Riv. Mat. Univ. Parma (4) 11 (1985), 287-298

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On the existence and uniqueness of solutions for problems in Kelvin-Voigt viscoelasticity (**)

1 - Introduction

In this paper we study the following initial-boundary value problem:

$$u_{tt}-\frac{\mathrm{d}}{\mathrm{d}x_i}a_i(x,t,u_{x_i})-\Delta_N u_t=f \qquad (x,t)\in\Omega\times(0,T), \tag{1.1}$$

$$u(x,0)=u_0(x)\,,\quad u_t(x,0)=u_1(x) \qquad x\in\Omega\,, \tag{1.2}$$

$$u(x,t)=0 \qquad \text{on } \partial\Omega\times[0,T]\,, \tag{1.3}$$

where summation over $i=1,2,\dots,N$ is understood, Δ_N is the N-dimensional Laplace operator and Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$.

For $N=1$ this problem arises in the purely longitudinal motion of a viscoelastic bar of Kelvin type where the stress σ on a cross section is given by $\sigma(x,t)=u_{xt}(x,t)+a(x,t,u_x(x,t))$. This model – first proposed in [7] – is the simplest one-dimensional model of a material whose stress depends on the history of motion.

Cauchy and mixed problem related to equation (1.1) have been treated by many authors assuming that the non-linear terms a_i were monotone in u_{x_i}. For $N=1$ Greenberg, Mac Camy and Mizel [5], [6], [7] showed the existence of a unique global smooth solution of (1.1)-(1.3) which is asymptotically stable. Pecher [9] obtained existence and uniqueness results for the Cauchy problem

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(**) Lavoro eseguito nell'ambito del G.N.F.M. (C.N.R.). — Ricevuto: 28-XI-1983.

in the case $N \leqslant 2$. Yet, in order to establish the existence of global or periodic solutions for arbitrary N some growth conditions besides monotonicity have to be imposed on each a_i (see [**2**]$_1$, [**4**], [**10**]). Also, an existence and uniqueness theorem for any N was proved by Clements [**2**]$_2$ assuming that the a_i were of « monomial » growth and monotonic in u_{x_i}.

Within a different approach quite similar results were obtained by Dafermos [**3**] and Andrews [**1**] in the one-dimensional case without imposing a monotonicity condition. Our goal is to extend their results to arbitrary N.

In this paper we prove the existence and uniqueness of solutions of the problem (1.1)-(1.3) *for any* N by assuming that each $a_i(x, t, u_{x_i})$ is sufficiently smooth in its arguments, but making *no monotonicity assumption*. As we shall show this makes the problem of asymptotic behaviour rather more interesting (for $N = 1$ see [**3**]). In particular, each a_i is required to be uniformly Lipschitz continuous in u_{x_i} and its dependence on x and t is restricted by certain requirements of boundedness. Under these assumptions a compactness argument is used to prove the existence of a unique global « strong » solution. Moreover, it turns out that the solution is asymptotically stable in the sense that the velocity, the velocity gradient and the acceleration decay to zero while the deformation gradient and the stress remain uniformly bounded as time grows to infinity.

2 - Existence and uniqueness of solutions

Let Ω be a bounded domain in R^N with sufficiently smooth boundary $\partial\Omega$ and let $(x_1, x_2, \dots, x_N) \in R^N$ be denoted by x.

For $1 \leqslant p \leqslant \infty$ let $L^p(\Omega)$ be the usual real Lebesgue space with norm

$$\|u\|_{0,p} = \Big(\int_\Omega |u(x)|^p \,\mathrm{d}x\Big)^{1/p} < \infty \qquad 1 \leqslant p < \infty\,,$$
$$\|u\|_{0,\infty} = \operatorname*{essup}_{\Omega} |u(x)| < \infty \qquad p = \infty\,.$$

In particular, $L^2(\Omega)$ is a Hilbert space with respect to the inner product $(u, v) = \int_\Omega u(x) v(x)\,\mathrm{d}x$. We shall use this notation also to denote the natural pairing of $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ where $1/p + 1/q = 1$. The Sobolev spaces $H^m(\Omega)$, $m \in N$, defined by

$$H^m(\Omega) = \{u \in L^2(\Omega) \colon D^\alpha u \equiv (\partial^{|\alpha|} u / \partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}) \in L^2(\Omega), \alpha_1 + \dots + \alpha_N = |\alpha| \leqslant m\}$$

are Banach spaces with respect to the norm

$$\|u\|_m^2 = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{0,2}^2\,.$$

By $H_0^m(\Omega)$ we mean the closure in the norm of $H^m(\Omega)$ of the smooth functions with compact support in Ω. For convenience we shall denote the norm $\|\cdot\|_{0,2}$ by $\|\cdot\|$ and the product $\sum_{i=1}^{N}(u_{x_i}, v_{x_i})$ by $((u, v))$. By means of the Poincaré inequality it turns out that the norm $|||u||| = \sqrt{((u, u))}$ is equivalent to $\|u\|_1$, in fact $\|u\| \leqslant k|||u|||$.

If X is a Banach space with norm $\|\cdot\|_X$ let X^* denote its dual and let $L^p(0, T; X)$, $1 \leqslant p \leqslant \infty$, denote the space of real measurable functions f: $(0, T) \to X$ with norm

$$\Big(\int_0^T \|f(t)\|_X^p \, dt\Big)^{1/p} < \infty \qquad 1 \leqslant p < \infty ,$$

$$\operatorname*{essup}_{(0,T)} \|f(t)\|_X < \infty \qquad p = \infty .$$

In order to prove existence and uniqueness of a generalized solution of (1.1)-(1.3) our assumptions are as follows.

(H.1) Each $a_i(x, t, \eta)$, $i = 1, 2, \ldots, N$, is a real valued function defined on $\Omega \times [0, T] \times R$ and once continuously differentiable in x, t and η.

(H.2) There exist a positive constant K_0 and some non negative functions $K_1, K_2 \in L^2(0, T)$ such that for all $x \in \Omega$, $t \in [0, T]$, $\eta \in R$ and each $i = 1, 2, .., N$

(i) $a_i(x, t, \eta)\eta \geqslant 0$, (ii) $|(\partial/\partial\eta) a_i(x, t, \eta)| \leqslant K_0$,

(iii) $|(\partial/\partial x_i) a_i(x, t, \eta)| \leqslant K_1(t)|\eta|$, (iv) $|(\partial/\partial t) a_i(x, t, \eta)| \leqslant K_2(t)|\eta|$.

(H.3) $f \in H^1(0, T; L^2(\Omega))$; $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$.

As a consequence the following result is established

Theorem 1. *If conditions* (H.1)-(H.3) *are satisfied there exists one and only one function u with*

$$u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) , \quad u_t \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) ,$$
$$u_{tt} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

such that $u(0) = u_0$, $u_t(0) = u_1$ a.e. on Ω and

$$u_{tt} - \frac{d}{dx_i} a_i(x, t, u_{x_i}) - \Delta_N u_t = f \qquad a.e. \text{ (}^1\text{)} .$$

(1) This means that the left hand side and the right hand side are equivalent a.e. on $(0, T)$ as functions from $(0, T)$ into $L^2(\Omega)$.

Proof of existence. Since Ω is a smooth bounded domain we can choose a countable set $\{w_j\}_{j\in N}$ of distinct elements of $C^2(\Omega)$ which are a basis of $H_0^1(\Omega)\cap H^2(\Omega)$ and are orthonormal in $L^2(\Omega)$. Let P_n be the projection in $L^2(\Omega)$ onto the subspace generated by the basis elements $w_1, w_2, \dots, w_n$. We now consider the following initial value problem

$$(2.1)\qquad \begin{aligned} &\big(u^n_{tt}(t), w_j\big) - \Big(\frac{\mathrm{d}}{\mathrm{d}x_i} a_i(x, t, u^n_{x_i}(t)), w_j\Big) - \big(\Delta_N u^n_t(t),\ w_j\big) = \big(f(t), w_j\big) \\ &u^n(0) = P_n u_0\,, \qquad u^n_t(0) = P_n u_1\,, \qquad\qquad j = 1, \dots, n\,, \end{aligned}$$

where $u^n \in P_n L^2(\Omega)$ for all t in $[0, T]$.

From the theory of ordinary differential equations and hypothesis (H.2) it follows that for each n there exists a solution $u^n = \sum_{k=1}^{n} c_{nk}(t) w_k$ which satisfies initial conditions a.e. on $[0, T_n]$ for some $T_n \in (0, T]$.

The following a priori bounds of u^n allow each T_n to be taken equal to T. The standard « energy » estimate is obtained from (2.1) in the usual way

$$(2.2)\qquad \begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|u^n_t(t)\|^2 + 2\int_\Omega \int_0^{u^n_{x_i}(t)} a_i(x, t, \eta)\,\mathrm{d}\eta\,\mathrm{d}x\Big\} + |||u^n_t(t)|||^2 \\ &\qquad = \int_\Omega \int_0^{u^n_{x_i}(t)} \frac{\partial}{\partial t}\ a_i(x, t, \eta)\,\mathrm{d}\eta\ \mathrm{d}x + \big(f(t), u^n_t(t)\big)\,. \end{aligned}$$

From conditions (H.2 (i)), (H.2 (ii)) and (H.2 (iv)) we deduce

$$0 \leqslant \int_0^\xi a_i(x, t, \eta)\,\mathrm{d}\eta \leqslant K_0\xi^2/2\,, \qquad \int_0^\xi \Big|\frac{\partial}{\partial t} a_i(x, t, \eta)\Big|\,\mathrm{d}\eta \leqslant K_2(t)\xi^2/2$$

for any $(x, t) \in \Omega \times [0, T]$ and any $\xi \in R$. Hence, adding $-(\Delta_N\ u^n(t), u^n_t(t))$ to both sides of (2.2) and integrating the result over $[0, t]$ it follows that

$$\|u^n_t(t)\|^2 + |||u^n(t)|||^2 + \int_0^t |||u^n_s|||^2\,\mathrm{d}s \leqslant \int_0^t \{\|u^n_s\|^2 + [1 + K_2(s)]|||u^n|||^2\}\,\mathrm{d}s + C_0\,.$$

Applying Gronwall's Lemma one obtains

$$(2.3)\qquad \|u^n_t(t)\|^2 + \|u^n(t)\|_1^2 + \int_0^t \|u^n_s\|_1^2\,\mathrm{d}s \leqslant C_1$$

independent of n for all $t \in [0, T]$.

Replacing w_j by $-\Delta_N u^n(t)$ in (2.1) and recalling that $\|u^n_{x_i x_i}(t)\| \leqslant \hat{K}\|\Delta_N u^n(t)\|$ for all t and n, from (H.2 (ii)), (H.2 (iii)) and the Cauchy inequality we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_N u^n(t)\|^2 \leqslant (1+\hat{K}K_0)\|\Delta_N u^n(t)\|^2 + \frac{1}{2}C_1 K_1^2(t) + \big(u^n_{tt}(t), \Delta_N u^n(t)\big) + \frac{1}{2}\|f\|^2 .$$

An integration over $(0,t)$ gives

$$\tfrac{1}{2}\|\Delta_N u^n(t)\|^2 \leqslant (1+\hat{K}K_0)\int_0^t \|\Delta_N u^n\|^2 \,\mathrm{d}s + \int_0^t |||u^n_s|||^2 \,\mathrm{d}s + \big(u^n_t(t), \Delta_N u^n(t)\big) + C_3 ,$$

so that from (2.3) and Gronwall's Lemma it follows

$$\|\Delta_N u^n(t)\|^2 \leqslant C_4 \quad \text{independent of } n . \tag{2.4}$$

By replacing w_j by $-\Delta_N u^n_t(t)$, (2.1) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||u^n_t(t)|||^2 + \|\Delta_N u^n_t(t)\|^2 \leqslant \big(f(t), \Delta_N u^n_t(t)\big) - \Big(\frac{\partial}{\partial\eta} a_i\big(x,t,u^n_{x_i}(t)\big) u^n_{x_i x_i}(t), \Delta_N u^n_t(t)\Big) - \Big(\frac{\partial}{\partial x_i} a_i\big(x,t,u^n_{x_i}(t)\big), \Delta_N u^n_t(t)\Big) ,$$

so that (H.2 (ii)), (H.2 (iii)), previous estimates and Schwarz's inequality yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||u^n_t(t)|||^2 + \|\Delta_N u^n_t(t)\|^2 \leqslant \big(\|f(t)\| + \sqrt{C_1}\,K_1(t) + C_5\big)\|\Delta_N u^n_t(t)\| .$$

Integrating over $(0,t)$ and using the Cauchy inequality we obtain

$$\|u^n_t(t)\|_1^2 + \int_0^t \|\Delta_N u^n_s\|^2 \,\mathrm{d}s \leqslant C_6 \quad \text{independent of } n . \tag{2.5}$$

Finally, for $h \in (0,T)$ and any function $\varphi(t)$ from $[0,T]$ into $L^2(\Omega)$ we set

$$\varphi_h(t) = \frac{1}{h}\big(\varphi(t+h) - \varphi(t)\big) \in L^2(\Omega) \qquad t \in [0, T-h] .$$

From (2.1) we obtain

$$\big(u^n_{htt}(t), w_j\big) - \Big(\frac{\mathrm{d}}{\mathrm{d}x_i} a_{ih}(t), w_j\Big) - \big(\Delta_N u^n_{ht}(t), w_j\big) = \big(f_h(t), w_j\big) , \qquad \text{where} \tag{2.6}$$

$$a_{ih}(t) = \frac{1}{h}\big[a_i\big(x, t+h, u^n_{x_i}(t+h)\big) - a_i\big(x,t,u^n_{x_i}(t)\big)\big] = \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda} a_i\big(x, \tau(\lambda), \eta^n(\lambda)\big)\,\mathrm{d}\lambda ,$$

$$\tau(\lambda) = \lambda(t+h) + (1-\lambda)t , \qquad \eta^n(\lambda) = \lambda u^n_{x_i}(t+h) + (1-\lambda)u^n_{x_i}(t) .$$

Then, each a_{ih} may be rewritten in the form

$$a_{ih} = A_i^{(\tau)} + A_i^{(\eta)} u^n_{hx_t} \qquad \text{with}$$

$$A_i^{(\tau)} = \int_0^1 \frac{\partial}{\partial\tau} a_i(x, \tau(\lambda), \eta^n(\lambda))\, d\lambda, \quad A_i^{(\eta)} = \int_0^1 \frac{\partial}{\partial\eta} a_i(x, \tau(\lambda), \eta^n(\lambda))\, d\lambda ,$$

and recalling (H.2 (ii)) and (H. 2 (iv)) we have

$$|A_i^{(\tau)}| \leqslant \int_0^1 K_2(\tau(\lambda))\, |\eta^n(\lambda)|\, d\lambda \qquad |A_i^{(\eta)}| \leqslant K_0 . \tag{2.7}$$

Now we replace w_j by $u^n_{ht}(t)$ in (2.6) thus obtaining

$$\frac{1}{2}\frac{d}{dt}\|u^n_{ht}\|^2 + |||u^n_{ht}|||^2 + (A_i^{(\tau)}, u^n_{hx_it}) + (A_i^{(\eta)} u^n_{hx_i}, u^n_{hx_it}) = (f_h, u^n_{ht}) .$$

From (2.7) and the Cauchy inequality

$$\frac{1}{2}\frac{d}{dt}\|u^n_{ht}\|^2 + |||u^n_{ht}|||^2 \leqslant \int_0^1 K_2^2(\tau(\lambda))\|\eta^n(\lambda)\|^2 d\lambda + K_0^2 |||u^n_h|||^2 + \frac{1}{2}|||u^n_{ht}|||^2 + K\|f_h\|^2 + \frac{1}{4K}\|u^n_{ht}\|^2 .$$

Applying the Poincaré inequality to the last term and integrating over $(0, t)$, $t \in [0, T-h]$, it follows

$$\|u^n_{ht}\|^2 + \tfrac{1}{2}\int_0^t |||u^n_{hs}|||^2 ds \leqslant 2\int_0^t\int_0^1 K_2^2(\tau(\lambda))\|\eta^n(\lambda)\|^2 d\lambda\, dt + 2K_0^2\int_0^t |||u^n_h|||^2 dt + 2K\int_0^t \|f_h\|^2 dt + \|u^n_{ht}(0)\|^2,$$

so that letting h go to zero we obtain

$$\|u^n_{tt}(t)\|^2 + \tfrac{1}{2}\int_0^t |||u^n_{ss}|||^2 ds \leqslant 4\int_0^t K_2^2(s)|||u^n|||^2 ds + 2K_0^2\int_0^t |||u^n_s|||^2 ds + 2K\int_0^t \|f_s\|^2 ds + \|u^n_{tt}(0)\|^2 . \tag{2.8}$$

On the other hand, replacing w_j by $u^n_{tt}(0)$ in (2.1) and taking $t = 0$, from the Cauchy inequality we have

$$\tfrac{1}{4}\|u^n_{tt}(0)\|^2 \leqslant \hat{K}^2 K_0^2 \|\Delta u_0\|^2 + K_1^2(0)|||u_0|||^2 + \|\Delta_N u_1\|^2 + \|f(0)\|^2$$

and by virtue of (H.3) the left hand side is bounded for all n. Hence, with the help of previous estimates we conclude that

$$\|u_{tt}^n(t)\|^2 + \int_0^t \|u_{ss}^n\|_1^2\,ds \leqslant C_7 \qquad \text{independent of } n\,. \tag{2.9}$$

As a consequence of (2.3)-(2.5) the following bounds of a_i and its derivatives hold

$$\begin{gathered}\int_0^t \|a_i(x, s, u_{x_i}^n)\|^2\,ds \leqslant C_8\,, \qquad \int_0^t \Big\|\frac{\partial}{\partial x_i}\, a_i(x, s, u_{x_i}^n)\Big\|^2\,ds \leqslant C_9\,,\\ \int_0^t \Big\|\frac{d}{dx_i}\, a_i(x, s, u_{x_i}^n)\Big\|^2\,ds \leqslant C_{10}\end{gathered} \qquad i = 1, \ldots, N\,. \tag{2.10}$$

The constants $C_0, C_1, \ldots, C_{10}$ which enter in the above estimates depend only on f, u_0, u_1, K_1, K_2 and T. Hence, the functions $c_{nj}(t) = (u^n(t), w_j)$ and $c'_{nj}(t) = (u_t^n(t), w_j)$, $j = 1, \ldots, n$, are uniformly bounded. Moreover, it turns out that they are equicontinuous for fixed j and arbitrary $n \geqslant j$ (see [**3**], p. 185). By the Ascoli theorem for each j we can select two subsequences, $c_{\nu j}(t)$ and $c'_{\nu j}(t)$, that converge to some continuous functions $c_j(t)$ and $d_j(t)$ uniformly on $[0, T]$ as $\nu \to \infty$. Then

$$\left.\begin{aligned} u^\nu &\to u\\ u_t^\nu &\to \tilde{u}\end{aligned}\right\} \quad \text{weakly in } L^2(\Omega), \text{ uniformly on } (0, T)\,,$$

where $u(t) = \sum_{j=1}^\infty c_j(t) w_j$ and $\tilde{u}(t) = \sum_{j=1}^\infty d_j(t) w_j$. It follows easily that $\tilde{u} = u_t$ in the weak sense. Now, taking further subsequences if necessary, estimates (2.3)-(2.5) and (2.9) lead to the following results as $\nu \to \infty$:

$$\begin{aligned}
u^\nu &\to u && \text{weak * in } L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))\,,\\
u_{x_i}^\nu &\to u_{x_i} && \text{weak * in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(\Omega \times (0, T))\,,\\
u_{x_i x_i}^\nu &\to u_{x_i x_i} && \text{weak * in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(\Omega \times (0, T))\,,\\
u_t^\nu &\to u_t && \text{weak * in } L^\infty(0, T; H_0^1(\Omega))\,,\\
\Delta_N u_t^\nu &\to \Delta_N u_t && \text{weakly in } L^2(\Omega \times (0, T))\,,\\
u_{tt}^\nu &\to u_{tt} && \text{weak * in } L^\infty(0, T; L^2(\Omega))\,,\\
u_{x_i tt}^\nu &\to u_{x_i tt} && \text{weakly in } L^2(\Omega \times (0, T))\,,
\end{aligned}$$

where all derivatives are considered in the usual weak or distribution sense. By means of a Sobolev embedding theorem (see [**8**], p. 72) and previous bounds we obtain

$$
(2.11)\qquad
\begin{aligned}
&u^\nu \to u\,, \quad u^\nu_{x_i} \to u_{x_i} \quad \text{strongly in } L^2(\Omega), \text{ uniformly on } (0, T)\,,\\
&u^\nu_t \to u_t\,, \quad u^\nu_{x_i t} \to u_{x_i t} \quad \text{strongly in } L^2(\Omega\times(0, T))\,.
\end{aligned}
$$

As a result we have $u(0) = u_0$ and $u_t(0) = u_1$ a.e. on Ω.

With the help of (H.1), (H.2) and above estimates, $(2.11)_2$ yields

$$
(2.12)\qquad
\begin{aligned}
&a_i(x, t, u^\nu_{x_i}) \to a_i(x, t, u_{x_i})\\
&\frac{\partial}{\partial x_i} a_i(x, t, u^\nu_{x_i}) \to \frac{\partial}{\partial x_i} a_i(x, t, u_{x_i})
\end{aligned}
\qquad i = 1, 2, \ldots, N
$$

a.e. on $\Omega\times(0, T)$ by continuity and strongly in $L^2(\Omega\times(0, T))$ as $\nu \to \infty$.

Moreover, we have $(\partial/\partial\eta)a_i(x, t, u^\nu_{x_i}) \to (\partial/\partial\eta)a_i(x, t, u_{x_i})$ a.e. on $\Omega\times(0, T)$ by continuity and for any function $w \in L^2(\Omega\times(0, T))$

$$
(2.13)\qquad \frac{\partial}{\partial\eta}\, a_i(x, t, u^\nu_{x_i})w \to \frac{\partial}{\partial\eta} a_i(x, t, u_{x_i})w \qquad \text{as } \nu \to \infty\,,
$$

strongly in $L^2(\Omega\times(0, T))$ by Lebesgue dominated convergence. Taking a suitable subsequence of u^ν if necessary, from (2.10) if follows

$$
(2.14)\qquad \frac{\mathrm{d}}{\mathrm{d}x_i}\, a_i(x, t, u^\nu_{x_i}) \to A_i(t) \qquad i = 1, 2, \ldots, N
$$

weakly in $L^2(\Omega\times(0, T))$. Then, letting ν go to infinity in the following identity

$$
\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}x_i}\, a_i(x, t, u^\nu_{x_i}), w\Big)\,\mathrm{d}t = \int_0^t \Big(u^\nu_{x_i x_i}\,,\ \frac{\partial}{\partial\eta}\, a_i(x, t, u^\nu_{x_i})w\Big)\,\mathrm{d}t + \int_0^t \Big(\frac{\partial}{\partial x_i}\, a_i(x,\ t,\ u^\nu_{x_i}), w\Big)\,\mathrm{d}t
$$

previous results yield

$$
\int_0^t \big(A_i(t), w\big)\,\mathrm{d}t = \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}x_i}\, a_i(x, t, u_{x_i}), w\Big)\,\mathrm{d}t
$$

for any $w \in L^2(\Omega\times(0, T))$. Combining the above informations we have that passage to the limit in (2.1) as $\nu \to \infty$ gives the required result.

Proof of uniqueness. Let $u(t)$ and $v(t)$ be two solutions of (1.1)-(1.3) and let $w = u - v$. It turns out that $w(0) = 0$, $w_t(0) = 0$ a.e. on Ω and

$$w_{tt} - \Delta_N w_t = \frac{\mathrm{d}}{\mathrm{d}x_i}[a_i(x, t, u_{x_i}) - a_i(x, t, v_{x_i})]$$

a.e. on $(0, T)$ as functions of $L^2(\Omega)$. Subtracting $\Delta_N w$ from both sides and multiplying by w_t the result, an integration over Ω yields

$$(2.15)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|w_t\|^2 + |||w|||^2\} + |||w_t|||^2 = \big(a_i(x, t, u_{x_i}) - a_i(x, t, v_{x_i}), w_{x_it}\big) + (w_{x_i}, w_{x_it})$$

$$\leqslant \tfrac{1}{2}\int_\Omega |a_i(x, t, u_{x_i}) - a_i(x, t, v_{x_i})|^2 \,\mathrm{d}x + \tfrac{1}{2}|||w|||^2 + |||w_t|||^2 .$$

With the help of (H.2 (ii)) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\|w_t\|^2 + |||w|||^2\} \leqslant (1 + K_0)|||w|||^2 \leqslant (1 + K_0^2)\{|||w|||^2 + \|w_t\|^2\} ,$$

so that from Gronwall's Lemma it follows

$$\|w_t\|^2 + \|w\|_1^2 \leqslant 0 , \qquad \text{which implies } w = 0 \text{ a.e. on } \Omega\times(0, T) .$$

3 - Asymptotic stability of solutions

Assuming for simplicity that $f = 0$, we shall investigate the asymptotic stability of solutions of (1.1)-(1.3). To this end previous conditions have to be modified replacing $[0, T]$ by $[0, \infty)$ and $L^2(0, T)$ by $L^2(0, \infty)$ in (H.1) and (H.2). Moreover, if we define the « elastic » energy by

$$(3.1)\qquad W(x, t, \xi_1, \xi_2, \ldots, \xi_N) = \sum_{i=1}^{N}\int_0^{\xi_i} a_i(x, t, \eta)\,\mathrm{d}\eta ,$$

it is necessary that W behaves properly as $t \to \infty$. We now state the further assumption

$$(\mathrm{H.2v})\qquad [\frac{\partial}{\partial t} a_i(x, t, \eta)]\,\eta \leqslant 0 \qquad \text{that implies } \frac{\partial}{\partial t} W(x, t, \xi_1, \ldots, \xi_N) \leqslant 0 .$$

Note that (H.2 (i)) leads to

$$(3.2)\qquad W(x, t, \xi_1, \ldots, \xi_N) \geqslant 0 .$$

It is clear that the existence of static solutions of (1.1)-(1.3) depends exclusively on the roots of the functions a_i. Trivially, by (H.2 (i)) $a_i(x, t, 0) = 0$,

$i = 1, \ldots, N$, for any $(x, t) \in \Omega \times [0, \infty)$; but the problem of asymptotic stability is particularly interesting in this case because each a_i may possess additional zeros. We will prove the following

Theorem 2. *Under conditions* (H.1), (H.2 (i-v)), (H.3) *the solution of* (1.1)-(1.3) *with* $f = 0$ *is asymptotically stable in the following sense*

$$\begin{aligned} &u_t \to 0 && \textit{strongly in } H_0^1(\Omega) \textit{ as } t \to \infty\,, \\ &u_{tt} \to 0 && \textit{strongly in } L^2(\Omega) \textit{ as } t \to \infty\,, \\ &\|u\|_1 && \textit{is uniformly bounded on } [0, \infty)\,. \end{aligned}$$

Furthermore, $\int_\Omega W(x, t, u_{x_1}(t), u_{x_2}(t), \ldots, u_{x_N}(t))\,\mathrm{d}x$ *converges as* $t \to \infty$.

Proof. We multiply (1.1) by u_t and we integrate the result over $\Omega \times (0, t)$, $t \in [0, \infty)$. The following « energy » estimate close to (2.2) is derived as in Theorem 1 with the help of (3.1) and (H.2 (v))

$$\left[\tfrac{1}{2}\|u_t\|^2 + \int_\Omega W\,\mathrm{d}x\right]_0^t + \int_0^t |||u_s|||^2\,\mathrm{d}s \leqslant 0\,. \tag{3.3}$$

Thus, from (3.2) it follows that

$$\|u_t(t)\|^2 + \int_0^t |||u_s|||^2\,\mathrm{d}s \leqslant M_0^2 \quad \text{independent of } t\,. \tag{3.4}$$

Multiplying (1.1) by u, integrating over $\Omega \times (0, t)$ and using (H.2 (i)) we end up with

$$\int_0^t \left[(u_{ss}, u) + \frac{\mathrm{d}}{\mathrm{d}s}\,|||u|||^2\right]\,\mathrm{d}s \leqslant 0\,.$$

After an integration by parts

$$\left[(u_s, u) + |||u|||^2\right]_0^t \leqslant \int_0^t \|u_s\|^2\,\mathrm{d}s\,.$$

Recalling (3.4), the Poincaré inequality yields

$$|||u(t)|||^2 \leqslant \|u_t(t)\| \cdot \|u(t)\| + M_1 \leqslant K M_0 |||u(t)||| + M_1 \qquad \text{which implies}$$

$$|||u(t)||| \leqslant M_2 \quad \text{independent of } t\,. \tag{3.5}$$

That is $\|u(t)\|_1$ is uniformly bounded on $(0, \infty)$.

By the same method used to obtain (2.8), from (1.1) we deduce the following inequality

$$\|u_{tt}(t)\|^2 + \tfrac{1}{2}\int_0^t |||u_{ss}|||^2\,\mathrm{d}s \leqslant 4\int_0^t K_2^2(s)|||u|||^2\,\mathrm{d}s + 2K_0^2\int_0^t |||u_s|||^2\,\mathrm{d}s + \|u_{tt}(0)\|^2 . \tag{3.6}$$

Recalling (3.4) and (3.5), it follows that

$$\|u_{tt}(t)\|^2 + \tfrac{1}{2}\int_0^t |||u_{ss}|||^2\,\mathrm{d}s \leqslant M_3 \quad \text{independent of } t . \tag{3.7}$$

Note now that

$$|||u_t|||^2 \in L^1(0, \infty) , \quad \frac{\mathrm{d}}{\mathrm{d}t}|||u_t|||^2 = 2\big((u_t, u_{tt})\big) \in L^1(0, \infty) , \tag{3.8}$$

by virtue of (3.4) and (3.7). Thus, $\|u_t(t)\|_1 \to 0$ as $t \to \infty$.

Moreover, (3.7) implies that

$$|||u_{tt}|||^2 \in L^1(0, \infty) \tag{3.9}$$

and by the Poincaré inequality $\|u_{tt}\|^2 \in L^1(0, \infty)$.

Hence, there exists a sequence of time points $\{t_i\}_{i\in N}$ such that

$$t_i \to \infty \quad \text{and} \quad \|u_{tt}(t_i)\|^2 \to 0 \quad \text{as } i \to \infty . \tag{3.10}$$

Replacing $(0, t)$ by (t_i, t), $t \geqslant t_i$, and recalling (3.5), inequality (3.6) takes the form

$$\|u_{tt}(t)\|^2 + \tfrac{1}{2}\int_{t_i}^t |||u_{ss}|||^2\,\mathrm{d}s \leqslant 4M_2^2\int_{t_i}^t K_2^2(s)\,\mathrm{d}s + 2K_0^2\int_{t_i}^t |||u_s|||^2\,\mathrm{d}s + \|u_{tt}(t_i)\|^2 .$$

Now, letting i and t go to ∞, with the help of (3.8), (3.9), (3.10) we conclude that $\|u_{tt}\|^2 \to 0$ as $t \to \infty$.

Finally, from the « energy » estimate (3.3) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant 0 \quad t \in (0, \infty) , \quad \text{with } E(t) = \frac{1}{2}\|u_t(t)\|^2 + \int_\Omega W\big(x, t, u_{x_1}(t), \dots, u_{x_N}(t)\big)\,\mathrm{d}x .$$

That is, $E(t)$ is a non-negative non-increasing function. Thus, as $t \to \infty$, $E(t)$ converges to some non negative value E_∞ and by virtue of the above results $\|W\|_{0,1}$ converges to the same limit.

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Sunto

In questa nota si stabiliscono condizioni sufficienti sull'operatore differenziale $A(u) = -\sum_{i=1}^{N} (\mathrm{d}/\mathrm{d}x_i) a_i(x, t, u_{x_i})$ *affinchè il problema ai valori iniziali ed al contorno per l'equazione* $u_{tt} + A(u) - \Delta_N u_t = f$ *ammetta una unica soluzione globale «forte» in domini limitati di* R^N*, qualunque sia la dimensione* N*. In particolare tali condizioni non implicano che* A *sia monotono e ciò ha reso più interessante lo studio della stabilità asintotica. Problemi di questo tipo si incontrano nell'esame del moto dei solidi viscoelastici di Kelvin-Voigt.*

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