Intersection sets, three-character multisets and associated codes

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Abstract

In this article we construct new minimal intersection sets in $\mathrm{AG}(r,q^2)$ sporting three intersection numbers with hyperplanes; we then use these sets to obtain linear error correcting codes with few weights, whose weight enumerator we also determine. Furthermore, we provide a new family of three-character multisets in $\mathrm{PG}(r,q^2)$ with r even and we also compute their weight distribution.

Keywords: Quadric, Hermitian variety, three-character set, multiset, error correcting code, weight enumerator
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1 Introduction

Throughout this paper q is taken to be an arbitrary prime power. A set of points S in the projective space $\mathrm{PG}(r,q^2)$ or in the affine space $\mathrm{AG}(r,q^2)$ is a *t-intersection set* or a *t-fold blocking set with respect to hyperplanes* if every hyperplane contains at least $t>0$ points of S. A point P of a t-intersection set S is said to be *essential* if $S\setminus\{P\}$ is not a t-intersection set. When all points of S are essential then S is *minimal.*

An intersection set S in $\mathrm{PG}(r,q^2)$ or in $\mathrm{AG}(r,q^2)$ is an *m-character set* if the size of the intersection of S with any hyperplane might assume just one out of m possible different values called *the characters* of S.

Sets with few characters are connected with many theoretical and applied areas such as coding theory, strongly regular graphs, association schemes, optimal multiple coverings, secret sharing; see in particular [7, 9, 10, 11, 12, 13, 16, 17]

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for applications of 2– and 3–character sets. For an extensive survey of results on three-character sets, see also [15] and the references therein.

A *multiset* in $\mathrm{PG}(r, q^2)$ is a mapping $M : \mathrm{PG}(r, q^2) \to \mathbb{N}$ from the points of $\mathrm{PG}(r, q^2)$ into non-negative integers. The *points* of a multiset are the points P of $\mathrm{PG}(r, q^2)$ with multiplicity $M(P) > 0$. Certain multisets arise in various classification problems for optimal linear codes of higher dimension; see [17, 18].

We recall how a linear code in q^2 symbols is generated from a (multi)-set $\mathcal{V}$ of points in $\mathrm{PG}(r, q^2)$. Fix a reference frame in $\mathrm{PG}(r, q^2)$ and construct a matrix G by taking as columns the coordinates of the points of $\mathcal{V}$ suitably normalized. The code $\mathcal{C}$ having G as generator matrix is called the *code generated from* $\mathcal{V}$.

In the case in which $\mathcal{V}$ is a set of points, that is G does not contain columns which are scalar multiples of each other, then $\mathcal{C}$ is the *projective code generated from* $\mathcal{V}$. The spectrum of the intersections of $\mathcal{V}$ with the hyperplanes of $\mathrm{PG}(r, q^2)$ provides the list of the weights of the associated code; we refer to [21] for further details on this geometric approach to codes.

As the order of the points in $\mathcal{V}$ or their normalization change, it is potentially possible to construct different codes from the same set of points. However, all of these are monomially equivalent; thus, in the following discussion we shall speak of *the* code associated to a multiset; see [14].

The present paper is organized as follows. In Section 2 we recall a non-standard model of $\mathrm{PG}(r, q^2)$ which will be useful for our constructions. In Section 3 we consider certain affine sets of $\mathrm{AG}(r, q^2)$ which allows to construct in theorems 3.1 and 3.2 interesting geometric objects with three characters. These sets are then applied in Section 4 to obtain linear error correcting codes with four weights whose weight enumerator we fully determine in Theorem 4.2. Finally, in Section 5 we consider one-point extensions of the sets obtained in Theorem 3.2 and obtain 3-character multiset in $\mathrm{PG}(r, q^2)$, for any r even in Theorem 5.1. This leads to 3-weight codes whose weights and weight distribution we compute in Theorem 5.2 and in Theorem 5.3, respectively.

We point out that study of the weights is important, since they measure the efficiency of the code and their knowledge is useful for decoding.

The codes we shall obtain in the present paper are all q-*divisible* that is they are q-ary code whose all non-zero weights are divisible by q; see [22].

2 Preliminaries

It is well known that all non–degenerate Hermitian varieties of $\mathrm{PG}(r, q^2)$ are projectively equivalent and that they sport just two intersection numbers with hyperplanes; see [20]. Thus, non–degenerate Hermitian varieties are two-character set. Quasi–Hermitian varieties $\mathcal{V}$ of $\mathrm{PG}(r, q^2)$ are combinatorial objects which have the same size and the same intersection numbers with hyperplanes as (non–degenerate) Hermitian varieties $\mathcal{H}$.

In [1, 2] new infinite families of quasi–Hermitian varieties have been constructed by modifying some point-hyperplane incidences in $\mathrm{PG}(r, q^2)$. To this purpose, the authors kept the point-set of $\mathrm{PG}(r, q^2)$ but altered the geometry by suitably replacing the subspaces of higher type.

The following non-standard model Π of $\mathrm{PG}(r, q^2)$, originally introduced in [2], leads to an extension to higher dimensional spaces of Buekenhout-Metz unitals and it shall also be relevant for the current work.

Fix a non-zero element $a \in \mathrm{GF}(q^2)$. For any choice $\mathbf{m} = (m_1, \ldots, m_{r-1}) \in \mathrm{GF}(q^2)^{r-1}$ and $d \in \mathrm{GF}(q^2)$ let $\mathcal{Q}_a(\mathbf{m}, d)$ denote the quadric of affine equation

$$x_r = a(x_1^2 + \ldots + x_{r-1}^2) + m_1x_1 + \ldots + m_{r-1}x_{r-1} + d. \tag{1}$$

Consider now the birational transform $\mathrm{AG}(r, q^2) \to \mathrm{AG}(r, q^2)$ given by

$$\varphi_a : (x_1, \ldots, x_{r-1}, x_r) \mapsto (x_1, \ldots, x_{r-1}, x_r - a(x_1^2 + \ldots + x_{r-1}^2)).$$

We can define a new geometry Π_a whose t-dimensional subspaces are the image under φ_a of the subspaces of $\mathrm{AG}(r, q^2)$ of dimension t for $0 \leq t \leq r-1$. As φ_a is bijective, Π_a is isomorphic to $\mathrm{AG}(r, q^2)$. In particular, the set of the hyperplanes of Π_a corresponds to the set of all hyperplanes of $\mathrm{AG}(r, q^2)$ through $P_\infty(0, 0, \ldots, 0, 1)$ together with all of the quadrics $\mathcal{Q}_a(\mathbf{m}, d)$. Completing Π_a with its points at infinity in the usual way we obtain a projective space isomorphic to $\mathrm{PG}(r, q^2)$.

In [1], an extension of Buekenhout-Tits unitals is considered, leading to non-isomorphic families of quasi-Hermitian varieties for q an odd power of 2. However, we shall not be concerned any further with this second construction in the present paper.

In order to be able classify quadrics defined over finite fields we shall extensively use Theorem [19, Theorem 1.2]; some special care is necessary when q is even. We refer to Chapter 1 of [19] for the complete details. Here, we just recall that for any quadric $\mathcal{Q}$ in $\mathrm{PG}(r, q^2)$, the radical $\mathrm{Rad}(\mathcal{Q})$ of $\mathcal{Q}$ is the subspace of $\mathrm{PG}(r, q^2)$ given by

$$\mathrm{Rad}(\mathcal{Q}) := \{x \in \mathcal{Q} : \forall y \in \mathcal{Q}, \langle x, y\rangle \subseteq \mathcal{Q}\},$$

where by $\langle x, y\rangle$ we denote the line through x and y.

3 3-character sets in $\mathrm{AG}(r, q^2)$

In this section we construct an infinite family of minimal intersection sets in $\mathrm{AG}(r, q^2)$ that sport just three intersection numbers. Fix a projective frame in $\mathrm{PG}(r, q^2)$ and assume the space to have homogeneous coordinates $(X_0, X_1, \ldots, X_r)$.

r	q	$4a^{q+1}+(b^q-b)^2$	$\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2)$
odd	odd	non-zero	
even	odd	non-square in GF(q)	
odd	even		Any
even	even		0

Table 1: Summary of the cases considered in [2, Theorem 3.1]

Let $\mathrm{AG}(r,q^2)$ be the affine space obtained by taking as hyperplane at infinity Π_∞ of $\mathrm{PG}(r,q^2)$ that of equation $X_0=0$. Then, the points of $\mathrm{AG}(r,q^2)$ have affine coordinates $(x_1,x_2,\ldots,x_r)$ where $x_i=X_i/X_0$ for $i\in\{1,\ldots,r\}$.

Consider now the non–degenerate Hermitian variety $\mathcal{H}$ with affine equation

$$x_r^q-x_r=(b^q-b)(x_1^{q+1}+\ldots+x_{r-1}^{q+1}), \tag{2}$$

where $b\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$. The set of the points at infinity of $\mathcal{H}$ is

$$\mathcal{F}=\{(0,x_1,\ldots,x_r)|x_1^{q+1}+\ldots+x_{r-1}^{q+1}=0\}; \tag{3}$$

that is $\mathcal{F}$ is a Hermitian cone of $\mathrm{PG}(r-1,q^2)$, projecting a Hermitian variety of $\mathrm{PG}(r-2,q^2)$ from the point $P_\infty(0,\ldots,0,1)$. In particular, the hyperplane Π_∞ is tangent to $\mathcal{H}$ at P_∞.

For any $a\in\mathrm{GF}(q^2)^*$ and $b\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$, let $\mathcal{B}(a,b)$ be the affine algebraic set of equation

$$x_r^q-x_r+a^q(x_1^{2q}+\ldots+x_{r-1}^{2q})-a(x_1^2+\ldots+x_{r-1}^2)=(b^q-b)(x_1^{q+1}+\ldots+x_{r-1}^{q+1}). \tag{4}$$

It is shown in [2] that $\mathcal{B}(a,b)$, together with the points at infinity of $\mathcal{H}$, as given by (3), is a quasi–Hermitian variety $\mathcal{V}$ of $\mathrm{PG}(r,q^2)$ provided that the following algebraic conditions are satisfied: for q odd, r is odd and $4a^{q+1}+(b^q-b)^2\neq 0$, or r is even and $4a^{q+1}+(b^q-b)^2$ is a non–square in $\mathrm{GF}(q)$; for q even, r is odd, or r is even and $\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2)=0$. Here Tr_q with $q=2^h$, denotes the absolute trace $\mathrm{GF}(q)\to\mathrm{GF}(2)$ which maps $x\in\mathrm{GF}(q)$ to $x+x^2+x^{2^2}+\ldots+x^{2^{h-1}}$.

We recall that for $r=2$ the condition that $4a^{q+1}+(b^q-b)^2$ is a non-square in $\mathrm{GF}(q)$ for q odd or $b\notin\mathrm{GF}(q)$ and $\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2)=0$ for q even is known as Ebert's discriminant condition see [5, 8].

We shall study the point-set $\mathcal{B}(a,b)$ when complementary of conditions of those mentioned above hold.

We are going to prove the following results.

Theorem 3.1. *Suppose q to be an odd prime-power and $4a^{q+1}+(b^q-b)^2=0$. Then, $\mathcal{B}(a,b)$ is a set of q^{2r-1} points of $\mathrm{AG}(r,q^2)$ with characters:*

r	q	$4a^{q+1}+(b^q-b)^2$	$Tr(a^{q+1}/(b^q+b)^2)$
odd	odd	0	
even	odd	0	
even	odd	non-zero square in GF(q)	
even	even		1

Table 2: Summary of the cases considered in Theorem 3.1 and Theorem 3.2

r	q	Case
$r \equiv 1 \pmod 4$	$q \equiv 1 \pmod 4$	1)
$r \equiv 1 \pmod 4$	$q \equiv 3 \pmod 4$	1)
$r \equiv 3 \pmod 4$	$q \equiv 1 \pmod 4$	1)
$r \equiv 3 \pmod 4$	$q \equiv 3 \pmod 4$	2)
$r \equiv 0 \pmod 2$	$q \equiv 1 \pmod 2$	3)

Table 3: Cases for Theorem 3.1

1. *for* $r \equiv 1 \pmod 4$ *or* r *odd and* $q \equiv 1 \pmod 4$

$$q^{2r-3} - q^{(3r-5)/2}, q^{2r-3}, q^{2r-3} - q^{(3r-5)/2} + q^{3(r-1)/2};$$

2. *for* $r \equiv 3 \pmod 4$ *and* $q \equiv 3 \pmod 4$

$$q^{2r-3} + q^{(3r-5)/2} - q^{3(r-1)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-5)/2};$$

3. *for* r *even,*

$$q^{2r-3} - q^{(3r-4)/2}, q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}.$$

Furthermore, $\mathcal{B}(a,b)$ *is a minimal intersection set with respect to hyperplanes for* $r > 2$.

Theorem 3.2. *Let* r *be even. Suppose that either* q *is odd with* $4a^{q+1} + (b^q - b)^2$ *a non–zero square in* $\mathrm{GF}(q)$ *or* q *is even and* $\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2) = 1$. *Then,* $\mathcal{B}(a,b)$ *is a set of* q^{2r-1} *points of* $\mathrm{AG}(r, q^2)$ *with characters*

$$q^{2r-3} - q^{r-2}, q^{2r-3}, q^{2r-3} - q^{r-2} + q^{r-1}.$$

$\mathcal{B}(a,b)$ *is also a minimal intersection set with respect to hyperplanes.*

As it can be seen from Tables 1 and 2, all of the possibilities have been accounted for. For the convenience of the reader, we also summarize the subcases of Theorem 3.1 in Table 3.

3.1 Proof of Theorem 3.1

Assume $\mathcal{B}(a,b)$ to have Equation (4). It is straightforward to see that $\mathcal{B}(a,b)$ coincides with the affine part of the Hermitian variety $\mathcal{H}$ of equation (2) in the space Π_a; hence, any hyperplane π_{P_∞} of $\mathrm{PG}(r,q^2)$ passing through P_∞ meets $\mathcal{B}(a,b)$ in $|\mathcal{H}\cap\pi_{P_\infty}|=q^{2r-3}$ points.

Now we are interested in the possible intersection sizes of $\mathcal{B}(a,b)$ with a generic hyperplane

$$\pi : x_r = m_1x_1+\cdots+m_{r-1}x_{r-1}+d,$$

of $\mathrm{AG}(r,q^2)$ with coefficients $m_1,\ldots,m_r,d\in\mathrm{GF}(q^2)$. This is the same as to study the intersection of $\mathcal{H}$ with the quadrics $\mathcal{Q}_a(\mathbf{m},d)$ and hence we have to solve the system

$$\begin{cases} x_r^q - x_r = (b^q-b)(x_1^{q+1}+\ldots+x_{r-1}^{q+1}) \\ a(x_1^2+\ldots+x_{r-1}^2)+m_1x_1+\ldots+m_{r-1}x_{r-1}-x_r+d=0 \end{cases} \tag{5}$$

Recovering x_r from the second equation in (5) and replacing its value in the first equation, we obtain the following

$$\begin{aligned} &a^q(x_1^{2q}+\ldots+x_{r-1}^{2q})+(b-b^q)(x_1^{q+1}+\ldots+x_{r-1}^{q+1})+m_1^qx_1^q+\\ &+\ldots+m_{r-1}^qx_{r-1}^q+d^q-a(x_1^2+\ldots+x_{r-1}^2)-m_1x_1+\ldots+\\ &-m_{r-1}x_{r-1}-d=0. \end{aligned} \tag{6}$$

As q is odd, there is $\varepsilon\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$ such that $\varepsilon^q=-\varepsilon$. For any $x\in\mathrm{GF}(q^2)$ write $x=x^0+\varepsilon x^1$ with $x^0,x^1\in\mathrm{GF}(q)$; in this way the previous Equation (6) becomes

$$\sum_{i=1}^{r-1}\big((b^1+a^1)\varepsilon^2(x_i^1)^2+2a^0x_i^0x_i^1+(a^1-b^1)(x_1^0)^2\big)+\sum_{i=1}^{r-1}(m_i^0x_i^1+m_i^1x_i^0)+d^1=0; \tag{7}$$

that is the number N of affine points which lie in $\mathcal{B}(a,b)\cap\pi$ is the same as the number of points of the affine quadric $\mathcal{Q}$ of $\mathrm{AG}(2r-2,q)$ of Equation (7). Following the approach of [2], in order to compute N, we first count the number of points of the quadric at infinity $\mathcal{Q}_\infty := \mathcal{Q}\cap\Pi_\infty$ of $\mathcal{Q}$ and then we determine $N=|\mathcal{Q}|-|\mathcal{Q}_\infty|$. Observe that the quadric $\mathcal{Q}_\infty$ of $\mathrm{PG}(2r-3,q)$ has a block matrix of the form

$$A_\infty=\begin{pmatrix} (a^1-b^1) & a^0 & & & \\ a^0 & (b^1+a^1)\varepsilon^2 & & & \\ & & \ddots & & \\ & & & (a^1-b^1) & a^0 \\ & & & a^0 & (b^1+a^1)\varepsilon^2 \end{pmatrix} \tag{8}$$

with determinant

$$\det A_\infty = (\varepsilon^2[(a^1)^2 - (b^1)^2] - (a^0)^2)^{r-1}.$$

Since $(a^0)^2 - \varepsilon^2[(a^1)^2 - (b^1)^2] = a^{q+1} + (b^q - b)^2/4$ and $4a^{q+1} + (b^q - b)^2 = 0$ we have $\det A_\infty = 0$. This means

$$\det \begin{pmatrix} (a^1 - b^1) & a^0 \\ a^0 & (a^1 + b^1)\varepsilon^2 \end{pmatrix} = 0,$$

that is, each of the 2×2 blocks on the main diagonal of A_∞ has rank 1. Consequently, the rank of the matrix A_∞ is exactly $r - 1$.

If $a^1 = b^1$, then $a^0 = 0$, the matrix A_∞ is diagonal and the quadric $\mathcal{Q}_\infty$ is projectively equivalent to

$$(x_1^1)^2 + (x_2^1)^2 + \cdots + (x_{r-1}^1)^2 = 0.$$

Otherwise, take

$$M = \begin{pmatrix} 1 & 0 & & & \\ -a^0/(a^1 - b^1) & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & -a^0/(a^1 - b^1) & 1 \end{pmatrix};$$

a direct computation proves that

$$M^T A_\infty M = \begin{pmatrix} a^1 - b^1 & 0 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & a^1 - b^1 & 0 \\ & & & 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{Q}_\infty$ is projectively equivalent to the quadric of rank $r - 1$ with equation

$$(x_1^0)^2 + (x_2^0)^2 + \cdots + (x_{r-1}^0)^2 = 0.$$

For r odd, we have that $\mathcal{Q}_\infty$ is either

- a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2, q)$ and basis a hyperbolic quadric $Q^+(r-2, q)$ if $q \equiv 1 \pmod 4$ or $r \equiv 1 \pmod 4$, or
- a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2, q)$ and basis an elliptic quadric $Q^-(r-2, q)$ if $q \equiv 3 \pmod 4$ and $r \equiv 3 \pmod 4$; see [19, Theorem 1.2].

For r even, $\mathcal{Q}_\infty$ is a cone with vertex $\mathrm{Rad}(\mathcal{Q}_\infty) \simeq \mathrm{PG}(r-2,q)$ and basis a parabolic quadric $Q(r-2,q)$.

We now move to investigate the quadric $\mathcal{Q}$. Clearly, the rank of its matrix is either $r-1$, r or $r+1$.

Write $\Pi_\infty = \Sigma \oplus \mathrm{Rad}(\mathcal{Q}_\infty)$. As Σ is disjoint from the radical of the quadratic form inducing $\mathcal{Q}_\infty$, we have that $\mathcal{Q}'_\infty := \Sigma \cap \mathcal{Q}_\infty$ is a nondegenerate quadric (either hyperbolic, elliptic or parabolic according to the various conditions).

When $\mathcal{Q}$ has the same rank $r-1$ as $\mathcal{Q}_\infty$, we have

$$\dim \mathrm{Rad}(\mathcal{Q}) = \dim \mathrm{Rad}(\mathcal{Q}_\infty) + 1.$$

Observe that $\mathrm{Rad}(\mathcal{Q}) \cap \Pi_\infty \leq \mathrm{Rad}(\mathcal{Q}_\infty)$. Thus, $\mathrm{Rad}(\mathcal{Q}) \cap \Sigma = \{\mathbf{0}\}$ and Σ is also a direct complement of $\mathrm{Rad}(\mathcal{Q})$. It follows that $\mathcal{Q}$ is cone of vertex a $\mathrm{PG}(r-1,q)$ and basis a quadric of the same kind as $\mathcal{Q}'_\infty$. If $\mathcal{Q}$ has rank $r+1$, then the hyperplane at infinity is tangent to $\mathcal{Q}$; in particular $\mathcal{Q}$ must have as radical a $\mathrm{PG}(r-3,q)$; by [19, Lemma 1.22], the basis $\mathcal{Q}''$ of $\mathcal{Q}$ must have the same character (elliptic, parabolic or hyperbolic) as $\mathcal{Q}'_\infty$.

In the case in which the matrix of $\mathcal{Q}$ has rank r, $\mathrm{Rad}(\mathcal{Q}) = \mathrm{Rad}(\mathcal{Q}_\infty)$ and $\mathcal{Q}$ is a cone of vertex a $\mathrm{PG}(r-2,q)$ and basis a parabolic quadric $Q(r-1,q)$ for r odd or $\mathcal{Q}$ is a cone of vertex a $\mathrm{PG}(r-2,q)$ and basis a hyperbolic quadric $Q^+(r-1,q)$ or an elliptic quadric $Q^-(r-1,q)$ for r even. We can now write the complete list of sizes for r odd:

$$|\mathcal{Q}_\infty| = \frac{q^{2r-3}-1}{q-1} \pm q^{(3r-5)/2};$$

in case $\mathrm{rank}(\mathcal{Q}) = r-1$, then

$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1} \pm q^{3(r-1)/2};$$

in case $\mathrm{rank}(\mathcal{Q}) = r$,

$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1};$$

in case $\mathrm{rank}(\mathcal{Q}) = r+1$, then

$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1} \pm q^{(3r-5)/2}.$$

In particular, the possible values for $N = |\mathcal{Q}| - |\mathcal{Q}_\infty|$ are

$$q^{2r-3}, q^{2r-3} + q^{3(r-1)/2} - q^{(3r-5)/2}, q^{2r-3} - q^{(3r-5)/2}$$

for $q \equiv 1 \pmod 4$ or $r \equiv 1 \pmod 4$ and

$$q^{2r-3} - q^{3(r-1)/2} + q^{(3r-5)/2}, q^{2r-3} + q^{(3r-5)/2}$$

for $q \equiv 3 \pmod 4$ and $r \equiv 3 \pmod 4$.
When r is even we get:

$$|\mathcal{Q}_\infty| = \frac{q^{2r-3}-1}{q-1};$$

in case $\mathrm{rank}(\mathcal{Q}) = r-1$ or $\mathrm{rank}(\mathcal{Q}) = r+1$, then

$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1};$$

in case $\mathrm{rank}(\mathcal{Q}) = r$,

$$|\mathcal{Q}| = \frac{q^{2r-2}-1}{q-1} \pm q^{(3r-4)/2}.$$

Thus, the possible list of cardinalities for $N = |\mathcal{Q}| - |\mathcal{Q}_\infty|$ is

$$q^{2r-3}, q^{2r-3} + q^{(3r-4)/2}, q^{2r-3} - q^{(3r-4)/2}.$$

Now we are going to show that $\mathcal{B}(a,b)$ is a minimal intersection set. First of all, we prove that for any $P \in \mathcal{B}(a,b)$ there exists a subspace $\Lambda_n(P) \simeq \mathrm{AG}(n,q^2)$, $1 \le n \le r-1$ through P such that $|\mathcal{B}(a,b) \cap \Lambda_n(P)| \le q^{2n-1} - q^{n-1}$. The argument is by induction on n. Assume $n=1$. Then, for any $P \in \mathcal{B}(a,b)$ there exists at least one line ℓ through P such that $|\ell \cap \mathcal{B}(a,b)| < q$, otherwise $\mathcal{B}(a,b)$ would contain more than q^{2r-1} points. Suppose now that the result holds for $n = 1, \ldots, r-2$, take $P \in \mathcal{B}(a,b)$ and suppose that any hyperplane π through P meets $\mathcal{B}(a,b)$ in at least q^{2r-3} points. By induction, there exists a subspace $\pi' := \Lambda_{r-2}(P) \simeq \mathrm{AG}(r-2,q^2)$ through P meeting $\mathcal{B}(a,b)$ in at most $q^{2r-5} - q^{r-3}$ points. By considering all hyperplanes containing π' we get $|\mathcal{B}(a,b)| \ge (q^2+1)(q^{2r-3} - q^{2r-5} + q^{r-3}) + q^{2r-5} - q^{r-3} > q^{2r-1}$, a contradiction. Thus, through any $P \in \mathcal{B}(a,b)$ there exists a hyperplane meeting $\mathcal{B}(a,b)$ in $(q^{2r-3} - q^{(3r-5)/2})$ points for r odd or $(q^{2r-3} - q^{(3r-4)/2})$ for r even. This implies that $\mathcal{B}(a,b)$ is a minimal intersection set for any $r > 2$.

Remark 3.3. The quadric $\mathcal{Q}_a(\mathbf{m},d)$ of Equation (1) shares its tangent hyperplane at P_∞ with the Hermitian variety (2).

The problem of the intersection of the Hermitian variety $\mathcal{H}$ with irreducible quadrics $\mathcal{Q}$ having the same tangent plane at a common point $P \in \mathcal{Q} \cap \mathcal{H}$ has been considered in detail for $r=3$ in [3, 4].

3.2 Proof of Theorem 3.2

First consider the case q odd. Arguing as in the proof of Theorem 3.1 we have that any hyperplane π_{P_∞} of $\mathrm{PG}(r,q^2)$ passing through P_∞ meets $\mathcal{B}(a,b)$ in q^{2r-3} points.

In order to determine the possible intersection sizes of $\mathcal{B}(a,b)$ with a hyperplane which does not pass through P_∞, say $\pi : x_r = m_1x_1 + \cdots + m_{r-1}x_{r-1} + d$, we need to compute the number N of affine points of the quadric $\mathcal{Q}$ in $\mathrm{AG}(2r-2,q)$ with equation (7). We first discuss the nature of $\mathcal{Q}_\infty = \mathcal{Q} \cap \Pi_\infty$ whose associated matrix A_∞ is of the form (8).

Observe that, under our assumptions, for q odd $(-1)^{r-1}\det A_\infty$ is always a square in $\mathrm{GF}(q)$; hence, $\mathcal{Q}_\infty$ is a hyperbolic quadric of $\mathrm{PG}(2r-3)$.

For q even, choose $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ such that $\varepsilon^2 + \varepsilon + \nu = 0$, for some $\nu \in \mathrm{GF}(q) \setminus \{1\}$ with $\mathrm{Tr}_q(\nu) = 1$. Then, $\varepsilon^{2q} + \varepsilon^q + \nu = 0$. Therefore, $(\varepsilon^q + \varepsilon)^2 + (\varepsilon^q + \varepsilon) = 0$, whence $\varepsilon^q + \varepsilon + 1 = 0$. With this choice of ε, the system given by (4) and (1) reads as

$$\begin{aligned}&(a^1+b^1)(x_1^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_1^1)^2 + b^1x_1^0x_1^1 + m_1^1x_1^0 + (m_1^0+m_1^1)x_1^1\\ &+\ldots+(a^1+b^1)(x_{r-1}^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_{r-1}^1)^2 + b^1x_{r-1}^0x_{r-1}^1\\ &+m_{r-1}^1x_{r-1}^0 + (m_{r-1}^0+m_{r-1}^1)x_{r-1}^1 + d^1 = 0.\end{aligned} \tag{9}$$

The discussion of the (possibly degenerate) quadric $\mathcal{Q}$ of Equation (9) may be carried out in close analogy to what has been done before.

Observe however that, as also pointed out in the remark before [19, Theorem 1.2], some caution is needed when quadrics are studied and classified in even characteristic. Indeed, let

$$A_\infty = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}, \text{ where } T = \begin{pmatrix} 2(a^1+b^1) & b_1 \\ b_1 & 2((a^0+a^1)+\nu(a^1+b^1)) \end{pmatrix}$$

be the formal matrix associated to the quadric $\mathcal{Q}_\infty$ of equation

$$(a^1+b^1)(x_1^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_1^1)^2 + b^1x_1^0x_1^1 + \ldots$$

$$+(a^1+b^1)(x_{r-1}^0)^2 + [(a^0+a^1)+\nu(a^1+b^1)](x_{r-1}^1)^2 + b^1x_{r-1}^0x_{r-1}^1 = 0.$$

Its determinant is equal to

$$\det A_\infty = [4(a^1+b^1)(a^0+a^1+\nu(a^1+b^1)) + (b^1)^2]^{r-1}.$$

In order to encompass the case q even, $\det A_\infty$ needs to be regarded as a polynomial function in the ring $\mathbb{Z}[z_0,z_1,z_2,z_3]$ where the terms (a^0,a^1,b^0,b^1) are replaced by indeterminates z_0,z_1,z_2,z_3; then we regard it over $\mathrm{GF}(q)$ for $(z_0,z_1,z_2,z_3) = (a^0,a^1,b^0,b^1)$. This gives $\det A_\infty = b_1^{2(r-1)}$. Here $b_1 \neq 0$ since, by our assumption, $b^q \neq b$. From [19, Theorem 1.2 (i)], the quadric $\mathcal{Q}_\infty$ must be non-degenerate.

Furthermore, by [19, Theorem 1.2 (ii)] the nature of $\mathcal{Q}_\infty$ can be ascertained as follows. Let B the matrix

$$B = \begin{pmatrix} 0 & b^1 & & & \\ -b^1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & b_1 \\ & & & -b^1 & 0 \end{pmatrix} \tag{10}$$

and define

$$\alpha = \frac{\det B - (-1)^{r-1} \det A_\infty}{4 \det B}.$$

A straightforward computation shows that

$$\alpha = \frac{(b^1)^{2(r-1)} + (4(a^1+b^1)(a^0+a^1+\nu(a^1+b^1)) + (b^1)^2)^{r-1}}{4\,(b^1)^{2(r-1)}}.$$

Here α has to be regarded as the quotient of two polynomials in $\mathbb{Z}[z_0, z_1, z_2, z_3]$ where the terms (a^0, a^1, b^0, b^1) are replaced by indeterminates z_0, z_1, z_2, z_3 and, then, evaluated it over $\mathrm{GF}(q)$ for $(z_0, z_1, z_2, z_3) = (a^0, a^1, b^0, b^1)$. In particular, we get

$$\alpha = \frac{(a^1+b^1)(a^0+a^1+\nu(a^1+b^1))}{(b^1)^2}.$$

Arguing as in [2, p. 439], we see that $\mathrm{Tr}_q(\alpha) = 0$ and, hence, $\mathcal{Q}_\infty$ is hyperbolic also for q even.

We investigate the possible nature of $\mathcal{Q}$ in either case q odd and q even. Suppose $\mathcal{Q}$ to be non-singular; then $\mathcal{Q}$ is a parabolic quadric and

$$N = |\mathcal{Q}| - |\mathcal{Q}_\infty| = \frac{(q^{r-1}+1)(q^{r-1}-1)}{q-1} - \frac{(q^{r-1}+1)(q^{r-2}-1)}{q-1} = q^{r-2}(q^{r-1}+1).$$

If $\mathcal{Q}$ is singular, then $\mathcal{Q}$ is a cone with vertex a point and basis a hyperbolic quadric; thus

$$N = |\mathcal{Q}| - |\mathcal{Q}_\infty| = \frac{q(q^{r-1}+1)(q^{r-2}-1)}{q-1} - \frac{(q^{r-1}+1)(q^{r-2}-1)}{q-1} + 1 =$$

$$= q^{r-2}(q^{r-1}+1) - q^{r-1}.$$

This gives the possible intersection numbers.

Finally, in order to show that $\mathcal{B}(a, b)$ is a minimal $(q^{2r-3} - q^{r-2})$–fold blocking set we can use the same techniques as those adopted to prove that $\mathcal{B}(a, b)$ is a minimal blocking set in Theorem 3.1.

4 4-weight q-ary codes

Throughout this section q is and odd prime power and $4a^{q+1} + (b^q - b)^2 = 0$ for any $a \in \mathrm{GF}(q^2)^*$ and $b \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. Let $\mathcal{B}(a,b)$ the affine set of equation (4).We are going to determine the parameters of the projective code generated from $\mathcal{B}(a,b)$ as explained in the Introduction.

We begin by determining how many hyperplanes meet $\mathcal{B}(a,b)$ in a prescribed number of points.

Lemma 4.1. *The number of hyperplanes N_j meeting $\mathcal{B}(a,b)$ in exactly j points are as follows:*

(a) *For $r \equiv 1 \pmod 4$, or r odd and $q \equiv 1 \pmod 4$*

$$N_{q^{2r-3}+q^{3(r-1)/2}-q^{(3r-5)/2}} = q^r, \qquad N_{q^{2r-3}} = \frac{q^{2r}-1}{q^2-1} - 1 + q^{2r} - q^{r+1},$$

$$N_{q^{2r-3}-q^{(3r-5)/2}} = q^{r+1} - q^r.$$

(b) *For $r \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$*

$$N_{q^{2r-3}+q^{(3r-5)/2}} = q^r, \qquad N_{q^{2r-3}} = \frac{q^{2r}-1}{q^2-1} - 1 + q^{2r} - q^{r+1},$$

$$N_{q^{2r-3}-q^{3(r-1)/2}+q^{(3r-5)/2}} = q^{r+1} - q^r.$$

(c) *For r even,*

$$N_{q^{2r-3}-q^{(3r-4)/2}} = \frac{1}{2}(q^{r+1} - q^r) \qquad N_{q^{2r-3}} = q^r + \frac{q^{2r}-1}{q^2-1} - 1 + q^{2r} - q^{r+1},$$

$$N_{q^{2r-3}+q^{3(r-4)/2}} = \frac{1}{2}(q^{r+1} - q^r).$$

Proof. From the proof of Theorem 3.2 it follows that in order to prove Cases (a) and (b) we need to count the number of vectors $v := (m_1^0, m_1^1, \ldots, m_{r-1}^0, m_{r-1}^1, d_1) \in \mathrm{GF}(q)^{2r-1}$ such that the matrix

$$A := \begin{pmatrix} & & & & m_1^0 \\ & & & & m_1^1 \\ & & A_\infty & & \vdots \\ & & & & m_{r-1}^1 \\ m_1^0 & m_1^1 & \cdots & m_{r-1}^1 & d_1 \end{pmatrix}$$

of $\mathcal{Q}$ with equation (7) has respectively rank $r-1$, r or $r+1$.

We observe that A has rank $r-1$ if, and only if, there exist a scalar λ such that for all $i=1,\ldots,r-1$ we have $m_i^1=\lambda m_i^0$; also, the value of d_1 turns out to be uniquely determined. Thus, the number of distinct possibilities for the parameters $m_1,\ldots,m_{r-1},d$ is exactly q^r. The rank of the matrix of $\mathcal{Q}$ is at least r in the remaining $q^{2r}-q^r$ cases. Suppose it to be $r+1$. This means that the column $(m_1^0,m_1^1,\ldots,m_{r-1}^0,m_{r-1}^1)^T$ is linearly independent from the columns of A_∞; so, there are $q^{2r-2}-q^{r-1}$ ways to choose $m_1^0,\ldots,m_{r-1}^1$. Furthermore, for any such choice the vector $v=(m_1^0,\ldots,m_{r-1}^1,d_1)$ is also independent from the first $2r-2$ rows of A. So the overall number of planes with such property is $q^2(q^{2r-2}-q^{r-1})=q^{2r}-q^{r+1}$. The remaining $q^{r+1}-q^r$ choices yield a matrix of rank r.

In Case (c), again from the proof of Theorem 3.2 when r is even, we need to count how often $\mathcal{Q}$ with equation (7) turns out to be elliptic rather than hyperbolic. For any choice of the parameters $m_1,\ldots,m_{r-1},d$ there is exactly one quadric $\mathcal{Q}$ to consider. As $\mathcal{Q}_\infty$ is always a parabolic quadric, we can assume it to be fixed. Denote by $\sigma^0,\sigma^+,\sigma^-$ respectively the number of quadrics $\mathcal{Q}$ which are parabolic, elliptic or hyperbolic. Clearly σ_0 corresponds to the case in which $\mathrm{rank}(\mathcal{Q})=\mathrm{rank}(\mathcal{Q}_\infty)$ or $\mathrm{rank}(\mathcal{Q})=\mathrm{rank}(\mathcal{Q}_\infty)+2$. We have

$$\sigma^++\sigma^0+\sigma^-=q^{2r},\qquad \sigma^0=q^{2r}-q^{r+1}+q^r.$$

Each point of $\mathcal{B}(a,b)$ lies on $\frac{q^{2r}-1}{q^2-1}$ hyperplanes; of these $\frac{q^{2r-2}-1}{q^2-1}$ pass through P_∞ (and they must be discounted). Thus, we get

$$q^{2r-2}|\mathcal{B}|=q^{4r-3}=\sigma^0q^{2r-3}+\sigma^+(q^{2r-3}+q^{(3r-4)/2})+\sigma^-(q^{2r-3}-q^{(3r-4)/2})= \\ q^{2r-3}(\sigma^0+\sigma^++\sigma^-)+q^{(3r-4)/2}(\sigma^+-\sigma^-)=q^{4r-3}+(\sigma^+-\sigma^-)q^{(3r-4)/2}.$$

Hence, $\sigma^+=\sigma^-=\frac{1}{2}(q^{r+1}-q^r)$. □

We can now prove the main theorem of this section.

Theorem 4.2. *The points of $\mathcal{B}(a,b)$ determine a projective code $\mathcal{C}$ of length $n=q^{2r-1}$, dimension $k=r+1$ and weight enumerator $w(x):=\sum_i A_ix^i$ where*

$$A_0=1,\qquad A_{q^{2r-1}}=(q^2-1)$$

and all of the remaining A_i's are 0 with the exception of

- *for $r\equiv 1 \pmod 4$ or r odd and $q\equiv 1 \pmod 4$,*

$$A_{q^{2r-1}-q^{2r-3}-q^{3(r-1)/2}+q^{(3r-5)/2}}=(q^{r+1}-q^r)(q^2-1)$$

$$A_{q^{2r-1}-q^{2r-3}}=q^{2r}-q^2+(q^{2r}-q^{r+1})(q^2-1),\quad A_{q^{2r-1}-q^{2r-3}+q^{(3r-5)/2}}=q^{r+2}-q^r;$$

- *for* $r \equiv 3 \pmod 4$ *and* $q \equiv 3 \pmod 4$,

$$A_{q^{2r-1}-q^{2r-3}-q^{(3r-5)/2}} = (q^{r+1}-q^r)(q^2-1), \quad A_{q^{2r-1}-q^{2r-3}} = q^{2r}-q^2-q^{r+1}(q^2-1)$$

$$A_{q^{2r-1}-q^{2r-3}+q^{3(r-1)/2}-q^{(3r-5)/2}} = q^{r+2}-q^r;$$

- *for* r *even,*

$$A_{q^{2r-1}-q^{2r-3}+q^{(3r-4)/2}} = A_{q^{2r-1}-q^{2r-3}-q^{(3r-4)/2}} = \frac{1}{2}(q^{r+1}-q^r)(q^2-1),$$

$$A_{q^{2r-1}-q^{2r-3}} = q^{2r}+q^{r+2}-q^r-q^2+(q^{2r}-q^{r+1})(q^2-1).$$

In particular, each of these codes has exactly 4 *non-zero weights.*

Proof. If we regard $\mathcal{B}(a,b)$ as a set of points in $\mathrm{PG}(r,q^2)$, then we can consider the projective code $\mathcal{C}$ of length q^{2r-1} and dimension $r+1$ generated from $\mathcal{B}(a,b)$. Denote by A_j the number of codewords of $\mathcal{C}$ of weight j. Observe that a hyperplane π meeting $\mathcal{B}(a,b)$ in n points always determines (q^2-1) codewords of weight $q^{2r-1}-n$. As the hyperplane at infinity is disjoint from $\mathcal{B}(a,b)$, we have

$$A_{q^{2r-1}} = (q^2-1).$$

The remaining weights follow from Lemma4.1.

□

5 3-character multisets in $\mathrm{PG}(r,q^2)$, r even and 3-weight codes

We keep all previous notation. In [6, Theorem 4.1] it is shown that for $r=2$, q odd and $4a^{q+1}+(b^q-b)^2 \neq 0$ or $r=2$, q even and $\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2)=1$, the set $\mathcal{B}(a,b)$ can be completed to a 2-character multiset $\overline{\mathcal{B}}(a,b)$ yielding a two-weight code.

Here we prove that using a similar technique we can construct two infinite families of 3-character multiset of $\mathrm{PG}(r,q^2)$ generating three-weight codes. The construction is as follows.

Let r be even. Suppose that either q is odd with $4a^{q+1}+(b^q-b)^2$ a non-zero square in $\mathrm{GF}(q)$ or q is even and $\mathrm{Tr}_q(a^{q+1}/(b^q+b)^2)=1$. From Theorem 3.2 $\mathcal{B}(a,b)$ is a set of q^{2r-1} points of $\mathrm{AG}(r,q^2)$ with characters $q^{2r-3}-q^{r-2}$, q^{2r-3}, $q^{2r-3}-q^{r-2}+q^{r-1}$.

Now consider the multiset $\overline{\mathcal{B}}(a,b)$ in $\mathrm{PG}(r,q^2)$ arising from $\mathcal{B}(a,b)$ by assigning multiplicity larger than 1 to the point P_∞.

More in detail the points of the 3-character multiset $\overline{\mathcal{B}}(a,b)$ are exactly those of $\mathcal{B}(a,b) \cup \{P_\infty\}$ where each affine point of $\mathcal{B}(a,b)$ has multiplicity one, and P_∞ has multiplicity j.

In this way $\overline{\mathcal{B}}(a,b)$ turns out to have the following characters:

$$j, q^{2r-3}+j, q^{2r-3}-q^{r-2}, q^{2r-3}-q^{r-2}+q^{r-1}.$$

We immediately get

Theorem 5.1. *The multiset* $\overline{\mathcal{B}}(a,b) = \mathcal{B}(a,b) \cup \{P_\infty\}$ *where each affine point of* $\mathcal{B}(a,b)$ *has multiplicity one, and* P_∞ *has multiplicity* j, $j \in \{q^{r-1}-q^{r-2}, q^{2r-3}-q^{r-2}\}$ *is a* 3*-character multiset of* $\mathrm{PG}(r,q^2)$.

Theorem 5.2. *The linear code* $\mathcal{C}$ *generated from* $\overline{\mathcal{B}}(a,b)$ *is a* $[q^{2r-1}+j, r+1]_{q^2}$*-code with weights*

$$q^{2r-1}, q^{2r-1}-q^{2r-3}, q^{2r-1}-q^{2r-3}+q^{r-2}+j, q^{2r-1}-q^{2r-3}+q^{r-2}-q^{r-1}+j$$

Furthermore for $j = q^{r-1}-q^{r-2}$ *or* $j = q^{2r-3}-q^{r-2}$ $\mathcal{C}$ *is a* 3*-weight code.*

Now we are going to determine the weight enumerator of $\mathcal{C}$ for $j = q^{r-1}-q^{r-2}$ or $j = q^{2r-3}-q^{r-2}$. We prove the following

Theorem 5.3. *Let* $\mathcal{C}$ *be the linear code generated from* $\overline{\mathcal{B}}(a,b)$*. The weight enumerator of* $\mathcal{C}$ *is* $w(x) := \sum_i A_i x^i$ *where*

$$A_0 = 1, \qquad A_{q^{2r-1}} = (q^2-1)$$

and all of the remaining A_i*'s are* 0 *with the exception of*

$$A_{q^{2r-1}-q^{2r-3}} = q^{2r}-1+q^{2r-1}(q^2-1), \quad A_{q^{2r-1}-q^{2r-3}+q^{r-1}} = (q^{2r}-q^{2r-1}-1)(q^2-1),$$

for $j = q^{r-1}-q^{r-2}$*, or*

$$A_0 = 1, \qquad A_{q^{2r-1}} = (q^{2r}-1)$$

and all of the remaining A_i*'s are* 0 *with the exception of*

$$A_{q^{2r-1}-q^{2r-3}} = q^{2r-1}(q^2-1), \quad A_{q^{2r-1}-q^{r-1}} = q^{2r-1}(q-1)(q^2-1),$$

for $j = q^{2r-3}-q^{r-2}$.

Proof. Let denote by N_i the number of hyperplanes meeting $\overline{\mathcal{B}}(a,b)$ in i points. For $j = q^{r-1}-q^{r-2}$ we have just to observe that the only hyperplane meeting $\overline{\mathcal{B}}(a,b)$ in j points is that at infinity; thus $N_{q^{r-1}-q^{r-2}} = 1$; the hyperplanes meeting $\overline{\mathcal{B}}(a,b)$

in $q^{2r-3}-q^{r-2}+q^{r-1}$ points are the hyperplanes passing through P_∞ together with the hyperplanes for which the corresponding quadric $\mathcal{Q}$ of equation (9) is singular. Therefore,

$$N_{q^{2r-3}-q^{r-2}+q^{r-1}} = \frac{q^{2r}-1}{q^2-1} + q^{2r-1}.$$

The remaining hyperplanes intersect $\overline{\mathcal{B}}(a,b)$ in $q^{2r-3}-q^{r-2}$ points and hence

$$N_{q^{2r-3}-q^{r-2}} = q^{2r}-q^{2r-1}-1.$$

For $j = q^{2r-3}-q^{r-2}$ a similar argument gives

$$N_{2q^{2r-3}-q^{r-2}} = \frac{q^{2r}-1}{q^2-1}, \qquad N_{q^{2r-3}-q^{r-2}+q^{r-1}} = q^{2r-1},$$

$$N_{q^{2r-3}-q^{r-2}} = q^{2r}-q^{2r-1}$$

and our theorem follows.

□

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