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Line polar Grassmann codes of orthogonal type

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ABSTRACT

Polar Grassmann codes of orthogonal type have been introduced in [1]. They are punctured versions of the Grassmann code arising from the projective system defined by the Plücker embedding of a polar Grassmannian of orthogonal type. In the present paper we fully determine the minimum distance of line polar Grassmann codes of orthogonal type for q odd.

1. Introduction

Codes $\mathcal{C}_{m,k}$ arising from the Plücker embedding of the k-Grassmannians of m-dimensional vector spaces have been widely investigated since their first introduction in [10,11]. They are a remarkable generalization of Reed–Muller codes of the first order and their monomial automorphism groups and minimum weights are well understood, see [4–6,8].

In [1], the first two authors of the present paper introduced some new codes $\mathcal{P}_{n,k}$ arising from embeddings of orthogonal Grassmannians $\Delta_{n,k}$. These codes correspond to the projective system determined by the Plücker embedding of the Grassmannian $\Delta_{n,k}$ representing all totally singular k-spaces with respect to some non-degenerate quadratic form η defined on a vector space $V(2n+1,q)$ of dimension $2n+1$ over a finite field $\mathbb{F}_q$. An orthogonal Grassmann code $\mathcal{P}_{n,k}$ can be obtained from the ordinary Grassmann code $\mathcal{C}_{2n+1,k}$ by just deleting all the columns corresponding to k-spaces which are non-singular with respect to η; it is thus a punctured version of $\mathcal{C}_{2n+1,k}$. For q odd, the dimension of $\mathcal{P}_{n,k}$ is the same as that of $\mathcal{G}_{2n+1,k}$, see [1]. The minimum distance $d_{\min}$ of $\mathcal{P}_{n,k}$ is always bounded away from 1. Actually, it has been shown in [1] that for q odd, $d_{\min} \geq q^{k(n-k)+1} + q^{k(n-k)} - q$. By itself, this proves that the redundancy of these codes is somehow better than that of $\mathcal{C}_{2n+1,k}$.

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In the present paper we prove the following theorem, fully determining all the parameters for the case of line orthogonal Grassmann codes (that is orthogonal polar Grassmann codes with $k = 2$) for q odd.

Main Theorem. *For q odd, the minimum distance $d_{\min}$ of the orthogonal Grassmann code $\mathcal{P}_{n,2}$ is*

$$d_{\min} = q^{4n-5} - q^{3n-4}.$$

Furthermore, for $n > 2$ all words of minimum weight are projectively equivalent; for $n = 2$ there are two different classes of projectively equivalent minimum weight codewords.

Hence, we have the following.

Corollary 1.1. *For q odd, line polar Grassmann codes of orthogonal type are $[N, K, d_{\min}]$-projective codes with*

$$N = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q^2 - 1)(q - 1)}, \quad K = \binom{2n+1}{2}, \quad d_{\min} = q^{4n-5} - q^{3n-4}.$$

1.1. Organization of the paper

In Section 2 we recall some well-known facts on projective systems and related codes, as well as the notion of polar Grassmannian of orthogonal type. In Section 3 we prove our main theorem.

2. Preliminaries

2.1. Projective systems and Grassmann codes

Let W be a vector space. An $[N, K, d_{\min}]_q$ projective system $\Omega \subseteq \mathrm{PG}(W)$ is a set of N points spanning $\mathrm{PG}(K-1, q) \leq \mathrm{PG}(W)$ such that there is a hyperplane Σ of $\mathrm{PG}(K-1, q)$ with $\#(\Omega \setminus \Sigma) = d_{\min}$ and for any hyperplane Σ' of $\mathrm{PG}(K-1, q)$,

$$\#(\Omega \setminus \Sigma') \geq d_{\min}.$$

Existence of $[N, K, d_{\min}]_q$ projective systems is equivalent to that of projective linear codes with the same parameters; see, for instance, [12]. Indeed, let Ω be a projective system and denote by G a matrix whose columns $G_1, \ldots, G_N$ are the coordinates of representatives of the points of Ω with respect to some fixed reference system. Then, G is the generator matrix of an $[N, K, d_{\min}]$ code over $\mathbb{F}_q$, say $\mathcal{C} = \mathcal{C}(\Omega)$. The code $\mathcal{C}(\Omega)$ is not, in general, uniquely determined, but it is unique up to code equivalence. We shall thus speak, with a slight abuse of language, of *the* code defined by Ω.

As any word c of $\mathcal{C}(\Omega)$ is of the form $c = mG$ for some row vector $m \in \mathbb{F}_q^K$, it is straightforward to see that the number of zeroes in c is the same as the number of points of Ω lying on the hyperplane Π_c of equation $m \cdot x = 0$, where $m \cdot x = \sum_{i=1}^K m_i x_i$ and $m = (m_i)_1^K$, $x = (x_i)_1^K$. The weight (i.e. the number of non-zero components) of c is then

$$\mathrm{wt}(c) := |\Omega| - |\Omega \cap \Pi_c|. \tag{1}$$

Thus, the minimum distance $d_{\min}$ of $\mathcal{C}$ is

$$d_{\min} = |\Omega| - f_{\max}, \quad \text{where} \quad f_{\max} = \max_{\substack{\Sigma \leq \mathrm{PG}(K-1,q) \\ \dim \Sigma = K-2}} |\Omega \cap \Sigma|. \tag{2}$$

We point out that any projective code $\mathcal{C}(\Omega)$ can also be regarded, equivalently, as an evaluation code over Ω of degree 1. In particular, when Ω spans the whole of $\mathrm{PG}(K-1,q)=\mathrm{PG}(W)$, then there is a bijection, induced by the standard inner product of W, between the points of the dual vector space W^* and the codewords c of $\mathcal{C}(\Omega)$.

Let $\mathcal{G}_{2n+1,k}$ be the Grassmannian of the k-subspaces of a vector space $V:=V(2n+1,q)$, with $k\leq n$ and let $\eta: V\to\mathbb{F}_q$ be a non-degenerate quadratic form over V.

Denote by $\varepsilon_k:\mathcal{G}_{2n+1,k}\to\mathrm{PG}(\bigwedge^k V)$ the usual Plücker embedding

$$\varepsilon_k:\mathrm{Span}(v_1,\dots,v_k)\to\mathrm{Span}(v_1\wedge\cdots\wedge v_k).$$

The orthogonal Grassmannian $\Delta_{n,k}$ is a geometry having as points the k-subspaces of V totally singular for η. Let $\varepsilon_k(\mathcal{G}_{2n+1,k}):=\{\varepsilon_k(X_k)\colon X_k \text{ is a point of } \mathcal{G}_{2n+1,k}\}$ and $\varepsilon_k(\Delta_{n,k})=\{\varepsilon_k(\bar{X}_k)\colon \bar{X}_k \text{ is a point of } \Delta_{n,k}\}$. Clearly, we have $\varepsilon_k(\Delta_{n,k})\subseteq\varepsilon_k(\mathcal{G}_{2n+1,k})\subseteq\mathrm{PG}(\bigwedge^k V)$. Throughout this paper we shall denote by $\mathcal{P}_{n,k}$ the code arising from the projective system $\varepsilon_k(\Delta_{n,k})$. By [3, Theorem 1.1], if $n\geq 2$ and $k\in\{1,\dots,n\}$, then $\dim\mathrm{Span}(\varepsilon_k(\Delta_{n,k}))=\binom{2n+1}{k}$ for q odd, while $\dim\mathrm{Span}(\varepsilon_k(\Delta_{n,k}))=\binom{2n+1}{k}-\binom{2n+1}{k-2}$ when q is even.

We recall that for $k<n$, any line of $\Delta_{n,k}$ is also a line of $\mathcal{G}_{2n+1,k}$. For $k=n$, the lines of $\Delta_{n,n}$ are not lines of $\mathcal{G}_{2n+1,n}$; indeed, in this case $\varepsilon_n|_{\Delta_{n,n}}\colon\Delta_{n,n}\to\mathrm{PG}(\bigwedge^n V)$ maps the lines of $\Delta_{n,n}$ onto non-singular conics of $\mathrm{PG}(\bigwedge^n V)$.

The projective system identified by $\varepsilon_k(\Delta_{n,k})$ determines a code of length $N=\prod_{i=0}^{k-1}\frac{q^{2(n-i)}-1}{q^{i+1}-1}$ and dimension $K=\binom{2n+1}{k}$ or $K=\binom{2n+1}{k}-\binom{2n+1}{k-2}$ according to whether q is odd or even. The following universal property provides a well-known characterization of alternating multilinear forms; see for instance [9, Theorem 14.23].

Theorem 2.1. *Let V and U be vector spaces over the same field. A map $f: V^k\longrightarrow U$ is alternating k-linear if and only if there is a linear map $\overline{f}:\bigwedge^k V\longrightarrow U$ with $\overline{f}(v_1\wedge v_2\wedge\cdots\wedge v_k)=f(v_1,v_2,\dots,v_k)$. The map $\overline{f}$ is uniquely determined.*

In general, the dual space $(\bigwedge^k V)^*\cong\bigwedge^k V^*$ of $\bigwedge^k V$ is isomorphic to the space of all k-linear alternating forms of V. For any given non-null vector $\mathbf{v}\in\bigwedge^{2n+1}V\cong V(1,q)\cong\mathbb{F}_q$, we have an isomorphism $\jmath_{\mathbf{v}}:\bigwedge^{2n+1-k}V\to(\bigwedge^k V)^*$ defined by $\jmath_{\mathbf{v}}(\omega)(x)=c$ for any $\omega\in\bigwedge^{2n+1-k}V$ and $x\in\bigwedge^k V$, where $c\in\mathbb{F}_q$ is such that $\omega\wedge x=c\mathbf{v}$. Clearly, as $\mathbf{v}\neq 0$ varies in $\bigwedge^{2n+1}V$ we obtain different isomorphisms. For the sake of simplicity, we will say that $\omega\in\bigwedge^{2n+1-k}V$ acts on $x\in\bigwedge^k V$ as $\omega\wedge x$.

For any $k=1,\dots,2n$ and $\varphi\in(\bigwedge^k V)^*, v\in\bigwedge^k V$, we shall use the symbol $\langle\varphi,v\rangle$ to denote the bilinear pairing

$$(\bigwedge^k V)^*\times(\bigwedge^k V)\to\mathbb{F}_q,\ \ \langle\varphi,v\rangle=\varphi(v).$$

Since the codewords of $\mathcal{P}_{n,k}$ bijectively correspond to functionals on $\bigwedge^k V$, we can regard a codeword as an element of $(\bigwedge^k V)^*\cong\bigwedge^k V^*$.

In this paper we are concerned with line Grassmannians, that is we assume $k=2$.

By Theorem 2.1, we shall implicitly identify any functional $\varphi\in(\bigwedge^2 V)^*$ with the (necessarily degenerate) alternating bilinear form

$$\begin{cases} V\times V\to\mathbb{F}_q\\ (x,y)\to\varphi(x\wedge y).\end{cases}$$

The *radical* of φ is the set

$$\mathrm{Rad}(\varphi):=\{v\in V: \forall w\in V, \varphi(v,w)=0\}.$$

This is always a vector space and its codimension in V is even. As $\dim V$ is odd, $2n-1 \geq \dim \mathrm{Rad}(\varphi) \geq 1$ for $\varphi \neq 0$.

We point out that it has been proved in [8] that the minimum weight codewords of the line projective Grassmann code $\mathcal{C}_{2n+1,2}$ correspond to points of $\varepsilon_{2n-1}(\mathcal{G}_{2n+1,2n-1})$; these can be regarded as non-null bilinear alternating forms of V of maximum radical. Actually, non-null bilinear forms of maximum radical may yield minimum weight codewords also for Symplectic Polar Grassmann Codes, see [2].

In the case of orthogonal line Grassmannians, not all points of $\mathcal{G}_{2n+1,2n-1}$ yield codewords of $\mathcal{P}_{n,2}$ of minimum weight. However, as a consequence of the proof of our main result, we shall see that for $n > 2$ all the codewords of minimum weight of $\mathcal{P}_{n,2}$ do indeed correspond to some $(2n-1)$-dimensional subspaces of V, that is to say, to bilinear alternating forms of maximum radical. In the case $n = 2$, there are two classes of minimum weight codewords: one corresponding to bilinear alternating forms of maximum radical and another corresponding to certain bilinear alternating forms with radical of dimension 1.

2.2. A recursive condition

Since $\bigwedge^k V^* \cong (\bigwedge^k V)^* \cong \bigwedge^{2n+1-k} V$, for any $\varphi \in (\bigwedge^k V)^*$ there is an element $\widehat{\varphi} \in \bigwedge^{2n+1-k} V$ such that

$$\langle \varphi, x \rangle = \widehat{\varphi} \wedge x, \quad \forall x \in \bigwedge^k V.$$

Fix now $u \in V$ and $\varphi \in (\bigwedge^k V)^*$. Then, there is a unique element $\varphi_u \in \bigwedge^{k-1} V^*$ such that $\widehat{\varphi}_u = \widehat{\varphi} \wedge u \in \bigwedge^{2n+2-k} V$.

Let $\mathcal{Q}$ be the parabolic quadric defined by the (non-degenerate) quadratic form η. For any $u \in \mathcal{Q}$, put $V_u := u^{\perp_{\mathcal{Q}}}/\mathrm{Span}(u)$. Observe that as $\langle \varphi_u, u \wedge w \rangle = \widehat{\varphi} \wedge u \wedge u \wedge w = 0$ for any $u \wedge w \in \bigwedge^{k-1} V$, the functional

$$\overline{\varphi}_u : \begin{cases} \bigwedge^{k-1} V_u \to \mathbb{F}_q \\ x + (u \bigwedge^{k-2} V) \to \varphi_u(x) \end{cases}$$

with $x \in \bigwedge^{k-1} V$ and $u \bigwedge^{k-2} V := \{u \wedge y : y \in \bigwedge^{k-2} V\}$ is well defined. Furthermore, V_u is endowed with the quadratic form $\eta_u : x + \mathrm{Span}(u) \to \eta(x)$. Clearly, $\dim V_u = 2n-1$. It is well known that the set of all totally singular points for η_u is a parabolic quadric of rank $n-1$ in V_u which we shall denote by $\mathrm{Res}_{\mathcal{Q}} u$. In other words the points of $\mathrm{Res}_{\mathcal{Q}} u$ are the lines of $\mathcal{Q}$ through u.

We are now ready to deduce a recursive relation on the weight of codewords, in the spirit of [8].

Lemma 2.2. *Let $\varphi \in \bigwedge^k V^*$. Then,*

$$\mathrm{wt}(\varphi) = \frac{1}{q^k - 1} \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \mathrm{wt}(\overline{\varphi}_u).$$

Proof. Recall that

$$\begin{aligned} \mathrm{wt}(\varphi) &= \#\{\mathrm{Span}(v_1, \ldots, v_k) \colon \langle \varphi, v_1 \wedge \cdots \wedge v_k \rangle \neq 0, \mathrm{Span}(v_1, \ldots, v_k) \in \Delta_{n,k}\} \\ &= \frac{1}{|\mathrm{GL}_k(q)|} \#\{(v_1, \ldots, v_k) \colon \langle \varphi, v_1 \wedge \cdots \wedge v_k \rangle \neq 0, \mathrm{Span}(v_1, \ldots, v_k) \in \Delta_{n,k}\}, \end{aligned} \tag{3}$$

where the list $(v_1, \ldots, v_k)$ is an ordered basis of $\mathrm{Span}(v_1, \ldots, v_k) \subset \mathcal{Q}$.

For any point $u \in \mathcal{Q}$, we have $\mathrm{Span}(u, v_2, \ldots, v_k) \in \Delta_{n,k}$ if and only if $\mathrm{Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)$, where $\Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)$ is the $(k-1)$-Grassmannian of $\mathrm{Res}_{\mathcal{Q}} u$ and by the symbol $\mathrm{Span}_u(v_2, \ldots, v_k)$ we mean $\mathrm{Span}(u, v_2, \ldots, v_k)/\mathrm{Span}(u)$. Furthermore, given a space $\mathrm{Span}_u(v_2, \ldots, v_k) \in$

$\Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)$, any of the q^{k-1} lists $(u, v_2 + \alpha_2 u, \dots, v_k + \alpha_k u)$ is a basis for the same totally singular k-space through u, namely $\mathrm{Span}(u, v_2, \dots, v_k)$. Conversely, given any totally singular k-space $W \in \Delta_{n,k}$ with $u \in W$, there are $v_2, \dots v_k \in \mathrm{Res}_{\mathcal{Q}} u$ such that $W = \mathrm{Span}(u, v_2, \dots, v_k)$ and $\mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)$. Let

$$\Omega_u := \{(u, v_2 + \alpha_2 u, \dots, v_k + \alpha_k u) \colon \langle \varphi, u \wedge v_2 \wedge \dots \wedge v_k \rangle \neq 0, \\ \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u), \alpha_2, \dots, \alpha_k \in \mathbb{F}\}.$$

Then, we have the following disjoint union

$$\{(v_1, \dots, v_k) \colon \langle \varphi, v_1 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}(v_1, \dots, v_k) \in \Delta_{n,k}\} = \bigcup_{u \in \mathcal{Q}} \Omega_u. \tag{4}$$

Observe that if u is not singular, then, $\Omega_u = \emptyset$, as $\mathrm{Span}(u, v_2, \dots, v_k) \nsubseteq \mathcal{Q}$; likewise, if $\overline{\varphi}_u = 0$, then, $\langle \overline{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle = 0$ for any $v_2, \dots, v_k$ and, consequently, $\Omega_u = \emptyset$.

The coefficients α_i, $2 \leq i \leq k$, are arbitrary in $\mathbb{F}$; thus,

$$\#\Omega_u = q^{k-1} \#\{(u, v_2, \dots, v_k) : \langle \overline{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)\}.$$

Hence,

$$|\mathrm{GL}_k(q)| \mathrm{wt}(\varphi) = \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \#\Omega_u =$$

$$= q^{k-1} \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \#\{(u, v_2, \dots, v_k) \colon \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)\}. \tag{5}$$

Since u is fixed,

$$\#\{(u, v_2, \dots, v_k) \colon \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)\} \\ = \#\{(v_2, \dots, v_k) \colon \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)\}.$$

On the other hand, by (3) and by the definition of $\overline{\varphi}_u$,

$$|\mathrm{GL}_{k-1}(q)| \mathrm{wt}(\overline{\varphi}_u) = \#\{(v_2, \dots, v_k) \colon \langle \overline{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \mathrm{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\mathrm{Res}_{\mathcal{Q}} u)\};$$

thus,

$$\mathrm{wt}(\varphi) = q^{k-1} \frac{|\mathrm{GL}_{k-1}(q)|}{|\mathrm{GL}_k(q)|} \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \mathrm{wt}(\overline{\varphi}_u) = \frac{1}{q^k - 1} \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \mathrm{wt}(\overline{\varphi}_u). \qquad \square \tag{6}$$

3. Proof of the Main Theorem

As $\dim V$ is odd, all non-degenerate quadratic forms on V are projectively equivalent. For the purposes of the present paper we can assume without loss of generality that a basis $(e_1, \dots, e_{2n+1})$ has been fixed such that

$$\eta(x) := \sum_{i=1}^{n} x_{2i-1} x_{2i} + x_{2n+1}^2. \tag{7}$$

Let $\beta(x,y) := \eta(x+y) - \eta(x) - \eta(y)$ be the bilinear form associated with η. As in Section 2.2, denote by $\mathcal{Q}$ the set of the non-zero totally singular vectors for η. Clearly, for any k-dimensional vector subspace W of V, then $W \in \Delta_{n,k}$ if and only if $W \subseteq \mathcal{Q}$.

Henceforth we shall work under the assumption $k = 2$. Denote by φ an arbitrary alternating bilinear form defined on V and let M and S be the matrices representing respectively β and φ with respect to the basis $(e_1, \dots, e_{2n+1})$ of V. Write $\perp_{\mathcal{Q}}$ for the orthogonal relation induced by η and $\perp_W$ for the (degenerate) symplectic relation induced by φ. In particular, for $v \in V$, the symbols $v^{\perp_{\mathcal{Q}}}$ and $v^{\perp_W}$ will respectively denote the space orthogonal to v with respect to β and φ. Likewise, when X is a subspace of V, the notations $X^{\perp_{\mathcal{Q}}}$ and $X^{\perp_W}$ will be used to denote the spaces orthogonal to X with respect to β and φ. We shall say that a subspace X is *totally singular* if $X \leq X^{\perp_{\mathcal{Q}}}$ and *totally isotropic* if $X \leq X^{\perp_W}$.

Lemma 3.1. *Let $\mathcal{Q}$ be a parabolic quadric with equation of the form (7), and let $p \in V$, $p \neq 0$. Denote by ρ a codeword corresponding to the hyperplane $p^{\perp_{\mathcal{Q}}}$. Then,*

$$\mathrm{wt}(\rho) = \begin{cases} q^{2n-1} & \text{if } \eta(p) = 0 \\ q^{2n-1} - q^{n-1} & \text{if } \eta(p) \text{ is a non-zero square} \\ q^{2n-1} + q^{n-1} & \text{if } \eta(p) \text{ is a non-square.} \end{cases}$$

Proof. If $\eta(p) = 0$, then $p \in \mathcal{Q}$ and $p^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$ is a cone with basis a parabolic quadric of rank $n-1$; it has $1 + (q^{2n-1} - q)/(q-1)$ projective points, see [7]. The value of $\mathrm{wt}(\rho)$ now directly follows from (1).

Suppose now p to be external to $\mathcal{Q}$, that is $p^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$ is a hyperbolic quadric; it is immediate to see that in this case $\mathrm{wt}(\rho) = q^{2n-1} - q^{n-1}$. Likewise, when p is internal to $\mathcal{Q}$, $\mathrm{wt}(\rho) = q^{2n-1} + q^{n-1}$.

The orthogonal group $O(V)$ stabilizing the quadric $\mathcal{Q}$ has 3 orbits on the points of V; these correspond respectively to totally singular, external and internal points to $\mathcal{Q}$. By construction, all elements in the same orbit are isomorphic 1-dimensional quadratic spaces. In other words, the quadratic class of $\eta(p)$ is constant on each of these orbits. In particular, the point e_{2n+1} is external to $\mathcal{Q}$ and $\eta(e_{2n+1}) = 1$ is a square. Thus we have that external points to $\mathcal{Q}$ correspond to those p for which $\eta(p)$ is a square, $\eta(p) \neq 0$ and internal points correspond to those for which $\eta(p)$ is a non-square. □

3.1. Some linear algebra

Lemma 3.2.

1. *For any $v \in V$, $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$ if and only if v is an eigenvector of non-zero eigenvalue of $T := M^{-1}S$.*
2. *The radical $\mathrm{Rad}(\varphi)$ of φ corresponds to the eigenspace of T of eigenvalue 0.*

Proof. 1. Observe that $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$ if and only if the equations $x^T M v = 0$ and $x^T S v = 0$ are equivalent for any $x \in V$. This means that there exists an element $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that $Sv = \lambda M v$. As M is non-singular, the latter says that v is an eigenvector of non-zero eigenvalue λ for T.

2. Let v be an eigenvector of T of eigenvalue 0. Then $M^{-1}Sv = 0$, hence $Sv = 0$ and $x^T S v = 0$ for every $x \in V$, that is $v^{\perp_W} = V$. This means $v \in \mathrm{Rad}(\varphi)$. □

We can now characterize the eigenspaces of T.

Lemma 3.3. *Let μ be a non-zero eigenvalue of T and V_μ be the corresponding eigenspace. Then,*

(1) *$\forall v \in V_\mu$ and $r \in \mathrm{Rad}(\varphi)$, $r \perp_{\mathcal{Q}} v$. Hence, $V_\mu \leq r^{\perp_{\mathcal{Q}}}$.*
(2) *The eigenspace V_μ is both totally isotropic for φ and totally singular for η.*

(3) *Let $\lambda, \mu \neq 0$ be two not necessarily distinct eigenvalues of T and u, v be two corresponding eigenvectors. Then, one of the following holds:*
   (a) *$u \perp_{\mathcal{Q}} v$ and $u \perp_W v$.*
   (b) *$\mu = -\lambda$.*

(4) *If λ is an eigenvalue of T then $-\lambda$ is an eigenvalue of T.*

Proof. 1. Take $v \in V_\mu$. As $Tv = M^{-1}Sv = \mu v$ we also have $\mu v^T = v^T S^T M^{-T}$. So, $v^T M^T = \mu^{-1} v^T S^T$. Let $r \in \mathrm{Rad}(\varphi)$. Then, as $S^T = -S$, $v^T M r = \mu^{-1} v^T S^T r$ and $v^T S r = 0$ for any v, we have $v^T M r = 0$, that is $r \perp_{\mathcal{Q}} v$.

2. Let $v \in V_\mu$. Then $M^{-1}Sv = \mu v$, which implies $Sv = \mu M v$. Hence, $v^T S v = \mu v^T M v$. Since $v^T S v = 0$ and $\mu \neq 0$, we also have $v^T M v = 0$, for every $v \in V_\mu$. Thus, V_μ is totally singular for η. Since V_μ is totally singular, for any $u \in V_\mu$ we have $u^T M v = 0$; so, $u^T S v = \mu u^T M v = 0$, that is V_μ is also totally isotropic.

3. Suppose that either $u \not\perp_{\mathcal{Q}} v$ or $u \not\perp_W v$. Since, by Lemma 3.2, $u^{\perp_{\mathcal{Q}}} = u^{\perp_W}$ and $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$, we have $Mu = \lambda^{-1} S u$ and $Mv = \mu^{-1} S v$. So, $u \not\perp_{\mathcal{Q}} v$ or $u \not\perp_W v$ implies $v^T M u \neq 0 \neq v^T S u$. Since $M^{-1} S u = \lambda u$ and $M^{-1} S v = \mu v$, we have

$$v^T S u = v^T S(\lambda^{-1} M^{-1} S u) = \lambda^{-1}(-M^{-1} S v)^T S u = -(\lambda^{-1}\mu) v^T S u;$$

hence, $-\lambda^{-1}\mu = 1$.

4. Let $\lambda \neq 0$ be an eigenvalue of T and x a corresponding eigenvector. Then $M^{-1} S x = \lambda x$ if and only if $S M^{-1} S x = \lambda S x$, which, in turn, is equivalent to $-(M^{-1}S)^T S x = \lambda S x$, that is $(M^{-1}S)^T (Sx) = -\lambda S x$. Since $\lambda \neq 0$, Sx is an eigenvector of $(M^{-1}S)^T$ of eigenvalue $-\lambda$. Clearly, $(M^{-1}S)^T$ and $M^{-1}S$ have the same eigenvalues, so $-\lambda$ is an eigenvalue of T. □

Corollary 3.4. *Let V_λ and V_μ be two eigenspaces of non-zero eigenvalues $\lambda \neq -\mu$. Then, $V_\lambda \oplus V_\mu$ is both totally singular and totally isotropic.*

3.2. Minimum weight codewords

Recall that $\varphi \in \bigwedge^2 V^*$ and, for any $u \in \mathcal{Q}$, $\overline{\varphi}_u \in V^*$. In particular, $\overline{\varphi}_u$ either determines a hyperplane of $V_u = u^{\perp_{\mathcal{Q}}}/\mathrm{Span}(u)$ or it is null on V_u.

Lemma 3.5. *$\overline{\varphi}_u = 0$ if and only if u is an eigenvector of T.*

Proof. By Lemma 3.2, u is an eigenvector of T if and only if $u^{\perp_{\mathcal{Q}}} \subseteq u^{\perp_W}$. By definition of $\perp_{\mathcal{Q}}$, for every $v \in u^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$, we have $\mathrm{Span}(u, v) \in \Delta_{n,2}$. However, as $v \in u^{\perp_W}$, also $\langle \varphi, u \wedge v \rangle = 0$. So, $\overline{\varphi}_u(v) = 0$, $\forall v \in u^{\perp_{\mathcal{Q}}}$. Thus, $\overline{\varphi}_u = 0$ on $\mathrm{Res}_{\mathcal{Q}} u$. Conversely, reading the argument backwards, we see that if $\overline{\varphi}_u = 0$ then u is eigenvector of T. □

We remark that $\varphi_u = 0$ if and only if $u \in \ker T$ (by Lemma 3.2(2)).

Lemma 3.6. *Suppose $u \in \mathcal{Q}$ not to be an eigenvector of T. Then,*

$$\mathrm{wt}(\overline{\varphi}_u) = \begin{cases} q^{2n-3} & \text{if } \eta(Tu) = 0 \\ q^{2n-3} - q^{n-2} & \text{if } \eta(Tu) \neq 0 \text{ is a square} \\ q^{2n-3} + q^{n-2} & \text{if } \eta(Tu) \text{ is a non-square.} \end{cases}$$

Proof. Let $a_u := Tu$ and let $\mathcal{Q}_u := a_u^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$. Note that $u \in \mathcal{Q}_u \cap u^{\perp_{\mathcal{Q}}}$. Indeed, $u^T M T u = u^T S u = 0$. So, $\mathrm{wt}(\overline{\varphi}_u) = \mathrm{wt}(\overline{\varphi}_{a_u})$. The quadric $\mathrm{Res}_{\mathcal{Q}_u} u := (\mathcal{Q}_u \cap u^{\perp_{\mathcal{Q}}})/\mathrm{Span}(u)$ is either hyperbolic, elliptic or degenerate according as a_u is external, internal or contained in $\mathcal{Q}$. The result now follows from Lemma 3.1. □

Define

$$\mathfrak{A}' := \{u : u \in \mathcal{Q} \text{ and } u \text{ non-eigenvector of } T\}, \qquad A' := \#\mathfrak{A}';$$
$$\mathfrak{B} := \{u : u \in \mathfrak{A}' \text{ and } Tu \in \mathcal{Q}\}, \qquad B := \#\mathfrak{B};$$
$$\mathfrak{C} := \{u : u \in \mathfrak{A}' \text{ and } \eta(Tu) \text{ is a non-square}\}, \qquad C := \#\mathfrak{C}.$$

By definition, both $\mathfrak{B}$ and $\mathfrak{C}$ are subset of $\mathfrak{A}'$. Using (6) we can write

$$\mathrm{wt}(\varphi) = \frac{q^{2n-3} - q^{n-2}}{q^2-1}A' + \frac{q^{n-2}}{q^2-1}B + \frac{2q^{n-2}}{q^2-1}C. \tag{8}$$

Put $A = q^{2n-2} - 1 - \#\{u : u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\}$; then, (8) becomes

$$\mathrm{wt}(\varphi) = q^{4n-5} - q^{3n-4} + \frac{q^{n-2}}{q^2-1}((q^{n-1}-1)A + B + 2C). \tag{9}$$

Clearly, $B, C \geq 0$. We investigate A more closely. Let $\mathrm{Spec}'(T)$ be the set of non-zero eigenvalues of T and let $V_\lambda = \ker(T - \lambda I)$ be the corresponding eigenspaces for $\lambda \in \mathrm{Spec}'(T)$. By Lemma 3.3, each space V_λ is totally singular; thus

$$A = q^{2n-2} - 1 - \sum_{\lambda \in \mathrm{Spec}'(T)} (\#V_\lambda - 1) - \#(\ker T \cap \mathcal{Q}). \tag{10}$$

Let $r \in \mathbb{N}$ be such that $\dim \mathrm{Rad}(\varphi) = \dim \ker T = 2(n-r)+1$, where by Theorem 2.1, we may regard φ as a bilinear alternating form.

The non-degenerate symmetric bilinear form β induces a symmetric bilinear form β^* on V^*, defined as $\beta^*(v_1^*, v_2^*) = \beta(v_1, v_2)$ where v_1^*, v_2^* are functionals determining respectively the hyperplanes $v_1^{\perp_{\mathcal{Q}}}$ and $v_2^{\perp_{\mathcal{Q}}}$. In particular, given the basis $(e_1, \dots, e_{2n+1})$ of V, the above correspondence determines a basis $(e^1, \dots, e^{2n+1})$ of V^*, where e^i, as a functional, describes the hyperplane $e_i^{\perp_{\mathcal{Q}}}$ for $1 \leq i \leq 2n+1$. As before, let also $O(V)$ be the orthogonal group stabilizing $\mathcal{Q}$. We have the following theorem.

Theorem 3.7. *For any $\varphi \in \bigwedge^2 V^*$ exactly one of the following conditions holds:*

(1) $r = 1$; then $\mathrm{wt}(\varphi) \geq q^{4n-5} - q^{3n-4}$ with equality occurring if and only if φ is in the $O(V)$-orbit of $e^1 \wedge e^{2n+1}$;
(2) $r > 1$ and $A > 0$: in this case $\mathrm{wt}(\varphi) > q^{4n-5} - q^{3n-4}$;
(3) $r > 1$ and $A < 0$: in this case $r = n = 2$ and φ is in the $O(V)$-orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$ with $\mathrm{wt}(\varphi) = q^3 - q^2$.

Proof. If $r = 1$, then $\dim \mathrm{Rad}(\varphi) = 2n - 1$. As $\varphi \in \bigwedge^2 V^*$ has tensor rank 1 (i.e. is *fully decomposable*), φ determines a unique 2-dimensional subspace W_φ of V^*. In particular, the subspace W_φ is endowed with the quadratic form obtained from the restriction of β^* to W_φ. There are just 5 types of 2-dimensional quadratic spaces; they correspond respectively to the forms $f_1(x,y) = 0$, $f_2(x,y) = y^2$, $f_3(x,y) = \varepsilon y^2$, $f_4(x,y) = x^2 - \varepsilon y^2$ and $f_5(x,y) = xy$, where ε is a non-square in $\mathbb{F}_q$ and the coordinates are with respect to a given reference system of W_φ.

For each f_i, $1 \leq i \leq 5$, there are some $\varphi_i \in \bigwedge^2 V^*$ such that $\beta^*|_{W_{\varphi_i}} \cong f_i$. Examples of such φ_i inducing, respectively, f_i for $i = 1, \dots, 5$ are the following: $\varphi_1 = e^1 \wedge e^3$, $\varphi_2 = e^1 \wedge e^{2n+1}$, $\varphi_3 = e^1 \wedge (e^3 + \varepsilon e^4)$, $\varphi_4 = e^{2n+1} \wedge (e^1 - \varepsilon e^2)$ and $\varphi_5 = e^1 \wedge e^2$.

Using Witt's extension theorem we see that there always is an isometry between a given W_φ and any of these spaces W_{φ_i} $(1 \le i \le 5)$ which can be extended to an element of $O(V)$. In other words any form with $r = 1$ is equivalent to one of the aforementioned five elements of $\bigwedge^2 V^*$.

A direct computation shows that the list of possible weights is as follows:

$$\begin{aligned}&\mathrm{wt}(e^1 \wedge e^2) = \mathrm{wt}(e^{2n+1} \wedge (e^1 - \varepsilon e^2)) = q^{4n-5} - q^{2n-3},\\ &\mathrm{wt}(e^1 \wedge e^3) = q^{4n-5}, \quad \mathrm{wt}(e^1 \wedge e^{2n+1}) = q^{4n-5} - q^{3n-4},\\ &\mathrm{wt}(e^1 \wedge (e^3 + \varepsilon e^4)) = q^{4n-5} + q^{3n-4}.\end{aligned}$$

As an example we will explicitly compute $\mathrm{wt}(e^1 \wedge e^2)$. The remaining cases are analogous. Since $\varphi_5 = e^1 \wedge e^2$, we have, by (3),

$$\mathrm{wt}(\varphi_5) = \#\{(v_1, v_2)\colon v_1, v_2 \in \{e_1, e_2\}^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}, \beta(e_1 + v_1, e_2 + v_2) = 0\}.$$

In particular, as

$$\beta(e_1 + v_1, e_2 + v_2) = \beta(e_1, v_2) + \beta(v_1, e_2) + \beta(e_1, e_2) + \beta(v_1, v_2) = 1 + \beta(v_1, v_2)$$

we have $\beta(v_1, v_2) = -1$. Observe that $\mathcal{Q}' := \{e_1, e_2\}^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$ is a non-singular parabolic quadric $\mathcal{Q}(2n-2, q)$ of rank $n-1$; thus it contains $(q^{2n-2}-1)$ non-zero vectors and we can choose v_1 in $(q^{2n-2}-1)$ ways. For each projective point $p \in \mathcal{Q}'$ with $p \notin v_1^{\perp_{\mathcal{Q}}}$, there is exactly one vector v_2 such that $v_2 \in p$ and $\beta(v_1, v_2) = -1$. The number of such points is

$$\#\mathcal{Q}' - \#(v_1^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}') = \frac{q^{2n-2} - 1}{q - 1} - (\frac{q^{2n-4} - 1}{q - 1} q + 1) = q^{2n-3}.$$

In particular, the overall weight of $\mathrm{wt}(\varphi_5)$ is

$$\mathrm{wt}(\varphi_5) := q^{2n-3}(q^{2n-2} - 1) = q^{4n-5} - q^{2n-3}.$$

The case $e^1 \wedge e^{2n+1}$ will yield words of minimum weight.

Suppose now $r > 1$. Clearly,

$$\#\ker(T) \cap \mathcal{Q} \le \#\ker(T) - 1 = q^{2n-2r+1} - 1.$$

Furthermore, if $\lambda \in \mathrm{Spec}'(T)$ then also $-\lambda \in \mathrm{Spec}'(T)$ by Lemma 3.3 (4). Thus, we can write $\mathrm{Spec}'(T) = \{\lambda_1, \ldots, \lambda_\ell\} \cup \{-\lambda_1, \ldots, -\lambda_\ell\}$ with $\lambda_i \ne \pm\lambda_j$ if $i \ne j$. By Corollary 3.4, the space $X^+ := \oplus_{i=1}^{\ell} V_{\lambda_i}$ is totally singular; hence, $\dim X^+ \le n$ and

$$\sum_{i=1}^{\ell} \#(V_{\lambda_i} \setminus \{0\}) \le \#X^+ - 1 \le q^n - 1;$$

likewise, considering $X^- := \oplus_{i=1}^{\ell} V_{-\lambda_i}$, we get $\sum_{i=1}^{\ell} \#(V_{-\lambda_i} \setminus \{0\}) \le q^n - 1$. Thus,

$$A \ge q^{2n-2} - q^{2n-2r+1} - 2(q^n - 1). \tag{11}$$

If $A > 0$, then $\mathrm{wt}(\varphi) > q^{4n-5} - q^{3n-4}$. We now distinguish two cases.

Suppose that $\mathrm{Rad}(\varphi)$ contains a singular vector u; then, by statement (1) of Lemma 3.2, $X^+ \oplus \mathrm{Span}(u)$ would then be a totally singular subspace; thus, $\dim X^\pm \le n - 1$ and

$$A \ge q^{2n-2} - q^{2n-2r+1} - 2(q^{n-1} - 1) > q^{n-1}(q^{n-1} - q^{n-2} - 2) \ge 0;$$

therefore, $A > 0$. By Chevalley–Warning theorem, as $2(n-r)+1 \geq 3$, the set $\mathrm{Rad}(\varphi) \cap \mathcal{Q}$ always contains a non-zero singular vector.

Suppose now that $\mathrm{Rad}(\varphi)$ does not contain any non-zero singular vector; then $n = r$ and, consequently, $A \geq q^{2n-2} - 1 - 2(q^n - 1)$ (where we have replaced by 1 the term $q^{2(n-r)+1}$ of (11), which was an upper bound for the number of singular vectors in $\mathrm{Rad}(\varphi)$). This latter quantity is positive unless $n = 2$.

Therefore, $A \leq 0$ and $r > 1$ can occur only for $r = n = 2$.

If $A = 0$, then

$$\#\{u \colon u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\} \cup \{0\} = q^2.$$

This happens only if there exists an eigenvalue $\lambda \neq 0$ such that $V_\lambda \subseteq \mathcal{Q}$ and $\dim(V_\lambda) = 2$. By Lemma 3.3(4), also $-\lambda$ is an eigenvalue, so $V_{-\lambda} \subseteq \mathcal{Q}$. Then,

$$\#\{u \colon u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\} \cup \{0\} > q^2, \quad \text{a contradiction.}$$

Hence, $A < 0$ and $r > 1$. In this case $\mathrm{Rad}(\varphi)$ would be a one dimensional subspace of V not contained in $\mathcal{Q}$. We claim that actually φ is in the $O(V^*)$-orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$. As before, let $\mathrm{Spec}'(T) = \{\lambda_1, \dots, \lambda_\ell\} \cup \{-\lambda_1, \dots, -\lambda_\ell\}$. Since X^+ is totally singular, $\dim X \leq 2$, whence $\ell \leq 2$. If $\ell = 2$, then $\dim X^+ = \dim X^- = 2$. Thus, all four eigenspaces $V_{\pm\lambda_i}$ have dimension 1 and $\sum_{\lambda \in \mathrm{Spec}'(T)} \#(V_\lambda \setminus \{0\}) = 4(q-1)$. It follows $A \geq q^2 - 1 - 4(q-1) = (q-2)^2 - 1 \geq 0$ and we are done. Therefore, $\ell \leq 1$. If $\ell = 0$, then $A \geq q^2 - 1 > 0$. Likewise, if $\ell = 1$ and $\dim V_{\lambda_1} = \dim V_{-\lambda_1} = 1$, then $A \geq q^2 - 1 - 2(q-1) > 0$. There remain to consider only the case $\ell = 1$ and $\dim V_\lambda = \dim V_{-\lambda} = 2$. Observe first that if there were a vector $b_3 \in V_{-\lambda} \cap V_\lambda^{\perp \mathcal{Q}}$, then $V_\lambda \oplus \mathrm{Span}(b_3)$ would be totally singular — a contradiction, as the rank of $\mathcal{Q}$ is 2. Therefore we can choose a basis $(b_1, b_2, \dots, b_5)$ for V such that $V_\lambda = \mathrm{Span}(b_1, b_3)$, $V_{-\lambda} = \mathrm{Span}(b_2, b_4)$, $\beta(b_2, b_1) = 1$, $\beta(b_3, b_2) = 0$, $\mathrm{Rad}(\varphi) = \mathrm{Span}(b_5)$ and $\mathrm{Span}(b_1, b_2, b_5)^{\perp \mathcal{Q}} = \mathrm{Span}(b_3, b_4)$. Indeed, we may assume that b_3, b_4 are a hyperbolic pair. By construction $\beta(Tb_4, b_i) = -\beta(b_4, Tb_i) = 0$ for $i = 1, 2, 4, 5$. Hence T has matrix $\mathrm{diag}(\lambda, -\lambda, \lambda, -\lambda, 0)$ with respect to this basis, that is $\varphi = b^1 \wedge b^2 + b^3 \wedge b^4$. We now compute $\mathrm{wt}(\varphi)$ directly, under the assumption $n = 2$ and obtain

$$\mathrm{wt}(\varphi) = q^3 - q^2.$$

This completes the proof of the Main Theorem. □

Corollary 3.8. *If $n > 2$ the codewords of minimum weight all lie on the orbit of $e^1 \wedge e^{2n+1}$ under the action of the orthogonal group $O(V)$. For $n = 2$ the minimum weight codewords either lie in the orbit of $e^1 \wedge e^5$ or in the orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$.*

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