of them is the so-called *spin embedding*, say it e_n^{spin}. It is projective, 2^n-dimensional and lives in the Weyl module for $\mathrm{O}(N, \mathbb{F})$ associated the n-th fundamental weight λ_n of the root system of type B_n. The second Weyl embedding, say it e_n^W, is $\binom{N}{n}$-dimensional and lives in the Weyl module associated to the weight $2\lambda_n$. This embedding is veronesean and is closely related with ε_n. In fact, when $\mathrm{char}(\mathbb{F}) \neq 2$ then $e_n^W = \varepsilon_n$ while, if $\mathrm{char}(\mathbb{F}) = 2$, then ε_n is a proper quotient of e_n^W (see [4] and [5]). We call e_n^W the *canonical veronesean embedding* of $\mathcal{Q}_n$.

An interesting relation exists between e_n^{spin} and e_n^W. Explicitly, let ν be the veronesean embedding of the codomain $\mathrm{PG}(2^n - 1, \mathbb{F})$ of e_n^{spin} in $\mathrm{PG}(\binom{2^n+1}{2} - 1, \mathbb{F})$, which bijectively maps $\mathrm{PG}(2^n - 1, \mathbb{F})$ onto the standard veronesean variety of $\mathrm{PG}(\binom{2^n+1}{2} - 1, \mathbb{F})$. Then $e_n^W = \nu \cdot e_n^{\mathrm{spin}}$ (see e.g. [4], [5]). In particular, when $e_n^W = \varepsilon_n$ (equivalently $\mathrm{char}(\mathbb{F}) \neq 2$), then $\varepsilon_n = \nu \cdot e_n^{\mathrm{spin}}$.

So far for the case $d_0 = 1$. We shall now consider the case $d_0 > 1$, but we firstly state some terminology which will facilitate the discussion of this case. We say that an injective mapping e from the point-set P of a point-line geometry Γ into the point-set of a projective geometry Σ is a *quadratic embedding* of *rank* $r \geq 2$ if e maps the lines of Γ onto non-degenerate quadrics of Witt index 1 spanning r-dimensional subspaces of Σ (and in addition $e(P)$ spans Σ). Thus, the veronesean embeddings as defined in [4], [5] and [18] are just the quadratic embeddings of rank 2. Morphisms of quadratic embeddings can be defined just as for veronesean embeddings (see [5], [18, Section 2.2]).

We can now assume $d_0 > 1$ with $d_0' = 1$ when $\mathrm{char}(\mathbb{F}) = 2$. As noticed in Section 1.1, the Grassmann embedding ε_n of $\mathcal{Q}_n$ is quadratic of rank $d_0 + 1$. We can still consider the extension $\widehat{q}$ of q as in Lemmas 7.8 and 7.9. The form $\widehat{q}$ is non-degenerate with Witt index $\widehat{n} = n + \lfloor d_0/2 \rfloor$, where $\lfloor . \rfloor$ stands for integral part, and anisotropic defect $\widehat{d_0}$ equal to 0 or 1 according to whether d_0 is even or odd. In any case $\widehat{n} > n$. So, the Weyl embedding $\widehat{e}_n^W$ of $\widehat{\mathcal{Q}}_n$ is projective, as well as the Grassmann embedding $\widehat{\varepsilon}_n$. The n-Grassmannian $\mathcal{Q}_n$ is not a subgeometry of $\widehat{\mathcal{Q}}_n$; nevertheless all of its points are points of $\widehat{\mathcal{Q}}_n$. So, Lemma 7.1 still allows to lift ε_n to an embedding $e_n^W : \mathcal{Q}_n \to \mathrm{PG}(V_n^W)$ through the morphism $\widehat{\varphi} : \widehat{e}_n^W \to \widehat{\varepsilon}_n$, for a suitable $\mathbb{F}$-subspace of $\widehat{V}_n$. However e_n^W cannot be projective. More explicitly, since $\widehat{\varphi}$ is linear, the lifting map behaves linearly, as shown in the proof of Theorem 7.4. This implies that the embedding e_n^W is quadratic with the same rank $d_0 + 1$ as ε_n. Moreover $\widehat{\varphi}$ maps $V_n^W \subset \widehat{V}_n$ onto V_n and induces a morphism $\varphi : e_n^W \to \varepsilon_n$ (an isomorphism if $\widehat{\varphi}$ is an isomorphism). We leave the proofs of these claims to the reader. We call e_n^W the *Weyl-like* embedding of $\mathcal{Q}_n$.

The case $d_0 = 2$ is particularly interesting. In this case $\widehat{\mathcal{Q}}_n = \mathcal{Q}_n(n+1, 0, 0; \widehat{\mathbb{F}})$ and $\widehat{\mathbb{F}}$, chosen of minimal degree over $\mathbb{F}$, is a separable quadratic extension of $\mathbb{F}$. Let $\widetilde{\mathcal{Q}}_n := \mathcal{Q}_n(n, 1, 0; \widehat{\mathbb{F}})$. Then $\widetilde{\mathcal{Q}}_n$ is contained in $\widehat{\mathcal{Q}}_n$ (but not as a subgeometry) while $\mathcal{Q}_n$ is a full subgeometry of $\widetilde{\mathcal{Q}}_n$ (see e.g. [12]; also [9, §3.4]). The geometry $\widetilde{\mathcal{Q}}_n$ admits both the spin embedding $\widetilde{e}_n^{\mathrm{spin}} : \widetilde{\mathcal{Q}}_n \to \mathrm{PG}(2^n - 1, \widehat{\mathbb{F}})$ and the canonical veronesean embedding $\widetilde{e}_n^W : \widetilde{\mathcal{Q}}_n \to \mathrm{PG}(\binom{2^n+1}{2} - 1, \widehat{\mathbb{F}})$. The spin embedding $\widetilde{e}_n^{\mathrm{spin}}$ induces a 2^n-dimensional projective embedding on $\mathcal{Q}_n$ (see e.g. [12]; [9, §4.2]). We call it the *spin-like* embedding of $\mathcal{Q}_n$ and use the symbol e_n^{spin} for it too.

Conjecture 7.13. *We conjecture that the Weyl embedding $\widehat{e}_n^W$ of $\widehat{\mathcal{Q}}_n$ induces on $\widetilde{\mathcal{Q}}_n$ its veronesean embedding $\widetilde{e}_n^W$, the codomain V_k^W of the Weyl-like embedding e_n^W of $\mathcal{Q}_n$ is the canonical Baer subgeometry of $\mathrm{PG}(\binom{2^n+1}{2} - 1, \widehat{\mathbb{F}})$ defined over $\mathbb{F}$ and $\widetilde{e}_n^W$ induces e_n^W on $\mathcal{Q}_n$. If so, then $e_n^W = \nu \cdot e_n^{\mathrm{spin}}$, where ν is the canonical veronesean embedding of $\mathrm{PG}(2^n - 1, \mathbb{F})$ in $\mathrm{PG}(\binom{2^n+1}{2} - 1, \mathbb{F})$. In particular, if $e_n^W = \varepsilon_n$ (as when $\mathrm{char}(\mathbb{F}) \neq 2$) then $\varepsilon_n = \nu \cdot e_n^{\mathrm{spin}}$.*

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