Richter-Peleg multi-utility representations of preorders

José Carlos R. Alcantud [a], Gianni Bosi [b,*], Magalì Zuanon [c]

[a] *Facultad de Economía y Empresa and Multidisciplinary Institute of Enterprise (IME), Universidad de Salamanca, E 37008 Salamanca, Spain.*

[b] *Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche, Università di Trieste, Piazzale Europa 1, 34127, Trieste, Italy.*

[c] *Dipartimento di Economia e Management, Università degli Studi di Brescia, Contrada Santa Chiara 50, 25122 Brescia, Italy*

Abstract

The existence of a Richter-Peleg multi-utility representation of a preorder by means of upper semicontinuous or continuous functions is discussed in connection with the existence of a Richter-Peleg utility representation. We give several applications that include the analysis of countable Richter-Peleg multi-utility representations.

Key words: Richter-Peleg multi-utility representation, Richter-Peleg representation

1 Introduction

The *multi-utility representation* of a not necessarily total preorder or quasi-ordering $\precsim$ on a decision space $\mathbf{X}$ characterizes the preorder by means of a family $\mathbf{V}$ of real-valued (isotonic) functions, in the sense that, for all elements $x, y \in \mathbf{X}$, $x \precsim y$ is required to be equivalent to $v(x) \leq v(y)$ for all functions $v \in \mathbf{V}$.

On the other hand, we recall that a function v on $\mathbf{X}$ is said to be a *Richter-Peleg utility representation* or *order-preserving function* for a preorder $\precsim$ on $\mathbf{X}$ if it is *increasing* (i.e., $x \precsim y$ implies that $v(x) \leq v(y)$ for all $x, y \in \mathbf{X}$) and in addition $x \prec y$ implies that $v(x) < v(y)$ for all $x, y \in \mathbf{X}$, where $\prec$ stands for the *strict part* of the preorder $\precsim$. While the mere existence of a Richter-Peleg

* Corresponding author. *E-mail address*: gianni.bosi@deams.units.it

utility v does not allow to characterize the preorder $\precsim$, it should be noted that if we are interested in finding a maximal element for $\precsim$, then such an element can be determined by maximizing v.

A *Richter-Peleg multi-utility representation* $\mathbf{V}$ for a preorder $\precsim$ on $\mathbf{X}$ is a multi-utility representation for $\precsim$ such that every function $v \in \mathbf{V}$ is a Richter-Peleg utility for $\precsim$. Such a representation is important since it is a synthesis that preserves the advantages of both approaches.

In this paper we prove that the existence of a single Richter-Peleg utility is necessary and sufficient for the existence of a Richter-Peleg multi-utility representation. A perfectly analogous result holds true when we require upper (or lower) semicontinuity of all the functions involved. We also show that the problem of obtaining a continuous Richter-Peleg multi-utility representation can be transformed to the problem of obtaining a continuous Richter-Peleg utility plus a continuous multi-utility representation. These results can be combined with the earlier findings on the existence of Richter-Peleg and multi-utility representations. For example, as a corollary of our main result, it follows that on a second countable topological space the existence of a continuous multi-utility representation implies the existence of a continuous Richter-Peleg multi-utility representation. Another notable corollary is that every preorder on a countable set has a (countable) Richter-Peleg multi-utility representation. Both of these observations follow from the fact that the existence of a countable multi-utility representation implies the existence of a Richter-Peleg utility.

As a disadvantage of our approach, we prove that it is impossible to represent a nontotal preorder on a connected topological space by means of a finite continuous Richter-Peleg multi-utility.

We recall that the main general contributions to the existence of (continuous) multi-utility representations were presented by Levin [11] and especially by Evren and Ok [8], who develop the ordinal theory of multi-utility representations (see also the more recent paper by Bosi and Herden [6]). The case of a finite representing family was studied by Ok [13] and more recently by Kaminski [10]. The notion of a Richter-Peleg multi-utility representation was first introduced and studied by Minguzzi [12], from a perspective different from ours.

The paper is organized as follows. Section 2 contains the definitions. Section 3 presents the main results, whose applications are developed in Section 4. Section 5 concerns the case of connected topological spaces with the illustration of the aforementioned restrictions. Section 6 finishes the paper with the conclusions.

2 Notation and definitions

Let $\mathbf{X}$ represent a *decision space* and $\precsim$ represent a *preorder*, also called *quasi-ordering* (reflexive, transitive binary relation) on $\mathbf{X}$. As usual, $\prec$ denotes the *strict part* of $\precsim$ and we use $x \precsim y$, resp. $x \prec y$, as a shorthand for $(x,y) \in \precsim$, resp. $(x,y) \in \prec$. The preorder $\precsim$ is *total* if for each $x, y \in \mathbf{X}$, either $x \precsim y$ or $y \precsim x$ holds true.
For every $x \in \mathbf{X}$ we set the following subsets of $\mathbf{X}$:

$$l(x) = \{y \in \mathbf{X} \mid y \prec x\}, \;\; r(x) = \{z \in \mathbf{X} \mid x \prec z\},$$

$$d(x) = \{y \in \mathbf{X} \mid y \precsim x\}, \;\; i(x) = \{z \in \mathbf{X} \mid x \precsim z\}.$$

A subset D of $\mathbf{X}$ is said to be *decreasing*, resp. *increasing*, if $d(x) \subset D$, resp. $i(x) \subset D$, for all $x \in D$.
We recall that $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is *isotonic* or *increasing* when, for each $x, y \in \mathbf{X}$, $x \precsim y \Rightarrow v(x) \leqslant v(y)$. Furthermore, v is *strictly isotonic* or *order preserving* if it is isotonic and in addition, for each $x, y \in \mathbf{X}$, $x \prec y \Rightarrow v(x) < v(y)$. Strictly isotonic functions on $(\mathbf{X}, \precsim)$ are also called *Richter-Peleg representations* of $\precsim$ (see e.g. Peleg [14] and Richter [15]). When $\precsim$ is total, any Richter-Peleg representation v of $\precsim$ is a standard utility representation: that is to say, it verifies, for each $x, y \in \mathbf{X}$, $x \precsim y \Leftrightarrow v(x) \leqslant v(y)$. It is obvious that every preorder with a utility representation is total.
Following the terminology adopted by Evren and Ok [8], we say that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is *upper, resp. lower, semicontinuous* if $i(x)$, resp. $d(x)$, is a closed subset of $\mathbf{X}$ for every $x \in \mathbf{X}$. And it is continuous if it is both upper and lower semicontinuous.
A *multi-utility representation* of the preordered space $(\mathbf{X}, \precsim)$ is a family $\mathbf{V}$ of functions $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$, with the property that for each $x, y \in \mathbf{X}$,

$$x \precsim y \Leftrightarrow [v(x) \leqslant v(y), \text{ for all } v \in \mathbf{V}] \tag{1}$$

We make note that each $v \in \mathbf{V}$ is an isotonic function when $\mathbf{V}$ is a multi-utility representation of $(\mathbf{X}, \precsim)$. If $\mathbf{V}$ is a countable, resp. finite, family then we say that $\mathbf{V}$ is a *countable*, resp. *finite*, *multi-utility representation* of $(\mathbf{X}, \precsim)$. When there is a topology τ on $\mathbf{X}$ and $\mathbf{V}$ is a family of upper semicontinuous/lower semicontinuous/continuous functions with the property that (1) holds for each $x, y \in \mathbf{X}$, then we say that $\mathbf{V}$ is an *upper semicontinuous/lower semicontinuous/continuous multi-utility representation* of $(\mathbf{X}, \precsim)$. Combinations of these concepts (e.g., countable continuous multi-utility representation) are naturally mentioned along the paper. If $\mathbf{V}$ is a multi-utility representation of $(\mathbf{X}, \precsim)$ then, for each $x, y \in \mathbf{X}$,

$$x \prec y \Leftrightarrow [v(x) \leqslant v(y) \text{ for all } v \in \mathbf{V}, \text{ and } v'(x) < v'(y) \text{ for some } v' \in \mathbf{V}] \tag{2}$$

The following result is often quoted along the paper:

Proposition 2.1 *(Evren and Ok [8, Proposition 2]) Every preorder (resp., upper semicontinuous preorder) on a set (resp., on a topological space) is representable by a multi-utility (resp., an upper semicontinuous multi-utility). If the set is countable then the preorder is representable by a countable multi-utility.*

Minguzzi [12, Section 5] introduces the following notion that we call *Richter-Peleg multi-utility representation.* A preordered set $(\mathbf{X}, \precsim)$ is *represented by a Richter-Peleg multi-utility* $\mathbf{V}$ if $\mathbf{V}$ is a family of strictly isotonic functions on $(\mathbf{X}, \precsim)$ such that for each $x, y \in \mathbf{X}$, property (1) holds true. Therefore Richter-Peleg multi-utility representations are multi-utility representations. From the fact that there are preorders without a Richter-Peleg representation we deduce:

Corollary 2.2 *There are preorders that cannot be represented by Richter-Peleg multi-utilities.*

In particular, the existence of multi-utility representations does not secure existence of Richter-Peleg multi-utility representations. The class of preordered sets for which Richter-Peleg multi-utility representations exist has not been identified yet. When there is a topology τ on $\mathbf{X}$, *upper semicontinuous/lower semicontinuous/continuous Richter-Peleg multi-utility representations* of $\precsim$ can be defined in a direct manner as above. Concepts like countable continuous Richter-Peleg multi-utility representations are naturally mentioned along the paper and their meaning is inherited from the formalizations above.

Remark 2.3 *It is immediate to check that a Richter-Peleg multi-utility representation* $\boldsymbol{V}$ *of a preordered set* $(\boldsymbol{X}, \precsim)$ *also characterizes the strict part* $\prec$ *of* $\precsim$*, in the sense that for each* $x, y \in \boldsymbol{X}$*,*

$$x \prec y \Leftrightarrow [v(x) < v(y), \text{ for all } v \in \boldsymbol{V}]. \tag{3}$$

With a Richter-Peleg multi-utility representation $\mathbf{V}$ we can attach a multi-self interpretation in the sense of Evren [7, Section 5]. Each $v \in \mathbf{V}$ gives what Evren calls “a description of a possible self of the agent defined by $\precsim$”. This means that every maximal element of v is a maximal element of $\precsim$ because $\mathbf{V}$ delivers strictly isotonic functions (or Richter-Peleg utilities) on $(\mathbf{X}, \precsim)$. However with a multi-utility representation $\mathbf{U}$ of the preorder, maximization of an individual $u \in \mathbf{U}$ does not generally produce maximal elements for the preorder. The reason is that now $x \prec y$ needs not enforce any condition on u.

3 Main results

The existence of a Richter-Peleg representation implies the existence of a Richter-Peleg multi-utility representation. Indeed, the following theorem holds.

Theorem 3.1 *Let $\precsim$ be a preorder on a set $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a Richter-Peleg multi-utility.

2. There is a Richter-Peleg representation of $\precsim$.

The equivalence remains true if there is a topology on $\boldsymbol{X}$ and we insert the term 'upper/lower semicontinuous' in each of the clauses of the statement.

Proof. Since the implication $1 \Rightarrow 2$ is obvious we only need to prove that $2 \Rightarrow 1$. Let $\mathbf{V}$ be a multi-utility representation of $\precsim$, and let f be a Richter-Peleg representation of $\precsim$. Then $\mathbf{U} = \{v + \alpha f : v \in \mathbf{V}, \alpha \in \mathbb{Q}, \alpha > 0\}$ is a Richter-Peleg multi-utility representation of $\precsim$.

This argument serves for the corresponding equivalence under upper/lower semicontinuity too. $\square$

Proposition 3.2 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a continuous Richter-Peleg multi-utility.

2. $\precsim$ can be represented by a continuous multi-utility, and there are continuous Richter-Peleg representations of $\precsim$.

Proof. The implication $1 \Rightarrow 2$ is trivial. The implication $2 \Rightarrow 1$ can be proven by mimicking the proof of Theorem 3.1. $\square$

As a consequence of Theorem 3.1 one can deduce that the notion of (resp., upper, lower semicontinuous) Richter-Peleg multi-utility is more demanding than the notion of (resp., upper, lower semicontinuous) multi-utility, and it is also more demanding than Richter-Peleg utility representation in the continuous case.

4 Applications

In this section we demonstrate how our main findings can be utilized to obtain further representation results.
In contrast to the general observations above, the existence of a countable multi-utility representation implies the existence of a countable Richter-Peleg multi-utility representation. Indeed, if $\mathbf{V} = \{v_1, v_2, ...\}$ is a multi-utility representation of $\precsim$, then the function $f := \sum_{n \in \mathbb{N}^+} 2^{-n} v_n$ is a Richter-Peleg representation of $\precsim$, where without loss of generality we assume that $\mathbf{V}$ consists of uniformly bounded functions. We can then invoke Theorem 3.1 to obtain a Richter-Peleg multi-utility representation of $\precsim$. It is also clear that this Richter-Peleg multi-utility representation will invoke countably many functions, and that a continuous analogue of this observation follows from Proposition 3.2. We thus have the following result.

Proposition 4.1 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a countable continuous Richter-Peleg multi-utility.

2. $\precsim$ can be represented by a countable continuous multi-utility.

The equivalence remains true if the term 'continuous' is deleted from each clause, or replaced with 'upper/lower semicontinuous'.

Following the proof of Proposition 2 by Evren and Ok [8], it can easily be shown that every preorder on a countable set admits a countable multi-utility representation. Thus the next result is a corollary of Proposition 4.1.

Corollary 4.2 *Let $\precsim$ be a preorder on a countable set $\boldsymbol{X}$. Then there are countable Richter-Peleg multi-utility representations of $\precsim$.*

Our next result shows that on second countable spaces, the existence of a continuous Richter-Peleg multi-utility representation is equivalent to that of a continuous multi-utility representation.

Proposition 4.3 *Suppose that a preorder $\precsim$ on a second countable topological space (X, τ) has a continuous multi-utility representation $\boldsymbol{V}$. Then $\precsim$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Proof. We benefit from a technique in Minguzzi [12, Theorem 5.5]. Define $G(\precsim) = \{(x, y) \in X \times X : x \precsim y\}$ and $G_v = \{(x, y) \in X \times X : v(x) \leq v(y)\}$ for each $v \in \mathbf{V}$. Then $G(\precsim) = \bigcap_{v \in \mathbf{V}} G_v$ and each G_v is closed by continuity of v. The product space $X \times X$ is second countable (Willard [17, 16E]) hence hereditary Lindelöff (Hocking and Young [9, Exercise 2-17]), which ensures the existence of a countable family $\mathbf{V}' \subseteq \mathbf{V}$ such that $G(\precsim) = \bigcap_{v \in \mathbf{V}'} G_v$. This means that $\mathbf{V}'$ is a countable continuous multi-utility representation of $\precsim$. In order to conclude we invoke Proposition 4.1. □

We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *weakly continuous* if for every pair $(x, y) \in \prec$ there exists a continuous increasing real-valued function f_{xy} on $(\mathbf{X}, \tau)$ such that $f_{xy}(x) < f_{xy}(y)$.
A preorder $\precsim$ on $(\mathbf{X}, \tau)$ is said to satisfy the *continuous analogue of the Dushnik and Miller theorem* (see Bosi and Herden [4,5]) if it is the intersection of all the continuous total preorders $\leq$ extending it (i.e., all the continuous total preorders $\leq$ such that $\precsim \subset \leq$ and $\prec \subset <$).
The next result shows that these two properties jointly imply the existence of a continuous Richter-Peleg multi-utility on second countable spaces.

Proposition 4.4 *Let $\precsim$ be a weakly continuous preorder on a second countable topological space $(\boldsymbol{X}, \tau)$. If $\precsim$ satisfies the continuous analogue of the Dushnik-Miller theorem, then $\precsim$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Proof. By Bosi and Herden [6, Proposition 3.4], there is a continuous multi-utility representation of $\precsim$. In addition, there is a continuous Richter-Peleg representation of $\precsim$ by Bosi et al. [3, Theorem 3.1]. Therefore the conclusion follows from Proposition 3.2 and Proposition 4.3. □

We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *closed* if $\precsim$ is a closed subset of $\mathbf{X} \times \mathbf{X}$ with respect to the product topology $\tau \times \tau$ on $\mathbf{X} \times \mathbf{X}$ that is induced by τ. The following corollary of Proposition 4.3 easily follows from Evren and Ok [8, Corollary 1], who proved that every closed preorder $\precsim$ on a locally compact metrizable topological space $(\mathbf{X}, \tau)$ has a continuous Richter-Peleg multi-utility representation $\mathbf{V}$

Corollary 4.5 *Let $(\mathbf{X}, \tau)$ be a locally compact metrizable topological space. If τ is a second countable topology, then every closed preorder $\precsim$ on $(\mathbf{X}, \tau)$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Banerjee and Dubey [2, Proposition 1] show that an *ethical social welfare relation*[1] (SWR) does not admit a Richter-Peleg representation. Then in their Theorem 2 they prove that no ethical SWR admits a countable multi-utility representation. An appeal to the above arguments according to which the existence of a countable (continuous) multi-utility implies the existence of a (continuous) Richter-Peleg utility permits to derive the latter result from the former immediately. However Alcantud and Dubey [1] show that there are

[1] A social welfare relation (i.e. a preorder on $[0,1]^{\mathbb{N}}$) is said to be *ethical* if it is *anonymous* and *strong Pareto.*

SWRs that have both multi-utility representations continuous with respect to the product topology (with the set of utilities being countable infinite) and Richter-Peleg representations. Theorem 3.1 ensures that such SWRs admit countable Richter-Peleg multi-utility representations continuous in the product topology.

5 Continuous Richter-Peleg multi-utilities on connected spaces

It is not immediate to obtain a Richter-Peleg multi-utility representation in which the representing set $\mathbf{V}$ is well behaved for the purposes of optimization. Unless the finiteness of the representing family would be useful for the purposes of optimization, we now proceed to identify a restriction on the preorders for which finite continuous Richter-Peleg multi-utility representations exist, This happens when the topology is connected. Proposition 5.2 below proves that under such conditions the preorder must be representable by continuous utilities, therefore it must be total. To state that result, we recall that a preorder $\precsim$ on a set $\mathbf{X}$ is a said to be *nontrivial* if there exist two elements $x, y \in \mathbf{X}$ such that $x \prec y$. The following lemma is well known and widely cited in the literature.

Lemma 5.1 (Schmeidler [16]) *Let $\precsim$ be a nontrivial preorder on a connected topological space (X, τ). If for every $x \in X$ the sets $d(x)$ and $i(x)$ are closed and the sets $l(x)$ and $r(x)$ are open, then the preorder $\precsim$ is total.*

Proposition 5.2 *If a nontrivial preorder $\precsim$ on a connected topological space $(\mathbf{X}, \tau)$ has a continuous Richter-Peleg multi-utility representation $\mathbf{V} = \{v_1, ..., v_n\}$ then $\precsim$ is total and every v_i is a continuous utility representation of $\precsim$.*

Proof. It suffices to check that $\precsim$ is total, because in that case any Richter-Peleg representation of $\precsim$ is a utility representation and each v_i is Richter-Peleg representation of $\precsim$ by assumption. It is immediate to check that if a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ has a continuous multi-utility representation then both $d(x)$ and $i(x)$ are closed subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$ (see e.g. Proposition 5 in Bosi and Herden [6] or Theorem 3.1 in Kaminski [10] for a restricted version). Therefore, by using Lemma 5.1, it suffices to show that under our assumptions, both $l(x)$ and $r(x)$ are open subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$. To prove this fact we observe that, from Remark 2.3,

$$
\begin{aligned}
l(x) &= \{y \in \mathbf{X} \mid y \prec x\} = \{y \in \mathbf{X} \mid v_i(y) < v_i(x), \text{ for all } i \in \{1, ..., n\}\} = \\
&= \bigcap_{i=1}^{n} v_i^{-1}(]-\infty, v_i(x)[), \text{ and} \\
r(x) &= \{y \in \mathbf{X} \mid x \prec y\} = \{y \in \mathbf{X} \mid v_i(x) < v_i(y), \text{ for all } i \in \{1, ..., n\}\} = \\
&= \bigcap_{i=1}^{n} v_i^{-1}(]v_i(x), +\infty[)
\end{aligned}
$$

for each $x \in \mathbf{X}$. From these equalities and continuity of the functions v_i, the conclusion follows immediately. □

6 Conclusions and final comments

In this paper, we have studied multi-utility representations that consist of Richter-Pelg utility functions. Our results show that, in general, this representation notion is more demanding that the notion of a multi-utility representation. Yet, the two representation notions turn out to be equivalent in many cases of interest. The advantage of the former representation notion is that any alternative that maximizes any one of the representing functions on a given choice set is guaranteed to be a maximal element of that set. On the other hand, when the space of alternatives is connected, this approach necessitates infinitely many utility functions to characterize an incomplete (nontotal) preorder. An interesting venue for further research can be the study of an alternative notion of a multi-utility representation that was recently proposed by Evren [7]. The distinctive feature of Evren's approach is that it does not necessitate the preorder to be closed even when the representing utility functions are continuous. Consequently, this approach is compatible with finitely many continuous Richter-Peleg utility functions even if the domain is a connected space.

Acknowledgment

We thank the Coordinating Editor and two anonymous referees for very detailed and perceptive comments. Financial support by the Spanish Ministerio de Ciencia e Innovación under Project ECO2012–31933 (J. C. R. Alcantud) is gratefully acknowledged. G. Bosi and M. Zuanon acknowledge financial support by Istituto Nazionale di Alta Matematica "F. Severi" (Italy).

References

[1] J.C.R. Alcantud and R.S. Dubey, No impatience without the completeness axiom: continuity and efficiency with an infinite horizon. Available at SSRN: http://ssrn.com/abstract=2404912 or http://dx.doi.org/10.2139/ssrn.2404912

[2] K. Banerjee and R.S. Dubey, On multi-utility representation of equitable intergenerational preferences. Econophysics and Economics of Games, Social Choices and Quantitative Techniques. New Economic Windows. Springer (2010), pp. 175-180.

[3] G. Bosi, A. Caterino and R. Ceppitelli, Existence of continuous utility functions for arbitrary binary relations: some sufficient conditions, *Tatra Mountains Mathematical Publications* **46** (2010), 1-13.

[4] G. Bosi and G. Herden, On a strong continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller, *Order* **22** (2005), 329-342.

[5] G. Bosi and G. Herden, On a possible continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller, *Order* **23** (2006), 271-296.

[6] G. Bosi and G. Herden, Continuous multi-utility representations of preorders, *Journal of Mathematical Economics* **48** (2012), 212-218.

[7] O. Evren, Scalarization methods and expected multi-utility representations, *Journal of Economic Theory* **151** (2014), 30-63.

[8] O. Evren and E.A. Ok, On the multi-utility representation of preference relations, *Journal of Mathematical Economics* **47** (2011), 554-563.

[9] J.G. Hocking and G.S. Young, Topology. Dover (1988).

[10] B. Kaminski, On quasi-orderings and multi-objective functions, *European Journal of Operational Research* **177** (2007), 1591-1598.

[11] V.L. Levin, The Monge-Kantorovich problems and stochastic preference relation, *Advances in Mathematical Economics* **3** (2001), 97-124.

[12] E. Minguzzi, Normally preordered spaces and utilities, *Order* **30** (2013), 137-150.

[13] E.A. Ok, Utility representation of an incomplete preference relation, *Journal of Economic Theory* **104** (2002), 429-449.

[14] B. Peleg, Utility functions for partially ordered topological spaces, *Econometrica* **38** (1970), 93-96.

[15] M. Richter, Revealed preference theory, *Econometrica* **34** (1966), 635-645.

[16] D. Schmeidler, A condition for the completeness of partial preference relations, *Econometrica* **3** (1971), 403-404.

[17] S. Willard, General Topology. Dover (2004).