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WELL-POSEDNESS OF THE LINEARIZED PLASMA-VACUUM INTERFACE PROBLEM IN IDEAL INCOMPRESSIBLE MHD

By

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Abstract. We study the free boundary problem for the plasma-vacuum interface in ideal incompressible magnetohydrodynamics (MHD). In the vacuum region the magnetic field is described by the div-curl system of pre-Maxwell dynamics, while at the interface the total pressure is continuous and the magnetic field is tangent to the boundary. Under a suitable stability condition satisfied at each point of the plasma-vacuum interface, we prove the well-posedness of the linearized problem in Sobolev spaces.

1. Introduction. We consider the equations of ideal incompressible magnetohydrodynamics (MHD), i.e., the equations governing the motion of a perfectly conducting inviscid incompressible plasma. In the case of homogeneous plasma (the density $\rho(t,x)\equiv \text{const}>0$) these equations in a dimensionless form are

$$\partial_t v+(v,\nabla)v-(H,\nabla)H+\nabla q=0\,, \tag{1.1a}$$

$$\partial_t H+(v,\nabla)H-(H,\nabla)v=0\,, \tag{1.1b}$$

$$\operatorname{div} v=0\,, \tag{1.1c}$$

where $v=v(t,x)=(v_1,v_2,v_3)$ denotes the plasma velocity, $H=H(t,x)=(H_1,H_2,H_3)$ the magnetic field (in Alfvén velocity units), $q=p+|H|^2/2$ the total pressure, and

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$p = p(t, x)$ the pressure (divided by ρ). As the unknown we fix the vector $U = (q, W)$ with $W = (v, H)$. System (1.1) is supplemented by the *divergence constraint*

$$\operatorname{div} H = 0 \tag{1.2}$$

on the initial data $W_{|t=0} = W_0$.

The classical plasma-vacuum interface problem models confined plasmas in a closed vessel (see, e.g., [8]). In this model the plasma is confined inside a perfectly conducting rigid wall and isolated from it by a vacuum region. Until recent times there were no well-posedness results for full (*non-stationary*) plasma-vacuum models. The linearized plasma-vacuum problem in ideal *compressible* MHD was studied in [13, 16], and the well-posedness of the original nonlinear free boundary problem was recently proved in [14] by the Nash-Moser method. Our main goal is to obtain an analogous result for the plasma-vacuum interface problem for the model of incompressible MHD which can be used when the characteristic plasma velocity is very small compared to the speed of sound. In this paper we concentrate on the corresponding linearized problem. It is noteworthy that the assumption in [13, 14, 16] that the plasma density is strictly positive up to the free boundary of the plasma region is automatically satisfied in incompressible MHD. However, the non-hyperbolicity of system (1.1) produces additional difficulties compared to the analysis in [13, 14, 16].

Regarding the case without magnetic fields, the well-posedness of the free boundary problem for incompressible Euler equations with a free interface that separates the fluid region from the vacuum was proved in [10, 6, 18] (see also [7] for a comprehensive review) under the condition $(\partial p/\partial n)|_\Gamma < 0$, where n is the outward normal to the interface Γ. In [10, 6] the fluid domain was assumed to be bounded whereas in [18] the problem was set up in an unbounded domain. For our plasma-vacuum problem (see its statement just below) we consider the case of an unbounded plasma domain and, as in [10], neglect the influence of gravity because it just contributes with a lower-order term in (1.1a).

Let $\Omega^+(t)$ and $\Omega^-(t)$ be space-time domains occupied by the plasma and the vacuum respectively. That is, in the domain $\Omega^+(t)$ we consider system (1.1) governing the motion of an ideal plasma and in the domain $\Omega^-(t)$ we have the elliptic (div-curl) system

$$\nabla \times \mathcal{H} = 0, \qquad \operatorname{div} \mathcal{H} = 0, \tag{1.3}$$

describing the vacuum magnetic field $\mathcal{H} = \mathcal{H}(t, x) = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \in \mathbb{R}^3$. Here, as in [3, 8], we consider so-called *pre-Maxwell dynamics*. That is, as usual in nonrelativistic MHD, we neglect the displacement current $(1/c)\, \partial_t E$, where c is the speed of the light and E is the electric field.

The boundary of the domain $\Omega^+(t)$ is a hypersurface $\Gamma(t) = \{\eta(t, x) = 0\}$ that is the interface between plasma and vacuum. It is to be determined and moves with the velocity of plasma particles at the boundary:

$$\partial_t \eta + (v, \nabla \eta) = 0 \quad \text{on } \Gamma(t) \tag{1.4}$$

(for all $t \in [0, T]$). As η is an unknown of the problem, this is a free-boundary problem.

For technical simplicity we assume that the space-time domain $\Omega^+(t)$ (the plasma region) and $\Omega^-(t)$ (the vacuum region) are unbounded and the interface $\Gamma(t)$ has the form of a graph: $x_1 = \varphi(t, x')$, $x' = (x_2, x_3)$. That is,

$$\Omega^{\pm}(t) = \{x_1 \gtrless \varphi(t, x')\} \tag{1.5}$$

and the function $\varphi(t, x')$ is to be determined. With the choice $\eta(t, x) = x_1 - \varphi(t, x')$, (1.4) becomes

$$\partial_t \varphi = (v, N) \quad \text{on } \Gamma(t), \tag{1.6}$$

where $N = \nabla \eta = (1, -\partial_2 \varphi, -\partial_3 \varphi)$.

The plasma variable U is connected with the vacuum magnetic field $\mathcal{H}$ through the relations (cf. [3, 8])

$$[q] = 0, \quad (H, N) = 0 \quad (\mathcal{H}, N) = 0, \quad \text{on } \Gamma(t), \tag{1.7}$$

where $[q] = q|_{\Gamma} - \frac{1}{2}|\mathcal{H}|^2_{|\Gamma}$ denotes the jump of the total pressure across the interface. These relations together with (1.6) are the boundary conditions at the interface $\Gamma(t)$.

From the mathematical point of view, a natural wish is to find conditions on the initial data

$$W(0, x) = W_0(x), \quad x \in \Omega^+(0), \qquad \eta(0, x) = \eta_0(x), \quad x \in \Gamma(0), \tag{1.8}$$

$$\mathcal{H}(0, x) = \mathcal{H}_0(x), \quad x \in \Omega^-(0), \tag{1.9}$$

providing the local-in-time existence and uniqueness of a solution $(U, \mathcal{H}, \eta)$ of problem (1.1), (1.3)–(1.9) in Sobolev spaces.

Remark 1.1. In fact, for both the "elliptic" unknowns q and $\mathcal{H}$ we do not need to pose initial data. That is, the initial data (1.9) are not quite necessary because the vector $\mathcal{H}_0$ is uniquely defined through η_0 from zero-order compatibility conditions. Indeed, after straightening the interface $\Gamma(0)$ one can show that the elliptic problem composed by system (1.3) and the last boundary condition in (1.7) considered at $t = 0$ has a unique solution $\mathcal{H}_0$ in Sobolev spaces (see [14] for more details).

Remark 1.2. As for current-vortex sheets, see [12], [15], we must regard the second boundary condition in (1.7) as the restriction on the initial data (1.8). More precisely, after straightening of the interface and in exactly the same manner as in [12], [15], we can prove that a solution of (1.1)–(1.7), (1.8), (1.9) (if it exists for all $t \in [0, T]$) satisfies

$$\operatorname{div} H = 0 \quad \text{in } \Omega^+(t) \quad \text{and} \quad (H, N) = 0 \quad \text{on } \Gamma(t)$$

for all $t \in [0, T]$, if the latter was satisfied at $t = 0$, i.e., for the initial data (1.8).

In the next section we first reduce the free boundary problem (1.1)–(1.7), (1.8), (1.9) to that in a fixed domain by a suitable straightening of the unknown interface; then we linearize the resulting problem around a basic state ("unperturbed flow"). Under a suitable stability condition[1] satisfied at each point of the unperturbed interface, we prove the well-posedness of the linearized problem in the Sobolev space H^1.

The rest of the paper is organized as follows. In Section 2 we obtain the linearized problem. In Section 3 we introduce the functional setting. In Section 4 we state the main result. In Section 5 we introduce a suitable "hyperbolic" regularization of the linearized problem. In Section 6 we derive a priori estimates for the regularized problem.

[1]Strictly speaking, in this paper by stability we mean the *well-posedness* of the problem resulting from the linearization about a given (generally speaking, non-stationary) basic state. This basic state is not necessarily a solution of the nonlinear problem.

In Section 7 we prove the well-posedness of the hyperbolic regularized problem. In Section 8 we prove the well-posedness of the original linearized problem in conormal Sobolev spaces (see Section 3 for their definition). At last, in Section 9, using as in [12] a current-vorticity-type linearized system, we estimate missing normal derivatives of the perturbations of the velocity and the plasma magnetic field and prove the well-posedness of the linearized problem in Sobolev spaces (more precisely, in weighted Sobolev spaces, see Section 3), as stated in Section 4.

1.1. *Reduction to a fixed domain.* We straighten the interface Γ by using the same change of independent variables as in [13], that is inspired, in its turn, by Lannes [9] (see also [4]). As in [13], we set

$$\Omega^{\pm} := \mathbb{R}^3 \cap \{\pm x_1 > 0\}\,, \qquad \Gamma := \mathbb{R}^3 \cap \{x_1 = 0\}\,. \tag{1.10}$$

We want to reduce the free boundary problem (1.1)–(1.7), (1.8), (1.9) to the fixed domains $\Omega^{\pm}$, by constructing a global diffeomorphism of $\mathbb{R}^3$, mapping $\Omega^{\pm}(t)$ onto $\Omega^{\pm}$ and $\Gamma(t)$ onto Γ at each time $t \in [0,T]$.

The construction is based on the following lemma that shows how to lift functions from Γ to $\mathbb{R}^3$; the key point is the regularization of one half derivative of the lifting function Ψ with respect to the given function φ on Γ.

Lemma 1.3. Let $m \geq 3$ be a fixed integer. For all $\epsilon > 0$ there exists a continuous linear map $\varphi \in H^{m-0.5}(\mathbb{R}^2) \mapsto \Psi \in H^m(\mathbb{R}^3)$ such that $\Psi(0,x') = \varphi(x')$, $\partial_1\Psi(0,x') = 0$ on Γ, and

$$\|\partial_1\Psi\|_{L^\infty(\mathbb{R}^3)} \leq \epsilon \|\varphi\|_{H^2(\mathbb{R}^2)}\,. \tag{1.11}$$

The following lemma gives the time-dependent version of Lemma 1.3.

Lemma 1.4. Let $m \geq 3$ be a fixed integer and let $T > 0$. For all $\epsilon > 0$ there exists a continuous linear map $\varphi \in \cap_{j=0}^{m-1}\mathcal{C}^j([0,T];H^{m-j-0.5}(\mathbb{R}^2)) \mapsto \Psi \in \cap_{j=0}^{m-1}\mathcal{C}^j([0,T];H^{m-j}(\mathbb{R}^3))$ such that $\Psi(t,0,x') = \varphi(t,x')$, $\partial_1\Psi(t,0,x') = 0$ on Γ, and

$$\|\partial_1\Psi\|_{\mathcal{C}([0,T];L^\infty(\mathbb{R}^3))} \leq \epsilon \|\varphi\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))}\,. \tag{1.12}$$

Furthermore, there exists a constant $C > 0$, that is independent of T and only depends on m, such that

$$\begin{aligned} &\forall\, \varphi \in \cap_{j=0}^{m-1}\mathcal{C}^j([0,T];H^{m-j-0.5}(\mathbb{R}^2))\,, \quad \forall\, t \in [0,T]\,, \\ &\qquad \|\partial_t^j\Psi(t,\cdot)\|_{H^{m-j}(\mathbb{R}^3)} \leq C\|\partial_t^j\varphi(t,\cdot)\|_{H^{m-j-0.5}(\mathbb{R}^2)}, \qquad j = 0,\dots,m-1\,. \end{aligned} \tag{1.13}$$

For the proof of Lemmata 1.3 and 1.4 the reader is referred to [13]. The diffeomorphism that reduces the free boundary problem (1.1)–(1.7), (1.8), (1.9) to the fixed domains $\Omega^{\pm}$ is given by the following lemma.

Lemma 1.5. Let $m \geq 3$ be an integer. For all $T > 0$ and for all $\varphi \in \cap_{j=0}^{m-1}\mathcal{C}^j([0,T];$ $H^{m-j-0.5}(\mathbb{R}^2))$ satisfying without loss of generality $\|\varphi\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))} \leq 1$, there exists a function $\Psi \in \cap_{j=0}^{m-1}\mathcal{C}^j([0,T];H^{m-j}(\mathbb{R}^3))$ such that the function

$$\Phi(t,x) := (x_1 + \Psi(t,x), x') = (\Phi_1(t,x), x')\,, \qquad (t,x) \in [0,T]\times\mathbb{R}^3 \tag{1.14}$$

defines a H^m-diffeomorphism of $\mathbb{R}^3$ for all $t \in [0,T]$. Moreover, there holds $\partial_t^j(\Phi - Id) \in \mathcal{C}([0,T]; H^{m-j}(\mathbb{R}^3))$ for $j = 0, \dots, m-1$, $\Phi(t,0,x') = (\varphi(t,x'),x')$, $\partial_1\Phi(t,0,x') = (1,0,0)$.

Proof. The proof follows from Lemma 1.4, because

$$\partial_1\Phi_1(t,x) = 1 + \partial_1\Psi(t,x) \geq 1 - \|\partial_1\Psi\|_{\mathcal{C}([0,T];L^\infty(\mathbb{R}^3))} \geq 1 - \epsilon\|\varphi\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))} \geq 1/2\,,$$

provided ϵ is taken sufficiently small, e.g. $\epsilon < 1/2$. The other properties of Φ follow directly from Lemma 1.4. □

It is straightforward to check that, at each $t \in [0,T]$, the diffeomorphism $\Phi(t,x)$, given in Lemma 1.5, maps the time-dependent domain $\Omega^\pm(t)$ onto the reference domain $\Omega^\pm$ and the unknown interface $\Gamma(t)$ onto Γ.

We introduce the change of unknown functions induced by (1.14), by setting

$$\widetilde{U}(t,x) := U(t,\Phi(t,x)), \quad \widetilde{\mathcal{H}}(t,x) := \mathcal{H}(t,\Phi(t,x))\,. \tag{1.15}$$

The vector-functions $\widetilde{U} = (\tilde{q},\tilde{v},\tilde{H})$ and $\widetilde{\mathcal{H}}$ are smooth in the half-spaces Ω^+ and Ω^- respectively. Dropping the tildes for convenience, the problem (1.1)–(1.7), (1.8), (1.9) can be restated in the fixed reference domains $\Omega^\pm$ as follows.

Plasma part. System (1.1) is reduced to the following

$$\begin{aligned} &\partial_t v + \frac{1}{\partial_1\Phi_1}\{(w,\nabla)v - (h,\nabla)H\} + \nabla_\Phi q = 0\,,\\ &\partial_t H + \frac{1}{\partial_1\Phi_1}\{(w,\nabla)H - (h,\nabla)v\} = 0\,,\\ &\mathrm{div}\ u = 0 \quad \text{in } [0,T]\times\Omega^+, \end{aligned} \tag{1.16}$$

where

$$u = (v_n, v_2\partial_1\Phi_1, v_3\partial_1\Phi_1), \quad v_n = (v,n), \quad n = (1,-\partial_2\Phi_1,-\partial_3\Phi_1) = (1,-\partial_2\Psi,-\partial_3\Psi),$$

$$w = u - (\partial_t\Phi_1,0,0) = u - (\partial_t\Psi,0,0), \quad h = (H_n, H_2\partial_1\Phi_1, H_3\partial_1\Phi_1), \quad H_n = (H,n),$$

$$\nabla_\Phi q = \Big(\frac{\partial_1 q}{\partial_1\Phi_1}, -\frac{\partial_2\Psi}{\partial_1\Phi_1}\partial_1 q + \partial_2 q, -\frac{\partial_3\Psi}{\partial_1\Phi_1}\partial_1 q + \partial_3 q\Big).$$

Here and below, vectors will be written indifferently in rows or columns in order to simplify the presentation.

System (1.16) can be shortly rewritten in the following matrix form

$$\mathbb{P}(U,\Psi) := \begin{pmatrix} \mathbb{L}(U,\Psi) \\ \mathrm{div}\, u \end{pmatrix} = 0 \quad \text{in } [0,T]\times\Omega^+\,, \tag{1.17}$$

with

$$\mathbb{L}(U,\Psi) = L(W,\Psi)U = L_1(W,\Psi)W + \begin{pmatrix} \nabla_\Phi q \\ 0 \end{pmatrix},$$

where

$$L_1(W,\Psi) = \partial_t + \widetilde{A}_1(W,\Psi)\partial_1 + A_2(W)\partial_2 + A_3(W)\partial_3 \tag{1.18}$$

and

$$\widetilde{A}_1(W,\Psi) = \frac{1}{\partial_1\Phi_1}\Big(A_1(W) - \sum_{k=2}^3 A_k(W)\partial_k\Psi - I_6\partial_t\Psi\Big),$$

$$A_k(W) = \begin{pmatrix} v_k I_3 & -H_k I_3 \\ -H_k I_3 & v_k I_3 \end{pmatrix} = I_3 \otimes \begin{pmatrix} v_k & -H_k \\ -H_k & v_k \end{pmatrix}, \qquad \text{with } k = 1, 2, 3 .$$

Vacuum part. The elliptic system (1.3) becomes

$$\mathbb{V}(\mathcal{H}, \Psi) = 0 \quad \text{in } [0, T] \times \Omega^-,$$

where

$$\mathbb{V}(\mathcal{H}, \Psi) = \begin{pmatrix} \nabla \times \mathfrak{H} \\ \operatorname{div} \mathfrak{h} \end{pmatrix} \tag{1.19}$$

and

$$\mathfrak{H} = (\mathcal{H}_1 \partial_1 \Phi_1, \mathcal{H}_{\tau_2}, \mathcal{H}_{\tau_3}), \quad \mathfrak{h} = (\mathcal{H}_n, \mathcal{H}_2 \partial_1 \Phi_1, \mathcal{H}_3 \partial_1 \Phi_1),$$
$$\mathcal{H}_{\tau_k} = \mathcal{H}_1 \partial_k \Psi + \mathcal{H}_k, \quad k = 2, 3, \qquad \mathcal{H}_n = (\mathcal{H}, n) .$$

Boundary Conditions. Conditions (1.6) and the first and third equations in (1.7) become

$$\mathbb{B}(U, \mathcal{H}, \varphi) = 0 \quad \text{on } [0, T] \times \Gamma,$$

where

$$\mathbb{B}(U, \mathcal{H}, \varphi) = \begin{pmatrix} \partial_t \varphi - v_N \\ [q] \\ \mathcal{H}_N \end{pmatrix} \tag{1.20}$$

and

$$[q] = q_{|\Gamma} - \frac{1}{2} |\mathcal{H}|^2_{|\Gamma}, \quad v_N = (v, N) , \quad \mathcal{H}_N = (H, N) , \quad N = (1, -\partial_2 \varphi, -\partial_3 \varphi) .$$

Notice that $v_{n|\Gamma} = v_N$, $\mathcal{H}_{n|\Gamma} = \mathcal{H}_N$.

Final System. To sum up, after the change of unknown functions (1.15), the free boundary problem (1.1), (1.3)–(1.9) is reduced to the following initial-boundary value problem

$$\mathbb{P}(U, \Psi) = 0 , \qquad \text{in } [0, T] \times \Omega^+ , \tag{1.21a}$$
$$\mathbb{V}(\mathcal{H}, \Psi) = 0 , \qquad \text{in } [0, T] \times \Omega^- , \tag{1.21b}$$
$$\mathbb{B}(U, \mathcal{H}, \varphi) = 0 , \quad \text{on } [0, T] \times \Gamma, \tag{1.21c}$$
$$W|_{t=0} = W_0 , \quad \text{in } \Omega^+, \qquad \mathcal{H}|_{t=0} = \mathcal{H}_0 , \quad \text{in } \Omega^- , \qquad \varphi|_{t=0} = \varphi_0 \quad \text{in } \mathbb{R}^2, \tag{1.21d}$$

where $\mathbb{P}(U, \Psi), \mathbb{V}(\mathcal{H}, \Psi), \mathbb{B}(U, \mathcal{H}, \varphi)$ are the operators defined in (1.17), (1.19), (1.20) respectively. We also did not include in our problem the equation

$$\operatorname{div} h = 0 \quad \text{in } [0, T] \times \Omega^+ \tag{1.22}$$

and the boundary condition

$$H_N = 0 \quad \text{on } [0, T] \times \Gamma , \tag{1.23}$$

because they are just restrictions on the initial data (1.21d). More precisely, referring to [15], [12] for the proof, we have the following lemma.

LEMMA 1.6. Let the initial data (1.21d) satisfy (1.22) and (1.23). If $(U, \mathcal{H}, \varphi)$ is a solution of problem (1.21a)–(1.21d), then this solution satisfies (1.22) and (1.23) for all $t \in [0, T]$.

2. Linearized problem.

2.1. *Basic state.* For $T>0$, let us set

$$Q_T^{\pm} := (-\infty, T] \times \Omega^{\pm}\,, \qquad \omega_T := (-\infty, T] \times \Gamma\,. \tag{2.1}$$

Let

$$(\widehat{U}(t,x), \widehat{\mathcal{H}}(t,x), \hat{\varphi}(t,x')) \tag{2.2}$$

be a given sufficiently smooth vector-function, respectively defined on Q_T^+, Q_T^-, ω_T, with $\widehat{U} = (\hat{q}, \hat{v}, \widehat{H})$, such that

$$\begin{aligned} &\|\widehat{U}\|_{W^{2,\infty}(Q_T^+)} + \|\partial_1 \widehat{U}\|_{W^{2,\infty}(Q_T^+)} + \|\widehat{\mathcal{H}}\|_{W^{2,\infty}(Q_T^-)} + \|\hat{\varphi}\|_{W^{3,\infty}([0,T]\times\mathbb{R}^2)} \le K, \\ &\|\widehat{\varphi}\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))} \le 1\,, \end{aligned} \tag{2.3}$$

where $K>0$ is a constant. Corresponding to $\widehat{\varphi}$, let the function $\widehat{\Psi}$ and the diffeomorphism $\widehat{\Phi}$ be constructed as in Lemmata 1.4 and 1.5 such that

$$\partial_1 \widehat{\Phi}_1 \ge 1/2\,.$$

We assume that the basic state (2.2) satisfies

$$\partial_t \widehat{H} + \frac{1}{\partial_1 \widehat{\Phi}_1} \Big\{ (\widehat{w}, \nabla)\widehat{H} - (\hat{h}, \nabla)\hat{v} \Big\} = 0, \quad \operatorname{div} \hat{u} = 0 \qquad \text{in } Q_T^+, \tag{2.4a}$$

$$\operatorname{div} \hat{\mathfrak{h}} = 0 \qquad \text{in } Q_T^-, \tag{2.4b}$$

$$\partial_t \hat{\varphi} - \hat{v}_{\hat{N}} = 0, \quad [\hat{q}] = 0, \quad \widehat{\mathcal{H}}_{\hat{N}} = 0 \qquad \text{on } \omega_T, \tag{2.4c}$$

where all the "hat" values are determined like corresponding values for $(U, \mathcal{H}, \varphi)$, i.e.

$$\widehat{\mathfrak{H}} = (\widehat{\mathcal{H}}_1 \partial_1 \widehat{\Phi}_1, \widehat{\mathcal{H}}_{\hat{\tau}_2}, \widehat{\mathcal{H}}_{\hat{\tau}_3}), \qquad \widehat{\mathcal{H}}_{\hat{\tau}_k} = \widehat{\mathcal{H}}_1 \partial_k \widehat{\Psi} + \widehat{\mathcal{H}}_k\,, \;\; k = 2,3\,,$$

$$\hat{\mathfrak{h}} = (\widehat{\mathcal{H}}_{\hat{n}}, \widehat{\mathcal{H}}_2 \partial_1 \widehat{\Phi}_1, \widehat{\mathcal{H}}_3 \partial_1 \widehat{\Phi}_1)\,, \qquad \widehat{\mathcal{H}}_{\hat{n}} = (\widehat{\mathcal{H}}, \hat{n})\,,$$

$$\hat{h} = (\widehat{H}_{\hat{n}}, \widehat{H}_2 \partial_1 \widehat{\Phi}_1, \widehat{H}_3 \partial_1 \widehat{\Phi}_1)\,, \qquad \widehat{H}_{\hat{n}} = (\widehat{H}, \hat{n})\,,$$

$$\hat{v}_{\hat{N}} = (\hat{v}, \hat{N}), \quad \widehat{\mathcal{H}}_{\hat{N}} = (\widehat{\mathcal{H}}, \hat{N})\,, \quad \hat{N} = (1, -\partial_2 \hat{\varphi}, -\partial_3 \hat{\varphi})\,, \quad \hat{n} = (1, -\partial_2 \widehat{\Psi}, -\partial_3 \widehat{\Psi})$$

and where

$$\hat{u} = (\hat{v}_{\hat{n}}, \hat{v}_2 \partial_1 \widehat{\Phi}_1, \hat{v}_3 \partial_1 \widehat{\Phi}_1), \quad \hat{v}_{\hat{n}} = (\hat{v}, \hat{n})\,, \quad \widehat{w} = \hat{u} - (\partial_t \widehat{\Psi}, 0, 0)\,.$$

Note that (2.3) yields

$$\|\nabla_{t,x} \widehat{\Psi}\|_{W^{2,\infty}([0,T]\times\mathbb{R}^3)} \le C,$$

where $\nabla_{t,x} = (\partial_t, \nabla)$ and $C = C(K) > 0$ is a constant depending on K.

It follows from (2.4a) that the constraints

$$\operatorname{div} \hat{h} = 0 \;\text{ in } Q_T^+\,, \qquad \widehat{H}_{\hat{N}} = 0 \;\text{ on } \omega_T \tag{2.5}$$

are satisfied for the basic state (2.2), if they hold at $t=0$ (see [15], [12] for the proof). Thus, for the basic state we also require the fulfillment of conditions (2.5) at $t=0$.

2.2. *Linearized problem.* The linearized equations for (1.21a)-(1.21c) read:

$$\mathbb{P}'(\widehat{U},\widehat{\Psi})(\delta U,\delta\Psi):=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{P}(U_\varepsilon,\Psi_\varepsilon)|_{\varepsilon=0}=f\quad\text{in } Q_T^+,$$

$$\mathbb{V}'(\widehat{\mathcal{H}},\widehat{\Psi})(\delta\mathcal{H},\delta\Psi):=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{V}(\mathcal{H}_\varepsilon,\Psi_\varepsilon)|_{\varepsilon=0}=\mathcal{F}\quad\text{in } Q_T^-,$$

$$\mathbb{B}'(\widehat{U},\widehat{\mathcal{H}},\hat{\varphi})(\delta U,\delta\mathcal{H},\delta\varphi):=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{B}(U_\varepsilon,\mathcal{H}_\varepsilon,\varphi_\varepsilon)|_{\varepsilon=0}=g\quad\text{on } \omega_T,$$

where $U_\varepsilon=\widehat{U}+\varepsilon\,\delta U$, $\mathcal{H}_\varepsilon=\widehat{\mathcal{H}}+\varepsilon\,\delta\mathcal{H}$, $\varphi_\varepsilon=\hat{\varphi}+\varepsilon\,\delta\varphi$; $\delta\Psi$ is constructed from $\delta\varphi$ as in Lemma 1.4 and $\Psi_\varepsilon=\hat{\Psi}+\varepsilon\delta\Psi$. Here we introduce the source terms $f=(f_1,\dots,f_7)$, $\mathcal{F}=(\chi,\Xi)$, $\chi=(\chi_1,\chi_2,\chi_3)$ and $g=(g_1,g_2,g_3)$ to make the interior equations and the boundary conditions inhomogeneous.

We compute the exact form of the linearized equations (below we drop δ):

$$\mathbb{P}'(\widehat{U},\widehat{\Psi})(U,\Psi)=\begin{pmatrix}L(\widehat{W},\widehat{\Psi})U\\ \operatorname{div}u\end{pmatrix}-\begin{pmatrix}\big\{L(\widehat{W},\widehat{\Psi})\Psi\big\}\dfrac{\partial_1\widehat{U}}{\partial_1\widehat{\Phi}_1}\\ \left(\nabla\times\begin{pmatrix}0\\ \hat{v}_3\\ -\hat{v}_2\end{pmatrix},\nabla\Psi\right)\end{pmatrix}=f,\tag{2.6}$$

$$\mathbb{V}'(\widehat{\mathcal{H}},\widehat{\Psi})(\mathcal{H},\Psi)=\mathbb{V}(\mathcal{H},\widehat{\Psi})+\begin{pmatrix}\nabla\widehat{\mathcal{H}}_1\times\nabla\Psi\\ \left(\nabla\times\begin{pmatrix}0\\ -\widehat{\mathcal{H}}_3\\ \widehat{\mathcal{H}}_2\end{pmatrix},\nabla\Psi\right)\end{pmatrix}=\mathcal{F},\tag{2.7}$$

$$\mathbb{B}'(\widehat{U},\widehat{\mathcal{H}},\hat{\varphi})(U,\mathcal{H},\varphi)=\begin{pmatrix}\partial_t\varphi+\hat{v}_2\partial_2\varphi+\hat{v}_3\partial_3\varphi-v_{\hat{N}}\\ q-(\widehat{\mathcal{H}},\mathcal{H})\\ \mathcal{H}_{\hat{N}}-\widehat{\mathcal{H}}_2\partial_2\varphi-\widehat{\mathcal{H}}_3\partial_3\varphi\end{pmatrix}=g\,,\tag{2.8}$$

where

$$L(\widehat{W},\widehat{\Psi})U=L_1(\widehat{W},\widehat{\Psi})W+\begin{pmatrix}\nabla_{\widehat{\Phi}}q\\ 0\end{pmatrix}+C(\widehat{W},\widehat{\Psi})W,$$

$$u=(v_{\hat{n}},v_2\partial_1\widehat{\Phi}_1,v_3\partial_1\widehat{\Phi}_1)\,,\quad v_{\hat{n}}=(v,\hat{n}),$$

$$\big\{L(\widehat{U},\widehat{\Psi})\Psi\big\}\frac{\partial_1\widehat{U}}{\partial_1\widehat{\Phi}_1}=L_1(\widehat{W},\widehat{\Psi})\Psi\,\frac{\partial_1\widehat{W}}{\partial_1\widehat{\Phi}_1}+\begin{pmatrix}\nabla_{\widehat{\Phi}}\Psi\\ 0\end{pmatrix}\frac{\partial_1\widehat{q}}{\partial_1\widehat{\Phi}_1},$$

$L_1(\widehat{W},\widehat{\Psi})$ being the differential operator defined in (1.18) (with $(W,\Psi)=(\widehat{W},\widehat{\Psi})$), and the matrix $C(\widehat{W},\widehat{\Psi})$ is determined as follows:

$$C(\widehat{W},\widehat{\Psi})W=\begin{pmatrix}C_1(\widehat{W},\widehat{\Psi})W\\ C_2(\widehat{W},\widehat{\Psi})W\end{pmatrix}=\frac{1}{\partial_1\widehat{\Phi}_1}\begin{pmatrix}(u,\nabla)\widehat{v}-(h,\nabla)\widehat{H}\\ (u,\nabla)\widehat{H}-(h,\nabla)\widehat{v}\end{pmatrix}.\tag{2.9}$$

In order to cancel out the first-order operators in Ψ from the operators $\mathbb{P}'(\widehat{U},\widehat{\Psi})$ and $\mathbb{V}'(\widehat{\mathcal{H}},\widehat{\Psi})$, as in [1], the linearized problem is rewritten in terms of the "good unknown"

$$\dot{U}:=U-\frac{\Psi}{\partial_1\widehat{\Phi}_1}\,\partial_1\widehat{U},\quad \dot{\mathcal{H}}:=\mathcal{H}-\frac{\Psi}{\partial_1\widehat{\Phi}_1}\,\partial_1\widehat{\mathcal{H}}.\tag{2.10}$$

Taking into account assumptions (2.4c) and (2.4b) and omitting detailed calculations, we rewrite our linearized equations (2.6)-(2.8) in terms of the new unknowns (2.10):

$$\begin{pmatrix} L(\widehat{W},\widehat{\Psi})\dot{U} + \dfrac{\Psi}{\partial_1\widehat{\Phi}_1}\,\partial_1\big\{\mathbb{L}(\widehat{U},\widehat{\Psi})\big\} \\ \operatorname{div}\dot{u} \end{pmatrix} = f\,, \quad \text{in } Q_T^+\,, \tag{2.11}$$

$$\mathbb{V}(\dot{\mathcal{H}},\widehat{\Psi}) + \frac{\Psi}{\partial_1\widehat{\Phi}_1}\,\partial_1\big\{\mathbb{V}(\widehat{\mathcal{H}},\widehat{\Psi})\big\} = \mathcal{F}\,, \quad \text{in } Q_T^-\,, \tag{2.12}$$

$$\mathbb{B}'(\widehat{U},\widehat{\mathcal{H}},\hat{\varphi})(\dot{U},\dot{\mathcal{H}},\varphi) := \begin{pmatrix} \partial_t\varphi + \hat{v}_2\partial_2\varphi + \hat{v}_3\partial_3\varphi - \dot{v}_{\hat{N}} - \varphi\,\partial_1\hat{v}_{\hat{N}} \\ \dot{q} - (\widehat{\mathcal{H}},\dot{\mathcal{H}}) + [\partial_1\hat{q}]\varphi \\ \dot{\mathcal{H}}_{\hat{N}} - \partial_2\big(\widehat{\mathcal{H}}_2\varphi\big) - \partial_3\big(\widehat{\mathcal{H}}_3\varphi\big) \end{pmatrix} = g, \quad \text{on } \omega_T\,, \tag{2.13}$$

where

$$\dot{u} = (\dot{v}_{\hat{n}}, \dot{v}_2\partial_1\widehat{\Phi}_1, \dot{v}_3\partial_1\widehat{\Phi}_1), \quad \dot{v}_{\hat{n}} = (\dot{v},\hat{n}), \quad \dot{v}_{\hat{N}} = (\dot{v},\hat{N})\,,$$

$$\dot{\mathcal{H}}_{\hat{N}} = (\dot{\mathcal{H}},\hat{N}), \quad [\partial_1\hat{q}] = \partial_1\hat{q}|_\Gamma - (\widehat{\mathcal{H}},\partial_1\widehat{\mathcal{H}})|_\Gamma.$$

While writing down the last boundary condition in (2.13) we used (2.4b) taken at $x_1 = 0$.

As in [1, 5, 15], we drop the zeroth-order term in Ψ in (2.11), (2.12) and consider the effective linear operators

$$\mathbb{P}'_e(\widehat{U},\widehat{\Psi})\dot{U} := \begin{pmatrix} \mathbb{L}'_e(\widehat{U},\widehat{\Psi})\dot{U} \\ \operatorname{div}\dot{u} \end{pmatrix} = f\,,$$

$$\mathbb{V}(\dot{\mathcal{H}},\widehat{\Psi}) = \begin{pmatrix} \nabla\times\dot{\mathfrak{H}} \\ \operatorname{div}\dot{\mathfrak{h}} \end{pmatrix} = \mathcal{F}\,,$$

where

$$\mathbb{L}'_e(\widehat{U},\widehat{\Psi})\dot{U} = L(\widehat{W},\widehat{\Psi})\dot{U} = L_1(\widehat{W},\widehat{\Psi})\dot{W} + \begin{pmatrix} \nabla_{\widehat{\Phi}}\dot{q} \\ 0 \end{pmatrix} + C(\widehat{W},\widehat{\Psi})\dot{W} \tag{2.14}$$

and

$$\dot{\mathfrak{H}} = (\dot{\mathcal{H}}_1\partial_1\widehat{\Phi}_1, \dot{\mathcal{H}}_{\hat{\tau}_2}, \dot{\mathcal{H}}_{\hat{\tau}_3}), \quad \dot{\mathfrak{h}} = (\dot{\mathcal{H}}_N, \dot{\mathcal{H}}_2\partial_1\widehat{\Phi}_1, \dot{\mathcal{H}}_3\partial_1\widehat{\Phi}_1),$$

$$\dot{\mathcal{H}}_N = \dot{\mathcal{H}}_1 - \dot{\mathcal{H}}_2\partial_2\widehat{\Psi} - \dot{\mathcal{H}}_3\partial_3\widehat{\Psi}, \quad \dot{\mathcal{H}}_{\hat{\tau}_i} = \dot{\mathcal{H}}_1\partial_i\widehat{\Psi} + \dot{\mathcal{H}}_i, \quad i = 2,3.$$

In the future nonlinear analysis by Nash-Moser iterations the dropped term in (2.11), (2.12) should be considered as an error term.

To sum up, under the preceding reductions the linearized problem reads as follows:

$$\mathbb{L}'_e(\widehat{U},\widehat{\Psi})\dot{U} = \begin{pmatrix} f_v \\ f_H \end{pmatrix}, \tag{2.15a}$$

$$\operatorname{div}\dot{u} = f_7\,, \qquad \text{in } Q_T^+\,, \tag{2.15b}$$

$$\nabla\times\mathfrak{H} = \chi\,, \quad \operatorname{div}\mathfrak{h} = \Xi\,, \qquad \text{in } Q_T^-\,, \tag{2.15c}$$

$$\partial_t\varphi = \dot{v}_{\hat{N}} - \hat{v}_2\partial_2\varphi - \hat{v}_3\partial_3\varphi + \varphi\partial_1\hat{v}_{\hat{N}} + g_1\,, \tag{2.15d}$$

$$\dot{q} = (\widehat{\mathcal{H}},\dot{\mathcal{H}}) - [\partial_1\hat{q}]\varphi + g_2\,, \tag{2.15e}$$

$$\dot{\mathcal{H}}_{\hat{N}} = \partial_2(\widehat{\mathcal{H}}_2\varphi) + \partial_3(\widehat{\mathcal{H}}_3\varphi) + g_3\,, \qquad \text{on } \omega_T\,, \tag{2.15f}$$

$$(\dot{W},\dot{\mathcal{H}},\varphi) = 0\,, \qquad \text{for } t<0\,, \tag{2.15g}$$

where we have used the notations $f = (f_v, f_H, f_7)$, $f_v = (f_1, f_2, f_3)$, $f_H = (f_4, f_5, f_6)$, $\mathcal{F} = (\chi, \Xi)$ and $g = (g_1, g_2, g_3)$ for the source terms introduced in (2.6)–(2.8).

The source term χ of the first equation in (2.15c) should satisfy the constraint

$$\operatorname{div}\chi = 0\,. \tag{2.16}$$

Moreover, for the resolution of the elliptic problem (2.15c), (2.15f), the data Ξ and g_3 must satisfy the necessary compatibility condition

$$\int_{\Omega^-}\Xi\,dx = \int_{\Gamma} g_3\,dx'\,, \tag{2.17}$$

see [13]. We assume that the source terms (f,χ,Ξ) and the boundary data g vanish in the past and consider the case of zero initial data. The case of nonzero initial data is postponed to the nonlinear analysis.

2.3. *Reduction to homogeneous data.* We can reduce problem (2.15) to that with homogeneous data $f_H = 0$, $f_7 = 0$, $\mathcal{F} = 0$ and $g = 0$ (except $f_v \neq 0$) by the following steps.

2.3.1. *Plasma part ($f_7 = 0$ and $g_1 = 0$).* We decompose the velocity $\dot{v}$ as $\dot{v} = \dot{v}' + \tilde{v}$ and the front φ as $\varphi = \varphi' + \tilde{\varphi}$, where $\tilde{v}$ and $\tilde{\varphi}$ are such that

$$\operatorname{div}\tilde{u} = f_7 \qquad \text{and} \qquad \partial_t\tilde{\varphi} = \tilde{v}_{\hat{N}} - \hat{v}_2\partial_2\tilde{\varphi} - \hat{v}_3\partial_3\tilde{\varphi} + \tilde{\varphi}\partial_1\hat{v}_{\hat{N}} + g_1$$

(i.e. $\tilde{v}$ satisfies (2.15b) and $\tilde{\varphi}$ satisfies (2.15d) with $\tilde{v}_{\hat{N}}$ instead of $\dot{v}_{\hat{N}}$). Then $\dot{v}'$ solves the homogeneous equation

$$\operatorname{div}\dot{u}' = 0 \quad \text{in } Q_T^+,$$

with $\dot{u}' = (\dot{v}'_{\hat{n}}, \dot{v}'_2\partial_1\widehat{\Phi}_1, \dot{v}'_3\partial_1\widehat{\Phi}_1)$ and $\dot{v}'_{\hat{n}} := \dot{v}'_1 - \dot{v}'_2\partial_2\widehat{\Psi} - \dot{v}'_3\partial_3\widehat{\Psi}$, and φ' is such that

$$\partial_t\varphi' = v'_{\hat{N}} - \hat{v}_2\partial_2\varphi' - \hat{v}_3\partial_3\varphi' + \varphi'\partial_1\hat{v}_{\hat{N}}.$$

Hence, $(\dot{q}, \dot{v}', \dot{H}, \varphi')$ satisfies system (2.15) with $f_7 = 0$ and $g_1 = 0$ and new data $f_v = f'_v$, $f_H = f'_H$, $g_2 = g'_2$ and $g_3 = g'_3$.

2.3.2. *Vacuum part ($\chi = 0$ and $\Xi = g_3' = 0$).* As in [13], we decompose the vacuum magnetic field $\dot{\mathcal{H}}$ as $\dot{\mathcal{H}} = \dot{\mathcal{H}}' + \widetilde{\mathcal{H}}$ (and accordingly $\dot{\mathfrak{H}} = \dot{\mathfrak{H}}' + \widetilde{\mathfrak{H}}$ and $\dot{\mathfrak{h}} = \dot{\mathfrak{h}}' + \widetilde{\mathfrak{h}}$), where $\widetilde{\mathcal{H}}$ is a solution, for each t, of the following elliptic problem

$$
\begin{aligned}
&\nabla \times \widetilde{\mathfrak{H}} = \chi\,, \quad \operatorname{div} \widetilde{\mathfrak{h}} = \Xi \quad \text{in } \Omega^-\,,\\
&\widetilde{\mathfrak{h}}_1 = \widetilde{\mathcal{H}}_{\hat{N}} = g_3' \quad \text{on } \Gamma\,.
\end{aligned}
\tag{2.18}
$$

Provided the data (χ, Ξ, g_3') vanish at infinity in an appropriate way and satisfy (2.16), (2.17) (with $g_3 = g_3'$), the classical results of the elliptic theory ensure the existence of a unique solution of (2.18) vanishing at infinity.

Once $\widetilde{\mathcal{H}}$ is given, we look for $\dot{\mathcal{H}}'$ as a solution to the problem

$$
\begin{aligned}
&\nabla \times \dot{\mathfrak{H}}' = 0\,, \quad \operatorname{div} \dot{\mathfrak{h}}' = 0\,, \quad \text{in } Q_T^-\,,\\
&\dot{q} = (\widehat{\mathcal{H}}, \dot{\mathcal{H}}') - [\partial_1 \widehat{q}]\varphi' + g_2''\,,\\
&\dot{\mathcal{H}}'_{\hat{N}} = \partial_2(\widehat{\mathcal{H}}_2 \varphi') + \partial_3(\widehat{\mathcal{H}}_3 \varphi') \quad \text{on } \omega_T\,,
\end{aligned}
\tag{2.19}
$$

where

$$
g_2'' = g_2' + (\widehat{\mathcal{H}}, \widetilde{\mathcal{H}})\,. \tag{2.20}
$$

If $\widetilde{\mathcal{H}}$ and $\dot{\mathcal{H}}'$ solve (2.18) and (2.19) respectively, then it is clear that $\dot{\mathcal{H}} = \dot{\mathcal{H}}' + \widetilde{\mathcal{H}}$ solves (2.15c), (2.15e) and (2.15f) with $\varphi = \varphi'$, $g_2 = g_2'$ and $g_3 = g_3'$.

Collecting the changes of unknowns performed above and dropping for convenience the primes in $\dot{v}'$, $\dot{\mathcal{H}}'$, φ', g_2'', f_v' and f_H', we obtain the linearized problem (2.15) with $f_7 = 0$, $\mathcal{F} = 0$ and $g_1 = g_3 = 0$:

$$
\begin{aligned}
&\mathbb{L}_e'(\widehat{U}, \widehat{\Psi})\dot{U} = \begin{pmatrix} f_v \\ f_H \end{pmatrix},\\
&\operatorname{div} \dot{u} = 0\,, \qquad \text{in } Q_T^+\,,\\
&\mathbb{V}(\dot{\mathcal{H}}, \widehat{\Psi}) = 0\,, \qquad \text{in } Q_T^-\,,\\
&\mathbb{B}'(\widehat{U}, \widehat{\mathcal{H}}, \hat{\varphi})(\dot{U}, \dot{\mathcal{H}}, \varphi) := \begin{pmatrix} 0 \\ g_2 \\ 0 \end{pmatrix}, \quad \text{on } \omega_T\,.
\end{aligned}
\tag{2.21}
$$

2.3.3. *Plasma-vacuum interface ($f_H = 0$ and $g_2 = 0$).* From system (2.21) we can deduce nonhomogeneous equations which are a linearized counterpart of the divergence constraint (1.22) and the "redundant" boundary condition (1.23). More precisely, with reference to [15, Proposition 2] and [12] for the proof, we have the following.

Lemma 2.1 ([15]). Let the basic state (2.2) satisfies assumptions (2.3)–(2.5). Then solutions of problem (2.21) satisfy

$$
\operatorname{div} \dot{h} = r \quad \text{in } Q_T^+, \tag{2.22}
$$

$$
\widehat{H}_2 \partial_2 \varphi + \widehat{H}_3 \partial_3 \varphi - \dot{H}_{\hat{N}} - \varphi\, \partial_1 \widehat{H}_{\hat{N}} = G \quad \text{on } \omega_T. \tag{2.23}
$$

Here

$$
\dot{h} = (\dot{H}_{\hat{n}}, \dot{H}_2 \partial_1 \widehat{\Phi}_1, \dot{H}_3 \partial_1 \widehat{\Phi}_1), \quad \dot{H}_{\hat{n}} = \dot{H}_1 - \dot{H}_2 \partial_2 \widehat{\Psi} - \dot{H}_3 \partial_3 \widehat{\Psi} \qquad (\dot{H}_{\hat{N}}|_{x_1=0} = \dot{H}_{\hat{n}}|_{x_1=0}).
$$

The functions $r = r(t,x)$ and $G = G(t,x')$, which vanish in the past, are determined by the source terms and the basic state as solutions to the linear inhomogeneous transport equations

$$\partial_t r + \hat{v}_2 \partial_2 r + \hat{v}_3 \partial_3 r + (\partial_2 \hat{v}_2 + \partial_3 \hat{v}_3)\, r = \mathrm{div} \mathfrak{f}_H \quad \text{in } Q_T^+, \tag{2.24}$$

where $\mathfrak{f}_H := (f_{H,\hat{n}}, \partial_1 \widehat{\Phi}_1 f_5, \partial_1 \widehat{\Phi}_1 f_6)$, $f_{H,\hat{n}} := (f_H, \hat{n}) = f_4 - \partial_2 \widehat{\Psi} f_5 - \partial_3 \widehat{\Psi} f_6$ and

$$\partial_t G + \hat{v}_2 \partial_2 G + \hat{v}_3 \partial_3 G + (\partial_2 \hat{v}_2 + \partial_3 \hat{v}_3)\, G = f_{H,\hat{n}\,|\,x_1=0} \quad \text{on } \omega_T\,. \tag{2.25}$$

Equations (2.24), (2.25) do not need boundary conditions at $\{x_1 = 0\}$.

Following [15], we now perform a further change of unknowns to make f_H and g_2 equal to zero (in view of Lemma 2.1, r and G in (2.22), (2.23) will become zero as well). Let $\chi \in C_0^\infty(\mathbb{R}_+)$ be a cut-off function equal to 1 on $[0,1]$. We define

$$\widetilde{q} = \chi(x_1) g_2 \tag{2.26}$$

and $\widetilde{H}$ solves the equation for $\dot{H}$ contained in (2.21) with $\dot{v} = 0$, namely

$$\partial_t \widetilde{H} + \frac{1}{\partial_1 \widehat{\Phi}_1} (\hat{w}, \nabla) \widetilde{H} + C_2(\widehat{W}, \widehat{\Psi}) \begin{pmatrix} 0 \\ \widetilde{H} \end{pmatrix} = f_H \qquad \text{in } Q_T^+\,. \tag{2.27}$$

We define the new unknowns

$$U^\natural := \begin{pmatrix} q^\natural \\ v^\natural \\ H^\natural \end{pmatrix} = \begin{pmatrix} \dot{q} - \widetilde{q} \\ \dot{v} \\ \dot{H} - \widetilde{H} \end{pmatrix}, \quad \mathcal{H}^\natural := \dot{\mathcal{H}}\,. \tag{2.28}$$

One can check that $(U^\natural, \mathcal{H}^\natural)$ satisfies problem (2.21) with $f_H = 0$ and $g_2 = 0$ (and a new f_v). Dropping for convenience the indices $^\natural$ in (2.28), the final form of our reduced linearized problem reads

$$\begin{aligned}
&\mathbb{L}_e'(\widehat{U}, \widehat{\Psi}) U = \begin{pmatrix} f_v \\ 0 \end{pmatrix}, \\
&\mathrm{div}\, u = 0\,, \qquad\qquad \text{in } Q_T^+\,, \\
&\mathbb{V}(\mathcal{H}, \widehat{\Psi}) = 0\,, \qquad \text{in } Q_T^-\,, \\
&\mathbb{B}'(\widehat{U}, \widehat{\mathcal{H}}, \hat{\varphi})(U, \mathcal{H}, \varphi) = 0\,, \quad \text{on } \omega_T\,.
\end{aligned} \tag{2.29}$$

Recall that the operators $\mathbb{L}'_e(\widehat{U},\widehat{\Psi})$, $\mathbb{V}(\mathcal{H},\widehat{\Psi})$ and $\mathbb{B}'(\widehat{U},\widehat{\mathcal{H}},\widehat{\varphi})$ are defined in (2.14), (1.19) and (2.13) respectively. We also write down problem (2.29) in the component-wise form

$$\partial_t v + \frac{1}{\partial_1\widehat{\Phi}_1}\Big\{(\hat{w},\nabla)v-(\hat{h},\nabla)H\Big\}+\nabla_{\widehat{\Phi}}q+C_1(\widehat{W},\widehat{\Psi})W=f_v\,, \tag{2.30a}$$

$$\partial_t H + \frac{1}{\partial_1\widehat{\Phi}_1}\Big\{(\hat{w},\nabla)H-(\hat{h},\nabla)v\Big\}+C_2(\widehat{W},\widehat{\Psi})W=0\,, \tag{2.30b}$$

$$\operatorname{div} u = 0 \qquad \text{in } Q_T^+\,, \tag{2.30c}$$

$$\nabla\times\mathfrak{H}=0\,,\quad \operatorname{div}\mathfrak{h}=0 \qquad \text{in } Q_T^-\,, \tag{2.30d}$$

$$\partial_t\varphi = v_{\hat{N}}-\hat{v}_2\partial_2\varphi-\hat{v}_3\partial_3\varphi+\varphi\partial_1\hat{v}_{\hat{N}}\,, \tag{2.30e}$$

$$q=(\widehat{\mathcal{H}},\mathcal{H})-[\partial_1\hat{q}]\varphi\,, \tag{2.30f}$$

$$\mathcal{H}_{\hat{N}}=\partial_2(\widehat{\mathcal{H}}_2\varphi)+\partial_3(\widehat{\mathcal{H}}_3\varphi) \qquad \text{on } \omega_T\,, \tag{2.30g}$$

$$(W,\mathcal{H},\varphi)=0 \qquad \text{for } t<0\,. \tag{2.30h}$$

Clearly, for problem (2.30) we get (2.22) and (2.23) with $r=0$ and $G=0$. That is, solutions to problem (2.30) satisfy

$$\operatorname{div} h = 0 \quad \text{in } Q_T^+, \tag{2.31}$$

$$H_{\hat{N}}=\widehat{H}_2\partial_2\varphi+\widehat{H}_3\partial_3\varphi-\varphi\,\partial_1\widehat{H}_{\hat{N}} \quad \text{on } \omega_T. \tag{2.32}$$

3. Function Spaces. The purpose of this section is to introduce the main function spaces to be used in the following and collect their basic properties.

Let us denote

$$Q^\pm := \mathbb{R}_t\times\Omega^\pm\,,\quad \omega:=\mathbb{R}_t\times\Gamma\,. \tag{3.1}$$

3.1. *Weighted Sobolev spaces.* For $\gamma\geq 1$ and $s\in\mathbb{R}$, we set

$$\lambda^{s,\gamma}(\xi):=(\gamma^2+|\xi|^2)^{s/2} \tag{3.2}$$

and, in particular, $\lambda^s:=\lambda^{s,1}$.

Throughout the paper, for real $\gamma\geq 1$, $H^s_\gamma(\mathbb{R}^n)$ will denote the Sobolev space of order s, equipped with the $\gamma-$depending norm $\|\cdot\|_{s,\gamma}$ defined by

$$\|u\|^2_{s,\gamma}:=(2\pi)^{-n}\int_{\mathbb{R}^n}\lambda^{2s,\gamma}(\xi)|\widehat{u}(\xi)|^2d\xi\,, \tag{3.3}$$

$\widehat{u}$ being the Fourier transform of u. The norms defined by (3.3), with different values of the parameter γ, are equivalent each other. For $\gamma=1$ we set for brevity $\|\cdot\|_s:=\|\cdot\|_{s,1}$ (and, accordingly, $H^s(\mathbb{R}^n):=H^s_1(\mathbb{R}^n)$ for the standard Sobolev space).

For $s\in\mathbb{N}$, the norm in (3.3) turns to be equivalent, *uniformly with respect to* γ, to the norm $\|\cdot\|_{H^s_\gamma(\mathbb{R}^n)}$ defined by

$$\|u\|^2_{H^s_\gamma(\mathbb{R}^n)}:=\sum_{|\alpha|\leq s}\gamma^{2(s-|\alpha|)}\|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)}\,, \tag{3.4}$$

where, for every multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we set $\partial^\alpha:=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$ and $|\alpha|:=\alpha_1+\cdots+\alpha_n$ as usual.

For functions defined over Q_T^+ we will consider the weighted Sobolev spaces $H^m_\gamma(Q_T^+)$ equipped with the natural γ-depending norm

$$\|u\|^2_{H^m_\gamma(Q_T^+)} := \sum_{|\alpha|\le m} \gamma^{2(m-|\alpha|)} \|\partial^\alpha u\|^2_{L^2(Q_T^+)}\,,$$

where $\partial^\alpha := \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ with $\partial_0 = \partial_t$. An useful remark is that

$$\|u\|_{s,\gamma} \le \gamma^{s-r}\|u\|_{r,\gamma}\,, \tag{3.5}$$

for arbitrary $s \le r$ and $\gamma \ge 1$.

Similar weighted Sobolev spaces will be considered for functions defined on Q^-

3.2. *Conormal Sobolev spaces.* Let us introduce some classes of function spaces of Sobolev type, defined over Q_T^+. Let $\sigma = \sigma(x_1)$ be a monotone increasing function in $C^\infty(\mathbb{R}_+)$, such that $\sigma(x_1) = x_1$ in a neighborhood of the origin and $\sigma(x_1) = 1$ for x_1 large enoungh. Then, for every multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$, the conormal derivative ∂^α_{tan} is defined by

$$\partial^\alpha_{tan} := \partial_0^{\alpha_0}(\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\,,$$

where $\partial_0 = \partial_t$.

Given an integer $m \ge 1$ the *conormal Sobolev space* $H^m_{tan}(Q_T^+)$ is defined as the set of functions $u \in L^2(Q_T^+)$ such that $\partial^\alpha_{tan} u \in L^2(Q_T^+)$, for all multi-indices α with $|\alpha| \le m$. Agreeing with the notations set for the usual Sobolev spaces, for $\gamma \ge 1$, $H^m_{tan,\gamma}(Q_T^+)$ will denote the conormal space of order m equipped with the $\gamma-$depending norm

$$\|u\|^2_{H^m_{tan,\gamma}(Q_T^+)} := \sum_{|\alpha|\le m} \gamma^{2(m-|\alpha|)} \|\partial^\alpha_{tan} u\|^2_{L^2(Q_T^+)} \tag{3.6}$$

and we have $H^m_{tan}(Q_T^+) := H^m_{tan,1}(Q_T^+)$.

Similar conormal Sobolev spaces with $\gamma-$depending norms will be considered for functions defined on Q^-. We will use the same notation for spaces of scalar and vector-valued functions.

3.3. *Homogeneous Sobolev space.* Because of the presence of the "elliptic" unknown q we will have also to use the homogeneous function space

$$\dot H^1(Q_T^+) := \{u \in L^1_{loc}(Q_T^+) \,|\, \nabla u \in L^2(Q_T^+)\}.$$

4. The main result. We are now in the position to state the main result of the paper. Recall that $U = (q, v, H)$, where we drop the dot from the variables for simplicity. The main result of the paper reads as follows.

THEOREM 4.1. Let $T > 0$. Let the basic state (2.2) satisfy assumptions (2.3)-(2.5) and

$$|\widehat{H} \times \widehat{\mathcal{H}}| \ge \delta > 0\,, \quad \text{on } \omega_T\,, \tag{4.1}$$

where δ is a fixed constant. Then there exists $\gamma_0 \ge 1$ such that for all $\gamma \ge \gamma_0$ and for all $f_{v,\gamma} \in H^1_\gamma(Q_T^+)$ vanishing in the past, namely for $t < 0$, problem (2.29) has a unique solution $(U_\gamma, \mathcal{H}_\gamma, \varphi_\gamma)$ such that $(q_\gamma, W_\gamma, \mathcal{H}_\gamma, \varphi_\gamma) \in \dot{H}^1(Q_T^+)\times H^1_\gamma(Q_T^+)\times H^1_\gamma(Q_T^-)\times H^1_\gamma(\omega_T)$ with the trace $(q_\gamma, u_{1,\gamma}, h_{1,\gamma}, \mathcal{H}_\gamma)|_{\omega_T} \in H^{1/2}_\gamma(\omega_T)$ and obeys the a priori estimate

$$\gamma\Big(\|W_\gamma\|^2_{H^1_\gamma(Q_T^+)} + \|\nabla q_\gamma\|^2_{L^2(Q_T^+)} + \|\mathcal{H}_\gamma\|^2_{H^1_\gamma(Q_T^-)} + \|\,(q_\gamma, u_{1,\gamma}, h_{1,\gamma}, \mathcal{H}_\gamma)\,|_{\omega_T}\|^2_{H^{1/2}_\gamma(\omega_T)}\Big) + \gamma^2\|\varphi_\gamma\|^2_{H^1_\gamma(\omega_T)} \le \frac{C}{\gamma}\|f_{v,\gamma}\|^2_{H^1_\gamma(Q_T^+)}\,, \tag{4.2}$$

where we have set $U_\gamma := e^{-\gamma t}U$, $\mathcal{H}_\gamma := e^{-\gamma t}\mathcal{H}$, $\varphi_\gamma := e^{-\gamma t}\varphi$ and so on, and where $C = C(K, T, \delta) > 0$ is a constant independent of the data f_v and the parameter γ.

5. Hyperbolic regularization of the reduced problem. Problem (2.29) (or (2.30)) is a nonstandard initial-boundary value problem. For its resolution we introduce a fully hyperbolic approximation. Concerning the plasma part, we replace the incompressible MHD equations with their "compressible" counterpart by introducing an evolution equation for the total pressure involving a small parameter ε which corresponds to the reciprocal of the sound speed in the fluid. As for the vacuum part, we consider a "hyperbolic" regularization of the elliptic system (2.30d) by introducing a new auxiliary unknown E which plays the role of the vacuum electric field, and the same small parameter of regularization ε as above is now associated with the physical parameter $1/c$, being c the speed of light. We also regularize the second boundary condition (2.30f) and introduce two boundary conditions for the unknown E.

Plasma part. Let us denote $U^\varepsilon = (q^\varepsilon, v^\varepsilon, H^\varepsilon)$ (we also set $W^\varepsilon = (v^\varepsilon, H^\varepsilon)$). The regularized system for the plasma part reads

$$\varepsilon^2\left\{\partial_t q^\varepsilon - (\partial_t\widehat{H}, H^\varepsilon) - (\widehat{H}, \partial_t H^\varepsilon) + \frac{1}{\partial_1\widehat{\Phi}_1}(\hat{w}, \nabla q^\varepsilon) - \frac{1}{\partial_1\widehat{\Phi}_1}\Big(\widehat{w}, (\nabla\widehat{H}, H^\varepsilon)\Big) - \frac{1}{\partial_1\widehat{\Phi}_1}\Big(\widehat{w}, (\widehat{H}, \nabla H^\varepsilon)\Big)\right\} + \frac{1}{\partial_1\widehat{\Phi}_1}\operatorname{div} u^\varepsilon = 0\,, \tag{5.1a}$$

$$\partial_t v^\varepsilon + \frac{1}{\partial_1\widehat{\Phi}_1}\Big\{(\hat{w}, \nabla)v^\varepsilon - (\hat{h}, \nabla)H^\varepsilon\Big\} + \nabla_{\widehat{\Phi}} q^\varepsilon + C_1(\widehat{W}, \widehat{\Psi})W^\varepsilon = f_v\,, \tag{5.1b}$$

$$\partial_t H^\varepsilon + \frac{1}{\partial_1\widehat{\Phi}_1}\Big\{(\hat{w}, \nabla)H^\varepsilon - (\hat{h}, \nabla)v^\varepsilon\Big\} + C_2(\widehat{W}, \widehat{\Psi})W^\varepsilon + \frac{\widehat{H}}{\partial_1\widehat{\Phi}_1}\operatorname{div} u^\varepsilon = 0 \quad \text{in } Q_T^+\,, \tag{5.1c}$$

where the matrices C_1 and C_2 were defined in (2.9), and u^ε is defined through v^ε like u is defined through v.

In the matrix form, system (5.1) can be shortly written as

$$\widehat{\mathcal{A}}_0^\varepsilon\partial_t U^\varepsilon + \sum_{j=1}^{3}\widehat{\mathcal{A}}_j^\varepsilon\partial_j U^\varepsilon + \widehat{\mathcal{C}}U^\varepsilon = F \qquad \text{in } Q_T^+, \tag{5.2}$$

where the matrix-valued coefficients $\widehat{\mathcal{A}}_j^\varepsilon$, $j=0,1,2,3$, and $\widehat{\mathcal{C}}$ are easily computed in terms of the basic state $(\widehat{W},\widehat{\Psi})$ and $F=(0,f_v,0)$. The latter system with $\varepsilon=1$ looks like the linearized system of *compressible* isentropic MHD equations reduced to a dimensionless form.

Vacuum part. Let us denote $V^\varepsilon=(\mathcal{H}^\varepsilon,E^\varepsilon)$. We consider the following regularized system for the unknown V^ε:

$$\varepsilon\,\partial_t\mathfrak{h}^\varepsilon+\nabla\times\mathfrak{E}^\varepsilon=0, \tag{5.3a}$$

$$\varepsilon\,\partial_t\mathfrak{e}^\varepsilon-\nabla\times\mathfrak{H}^\varepsilon=0\quad\text{in } Q_T^-, \tag{5.3b}$$

where

$$E^\varepsilon=(E_1^\varepsilon,E_2^\varepsilon,E_3^\varepsilon),\quad \mathfrak{E}^\varepsilon=(E_1^\varepsilon\partial_1\widehat{\Phi}_1,E_{\hat\tau_2}^\varepsilon,E_{\hat\tau_3}^\varepsilon),$$

$$\mathfrak{e}^\varepsilon=(E_{\hat n}^\varepsilon,E_2^\varepsilon\partial_1\widehat{\Phi}_1,E_3^\varepsilon\partial_1\widehat{\Phi}_1),\quad E_{\hat n}^\varepsilon=E_1^\varepsilon-E_2^\varepsilon\partial_2\widehat{\Psi}-E_3^\varepsilon\partial_3\widehat{\Psi},\quad E_{\hat\tau_k}^\varepsilon=E_1^\varepsilon\partial_k\widehat{\Psi}+E_k^\varepsilon,\ k=2,3\,.$$

All the other notations for $\mathcal{H}^\varepsilon$ (i.e. $\mathfrak{h}^\varepsilon$ and $\mathfrak{H}^\varepsilon$) are analogous of those for $\mathcal{H}$.

We rewrite (5.3) in the matrix form

$$\partial_tV^\varepsilon+\widetilde{B}_1^\varepsilon\partial_1V^\varepsilon+\sum_{k=2}^3B_k^\varepsilon\partial_kV^\varepsilon+\widehat{B}_4V^\varepsilon=0,$$

where

$$\widetilde{B}_1^\varepsilon=\frac{1}{\partial_1\widehat{\Phi}_1}\Big(B_1^\varepsilon-\sum_{k=2}^3B_k^\varepsilon\partial_k\widehat{\Psi}\Big),\quad \widehat{B}_4=I_2\otimes\widehat{B}, \tag{5.4}$$

$$\widehat{B}=\begin{pmatrix} 0 & -\partial_t\partial_2\widehat{\Psi}+\partial_2\widehat{\Psi}\dfrac{\partial_t\partial_1\widehat{\Psi}}{\partial_1\widehat{\Phi}_1} & -\partial_t\partial_3\widehat{\Psi}+\partial_3\widehat{\Psi}\dfrac{\partial_t\partial_1\widehat{\Psi}}{\partial_1\widehat{\Phi}_1}\\ 0 & \dfrac{\partial_t\partial_1\widehat{\Psi}}{\partial_1\widehat{\Phi}_1} & 0\\ 0 & 0 & \dfrac{\partial_t\partial_1\widehat{\Psi}}{\partial_1\widehat{\Phi}_1}\end{pmatrix}, \tag{5.5}$$

I_2 is the unit matrix of order 2, and the symmetric matrices B_j^ε $(j=1,2,3)$ coincide with the corresponding ones for the vacuum Maxwell equations if $\varepsilon=1$:

$$B_1^\varepsilon=\varepsilon^{-1}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\end{pmatrix},\quad B_2^\varepsilon=\varepsilon^{-1}\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\end{pmatrix},$$

$$B_3^\varepsilon=\varepsilon^{-1}\begin{pmatrix}0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}. \tag{5.6}$$

Boundary conditions. We couple equations (5.1) and (5.3) with the following regularized boundary conditions

$$\begin{aligned}
&\partial_t\varphi^\varepsilon = v^\varepsilon_{\hat N} - \hat v_2\partial_2\varphi^\varepsilon - \hat v_3\partial_3\varphi^\varepsilon + \varphi^\varepsilon\partial_1\hat v_{\hat N},\\
&q^\varepsilon = (\widehat{\mathcal H},\mathcal H^\varepsilon) - [\partial_1\hat q]\varphi^\varepsilon - \varepsilon(\widehat E, E^\varepsilon)\\
&E^\varepsilon_{\hat\tau_2} = \varepsilon\,\partial_t(\widehat{\mathcal H}_3\varphi^\varepsilon) - \varepsilon\,\partial_2(\widehat E_1\varphi^\varepsilon),\\
&E^\varepsilon_{\hat\tau_3} = -\varepsilon\,\partial_t(\widehat{\mathcal H}_2\varphi^\varepsilon) - \varepsilon\,\partial_3(\widehat E_1\varphi^\varepsilon) \qquad \text{on } \omega_T,
\end{aligned} \tag{5.7}$$

where $\widehat E = (\widehat E_1,\widehat E_2,\widehat E_3)$ and the coefficients $\widehat E_j$ are given functions which will be chosen later on. Again, $v^\varepsilon_{\hat N} = (v^\varepsilon,\hat N)$.

Final form of the regularized problem. Collecting the previous equations we obtain the regularized problem given by (5.1), (5.3) and (5.7).

5.1. *An equivalent formulation for the regularized problem.*

5.1.1. *Plasma part.* We derive an equivalent form for system (5.2) in two steps. First we write down this system in terms of the new unknown $q'^{\,\varepsilon} = \varepsilon q^\varepsilon$ and then we pass to the "curved unknowns" $u^\varepsilon, h^\varepsilon$.

Step 1. To symmetrize system (5.2), we derive div u^ε from (5.1a) and rewrite the equation for the magnetic field in (5.1c) as

$$\begin{aligned}
\partial_t H^\varepsilon &+ \frac{1}{\partial_1\widehat\Phi_1}\Big\{(\hat w,\nabla)H^\varepsilon - (\hat h,\nabla)v^\varepsilon\Big\} + C_2(\widehat W,\widehat\Psi)W^\varepsilon\\
&- \varepsilon^2\widehat H\Big\{\partial_t q^\varepsilon - (\partial_t\widehat H, H^\varepsilon) - (\widehat H,\partial_t H^\varepsilon) + \frac{1}{\partial_1\widehat\Phi_1}(\hat w,\nabla q^\varepsilon)\\
&- \frac{1}{\partial_1\widehat\Phi_1}\Big(\widehat w,(\nabla\widehat H, H^\varepsilon)\Big) - \frac{1}{\partial_1\widehat\Phi_1}\Big(\widehat w,(\widehat H,\nabla H^\varepsilon)\Big)\Big\} = 0.
\end{aligned} \tag{5.8}$$

Substituting (5.8) in (5.1) gives a symmetric system. Unfortunately, the matrix-valued coefficient by the t–derivative of U^ε is not uniformly positive-definite with respect to ε that makes inconvenience because we are interested in obtaining an uniform in ε a priori estimate for smooth solutions of (5.1). Therefore, we make the change of unknown

$$q'^{\,\varepsilon} = \varepsilon q^\varepsilon\,, \tag{5.9}$$

and restate system (5.2) in terms of the new unknown $(q'^{\,\varepsilon}, v^\epsilon, H^\varepsilon)$. Just for simplicity we again denote $U^\varepsilon = (q'^{\,\varepsilon}, v^\epsilon, H^\varepsilon)$. In the matrix form, system (5.2) becomes

$$\widehat A_0\partial_t U^\varepsilon + \widetilde{\widehat A}{}^\varepsilon_1\partial_1 U^\varepsilon + \sum_{j=2}^{3}\widehat A^\varepsilon_j\partial_j U^\varepsilon + \widehat{\mathcal C}^\varepsilon U^\varepsilon = F \qquad \text{in } Q^+_T, \tag{5.10}$$

where the new coefficients are the symmetric matrices

$$\widehat A^\varepsilon_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & -\varepsilon\widehat H_1 & -\varepsilon\widehat H_2 & -\varepsilon\widehat H_3\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
-\varepsilon\widehat H_1 & 0 & 0 & 0 & 1+\varepsilon^2\widehat H_1^2 & \varepsilon^2\widehat H_1\widehat H_2 & \varepsilon^2\widehat H_1\widehat H_3\\
-\varepsilon\widehat H_2 & 0 & 0 & 0 & \varepsilon^2\widehat H_1\widehat H_2 & 1+\varepsilon^2\widehat H_2^2 & \varepsilon^2\widehat H_2\widehat H_3\\
-\varepsilon\widehat H_3 & 0 & 0 & 0 & \varepsilon^2\widehat H_1\widehat H_3 & \varepsilon^2\widehat H_2\widehat H_3 & 1+\varepsilon^2\widehat H_3^2
\end{pmatrix}$$

$$
= \begin{pmatrix} 1 & 0 & -\varepsilon \widehat{H}^T \\ 0 & I_3 & 0 \\ -\varepsilon \widehat{H} & 0 & I_3 + \varepsilon^2 \widehat{H} \otimes \widehat{H} \end{pmatrix},
$$

$$
\widehat{A}_1^\varepsilon = \begin{pmatrix}
\hat{v}_1 & \varepsilon^{-1} & 0 & 0 & -\varepsilon\widehat{v}_1\widehat{H}_1 & -\varepsilon\widehat{v}_1\widehat{H}_2 & -\varepsilon\widehat{v}_1\widehat{H}_3 \\
\varepsilon^{-1} & \hat{v}_1 & 0 & 0 & -\widehat{H}_1 & 0 & 0 \\
0 & 0 & \hat{v}_1 & 0 & 0 & -\widehat{H}_1 & 0 \\
0 & 0 & 0 & \hat{v}_1 & 0 & 0 & -\widehat{H}_1 \\
-\varepsilon\widehat{v}_1\widehat{H}_1 & -\widehat{H}_1 & 0 & 0 & \hat{v}_1 + \varepsilon^2\hat{v}_1\widehat{H}_1^2 & \varepsilon^2\hat{v}_1\widehat{H}_1\widehat{H}_2 & \varepsilon^2\hat{v}_1\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_1\widehat{H}_2 & 0 & -\widehat{H}_1 & 0 & \varepsilon^2\hat{v}_1\widehat{H}_1\widehat{H}_2 & \hat{v}_1 + \varepsilon^2\hat{v}_1\widehat{H}_2^2 & \varepsilon^2\hat{v}_1\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_1\widehat{H}_3 & 0 & 0 & -\widehat{H}_1 & \varepsilon^2\hat{v}_1\widehat{H}_1\widehat{H}_3 & \varepsilon^2\hat{v}_1\widehat{H}_2\widehat{H}_3 & \hat{v}_1 + \varepsilon^2\hat{v}_1\widehat{H}_3^2
\end{pmatrix},
$$

$$
\widehat{A}_2^\varepsilon = \begin{pmatrix}
\hat{v}_1 & 0 & \varepsilon^{-1} & 0 & -\varepsilon\widehat{v}_2\widehat{H}_1 & -\varepsilon\widehat{v}_2\widehat{H}_2 & -\varepsilon\widehat{v}_2\widehat{H}_3 \\
0 & \hat{v}_2 & 0 & 0 & -\widehat{H}_2 & 0 & 0 \\
\varepsilon^{-1} & 0 & \hat{v}_2 & 0 & 0 & -\widehat{H}_2 & 0 \\
0 & 0 & 0 & \hat{v}_2 & 0 & 0 & -\widehat{H}_2 \\
-\varepsilon\widehat{v}_2\widehat{H}_1 & -\widehat{H}_2 & 0 & 0 & \hat{v}_2 + \varepsilon^2\hat{v}_2\widehat{H}_1^2 & \varepsilon^2\hat{v}_2\widehat{H}_1\widehat{H}_2 & \varepsilon^2\hat{v}_2\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_2\widehat{H}_2 & 0 & -\widehat{H}_2 & 0 & \varepsilon^2\hat{v}_2\widehat{H}_1\widehat{H}_2 & \hat{v}_2 + \varepsilon^2\hat{v}_2\widehat{H}_2^2 & \varepsilon^2\hat{v}_2\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_2\widehat{H}_3 & 0 & 0 & -\widehat{H}_2 & \varepsilon^2\hat{v}_2\widehat{H}_1\widehat{H}_3 & \varepsilon^2\hat{v}_2\widehat{H}_2\widehat{H}_3 & \hat{v}_2 + \varepsilon^2\hat{v}_2\widehat{H}_3^2
\end{pmatrix},
$$

$$
\widehat{A}_3^\varepsilon = \begin{pmatrix}
\hat{v}_3 & 0 & 0 & \varepsilon^{-1} & -\varepsilon\widehat{v}_3\widehat{H}_1 & -\varepsilon\widehat{v}_3\widehat{H}_2 & -\varepsilon\widehat{v}_3\widehat{H}_3 \\
0 & \hat{v}_3 & 0 & 0 & -\widehat{H}_3 & 0 & 0 \\
0 & 0 & \hat{v}_3 & 0 & 0 & -\widehat{H}_3 & 0 \\
\varepsilon^{-1} & 0 & 0 & \hat{v}_3 & 0 & 0 & -\widehat{H}_3 \\
-\varepsilon\widehat{v}_3\widehat{H}_1 & -\widehat{H}_3 & 0 & 0 & \hat{v}_3 + \varepsilon^2\hat{v}_3\widehat{H}_1^2 & \varepsilon^2\hat{v}_3\widehat{H}_1\widehat{H}_2 & \varepsilon^2\hat{v}_3\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_3\widehat{H}_2 & 0 & -\widehat{H}_3 & 0 & \varepsilon^2\hat{v}_3\widehat{H}_1\widehat{H}_2 & \hat{v}_3 + \varepsilon^2\hat{v}_3\widehat{H}_2^2 & \varepsilon^2\hat{v}_3\widehat{H}_1\widehat{H}_3 \\
-\varepsilon\widehat{v}_3\widehat{H}_3 & 0 & 0 & -\widehat{H}_3 & \varepsilon^2\hat{v}_3\widehat{H}_1\widehat{H}_3 & \varepsilon^2\hat{v}_3\widehat{H}_2\widehat{H}_3 & \hat{v}_3 + \varepsilon^2\hat{v}_3\widehat{H}_3^2
\end{pmatrix},
$$

and the coefficient $\widetilde{\widehat{A}}_1^\varepsilon$ is

$$
\widetilde{\widehat{A}}_1^\varepsilon = \frac{1}{\partial_1 \widehat{\Phi}_1} \left(\widehat{A}_1^\varepsilon - \sum_{j=2}^{3} \partial_j \widehat{\Psi} \widehat{A}_j^\varepsilon - \partial_t \widehat{\Psi} \widehat{A}_0^\varepsilon \right),
$$

while

$$
\widehat{\mathcal{C}}^\varepsilon U^\varepsilon = \begin{pmatrix} -\varepsilon(\widehat{D}, H^\varepsilon) \\ C_1(\widehat{W}, \widehat{\Psi}) W^\varepsilon \\ C_2(\widehat{W}, \widehat{\Psi}) W^\varepsilon + \varepsilon^2 (\widehat{D}, H^\varepsilon) \widehat{H} \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ f_v \\ 0 \end{pmatrix},
$$

and

$$
\widehat{D} = \partial_t \widehat{H} + \frac{1}{\partial_1 \widehat{\Phi}_1} (\widehat{w}, \nabla) \widehat{H}\,.
$$

Note that the coefficients $\widehat{A}_j^\varepsilon$, for $j = 1, 2, 3$ and $\mathcal{C}^\varepsilon$ can be shortly rewritten as

$$
\widehat{A}_j^\varepsilon = \begin{pmatrix} \widehat{v}_j & \varepsilon^{-1} e_j^T & -\varepsilon \widehat{v}_j \widehat{H}^T \\ \varepsilon^{-1} e_j & \widehat{v}_j I_3 & -\widehat{H}_j I_3 \\ -\varepsilon \widehat{v}_j \widehat{H} & -\widehat{H}_j I_3 & \widehat{v}_j (I_3 + \varepsilon^2 \widehat{H} \otimes \widehat{H}) \end{pmatrix},
$$

with

$$e_j = \begin{pmatrix} \delta_{1,j} \\ \delta_{2,j} \\ \delta_{3,j} \end{pmatrix}, \quad \widehat{H} \otimes \widehat{H} = \widehat{H}\widehat{H}^T$$

and

$$\mathcal{C}^\varepsilon = \begin{pmatrix} 0 \\ C_1(\widehat{W}, \widehat{\Psi}) \\ C_2(\widehat{W}, \widehat{\Psi}) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\varepsilon \widehat{D}^T \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon^2 \widehat{H} \otimes \widehat{D} \end{pmatrix}.$$

Moreover, an explicit calculation gives for $\widetilde{\widehat{A}}{}_1^\varepsilon$ the following expression

$$\widetilde{\widehat{A}}{}_1^\varepsilon = \frac{1}{\partial_1 \widehat{\Phi}_1} \begin{pmatrix} \widehat{w}_1 & \varepsilon^{-1}\widehat{n}^T & -\varepsilon \widehat{w}_1 \widehat{H}^T \\ \varepsilon^{-1}\widehat{n} & \widehat{w}_1 I_3 & -\widehat{h}_1 I_3 \\ -\varepsilon \widehat{w}_1 \widehat{H} & -\widehat{h}_1 I_3 & \widehat{w}_1 (I_3 + \varepsilon^2 \widehat{H} \otimes \widehat{H}) \end{pmatrix}, \tag{5.11}$$

where we recall that

$$\widehat{n}^T = \begin{pmatrix} 1 & -\partial_2 \widehat{\Psi} & -\partial_3 \widehat{\Psi} \end{pmatrix}, \quad \widehat{w}_1 = \widehat{v}_{\hat{n}} - \partial_t \widehat{\Psi}, \quad \widehat{h}_1 = \widehat{H}_{\hat{n}}.$$

System (5.10) is symmetric hyperbolic because the matrix $\widehat{A}_0^\varepsilon$ is uniformly definite positive for ε sufficiently small. Unfortunately, the matrix in (5.11) contains the singular factor ε^{-1}. Fortunately, this potential difficulty will not prevent obtaining an uniform in ε a priori estimate.

Step 2. For overcoming the difficulty connected with the appearance of ε^{-1} in (5.11) we rewrite system (5.10) in terms of the new vector unknown $Y^\varepsilon = (q'^\varepsilon, u^\varepsilon, h^\varepsilon)$. Observing that $U^\varepsilon = JY^\varepsilon$, where the matrix J is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{J} & 0 \\ 0 & 0 & \widehat{J} \end{pmatrix}, \qquad \widehat{J} = \begin{pmatrix} 1 & \dfrac{\partial_2 \widehat{\Psi}}{\partial_1 \widehat{\Phi}_1} & \dfrac{\partial_3 \widehat{\Psi}}{\partial_1 \widehat{\Phi}_1} \\ 0 & \dfrac{1}{\partial_1 \widehat{\Phi}_1} & 0 \\ 0 & 0 & \dfrac{1}{\partial_1 \widehat{\Phi}_1} \end{pmatrix}, \tag{5.12}$$

we obtain the new system

$$\check{A}_0^\varepsilon \partial_t Y^\varepsilon + \sum_{j=1}^{3} \check{A}_j^\varepsilon \partial_j Y^\varepsilon + \check{A}_4^\varepsilon Y^\varepsilon = \widetilde{F}, \tag{5.13}$$

where

$$\begin{aligned} &\check{A}_0^\varepsilon = \partial_1 \widehat{\Phi}_1 J^T \widehat{A}_0^\varepsilon J, \qquad \check{A}_1^\varepsilon = \partial_1 \widehat{\Phi}_1 J^T \widetilde{\widehat{A}}{}_1^\varepsilon J, \qquad \check{A}_k^\varepsilon = \partial_1 \widehat{\Phi}_1 J^T \widehat{A}_k^\varepsilon J \quad (k = 2, 3), \\ &\check{A}_4^\varepsilon = \partial_1 \widehat{\Phi}_1 \left(J^T \widehat{A}_0^\varepsilon \partial_t J + J^T \widetilde{\widehat{A}}{}_1^\varepsilon \partial_1 J + \sum_{k=2}^{3} J^T \widehat{A}_k^\varepsilon \partial_k J + J^T \widehat{\mathcal{C}}^\varepsilon J \right), \end{aligned} \tag{5.14}$$

$$\widetilde{F} = \partial_1 \widehat{\Phi}_1 J^T F = \begin{pmatrix} 0 \\ \widetilde{f}_v \\ 0 \end{pmatrix}, \qquad \widetilde{f}_v = \partial_1 \widehat{\Phi}_1 \widetilde{J}^T f_v. \tag{5.15}$$

Direct calculations show that

$$\check{A}_j^\varepsilon = \widehat{\mathbb{A}}_j^\varepsilon + \varepsilon^{-1} \mathcal{E}_{1,j+1}, \qquad j = 1, 2, 3, \tag{5.16}$$

where

$$\widehat{\mathbb{A}}_j^{\varepsilon} = \begin{pmatrix} \widehat{w}_j & 0 & -\varepsilon\widehat{w}_j\widehat{H}^T\widehat{J} \\ 0 & \widehat{w}_j\widehat{J}^T\widehat{J} & -\widehat{h}_j\widehat{J}^T\widehat{J} \\ -\varepsilon\widehat{w}_j\widehat{J}^T\widehat{H} & -\widehat{h}_j\widehat{J}^T\widehat{J} & \widehat{w}_j\widehat{J}^T(I_3+\varepsilon^2\widehat{H}\otimes\widehat{H})\widehat{J} \end{pmatrix},$$
$$\mathcal{E}_{1,j+1} = \begin{pmatrix} 0 & e_j^T & 0 \\ e_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad j=1,2,3. \tag{5.17}$$

Compared to (5.10), the equivalent formulation (5.13) has the advantage that the factor ε^{-1} appears only by the constant matrices $\mathcal{E}_{1,j+1}$ and that the boundary matrix $\check{A}_1^{\varepsilon}$ takes the form

$$\check{A}_1^{\varepsilon} = \widehat{\mathbb{A}}_1^{\varepsilon} + \varepsilon^{-1}\mathcal{E}_{1,2}\,, \tag{5.18}$$

where

$$\widehat{\mathbb{A}}_1^{\varepsilon}|_{\omega_T} = 0 \tag{5.19}$$

(since $\widehat{w}_1|_{\omega_T} = \widehat{h}_1|_{\omega_T} = 0$, see (2.4c), (2.5)). Moreover, an explicit calculation shows that $\check{A}_0^{\varepsilon}$ and $\check{A}_4^{\varepsilon}$ do not contain the singular multiplier ε^{-1} (their elements are bounded as $\varepsilon \to 0$).

5.1.2. *Vacuum part.* System (5.3) can be written in terms of the "curved" unknown $\mathcal{W}^{\varepsilon} = (\mathfrak{H}^{\varepsilon}, \mathfrak{E}^{\varepsilon})$ as

$$B_0\partial_t\mathcal{W}^{\varepsilon} + \sum_{j=1}^{3} B_j^{\varepsilon}\partial_j\mathcal{W}^{\varepsilon} + B_4\mathcal{W}^{\varepsilon} = 0, \tag{5.20}$$

where

$$B_0 = \frac{1}{\partial_1\widehat{\Phi}_1}KK^T\,, \quad K = I_2\otimes\widehat{K}\,, \quad B_4 = \partial_t B_0\,, \tag{5.21}$$

$$\widehat{K} = \widehat{J}^{-1} = \begin{pmatrix} 1 & -\partial_2\widehat{\Psi} & -\partial_3\widehat{\Psi} \\ 0 & \partial_1\widehat{\Phi}_1 & 0 \\ 0 & 0 & \partial_1\widehat{\Phi}_1 \end{pmatrix}, \tag{5.22}$$

and the matrices $\widehat{J}$ and B_j^{ε} are defined in (5.12) and (5.6) respectively, see [13] for more details.

System (5.20) is symmetric hyperbolic. The main advantage of the usage of the variables $\mathcal{W}^{\varepsilon}$ rather than V^{ε} is that the matrices B_j^{ε} in (5.20) containing the singular multiplier ε^{-1} are constant.

5.1.3. *Boundary conditions.* We restate the boundary conditions above in terms of the unknown $(Y^{\varepsilon}, \mathcal{W}^{\varepsilon})$ by using the relations (recall that $\partial_1\widehat{\Phi}_1 = 1$ on ω_T)

$$(\widehat{\mathcal{H}}, \mathcal{H}^{\varepsilon}) = \widehat{\mathcal{H}}_{\hat{N}}\mathcal{H}_1^{\varepsilon} + \widehat{\mathcal{H}}_2\mathcal{H}_{\hat{\tau}_2}^{\varepsilon} + \widehat{\mathcal{H}}_3\mathcal{H}_{\hat{\tau}_3}^{\varepsilon} = (\widehat{\mathfrak{h}}, \mathfrak{H}^{\varepsilon})\,,$$
$$(\widehat{E}, E^{\varepsilon}) = \widehat{E}_{\hat{N}}E_1^{\varepsilon} + \widehat{E}_2E_{\hat{\tau}_2}^{\varepsilon} + \widehat{E}_3E_{\hat{\tau}_3}^{\varepsilon} = (\widehat{\mathfrak{e}}, \mathfrak{E}^{\varepsilon})\,. \tag{5.23}$$

Regarding the first line in (5.23), we notice that $\widehat{\mathfrak{h}}_1 = \widehat{\mathcal{H}}_{\hat N} = 0$ on ω_T, so that $\mathfrak{H}_1^\varepsilon$ does not appear in the boundary condition. Then the boundary conditions become

$$\begin{aligned}
&\partial_t\varphi^\varepsilon = u_1^\varepsilon - \hat v_2\partial_2\varphi^\varepsilon - \hat v_3\partial_3\varphi^\varepsilon + \varphi^\varepsilon\partial_1\hat v_{\hat N},\\
&\varepsilon^{-1}q'^{\,\varepsilon} = (\widehat{\mathfrak{h}}, \mathfrak{H}^\varepsilon) - [\partial_1\hat q]\varphi^\varepsilon - \varepsilon(\widehat{\mathfrak{e}}, \mathfrak{E}^\varepsilon)\\
&\mathfrak{E}_2^\varepsilon = \varepsilon\,\partial_t(\widehat{\mathcal{H}}_3\varphi^\varepsilon) - \varepsilon\,\partial_2(\widehat{E}_1\varphi^\varepsilon),\\
&\mathfrak{E}_3^\varepsilon = -\varepsilon\,\partial_t(\widehat{\mathcal{H}}_2\varphi^\varepsilon) - \varepsilon\,\partial_3(\widehat{E}_1\varphi^\varepsilon) \qquad \text{on } \omega_T\,.
\end{aligned} \tag{5.24}$$

5.1.4. *Full equivalent regularized problem.* To sum up, we consider the following regularized problem for the unknown $(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon)$:

$$\check A_0^\varepsilon\partial_t Y^\varepsilon + \sum_{j=1}^3 \check A_j^\varepsilon\partial_j Y^\varepsilon + \check A_4^\varepsilon Y^\varepsilon = \widetilde F \qquad \text{in } Q_T^+ \tag{5.25a}$$

$$B_0\partial_t\mathcal{W}^\varepsilon + \sum_{j=1}^3 B_j^\varepsilon\partial_j\mathcal{W}^\varepsilon + B_4\mathcal{W}^\varepsilon = 0 \qquad \text{in } Q_T^- \tag{5.25b}$$

$$\partial_t\varphi^\varepsilon = u_1^\varepsilon - \hat v_2\partial_2\varphi^\varepsilon - \hat v_3\partial_3\varphi^\varepsilon + \varphi^\varepsilon\partial_1\hat v_{\hat N}\,, \tag{5.25c}$$

$$\varepsilon^{-1}q'^{\,\varepsilon} = (\widehat{\mathfrak{h}}, \mathfrak{H}^\varepsilon) - [\partial_1\hat q]\varphi^\varepsilon - \varepsilon(\widehat{\mathfrak{e}}, \mathfrak{E}^\varepsilon)\,, \tag{5.25d}$$

$$\mathfrak{E}_2^\varepsilon = \varepsilon\,\partial_t(\widehat{\mathcal{H}}_3\varphi^\varepsilon) - \varepsilon\,\partial_2(\widehat{E}_1\varphi^\varepsilon)\,, \tag{5.25e}$$

$$\mathfrak{E}_3^\varepsilon = -\varepsilon\,\partial_t(\widehat{\mathcal{H}}_2\varphi^\varepsilon) - \varepsilon\,\partial_3(\widehat{E}_1\varphi^\varepsilon) \qquad \text{on } \omega_T, \tag{5.25f}$$

$$(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon) = 0 \qquad \text{for } t < 0. \tag{5.25g}$$

It is noteworthy that solutions to problem (5.25) satisfy

$$\operatorname{div} h^\varepsilon = 0 \qquad \text{in } Q_T^+, \tag{5.26a}$$

$$\operatorname{div}\mathfrak{h}^\varepsilon = 0, \quad \operatorname{div}\mathfrak{e}^\varepsilon = 0 \qquad \text{in } Q_T^-, \tag{5.26b}$$

$$H_{\hat N}^\varepsilon = \widehat H_2\partial_2\varphi^\varepsilon + \widehat H_3\partial_3\varphi^\varepsilon - \varphi^\varepsilon\partial_1\widehat H_{\hat N} \qquad \text{on } \omega_T, \tag{5.26c}$$

$$\mathcal{H}_{\hat N}^\varepsilon = \partial_2\big(\widehat{\mathcal{H}}_2\varphi^\varepsilon\big) + \partial_3\big(\widehat{\mathcal{H}}_3\varphi^\varepsilon\big) \qquad \text{on } \omega_T \tag{5.26d}$$

because (5.26) are just restrictions on the initial data which are automatically satisfied in view of (5.25g). Equations (5.26b) trivially follow from (5.25b) and (5.25g). Condition (5.26d) is obtained by considering the first scalar equation in (5.25b) at $x_1 = 0$ and taking into account (5.25c)-(5.25g). As we already noticed, (5.25a), (5.25b) is a symmetric hyperbolic system.

Remark 5.1. The invertible part of the boundary matrix of a system allows to control the trace at the boundary of the so called noncharacteristic component of the vector solution. Thus, with system (5.25a) (whose boundary matrix is $-\check A_1^\varepsilon|_{\omega_T} = -\varepsilon^{-1}\mathcal{E}_{1,2}$, because of (5.19)), we have the control of $q'^{\,\varepsilon}$, u_1^ε at the boundary; therefore the components of Y^ε appearing in the boundary conditions (5.25c), (5.25d) are well defined.

The same holds true for (5.25b), where we can get the control of $\mathfrak{H}_2^\varepsilon$, $\mathfrak{H}_3^\varepsilon$, $\mathfrak{E}_2^\varepsilon$, $\mathfrak{E}_3^\varepsilon$. The control of $\mathfrak{E}_1^\varepsilon$, which appears in (5.25d), is not given from system (5.25b), but from the constraints (5.26b), as will be shown later on. We recall that $\mathfrak{H}_1^\varepsilon$ does not appear in the boundary condition (5.25d), because $\widehat{\mathfrak{h}}_1 = \widehat H_{\hat N} = 0$ on ω_T.

Before studying problem (5.25) we should be sure that the number of boundary conditions in (5.25c)-(5.25f) is in agreement with the number of incoming characteristics for the hyperbolic systems (5.25a), (5.25b). Since one of the four boundary conditions in (5.25c)-(5.25f) is needed for determining the function $\varphi^\varepsilon(t,x')$, the common number of incoming characteristics should be three. Let us prove that this is true.

LEMMA 5.2. If $\varepsilon > 0$ is sufficiently small, system (5.25a) has one incoming characteristic for the boundary ω_T of the domain Q_T^+. If $\varepsilon > 0$ is sufficiently small, system (5.25b) has two incoming characteristics for the boundary ω_T of the domain Q_T^-.

Proof. In view of (5.18) and (5.19) we obtain

$$(\check{A}_1^\varepsilon Y^\varepsilon, Y^\varepsilon) = \varepsilon^{-1}(\mathcal{E}_{1,2}Y^\varepsilon, Y^\varepsilon) = 2\varepsilon^{-1}q'^{\,\varepsilon}u_1^\varepsilon \quad \text{on } \omega_T. \tag{5.27}$$

Hence, the boundary matrix $\check{A}_1^\varepsilon$ at the boundary ω_T has one negative eigenvalue $\lambda_- = -\varepsilon^{-1}$ ("incoming" in the domain Q_T^+) and one positive eigenvalue $\lambda_+ = \varepsilon^{-1}$, and other eigenvalues are zeros.

Let us consider system (5.25b). The boundary matrix B_1^ε has eigenvalues $\lambda_{1,2} = -\varepsilon^{-1}$, $\lambda_{3,4} = \varepsilon^{-1}$, $\lambda_{5,6} = 0$. Thus, system (5.25b) has two incoming characteristics in the domain Q_T^-. □

6. A BVP associated to the regularized hyperbolic problem: a priori estimates. Let $T > 0$. Let the basic state (2.2) satisfy assumptions (2.3)-(2.5) and (4.1). Our next goal is to prove the existence of solutions $(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon)$ to problem (5.25) and a uniform in ε a priori estimate in $H^1_{tan}(Q_T^+) \times H^1(Q_T^-) \times H^1(\omega_T)$. This will be done in several steps.

6.1. *The boundary value problem.* We assume that all the coefficients and data appearing in (5.25) are extended to the whole real line with respect to the time, and recall that $Q^\pm = \mathbb{R}_t \times \Omega^\pm$ and $\omega = \mathbb{R}_t \times \Gamma$ (see (3.1)).

The first step of our analysis is to prove a uniform in ϵ estimate for smooth solutions to the boundary value problem (5.25a)–(5.25g) in $Q^\pm$, i.e., to the problem

$$\begin{aligned}
&\check{A}_0^\varepsilon \partial_t Y^\varepsilon + \textstyle\sum_{j=1}^3 \check{A}_j^\varepsilon \partial_j Y^\varepsilon + \check{A}_4^\varepsilon Y^\varepsilon = \widetilde{F} \qquad \text{in } Q^+,\\
&B_0 \partial_t \mathcal{W}^\varepsilon + \textstyle\sum_{j=1}^3 B_j^\varepsilon \partial_j \mathcal{W}^\varepsilon + B_4 \mathcal{W}^\varepsilon = 0 \qquad \text{in } Q^-,\\
&\left.\begin{aligned}
&\partial_t\varphi^\varepsilon = u_1^\varepsilon - \hat{v}_2\partial_2\varphi^\varepsilon - \hat{v}_3\partial_3\varphi^\varepsilon + \varphi^\varepsilon\partial_1\hat{v}_{\hat{N}},\\
&\varepsilon^{-1}q'^{\,\varepsilon} = (\hat{\mathfrak{h}}, \mathfrak{H}^\varepsilon) - [\partial_1\hat{q}]\varphi^\varepsilon - \varepsilon(\hat{\mathfrak{e}}, \mathfrak{E}^\varepsilon)\\
&\mathfrak{E}_2^\varepsilon = \varepsilon\,\partial_t(\widehat{\mathcal{H}}_3\varphi^\varepsilon) - \varepsilon\,\partial_2(\widehat{E}_1\varphi^\varepsilon),\\
&\mathfrak{E}_3^\varepsilon = -\varepsilon\,\partial_t(\widehat{\mathcal{H}}_2\varphi^\varepsilon) - \varepsilon\,\partial_3(\widehat{E}_1\varphi^\varepsilon) \qquad \text{on } \omega,
\end{aligned}\right.\\
&(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon) = 0 \qquad \text{for } t < 0\,.
\end{aligned} \tag{6.1}$$

Recall that $Y^\varepsilon = (q'^{\,\varepsilon}, u^\varepsilon, h^\varepsilon)$ and $\mathcal{W}^\varepsilon = (\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon)$.

In this section, we prove a uniform in ε a priori estimate of smooth solutions of (6.1).

THEOREM 6.1. Let the basic state (2.2) satisfy assumptions (2.3)-(2.5) and (4.1) for all times. There exist $\varepsilon_0 > 0$, $\gamma_0 \geq 1$ such that if $0 < \varepsilon < \varepsilon_0$ and $\gamma \geq \gamma_0$, then all sufficiently smooth solutions $(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon)$ of problem (6.1) obey the estimate

$$\gamma\left(\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} + \|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)} + \|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{H^{1/2}_\gamma(\omega)} + \|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{H^{1/2}_\gamma(\omega)}\right) + \gamma^2\|\varphi^\varepsilon_\gamma\|^2_{H^1_\gamma(\omega)} \leq \frac{C}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}, \quad (6.2)$$

where we have set $Y^\varepsilon_\gamma = e^{-\gamma t}Y^\varepsilon$, $Y^\varepsilon_{n,\gamma} = e^{-\gamma t}(\varepsilon^{-1}q'^{\,\varepsilon}, u^\varepsilon_1, h^\varepsilon_1)$, $\mathcal{W}^\varepsilon_\gamma = e^{-\gamma t}\mathcal{W}^\varepsilon$, $\varphi^\varepsilon_\gamma = e^{-\gamma t}\varphi^\varepsilon$ and so on, and where $C = C(K, \delta) > 0$ is a constant independent of the data $\widetilde{F}$ and the parameters ε, γ.

In order to obtain the energy estimate (6.2), we use the same ideas as in [13] (see also [17]). We underline that the coefficients $\widehat{E}_j$ in the boundary conditions in (6.1) are still arbitrary functions whose choice will be crucial to make boundary conditions dissipative. Moreover, we have to be careful with lower order terms, because we must avoid the appearance of terms with ε^{-1} (otherwise our estimate will not be uniform in ε). Also for this reason we use the unknown $(Y^\varepsilon, \mathcal{W}^\varepsilon)$ rather than $(U^\varepsilon, V^\varepsilon)$.

For the proof of the energy estimate (6.2) we need a secondary symmetrization of the transformed Maxwell equations in vacuum (5.3).

6.2. *Secondary symmetrization for the vacuum part.* Let us perform a new symmetrization of the vacuum part (see [17]), that consists of replacing the original system (5.3) with the equivalent system

$$\begin{aligned}
&\widehat{K}^{-1}(\partial_t\mathfrak{h}^\varepsilon + \frac{1}{\varepsilon}\nabla\times\mathfrak{E}^\varepsilon) + \widehat{K}^{-1}(\partial_t\mathfrak{e}^\varepsilon - \frac{1}{\varepsilon}\nabla\times\mathfrak{H}^\varepsilon)\times\varepsilon\nu + \frac{\nu}{\partial_1\widehat{\Phi}_1}\operatorname{div}\mathfrak{h}^\varepsilon = 0,\\
&\widehat{K}^{-1}(\partial_t\mathfrak{e}^\varepsilon - \frac{1}{\varepsilon}\nabla\times\mathfrak{H}^\varepsilon) - \widehat{K}^{-1}(\partial_t\mathfrak{h}^\varepsilon + \frac{1}{\varepsilon}\nabla\times\mathfrak{E}^\varepsilon)\times\varepsilon\nu + \frac{\nu}{\partial_1\widehat{\Phi}_1}\operatorname{div}\mathfrak{e}^\varepsilon = 0,
\end{aligned} \quad (6.3)$$

where $\widehat{K}$ is defined in (5.22), while $\nu = (\nu_1, \nu_2, \nu_3)$ and $\nu_i = \nu_i(t, x)$ $(i = 1, 2, 3)$ are arbitrary functions that will be chosen in an appropriate way later on. We refer to [13, Lemma 16] for the detailed proof of the equivalence between systems (5.3) and (6.3), for an arbitrary $\nu \neq 0$.

Step 1. With respect to the variable $V^\varepsilon = (\mathcal{H}^\varepsilon, E^\varepsilon)$ system (6.3) reads

$$\mathfrak{B}^\varepsilon_0\partial_t V^\varepsilon + \widetilde{\mathfrak{B}}^\varepsilon_1\partial_1 V^\varepsilon + \sum_{k=2}^{3}\mathfrak{B}^\varepsilon_k\partial_k V^\varepsilon + \mathfrak{B}^\varepsilon_4 V^\varepsilon = 0, \quad (6.4)$$

where

$$\mathfrak{B}^\varepsilon_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \varepsilon\nu_3 & -\varepsilon\nu_2\\ 0 & 1 & 0 & -\varepsilon\nu_3 & 0 & \varepsilon\nu_1\\ 0 & 0 & 1 & \varepsilon\nu_2 & -\varepsilon\nu_1 & 0\\ 0 & -\varepsilon\nu_3 & \varepsilon\nu_2 & 1 & 0 & 0\\ \varepsilon\nu_3 & 0 & -\varepsilon\nu_1 & 0 & 1 & 0\\ -\varepsilon\nu_2 & \varepsilon\nu_1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_3 & \widehat{B^\varepsilon_0}\\ \widehat{B^\varepsilon_0}^T & I_3 \end{pmatrix},$$

$$\mathfrak{B}_1^\varepsilon = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 \\ \nu_2 & -\nu_1 & 0 & 0 & 0 & -\varepsilon^{-1} \\ \nu_3 & 0 & -\nu_1 & 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \\ 0 & 0 & \varepsilon^{-1} & \nu_2 & -\nu_1 & 0 \\ 0 & -\varepsilon^{-1} & 0 & \nu_3 & 0 & -\nu_1 \end{pmatrix} = \begin{pmatrix} \widehat{\mathfrak{B}_1} & \widehat{B_1^\varepsilon} \\ \widehat{B_1^\varepsilon}^T & \widehat{\mathfrak{B}_1} \end{pmatrix},$$

$$\mathfrak{B}_2^\varepsilon = \begin{pmatrix} -\nu_2 & \nu_1 & 0 & 0 & 0 & \varepsilon^{-1} \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 \\ 0 & \nu_3 & -\nu_2 & -\varepsilon^{-1} & 0 & 0 \\ 0 & 0 & -\varepsilon^{-1} & -\nu_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \\ \varepsilon^{-1} & 0 & 0 & 0 & \nu_3 & -\nu_2 \end{pmatrix} = \begin{pmatrix} \widehat{\mathfrak{B}_2} & \widehat{B_2^\varepsilon} \\ \widehat{B_2^\varepsilon}^T & \widehat{\mathfrak{B}_2} \end{pmatrix},$$

$$\mathfrak{B}_3^\varepsilon = \begin{pmatrix} -\nu_3 & 0 & \nu_1 & 0 & -\varepsilon^{-1} & 0 \\ 0 & -\nu_3 & \nu_2 & \varepsilon^{-1} & 0 & 0 \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 & -\nu_3 & 0 & \nu_1 \\ -\varepsilon^{-1} & 0 & 0 & 0 & -\nu_3 & \nu_2 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \begin{pmatrix} \widehat{\mathfrak{B}_3} & \widehat{B_3^\varepsilon} \\ \widehat{B_3^\varepsilon}^T & \widehat{\mathfrak{B}_3} \end{pmatrix},$$

$$\widetilde{\mathfrak{B}}_1^\varepsilon = \frac{1}{\partial_1\widehat{\Phi}_1}\Big(\mathfrak{B}_1^\varepsilon - \sum_{k=2}^3 \mathfrak{B}_k^\varepsilon \partial_k\widehat{\Psi}\Big),$$

$$\mathfrak{B}_4^\varepsilon = \mathfrak{B}_0^\varepsilon \widehat{B}_4,$$

where $\widehat{B}_4$ is defined in (5.4).

Note that

$$\widehat{\mathfrak{B}_j}^T = \widehat{\mathfrak{B}_j} \quad (j=1,2,3), \qquad \widehat{B_j^\varepsilon}^T = -\widehat{B_j^\varepsilon} \quad (j=0,1,2,3), \qquad \mathfrak{B}_4^\varepsilon|_{\varepsilon=0} = \widehat{B}_4.$$

Step 2. Again, to avoid the appearance of the "dangerous" multiplier ε^{-1} in the energy integral for problem (6.1) we pass in system (6.4) from the unknown V^ε to the "curved" unknown $\mathcal{W}^\varepsilon = (\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon)$:

$$M_0^\varepsilon \partial_t \mathcal{W}^\varepsilon + \widetilde{M}_1^\varepsilon \partial_1 \mathcal{W}^\varepsilon + \sum_{k=2}^3 M_k^\varepsilon \partial_k \mathcal{W}^\varepsilon + M_4^\varepsilon \mathcal{W}^\varepsilon = 0, \tag{6.5}$$

where

$$\begin{aligned} &M_0^\varepsilon = -\frac{1}{\partial_1\widehat{\Phi}_1} K\mathfrak{B}_0^\varepsilon K^{\mathsf{T}} > 0, \\ &\widetilde{M}_1^\varepsilon = -\frac{1}{\partial_1\widehat{\Phi}_1} K\widetilde{\mathfrak{B}}_1^\varepsilon K^{\mathsf{T}}, \quad M_k^\varepsilon = -\frac{1}{\partial_1\widehat{\Phi}_1} K\mathfrak{B}_k^\varepsilon K^{\mathsf{T}} \quad (k=2,3), \\ &M_4^\varepsilon = \widehat{M}_4^\varepsilon + \widetilde{M}_4, \quad \widehat{M}_4^\varepsilon = -\frac{1}{\partial_1\widehat{\Phi}_1} K\mathfrak{B}_4^\varepsilon K^{\mathsf{T}} - K\mathfrak{B}_0^\varepsilon \partial_t(L^{-1}), \\ &\widetilde{M}_4 = -K\left(\widetilde{\mathfrak{B}}_1^\varepsilon \partial_1(L^{-1}) + \mathfrak{B}_2^\varepsilon \partial_2(L^{-1}) + \mathfrak{B}_3^\varepsilon \partial_3(L^{-1})\right), \end{aligned} \tag{6.6}$$

the matrices L and K are obtained from the relations

$$\mathcal{W}^\varepsilon = LV^\varepsilon, \quad L^{-1} = \frac{1}{\partial_1\widehat{\Phi}_1} K^{\mathsf{T}}, \quad K = I_2 \otimes \widehat{K},$$

and the matrix $\widehat{K}$ was defined in (5.22). The symmetric system (6.5) is hyperbolic if $M_0^\varepsilon > 0$, i.e.

$$\varepsilon|\nu| < 1\,.$$

The last inequality is satisfied for small ε.

We need to know the behavior of the above matrices in (6.6) as $\varepsilon \to 0$. To this end, we find that

$$\begin{aligned} &M_0^\varepsilon = O(1), \quad \widetilde{M}_1^\varepsilon = -B_1^\varepsilon + O(1)\,, \\ &M_k^\varepsilon = -B_k^\varepsilon + O(1) \quad (k=2,3), \quad M_4^\varepsilon = O(1)\,, \end{aligned} \tag{6.7}$$

where by $O(1)$ we denote a generic matrix bounded w.r.t. ε and the matrices B_j^ε were defined in (5.6). As the matrices M_0^ε and M_4^ε do not contain the multiplier ε^{-1}, their norms are bounded as $\varepsilon \to 0$. Recalling that the matrices B_j^ε are constant, we deduce as well that all the possible derivatives (with respect to t and x) of the matrices $\widetilde{M}_1^\varepsilon$, M_k^ε have bounded norms as $\varepsilon \to 0$.

6.3. *Final form of the regularized problem.* After all the changes of unknowns described above the regularized problem (6.1) takes the new form

$$\check{A}_0^\varepsilon \partial_t Y^\varepsilon + \sum_{j=1}^{3} \check{A}_j^\varepsilon \partial_j Y^\varepsilon + \check{A}_4^\varepsilon Y^\varepsilon = \widetilde{F} \qquad \text{in } Q^+\,, \tag{6.8a}$$

$$M_0^\varepsilon \partial_t \mathcal{W}^\varepsilon + \widetilde{M}_1^\varepsilon \partial_1 \mathcal{W}^\varepsilon + \sum_{k=2}^{3} M_k^\varepsilon \partial_k \mathcal{W}^\varepsilon + M_4^\varepsilon \mathcal{W}^\varepsilon = 0 \qquad \text{in } Q^-\,, \tag{6.8b}$$

$$\partial_t \varphi^\varepsilon = u_1^\varepsilon - \hat{v}_2 \partial_2 \varphi^\varepsilon - \hat{v}_3 \partial_3 \varphi^\varepsilon + \varphi^\varepsilon \partial_1 \hat{v}_{\hat{N}}\,, \tag{6.8c}$$

$$\varepsilon^{-1} q'^{\,\varepsilon} = (\hat{\mathfrak{h}}, \mathfrak{H}^\varepsilon) - [\partial_1 \hat{q}]\varphi^\varepsilon - \varepsilon(\hat{\mathfrak{e}}, \mathfrak{E}^\varepsilon)\,, \tag{6.8d}$$

$$\mathfrak{E}_2^\varepsilon = \varepsilon\, \partial_t(\widehat{\mathcal{H}}_3 \varphi^\varepsilon) - \varepsilon\, \partial_2(\widehat{E}_1 \varphi^\varepsilon)\,, \tag{6.8e}$$

$$\mathfrak{E}_3^\varepsilon = -\varepsilon\, \partial_t(\widehat{\mathcal{H}}_2 \varphi^\varepsilon) - \varepsilon\, \partial_3(\widehat{E}_1 \varphi^\varepsilon) \quad \text{on } \omega, \tag{6.8f}$$

$$(Y^\varepsilon, \mathcal{W}^\varepsilon, \varphi^\varepsilon) = 0 \qquad \text{for } t < 0, \tag{6.8g}$$

where for the readers convenience we recall that equation (6.8a) is the "compressible" regularization of the plasma system written in terms of the unknown $Y^\epsilon = (q'^{\,\varepsilon}, u^\varepsilon, h^\varepsilon)$ while equation (6.8b) is the "hyperbolic" regularization of the div-curl vacuum system written, after the secondary symmetrization, in terms of $\mathcal{W}^\varepsilon = (\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon)$.

6.4. *Proof of Theorem 6.1.* To obtain the a priori estimate (6.2) we apply the energy methods to the symmetric hyperbolic systems (6.8a), (6.8b). In the sequel $\gamma_0 \geq 1$ denotes a generic constant sufficiently large which may increase from formula to formula, and C is a generic constant that may change from line to line.

First of all let us restate systems (6.8a), (6.8b) in terms of the γ-weighted unknowns $Y^\varepsilon_\gamma, \mathcal{W}^\varepsilon_\gamma$. The equations take the equivalent form

$$\gamma \check{A}^\varepsilon_0 Y^\varepsilon_\gamma + \check{A}_0 \partial_t Y^\varepsilon_\gamma + \sum_{j=1}^3 \check{A}^\varepsilon_j \partial_j Y^\varepsilon_\gamma + \check{A}^\varepsilon_4 Y^\varepsilon_\gamma = \widetilde{F}_\gamma \qquad \text{in } Q^+ \,, \tag{6.9a}$$

$$\gamma M^\varepsilon_0 \mathcal{W}^\varepsilon_\gamma + M^\varepsilon_0 \partial_t \mathcal{W}^\varepsilon_\gamma + \widetilde{M}^\varepsilon_1 \partial_1 \mathcal{W}^\varepsilon_\gamma + \sum_{k=2}^3 M^\varepsilon_k \partial_k \mathcal{W}^\varepsilon_\gamma + M^\varepsilon_4 \mathcal{W}^\varepsilon_\gamma = 0 \qquad \text{in } Q^- \,. \tag{6.9b}$$

The arguments below are, with suitable modifications, analogous to those from [13]. However, for the readers convenience we do not drop them and start with some preparatory estimates.

Conormal derivative of the plasma unknown. First of all we estimate the conormal derivative $\sigma\partial_1$ of Y^ε. Applying to system (6.9a) the operator $\sigma\partial_1$, multiplying by $\sigma\partial_1 Y^\varepsilon_\gamma$ and integrating by parts over Q^+ gives the inequality

$$\begin{aligned} &\gamma \|\sigma\partial_1 Y^\varepsilon_\gamma\|^2_{L^2(Q^+)} \\ &\qquad \leq \frac{C}{\gamma} \left\{ \|\widetilde{F}_\gamma\|^2_{H^1_{tan,\,\gamma}(Q^+)} + \|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} + \|\varepsilon^{-1}\partial_1 \left(\mathcal{E}_{1,2} Y^\varepsilon_\gamma\right)\|^2_{L^2(Q^+)} \right\}, \end{aligned} \tag{6.10}$$

for $\gamma \geq \gamma_0$. On the other hand, directly from equation (6.9a) we get

$$\|\varepsilon^{-1}\partial_1 \left(\mathcal{E}_{1,2} Y^\varepsilon_\gamma\right)\|^2_{L^2(Q^+)} \leq C \left\{ \|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} \right\}, \tag{6.11}$$

where the constant C is independent of ε and γ (recall the definition of the matrix $\mathcal{E}_{1,2}$ in (5.17)). From (6.10), (6.11) we obtain

$$\gamma \|\sigma\partial_1 Y^\varepsilon_\gamma\|^2_{L^2(Q^+)} \leq \frac{C}{\gamma} \left\{ \|\widetilde{F}_\gamma\|^2_{H^1_{tan,\,\gamma}(Q^+)} + \|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} \right\}, \qquad \gamma \geq \gamma_0 \,, \tag{6.12}$$

where C is independent of ε and γ.

Normal derivative of the noncharacteristic part of the plasma unknown. Also, using the structure of the boundary matrix in (6.9a) (see (5.19)) and the divergence constraint (5.26a) allows us to get an estimate of the noncharacteristic part $Y^\varepsilon_{n,\gamma} = e^{-\gamma t}(\varepsilon^{-1}q'^{\,\varepsilon}, u^\varepsilon_1, h^\varepsilon_1)$ of the "plasma" unknown:

$$\|\partial_1 Y^\varepsilon_{n,\gamma}\|^2_{L^2(Q^+)} \leq C \left\{ \|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} \right\}, \tag{6.13}$$

where C is independent of ε and γ.

Normal derivative of the vacuum unknown. As in [13], from system (6.9b) and the divergence constraints (5.26b) we can express the normal derivative of all components of the "vacuum" unknown $\mathcal{W}^\varepsilon_\gamma$ through its tangential derivatives. This gives the estimate

$$\|\partial_1 \mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)} \leq C \left\{ \gamma^2 \|\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)} + \|\partial_t \mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)} + \sum_{k=2}^3 \|\partial_k \mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)} \right\}, \tag{6.14}$$

where C is independent of ε and γ, for all $\varepsilon \leq \varepsilon_0$.

L^2–estimate of the front. Writing the first boundary condition (6.8c) for φ_γ, that is

$$\gamma\varphi^\varepsilon_\gamma + \partial_t\varphi^\varepsilon_\gamma = u^\varepsilon_{1,\gamma} - \hat v_2\partial_2\varphi^\varepsilon_\gamma - \hat v_3\partial_3\varphi^\varepsilon_\gamma + \varphi^\varepsilon_\gamma\partial_1\hat v_{\hat N}\,,$$

multiplying it by $\varphi^\varepsilon_\gamma$ and integrating by parts over ω yields

$$\gamma\|\varphi^\varepsilon_\gamma\|^2_{L^2(\omega)} \le \frac{C}{\gamma}\|u^\varepsilon_{1,\gamma}\|^2_{L^2(\omega)}\,, \qquad \gamma\ge\gamma_0\,, \tag{6.15}$$

where C is independent of γ.

Tangential derivatives of the front. As in [13], assumption (4.1) on the basic state $(\widehat U,\widehat{\mathcal H})$ allows to solve the system of the boundary conditions (5.26c), (5.26d) and (6.8c) (stated in terms of φ_γ) as an algebraic system for the space-time gradient $\nabla_{t,x'}\varphi^\varepsilon_\gamma = (\partial_t\varphi^\varepsilon_\gamma, \partial_2\varphi^\varepsilon_\gamma, \partial_3\varphi^\varepsilon_\gamma)$

$$\nabla_{t,x'}\varphi^\varepsilon_\gamma = \widehat a_1 h^\varepsilon_{1,\gamma} + \widehat a_2\mathfrak h^\varepsilon_{1,\gamma} + \widehat a_3 u^\varepsilon_{1,\gamma} + \widehat a_4\varphi^\varepsilon_\gamma + \gamma\widehat a_5\varphi^\varepsilon_\gamma\,, \tag{6.16}$$

where the vector-functions $\widehat a_\alpha = \widehat a_\alpha(\widehat U|_\omega, \widehat{\mathcal H}|_\omega)$ could be written explicitly.[2] From (6.16) we may estimate $\nabla_{t,x'}\varphi^\varepsilon_\gamma$ through the trace on the boundary ω of the noncharacteristic part of the unknowns $(Y^\varepsilon_\gamma, \mathcal W^\varepsilon_\gamma)$ and $\varphi^\varepsilon_\gamma$ itself:

$$\|\nabla_{t,x'}\varphi^\varepsilon_\gamma\|_{L^2(\omega)} \le C\left\{\|Y^\varepsilon_{n,\gamma}|_\omega\|_{L^2(\omega)} + \|\mathcal W^\varepsilon_\gamma|_\omega\|_{L^2(\omega)} + \gamma\|\varphi^\varepsilon_\gamma\|_{L^2(\omega)}\right\}\,, \qquad \gamma\ge\gamma_0\,, \tag{6.17}$$

where C is independent of ε and γ.

L^2–estimate. Now we are going to derive an L^2–energy estimate for $(Y^\varepsilon, \mathcal W^\varepsilon)$. To this end, we multiply system (6.9a) by Y^ε_γ and (6.9b) by $\mathcal W^\varepsilon_\gamma$, integrate by parts over $Q^\pm$ to find

$$\begin{aligned}
\gamma\int_{Q^+}(\check A^\varepsilon_0 Y^\varepsilon_\gamma, Y^\varepsilon_\gamma)\,dx\,dt + \gamma\int_{Q^-}(M^\varepsilon_0\mathcal W^\varepsilon_\gamma, \mathcal W^\varepsilon_\gamma)\,dx\,dt + \int_\omega \mathcal A^\varepsilon\,dx'\,dt \\
= \frac12\int_{Q^+}\Big(\Big(\partial_t\check A^\varepsilon_0 + \sum_{j=1}^3\partial_j\check A^\varepsilon_j - 2\check A^\varepsilon_4\Big)Y^\varepsilon_\gamma, Y^\varepsilon_\gamma\Big)\,dx\,dt \\
+ \frac12\int_{Q^-}\Big(\Big(\partial_t M^\varepsilon_0 + \partial_1\widetilde{M}^\varepsilon_1 + \sum_{k=2}^3\partial_k M^\varepsilon_k - 2M^\varepsilon_4\Big)\mathcal W^\varepsilon_\gamma, \mathcal W^\varepsilon_\gamma\Big)\,dx\,dt \\
+ \int_{Q^+}(\widetilde F_\gamma, Y^\varepsilon_\gamma)\,dx\,dt\,,
\end{aligned} \tag{6.18}$$

where we have denoted

$$\mathcal A^\varepsilon = -\frac12(\check A^\varepsilon_1 Y^\varepsilon_\gamma, Y^\varepsilon_\gamma)|_\omega + \frac12(\widetilde{M}^\varepsilon_1\mathcal W^\varepsilon_\gamma, \mathcal W^\varepsilon_\gamma)|_\omega\,. \tag{6.19}$$

Recalling that $\check A^\varepsilon_0$ and M^ε_0 are positive definite matrices uniformly in ε for $\varepsilon\le\varepsilon_0$, using the Cauchy-Schwarz and Young inequalities, and the fact that

$$\partial_t\check A^\varepsilon_0 + \sum_{j=1}^3\partial_j\check A^\varepsilon_j - 2\check A^\varepsilon_4 = O(1), \quad \partial_t M^\varepsilon_0 + \partial_1\widetilde{M}^\varepsilon_1 + \sum_{k=2}^3\partial_k M^\varepsilon_k - 2M^\varepsilon_4 = O(1),$$

[2] Under the conditions about the basic state $\widehat H_{\hat N} = \widehat{\mathcal H}_{\hat N} = 0$ on ω, one computes $|\widehat H\times\widehat{\mathcal H}|^2 = (\widehat H_2\widehat{\mathcal H}_3 - \widehat H_3\widehat{\mathcal H}_2)^2\langle\nabla'\widehat\varphi\rangle^2$, where $\langle\nabla'\widehat\varphi\rangle = (1+|\partial_2\widehat\varphi|^2+|\partial_3\widehat\varphi|^2)^{1/2}$ and $\widehat H_2\widehat{\mathcal H}_3 - \widehat H_3\widehat{\mathcal H}_2$ is just the determinant of the 2×2 algebraic system for $\partial_2\varphi^\varepsilon$, $\partial_3\varphi^\varepsilon$ made from the boundary conditions (5.26c), (5.26d).

from (6.18) we derive the L^2 estimate

$$\gamma\|Y^\varepsilon_\gamma\|^2_{L^2(Q^+)} + \gamma\|\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q_-)} + \int_\omega \mathcal{A}^\varepsilon\, dx'\, dt \leq C\left\{\frac{1}{\gamma}\|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y^\varepsilon_\gamma\|^2_{L^2(Q^+)} + \|\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}\right\}, \quad (6.20)$$

where C is independent of ε and γ.

Using (5.27), we obtain

$$(\check{A}^\varepsilon_1 Y^\varepsilon_\gamma, Y^\varepsilon_\gamma) = \varepsilon^{-1}(\mathcal{E}_{1,2}Y^\varepsilon_\gamma, Y^\varepsilon_\gamma) = e^{-2\gamma t}2\varepsilon^{-1}q'^{\,\varepsilon}u^\varepsilon_1 \quad \text{on } \omega\,.$$

Following [13], we choose the functions ν_j in the secondary symmetrization (6.3) by setting

$$\nu_1 = \bar{v}_1 = \hat{v}_2\partial_2\hat{\varphi} + \hat{v}_3\partial_3\hat{\varphi}, \quad \nu_k = \hat{v}_k, \quad k = 2,3. \quad (6.21)$$

After long calculations we get

$$\mathcal{A}^\varepsilon = e^{-2\gamma t}\big\{-\varepsilon^{-1}q'^{\,\varepsilon}u^\varepsilon_1 + \varepsilon^{-1}(\mathfrak{H}^\varepsilon_3\mathfrak{E}^\varepsilon_2 - \mathfrak{H}^\varepsilon_2\mathfrak{E}^\varepsilon_3) + (\hat{v}_2\mathfrak{H}^\varepsilon_2 + \hat{v}_3\mathfrak{H}^\varepsilon_3)\mathcal{H}^\varepsilon_{\hat{N}} + (\hat{v}_2\mathfrak{E}^\varepsilon_2 + \hat{v}_3\mathfrak{E}^\varepsilon_3)E^\varepsilon_{\hat{N}}\big\} \quad \text{on } \omega\,. \quad (6.22)$$

Now we use the boundary conditions in (6.1) and the assumption $\widehat{\mathcal{H}}_{\hat{N}}|_\omega = 0$ for calculating the quadratic form $\mathcal{A}^\varepsilon$. Thanks to the multiplicative factor ε in the boundary conditions (6.8e), (6.8f), we get rid of the singular multiplier ε^{-1} appearing in the second term of the right-hand side of (6.22). The factor ε^{-1} multiplying $q'^{\,\varepsilon}$ in the right-hand side of (6.22) is not dangerous because it is included in the definition of the noncharacteristic component $Y^\varepsilon_n = (\varepsilon^{-1}q'^{\,\varepsilon}, u^\varepsilon_1, h^\varepsilon_1)$ of the vector function $Y^\varepsilon = (q'^{\,\varepsilon}, u^\varepsilon, h^\varepsilon)$ to be estimated (see (6.2) in Theorem 6.1).

After long calculations we get

$$\begin{aligned}\mathcal{A}^\varepsilon = e^{-2\gamma t}\Big\{&\big(\widehat{E}_1 + \hat{v}_2\widehat{\mathcal{H}}_3 - \hat{v}_3\widehat{\mathcal{H}}_2\big)\big(\varepsilon E^\varepsilon_{\hat{N}}\partial_t\varphi^\varepsilon + \mathfrak{H}^\varepsilon_2\partial_3\varphi^\varepsilon - \mathfrak{H}^\varepsilon_3\partial_2\varphi^\varepsilon\big)\\ &+ \varepsilon\big(\widehat{E}_{\hat{\tau}_2}E^\varepsilon_2 + \widehat{E}_{\hat{\tau}_3}E^\varepsilon_3\big)\big(\partial_t\varphi^\varepsilon + \hat{v}_2\partial_2\varphi^\varepsilon + \hat{v}_3\partial_3\varphi^\varepsilon\big)\\ &+ \varphi^\varepsilon\big\{[\partial_1\hat{q}]\,v^\varepsilon_{\hat{N}} - \partial_1\hat{v}_{\hat{N}}(\varepsilon^{-1}q'^{\,\varepsilon} + [\partial_1\hat{q}]\varphi^\varepsilon) + (\partial_t\widehat{\mathcal{H}}_3 - \partial_2\widehat{E}_1)(\mathfrak{H}^\varepsilon_3 + \varepsilon\hat{v}_2E^\varepsilon_{\hat{N}})\\ &+(\partial_t\widehat{\mathcal{H}}_2 + \partial_3\widehat{E}_1)(\mathfrak{H}^\varepsilon_2 - \varepsilon\hat{v}_3E^\varepsilon_{\hat{N}}) + (\partial_2\widehat{\mathcal{H}}_2 + \partial_3\widehat{\mathcal{H}}_3)(\hat{v}_2\mathfrak{H}^\varepsilon_2 + \hat{v}_3\mathfrak{H}^\varepsilon_3)\big\}\Big\}\,. \end{aligned} \quad (6.23)$$

Now we make the following choice of the coefficients $\widehat{E}_j$ in the boundary conditions (6.8d)-(6.8f):

$$\widehat{E} = -\bar{v}\times\widehat{\mathcal{H}}, \quad (6.24)$$

where $\bar{v} = (\bar{v}_1, \hat{v}_2, \hat{v}_3)$. For this choice

$$\widehat{E}_1 + \hat{v}_2\widehat{\mathcal{H}}_3 - \hat{v}_3\widehat{\mathcal{H}}_2 = 0, \quad \widehat{E}_{\hat{\tau}_2} = 0, \quad \widehat{E}_{\hat{\tau}_3} = 0,$$

and this leaves us with

$$
\begin{aligned}
\mathcal{A}^\varepsilon = e^{-2\gamma t}\big\{\varphi^\varepsilon\big\{[\partial_1\hat{q}]\,u_1^\varepsilon - \partial_1\hat{v}_{\hat{N}}(\varepsilon^{-1}q'^{\,\varepsilon} + [\partial_1\hat{q}]\varphi^\varepsilon)\\
+ (\partial_t\widehat{\mathcal{H}}_3 - \partial_2\widehat{E}_1)(\mathfrak{H}_3^\varepsilon + \varepsilon\hat{v}_2E^\varepsilon_{\hat{N}})\\
+(\partial_t\widehat{\mathcal{H}}_2 + \partial_3\widehat{E}_1)(\mathfrak{H}_2^\varepsilon - \varepsilon\hat{v}_3E^\varepsilon_{\hat{N}}) + (\partial_2\widehat{\mathcal{H}}_2 + \partial_3\widehat{\mathcal{H}}_3)(\hat{v}_2\mathfrak{H}_2^\varepsilon + \hat{v}_3\mathfrak{H}_3^\varepsilon)\big\}\Big\}\\
= \varphi_\gamma^\varepsilon\big\{[\partial_1\hat{q}]\,u_{1,\gamma}^\varepsilon - \partial_1\hat{v}_{\hat{N}}(\varepsilon^{-1}q'^{\,\varepsilon}_\gamma + [\partial_1\hat{q}]\varphi_\gamma^\varepsilon)\\
+ (\partial_t\widehat{\mathcal{H}}_3 - \partial_2\widehat{E}_1)(\mathfrak{H}_{3,\gamma}^\varepsilon + \varepsilon\hat{v}_2E^\varepsilon_{\hat{N},\gamma}) + (\partial_t\widehat{\mathcal{H}}_2 + \partial_3\widehat{E}_1)(\mathfrak{H}_{2,\gamma}^\varepsilon - \varepsilon\hat{v}_3E^\varepsilon_{\hat{N},\gamma})\\
+ (\partial_2\widehat{\mathcal{H}}_2 + \partial_3\widehat{\mathcal{H}}_3)(\hat{v}_2\mathfrak{H}_{2,\gamma}^\varepsilon + \hat{v}_3\mathfrak{H}_{3,\gamma}^\varepsilon)\big\}\,, \quad (6.25)
\end{aligned}
$$

where we restore the usage of the subscript γ.

Substituting (6.25) into (6.20) and using the Cauchy-Schwarz and Young inequalities, from (6.20) we get

$$
\begin{aligned}
\gamma\|Y_\gamma^\varepsilon\|^2_{L^2(Q^+)} + \gamma\|\mathcal{W}_\gamma^\varepsilon\|^2_{L^2(Q_-)} \le \frac{C}{\gamma}\Big\{\|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)}\\
+ \|\mathcal{W}_\gamma^\varepsilon|_\omega\|^2_{L^2(\omega)}\Big\} + C\left\{\|Y_\gamma^\varepsilon\|^2_{L^2(Q^+)} + \|\mathcal{W}_\gamma^\varepsilon\|^2_{L^2(Q^-)}\right\} + C\gamma\|\varphi_\gamma^\varepsilon\|^2_{L^2(\omega)}\,, \quad (6.26)
\end{aligned}
$$

where C is independent of ε and γ. Thus, if γ_0 is large enough, we obtain from (6.26) and (6.15)

$$
\begin{aligned}
&\gamma\|Y_\gamma^\varepsilon\|^2_{L^2(Q^+)} + \gamma\|\mathcal{W}_\gamma^\varepsilon\|^2_{L^2(Q_-)}\\
&\quad\le \frac{C}{\gamma}\left\{\|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)} + \|\mathcal{W}_\gamma^\varepsilon|_\omega\|^2_{L^2(\omega)}\right\}, \quad 0<\varepsilon\le\varepsilon_0\,,\ \gamma\ge\gamma_0\,, \quad (6.27)
\end{aligned}
$$

where C is independent of ε and γ.

Tangential derivatives. Now we are going to derive an a priori estimate for the tangential time-space derivatives of Y^ε, $\mathcal{W}^\varepsilon$. For simplicity, let us denote by $\mathbb{Y}_\gamma^\varepsilon$, $\mathbb{W}_\gamma^\varepsilon$ the vectors

$$
\mathbb{Y}_\gamma^\varepsilon = e^{-\gamma t}\begin{pmatrix}\partial_t Y^\varepsilon\\ \partial_2 Y^\varepsilon\\ \partial_3 Y^\varepsilon\end{pmatrix}, \quad \mathbb{W}_\gamma^\varepsilon = e^{-\gamma t}\begin{pmatrix}\partial_t \mathcal{W}^\varepsilon\\ \partial_2 \mathcal{W}^\varepsilon\\ \partial_3 \mathcal{W}^\varepsilon\end{pmatrix}. \quad (6.28)
$$

Below it will be sometimes convenient to write ∂_0 instead of ∂_t. Applying ∂_l to (6.9a), (6.9b) for $l = 0, 2, 3$, we easily find that the vector-functions $\mathbb{Y}_\gamma^\varepsilon$, $\mathbb{W}_\gamma^\varepsilon$ must solve the following system

$$
\begin{aligned}
&\gamma\check{\mathcal{A}}_0^\varepsilon\mathbb{Y}_\gamma^\varepsilon + \check{\mathcal{A}}_0^\varepsilon\partial_t\mathbb{Y}_\gamma^\varepsilon + \sum_{j=1}^3\check{\mathcal{A}}_j^\varepsilon\partial_j\mathbb{Y}_\gamma^\varepsilon + \check{\mathcal{A}}_4^\varepsilon\mathbb{Y}_\gamma^\varepsilon = \mathbb{F}_\gamma \qquad \text{in } Q^+,\\
&\gamma\mathcal{M}_0^\varepsilon\mathbb{W}_\gamma^\varepsilon + \mathcal{M}_0^\varepsilon\partial_t\mathbb{W}_\gamma^\varepsilon + \widetilde{\mathcal{M}}_1^\varepsilon\partial_1\mathbb{W}_\gamma^\varepsilon + \sum_{k=2}^3\mathcal{M}_k^\varepsilon\partial_k\mathbb{W}_\gamma^\varepsilon + \mathcal{M}_4^\varepsilon\mathbb{W}_\gamma^\varepsilon = \mathbb{G}_\gamma \qquad \text{in } Q^-\,,
\end{aligned} \quad (6.29)
$$

where

$$
\check{\mathcal{A}}_j^\varepsilon = \begin{pmatrix}\check{A}_j^\varepsilon & & \\ & \ddots & \\ & & \check{A}_j^\varepsilon\end{pmatrix}, \; j = 0, \cdots, 3\,,
$$

$$\breve{\mathcal{A}}_4^\varepsilon = \begin{pmatrix} \breve{A}_4^\varepsilon + \partial_t \breve{A}_0^\varepsilon & \partial_t \breve{A}_2^\varepsilon & \partial_t \breve{A}_3^\varepsilon \\ \partial_2 \breve{A}_0^\varepsilon & \breve{A}_4^\varepsilon + \partial_2 \breve{A}_2^\varepsilon & \partial_2 \breve{A}_3^\varepsilon \\ \partial_3 \breve{A}_0^\varepsilon & \partial_3 \breve{A}_2^\varepsilon & \breve{A}_4^\varepsilon + \partial_3 \breve{A}_3^\varepsilon \end{pmatrix},$$

$$\mathbb{F}_\gamma = e^{-\gamma t} \begin{pmatrix} \partial_t \widetilde{F} - \partial_t \breve{A}_1^\varepsilon \partial_1 Y^\varepsilon - \partial_t \breve{A}_4^\varepsilon Y^\varepsilon \\ \partial_2 \widetilde{F} - \partial_2 \breve{A}_1^\varepsilon \partial_1 Y^\varepsilon - \partial_2 \breve{A}_4^\varepsilon Y^\varepsilon \\ \partial_3 \widetilde{F} - \partial_3 \breve{A}_1^\varepsilon \partial_1 Y^\varepsilon - \partial_3 \breve{A}_4^\varepsilon Y^\varepsilon \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}_1^\varepsilon = \begin{pmatrix} \widetilde{M}_1^\varepsilon & & \\ & \ddots & \\ & & \widetilde{M}_1^\varepsilon \end{pmatrix}, \quad \mathcal{M}_l^\varepsilon = \begin{pmatrix} M_l^\varepsilon & & \\ & \ddots & \\ & & M_l^\varepsilon \end{pmatrix}, \quad l = 0, 2, 3,$$

$$\mathcal{M}_4^\varepsilon = \begin{pmatrix} M_4^\varepsilon + \partial_t M_0^\varepsilon & \partial_t M_2^\varepsilon & \partial_t M_3^\varepsilon \\ \partial_2 M_0^\varepsilon & M_4^\varepsilon + \partial_2 M_2^\varepsilon & \partial_2 M_3^\varepsilon \\ \partial_3 M_0^\varepsilon & \partial_3 M_2^\varepsilon & M_4^\varepsilon + \partial_3 M_3^\varepsilon \end{pmatrix},$$

$$\mathbb{G}_\gamma = e^{-\gamma t} \begin{pmatrix} -\partial_t \widetilde{M}_1^\varepsilon \partial_1 \mathcal{W}^\varepsilon - \partial_t M_4^\varepsilon \mathcal{W}^\varepsilon \\ -\partial_2 \widetilde{M}_1^\varepsilon \partial_1 \mathcal{W}^\varepsilon - \partial_2 M_4^\varepsilon \mathcal{W}^\varepsilon \\ -\partial_3 \widetilde{M}_1^\varepsilon \partial_1 \mathcal{W}^\varepsilon - \partial_3 M_4^\varepsilon \mathcal{W}^\varepsilon \end{pmatrix}.$$

Arguing as for (6.9), we derive from (6.29) that

$$\begin{aligned}
\gamma \sum_{l=0,2,3} \int_{Q^+} & e^{-2\gamma t} (\breve{A}_0^\varepsilon \partial_l Y^\varepsilon, \partial_l Y^\varepsilon)\, dx\, dt \\
& + \gamma \sum_{l=0,2,3} \int_{Q^-} e^{-2\gamma t} (M_0^\varepsilon \partial_l \mathcal{W}^\varepsilon, \partial_l \mathcal{W}^\varepsilon)\, dx\, dt \\
& + \sum_{l=0,2,3} \frac{1}{2} \int_\omega e^{-2\gamma t} \Big\{ (\widetilde{M}_1^\varepsilon \partial_l \mathcal{W}^\varepsilon, \partial_l \mathcal{W}^\varepsilon) - (\breve{A}_1^\varepsilon \partial_l Y^\varepsilon, \partial_l Y^\varepsilon) \Big\}\, dx'\, dt \\
& = \frac{1}{2} \int_{Q^+} \left(\left(\sum_{j=0}^{3} \partial_j \breve{\mathcal{A}}_j^\varepsilon - 2 \breve{\mathcal{A}}_4^\varepsilon \right) \mathbb{Y}_\gamma^\varepsilon, \mathbb{Y}_\gamma^\varepsilon \right) dx\, dt \\
& + \frac{1}{2} \int_{Q^-} \left(\left(\partial_t \mathcal{M}_0^\varepsilon + \partial_1 \widetilde{\mathcal{M}}_1^\varepsilon + \sum_{k=2}^{3} \partial_k \mathcal{M}_k^\varepsilon - 2 \mathcal{M}_4^\varepsilon \right) \mathbb{W}_\gamma^\varepsilon, \mathbb{W}_\gamma^\varepsilon \right) dx\, dt \\
& + \int_{Q^+} (\mathbb{F}_\gamma, \mathbb{Y}_\gamma^\varepsilon)\, dx\, dt + \int_{Q^-} (\mathbb{G}_\gamma, \mathbb{W}_\gamma^\varepsilon)\, dx\, dt . \qquad (6.30)
\end{aligned}$$

The source terms $\mathbb{F}_\gamma$, $\mathbb{G}_\gamma$ appearing in the right-hand sides of (6.29) contain the derivatives of the functions Y_γ^ε, $\mathcal{W}_\gamma^\varepsilon$.

Since the normal derivative of the full vacuum unknown $\mathcal{W}_\gamma^\varepsilon$ is estimated through its tangential derivatives by (6.14), using also the structure of the matrices M_j^ε (cf. (6.7)) we get

$$\begin{aligned}\int_{Q^-}\left(\mathbb{G}_\gamma,\mathbb{W}^\varepsilon_\gamma\right)\,dx\,dt &= \sum_{l=0,2,3}\int_{Q^-} e^{-2\gamma t}\partial_l\widetilde{M}^\varepsilon_1\partial_1\mathcal{W}^\varepsilon\partial_l\mathcal{W}^\varepsilon\,dx\,dt\\ &\quad-\sum_{l=0,2,3}\int_{Q^-} e^{-2\gamma t}\partial_l M^\varepsilon_4\mathcal{W}^\varepsilon\partial_l\mathcal{W}^\varepsilon\,dx\,dt\\ &\le C\left\{\gamma^2\|\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}+\sum_{l=0,2,3}\|\partial_l\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}\right\}\,, \qquad (6.31)\end{aligned}$$

where C is independent of ε and γ.

Since the normal derivative of only the noncharacteristic part of the plasma unknown Y^ε_γ is estimated in (6.13), the source term $\mathbb{F}_\gamma$ requires more attention. Firstly, from the definition of $\mathbb{F}_\gamma$ we get

$$\begin{aligned}\int_{Q^+}(\mathbb{F}_\gamma,\mathbb{Y}^\varepsilon_\gamma)\,dx\,dt &= \sum_{l=0,2,3}\int_{Q^+} e^{-2\gamma t}\partial_l\widetilde{F}\partial_l Y^\varepsilon\,dx\,dt\\ -\sum_{l=0,2,3}\int_{Q^+} e^{-2\gamma t}\partial_l\check{A}^\varepsilon_1\partial_1 Y^\varepsilon\partial_l Y^\varepsilon\,dx\,dt &- \sum_{l=0,2,3}\int_{Q^+} e^{-2\gamma t}\partial_l\check{A}^\varepsilon_4 Y^\varepsilon\partial_l Y^\varepsilon\,dx\,dt\,. \qquad (6.32)\end{aligned}$$

The first and last integrals in the right-hand side of (6.32) involve only tangential derivatives of Y^ε and $\widetilde{F}$. The second integral in the right-hand side of (6.32) contains the normal derivative of Y^ε. But, it follows from the special structure of the matrix $\check{A}^\varepsilon_1$ given in (5.18) that

$$\partial_l\check{A}^\varepsilon_1=\partial_l\widehat{\mathbb{A}}^\varepsilon_1,\quad l=0,2,3, \qquad (6.33)$$

and (5.19) implies

$$\|\partial_l\widehat{\mathbb{A}}^\varepsilon_1\partial_1 Y^\varepsilon\|_{L^2(Q^+)}\le C\|\sigma\partial_1 Y^\varepsilon\|_{L^2(Q^+)}. \qquad (6.34)$$

By the Cauchy-Schwarz and Young inequalities, (6.32)–(6.34) give the estimate

$$\int_{Q^+}(\mathbb{F}_\gamma,\mathbb{Y}^\varepsilon_\gamma)\,dx\,dt\le\frac{C}{\gamma}\sum_{l=0,2,3}\|\partial_l\widetilde{F}_\gamma\|^2_{L^2(Q^+)}+\frac{\gamma}{2}\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\,,\quad\gamma\ge\gamma_0\,, \qquad (6.35)$$

where C is independent of ε and γ.

Using the fact that the matrices $\check{A}^\varepsilon_0$ and M^ε_0 are positive definite uniformly in ε for $\varepsilon\le\varepsilon_0$, from (6.30), (6.31) and (6.35) we derive

$$\begin{aligned}\gamma\sum_{l=0,2,3}\|\partial_l Y^\varepsilon_\gamma\|^2_{L^2(Q^+)}+\gamma\sum_{l=0,2,3}\|\partial_l\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}&\\ +\sum_{l=0,2,3}\int_\omega\mathcal{A}^\varepsilon_l\,dx'\,dt &\le\frac{\gamma}{2}\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\\ +C\left\{\frac{1}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\gamma^2\|\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}+\sum_{l=0,2,3}\|\partial_l\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}\right\}&\,,\quad\gamma\ge\gamma_0\,, \qquad (6.36)\end{aligned}$$

where C is independent of ε and γ and, for each $l = 0, 2, 3$,

$$\mathcal{A}_l^\varepsilon = e^{-2\gamma t}\frac{1}{2}\left\{-(\check{A}_1^\varepsilon \partial_l Y^\varepsilon, \partial_l Y^\varepsilon)|_\omega + (\widetilde{M}_1^\varepsilon \partial_l \mathcal{W}^\varepsilon, \partial_l \mathcal{W}^\varepsilon)|_\omega\right\}$$

$$= e^{-2\gamma t}\frac{1}{2}\left\{-\varepsilon^{-1}(\mathcal{E}_{1,2}\partial_l Y^\varepsilon, \partial_l Y^\varepsilon)|_\omega + (\widetilde{M}_1^\varepsilon \partial_l \mathcal{W}^\varepsilon, \partial_l \mathcal{W}^\varepsilon)|_\omega\right\}$$

$$= e^{-2\gamma t}\left\{\frac{1}{2}(\widetilde{M}_1^\varepsilon \partial_l \mathcal{W}^\varepsilon, \partial_l \mathcal{W}^\varepsilon)|_\omega - \varepsilon^{-1}\partial_l q'^{\,\varepsilon}\partial_l u_1^\varepsilon|_\omega\right\}. \quad (6.37)$$

For the same choices as in (6.21) and (6.24), we obtain for $\mathcal{A}_l^\varepsilon$ the following expression

$$\begin{aligned}\mathcal{A}_l^\varepsilon = e^{-2\gamma t}\partial_l\varphi^\varepsilon\Big\{&[\partial_1\widehat{q}](\partial_l u_1^\varepsilon + \partial_1\widehat{v}_{\hat{N}}\partial_l\varphi^\varepsilon + \varepsilon^{-1}\partial_1\widehat{v}_{\hat{N}}\partial_l q'^{\,\varepsilon}\\ &+ (\widehat{v}_2\partial_l\mathfrak{H}_2^\varepsilon + \widehat{v}_3\partial_l\mathfrak{H}_3^\varepsilon)(\partial_2\widehat{\mathcal{H}}_2 + \partial_3\widehat{\mathcal{H}}_3)\\ &+ (\partial_t\widehat{\mathcal{H}}_2 + \partial_3\widehat{E}_1)(\partial_l\mathfrak{H}_2^\varepsilon - \varepsilon\widehat{v}_3\partial_l E_{\hat{N}}^\varepsilon)\\ &+ (\partial_t\widehat{\mathcal{H}}_3 - \partial_2\widehat{E}_1)(\partial_l\mathfrak{H}_3^\varepsilon + \varepsilon\widehat{v}_2\partial_l E_{\hat{N}}^\varepsilon)\Big\} + \text{l.o.t.}, \quad \text{on } \omega, \quad (6.38)\end{aligned}$$

where l.o.t. is the sum of lower-order terms. The presence of ε^{-1} in the right-hand side of (6.38) is not dangerous because $\varepsilon^{-1}q'^{\,\varepsilon}$ is a component of the noncharacteristic part Y_n^ε of the vector function Y^ε. Using (6.16), we reduce the terms involved in (6.38) to those like

$$\widehat{c}h_1^\varepsilon\partial_l u_1^\varepsilon, \quad \widehat{c}h_1^\varepsilon\partial_l\varphi^\varepsilon, \quad \widehat{c}h_1^\varepsilon\partial_l\mathfrak{H}_j^\varepsilon, \quad \widehat{c}h_1^\varepsilon\partial_l\mathfrak{E}_j^\varepsilon, \ldots \text{on } \omega,$$

terms as above with $\mathfrak{h}_1^\varepsilon$, u_1^ε instead of h_1^ε, or even "better" terms like

$$\gamma\widehat{c}\varphi^\varepsilon\partial_l u_1^\varepsilon, \quad \gamma\widehat{c}\varphi^\varepsilon\partial_l\varphi^\varepsilon.$$

Here and below, $\widehat{c}$ denotes a generic coefficient depending on the basic state (2.2). Integrating by parts, such "better" terms can be reduced to the above ones and terms of lower order.

Concerning the terms like $\widehat{c}h_1^\varepsilon\partial_l u^\varepsilon_{1|\,x_1=0}$, they are estimated by passing to the volume integral and again integrating by parts:

$$\int_\omega e^{-2\gamma t}\widehat{c}h_1^\varepsilon\partial_l u^\varepsilon_{1|\,x_1=0}\,dx'\,dt = -\int_{Q^+} e^{-2\gamma t}\partial_1\left(\tilde{c}h_1^\varepsilon\partial_l u_1^\varepsilon\right)\,dx\,dt$$

$$= \int_{Q^+} e^{-2\gamma t}\big\{(\partial_l\tilde{c})h_1^\varepsilon(\partial_1 u_1^\varepsilon) + \tilde{c}(\partial_l h_1^\varepsilon)\partial_1 u_1^\varepsilon - (\partial_1\tilde{c})h_1^\varepsilon\partial_l u_1^\varepsilon - \tilde{c}(\partial_1 h_1^\varepsilon)\partial_l u_1^\varepsilon\big\}\,dx\,dt,$$

where $\tilde{c}|_{x_1=0} = \widehat{c}$. To estimate the volume integrals in the right-hand side of the equality above we only need to control normal derivatives of the noncharacteristic unknown $Y_{n,\gamma}^\varepsilon$. Thus, applying the Cauchy-Schwarz inequality and using (6.13) gives

$$\left|\int_\omega e^{-2\gamma t}\widehat{c}h_1^\varepsilon\partial_l u^\varepsilon_{1|\,x_1=0}\,dx'\,dt\right| \leq C\left\{\|\widetilde{F}_\gamma\|^2_{L^2(Q^+)} + \|Y_\gamma^\varepsilon\|^2_{H^1_{tan,\gamma}(Q^+)}\right\}. \quad (6.39)$$

In the same way we estimate the other similar terms $\widehat{c}h_1^\varepsilon\partial_l\mathfrak{H}^\varepsilon_{j|\,x_1=0}$, $\widehat{c}h_1^\varepsilon\partial_l\mathfrak{E}^\varepsilon_{j|\,x_1=0}$, $\widehat{c}\mathfrak{h}_1^\varepsilon\partial_l u^\varepsilon_{1|\,x_1=0}$, $\widehat{c}\mathfrak{h}_1^\varepsilon\partial_l\mathfrak{E}^\varepsilon_{j|\,x_1=0}$, where, after an integration by parts, again we only need to estimate normal derivatives either of components of $Y^\varepsilon_{n,\gamma}$ or $\mathcal{W}^\varepsilon_\gamma$, by using (6.13) and (6.14).

We treat the terms like $\widehat{c}h^\varepsilon_{1|\,x_1=0}\partial_l\varphi^\varepsilon$ again by substituting (6.16):

$$\begin{aligned}&\left|\int_\omega e^{-2\gamma t}\widehat{c}h^\varepsilon_1\partial_l\varphi^\varepsilon\,dx'\,dt\right|\\&\qquad=\left|\int_\omega \widehat{c}h^\varepsilon_{1,\gamma}\left(\widehat{a}_1h^\varepsilon_{1,\gamma}+\widehat{a}_2\mathfrak{h}^\varepsilon_{1,\gamma}+\widehat{a}_3u^\varepsilon_{1,\gamma}+\widehat{a}_4\varphi^\varepsilon_\gamma+\gamma\widehat{a}_5\varphi^\varepsilon_\gamma\right)\,dx'\,dt\right|\\&\qquad\qquad\le C\left\{\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{L^2(\omega)}+\gamma^2\|\varphi^\varepsilon_\gamma\|^2_{L^2(\omega)}\right\}.\end{aligned}\tag{6.40}$$

Final estimate. Combining (6.36), (6.39), (6.40), (6.15) and similar estimates for the other terms in (6.38) yields

$$\begin{aligned}&\gamma\sum_{l=0,2,3}\|\partial_lY^\varepsilon_\gamma\|^2_{L^2(Q^+)}+\gamma\sum_{l=0,2,3}\|\partial_l\mathcal{W}^\varepsilon_\gamma\|^2_{L^2(Q^-)}\\&\qquad\le\frac{\gamma}{2}\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+C\bigg\{\frac{1}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)}\\&\qquad\qquad+\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{L^2(\omega)}\bigg\},\quad 0<\varepsilon<\varepsilon_0\,,\ \gamma\ge\gamma_0\,,\end{aligned}\tag{6.41}$$

where C is independent of ε, γ. Then adding (6.12), (6.14), (6.27), (6.41) we obtain

$$\begin{aligned}&\gamma\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\gamma\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)}\le C\bigg\{\frac{1}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\\&\quad+\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)}+\gamma\left(\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{L^2(\omega)}\right)\bigg\},\quad 0<\varepsilon<\varepsilon_0\,,\ \gamma\ge\gamma_0\,,\end{aligned}\tag{6.42}$$

where C is independent of ε, γ.

It remains to produce an estimate for the traces on ω of $Y^\varepsilon_{n,\gamma}$ and $\mathcal{W}^\varepsilon_\gamma$. This is done following the same arguments of [13].

LEMMA 6.2. The functions $Y^\varepsilon_{n,\gamma}$ and $\mathcal{W}^\varepsilon_\gamma$ satisfy

$$\begin{aligned}&\gamma\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{L^2(\omega)}+\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{H^{1/2}_\gamma(\omega)}\le C\left(\|\widetilde{F}_\gamma\|^2_{L^2(Q^+)}+\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\right),\\&\gamma\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{L^2(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{H^{1/2}_\gamma(\omega)}\le C\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)}\,.\end{aligned}\tag{6.43}$$

Substituting (6.43) in (6.42) and taking γ_0 sufficiently large yields

$$\begin{aligned}&\gamma\|Y^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\gamma\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)}\\&\qquad\le\frac{C}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\,,\quad 0<\varepsilon<\varepsilon_0\,,\quad\gamma\ge\gamma_0\,,\end{aligned}\tag{6.44}$$

where C is independent of ε, γ. Finally, from (6.15), (6.17), (6.43) we get

$$\gamma\left(\|Y^\varepsilon_{n,\gamma}|_\omega\|^2_{H^{1/2}_\gamma(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{H^{1/2}_\gamma(\omega)}\right)+\gamma^2\|\varphi^\varepsilon_\gamma\|^2_{H^1_\gamma(\omega)} \leq \frac{C}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\,,\quad 0<\varepsilon<\varepsilon_0\,,\quad \gamma\geq\gamma_0\,. \tag{6.45}$$

Adding (6.44), (6.45) gives (6.2). This completes the proof of Theorem 6.1.

Remark 6.3. For the sequel it is convenient to reformulate Theorem 6.1 in terms of the original variable q^ε for the BVP corresponding to (5.1), (5.3), (5.7) when the time belongs to $\mathbb{R}$, i.e., for the problem

$$\begin{cases}\varepsilon^2\left\{\partial_t q^\varepsilon-(\partial_t\widehat{H},H^\varepsilon)-(\widehat{H},\partial_t H^\varepsilon)+\dfrac{1}{\partial_1\widehat{\Phi}_1}(\hat{w},\nabla q^\varepsilon)\right.\\ \qquad\left.-\dfrac{1}{\partial_1\widehat{\Phi}_1}\left(\widehat{w},(\nabla\widehat{H},H^\varepsilon)\right)-\dfrac{1}{\partial_1\widehat{\Phi}_1}\left(\widehat{w},(\widehat{H},\nabla H^\varepsilon)\right)\right\}+\dfrac{1}{\partial_1\widehat{\Phi}_1}\operatorname{div}u^\varepsilon=0\,,\\ \partial_t v^\varepsilon+\dfrac{1}{\partial_1\widehat{\Phi}_1}\left\{(\hat{w},\nabla)v^\varepsilon-(\hat{h},\nabla)H^\varepsilon\right\}+\nabla_{\widehat{\Phi}}q^\varepsilon+C_1(\widehat{W},\widehat{\Psi})W^\varepsilon=f_v\,,\\ \partial_t H^\varepsilon+\dfrac{1}{\partial_1\widehat{\Phi}_1}\left\{(\hat{w},\nabla)H^\varepsilon-(\hat{h},\nabla)v^\varepsilon\right\}\\ \qquad\qquad+C_2(\widehat{W},\widehat{\Psi})W^\varepsilon+\dfrac{\widehat{H}}{\partial_1\widehat{\Phi}_1}\operatorname{div}u^\varepsilon=0\quad\text{in }Q^+\,,\\ \varepsilon\,\partial_t\mathfrak{h}^\varepsilon+\nabla\times\mathfrak{E}^\varepsilon=0\\ \varepsilon\,\partial_t\mathfrak{e}^\varepsilon-\nabla\times\mathfrak{H}^\varepsilon=0\quad\text{in }Q^-,\\ \partial_t\varphi^\varepsilon=u_1^\varepsilon-\hat{v}_2\partial_2\varphi^\varepsilon-\hat{v}_3\partial_3\varphi^\varepsilon+\varphi^\varepsilon\partial_1\hat{v}_{\hat{N}},\\ q^\varepsilon=(\widehat{\mathfrak{h}},\mathfrak{H}^\varepsilon)-[\partial_1\hat{q}]\varphi^\varepsilon-\varepsilon(\widehat{\mathfrak{e}},\mathfrak{E}^\varepsilon),\\ \mathfrak{E}^\varepsilon_2=\varepsilon\,\partial_t(\widehat{\mathcal{H}}_3\varphi^\varepsilon)-\varepsilon\,\partial_2(\widehat{E}_1\varphi^\varepsilon),\\ \mathfrak{E}^\varepsilon_3=-\varepsilon\,\partial_t(\widehat{\mathcal{H}}_2\varphi^\varepsilon)-\varepsilon\,\partial_3(\widehat{E}_1\varphi^\varepsilon)\quad\text{on }\omega\,.\end{cases} \tag{6.46}$$

Theorem 6.4. Let the basic state (2.2) satisfies assumptions (2.3)-(2.5) and (4.1) for all times. There exist $\varepsilon_0>0$, $\gamma_0\geq1$ such that if $0<\varepsilon<\varepsilon_0$ and $\gamma\geq\gamma_0$ then all sufficiently smooth solutions $(q^\varepsilon,u^\varepsilon,h^\varepsilon,\mathcal{W}^\varepsilon,\varphi^\varepsilon)$ of problem (6.46) obey the estimate

$$\gamma\Big(\|\varepsilon q^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}+\|(u^\varepsilon_\gamma,h^\varepsilon_\gamma)\|^2_{H^1_{tan,\gamma}(Q^+)}+\|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-)} +\|(q^\varepsilon_\gamma,u^\varepsilon_{1,\gamma},h^\varepsilon_{1,\gamma})|_\omega\|^2_{H^{1/2}_\gamma(\omega)}+\|\mathcal{W}^\varepsilon_\gamma|_\omega\|^2_{H^{1/2}_\gamma(\omega)}\Big) +\gamma\|\nabla q^\varepsilon_\gamma\|^2_{L^2(Q^+)}+\gamma^2\|\varphi^\varepsilon_\gamma\|^2_{H^1_\gamma(\omega)}\leq\frac{C}{\gamma}\|\widetilde{F}_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)}\,, \tag{6.47}$$

where we have set $\mathcal{W}^\varepsilon_\gamma=e^{-\gamma t}\mathcal{W}^\varepsilon$, $\varphi^\varepsilon_\gamma=e^{-\gamma t}\varphi^\varepsilon$ and so on, and where $C=C(K,\delta)>0$ is a constant independent of the data $\widetilde{F}$ and the parameters ε, γ.

Proof. From the regularized interior equation $(6.46)_2$ we can express $\nabla_{\widehat{\Phi}} q^\varepsilon$ through conormal derivatives of (u,h) (since $\widehat{w}_1=\widehat{h}_1=0$ on ω) by

$$\nabla_{\widehat{\Phi}} q^\varepsilon = f_v - \partial_t v^\varepsilon - \frac{1}{\partial_1\widehat{\Phi}_1}\left\{(\hat{w},\nabla)v^\varepsilon - (\hat{h},\nabla)H^\varepsilon\right\} - C_1(\widehat{W},\widehat{\Psi})W^\varepsilon\,,$$

and we use

$$\nabla q^\varepsilon = \partial_1\widehat{\Phi}_1\widehat{J}^T\nabla_{\widehat{\Phi}} q^\varepsilon.$$

The rest of the estimate (6.47) comes from (6.2). □

7. Well posedness of the hyperbolic regularized problem. In this section, we focus on the existence of the solution to the regularized problem (5.25). Here, we follow a general strategy that is usual for initial-boundary value problems for linear hyperbolic systems (see e.g. [2], [13]). One firstly reduces the time-dependent problem (5.25) to the boundary value problem (6.1) (where the time spans the whole real line) by a suitable time-extension of the data $\widetilde{F}$.[3] Then one proves the existence of the solution of the boundary value problem (6.1) with such an extended data. The restriction to the time interval $(-\infty,T]$ of the solution to (6.1) will provide the solution of problem (5.25), (5.26).

As a first step, we prove the existence of the solution of (6.1). Here we rely on the result obtained by Secchi and Trakhinin in [13], where the plasma-vacuum problem for the compressible MHD equations was studied. Indeed, for a fixed ε, problem (6.1) coincides with that in [13] up to the passage to new "scaled" unknowns.[4] By applying [13, Theorem 15] to (6.1) for fixed ε $(0<\varepsilon<\varepsilon_0)$, we get the following result.

Lemma 7.1. There exist $\gamma_0\geq 1$, $\varepsilon_0>0$ such that for all $\gamma\geq\gamma_0$, $0<\varepsilon<\varepsilon_0$ and $\widetilde{F}_\gamma=(0,\widetilde{f}_{v,\gamma},0)\in H^1_{tan,\gamma}(Q^+)$, vanishing in the past, problem (6.1) has a unique solution $(Y^\varepsilon_\gamma,\mathcal{W}^\varepsilon_\gamma,\varphi^\varepsilon_\gamma)\in H^1_{tan,\gamma}(Q^+)\times H^1_\gamma(Q^-)\times H^{3/2}_\gamma(\omega)$ with $(Y^\varepsilon_{n,\gamma},\mathcal{W}^\varepsilon_\gamma)|_\omega\in H^{1/2}_\gamma(\omega)$.

Remark 7.2. In view of the proof of Theorem 4.1, that will be given in the Section 8, it is convenient also to restate Lemma 7.1 in terms of the variable q^ε for the regularized BVP (6.46).

Lemma 7.3. There exist $\gamma_0\geq 1$, $\varepsilon_0>0$ such that for all $\gamma\geq\gamma_0$, $0<\varepsilon<\varepsilon_0$ and $\widetilde{F}_\gamma=(0,\widetilde{f}_{v,\gamma},0)\in H^1_{tan,\gamma}(Q^+)$, vanishing in the past, problem (6.46) has a unique solution $(q^\varepsilon_\gamma,u^\varepsilon_\gamma,h^\varepsilon_\gamma,\mathcal{W}^\varepsilon_\gamma,\varphi^\varepsilon_\gamma)\in\dot{H}^1_\gamma(Q^+)\times H^1_{tan,\gamma}(Q^+)\times H^1_\gamma(Q^-)\times H^{3/2}_\gamma(\omega)$ with $\varepsilon q^\varepsilon_\gamma\in H^1_{tan,\gamma}(Q^+)$ and $(q^\varepsilon_\gamma,u^\varepsilon_{1,\gamma},h^\varepsilon_{1,\gamma},\mathcal{W}^\varepsilon_\gamma)|_\omega\in H^{1/2}_\gamma(\omega)$.

As we announced before, the existence of a unique solution to the evolution problem (5.1), (5.3), (5.7) for fixed $0<\varepsilon<\varepsilon_0$ and given data $\widetilde{F}_\gamma\in H^1_{tan,\gamma}(Q^+_T)$, vanishing in the past, comes directly from Lemma 7.3 applied to the time-extension of $\widetilde{F}_\gamma$. Since

[3]The extension of the data $\widetilde{F}$ beyond T is made by the use of reflection methods of Lions-Magenes, see [11] and [2] for details. This kind of time-extension keeps the regularity of original data on $(-\infty,T]$. In particular, it defines a continuous operator from $H^1_{tan,\gamma}(Q^+_T)$ into $H^1_{tan,\gamma}(Q^+)$, uniformly with respect to γ.

[4]More precisely, they coincide if without loss of generality we set $\varepsilon=1$ and reduce the regularized linear problem in [13] to a suitable dimensionless form.

the solution of the BVP (6.46) enjoys the a priori estimate (6.47), then the solution $(q^\varepsilon_\gamma, u^\varepsilon_\gamma, h^\varepsilon_\gamma, \mathcal{W}^\varepsilon_\gamma, \varphi^\varepsilon_\gamma) \in \dot H^1_\gamma(Q^+_T) \times H^1_{tan,\gamma}(Q^+_T) \times H^1_\gamma(Q^-_T) \times H^{3/2}_\gamma(\omega_T)$ to (5.1), (5.3), (5.7) satisfies the following estimate

$$\gamma\Big(\|\varepsilon q^\varepsilon_\gamma\|^2_{H^1_{tan,\gamma}(Q^+_T)} + \|(u^\varepsilon_\gamma, h^\varepsilon_\gamma)\|^2_{H^1_{tan,\gamma}(Q^+_T)} + \|\mathcal{W}^\varepsilon_\gamma\|^2_{H^1_\gamma(Q^-_T)} + \|(q^\varepsilon_\gamma, u^\varepsilon_{1,\gamma}, h^\varepsilon_{1,\gamma})|_{\omega_T}\|^2_{H^{1/2}_\gamma(\omega_T)} + \|\mathcal{W}^\varepsilon_\gamma|_{\omega_T}\|^2_{H^{1/2}_\gamma(\omega_T)}\Big) + \gamma\|\nabla q^\varepsilon_\gamma\|^2_{L^2(Q^+_T)} + \gamma^2\|\varphi^\varepsilon_\gamma\|^2_{H^1_\gamma(\omega_T)} \le \frac{C}{\gamma}\|\widetilde F_\gamma\|^2_{H^1_{tan,\gamma}(Q^+_T)}\,, \tag{7.1}$$

where the constant $C = C_T$ is independent of ε and γ.

8. Well-posedness of the original linearized problem in conormal Sobolev spaces. We first prove the well-posedness of problem (2.29) in conormal Sobolev spaces.

LEMMA 8.1. Let $T > 0$. Let the basic state (2.2) satisfy assumptions (2.3)-(2.5) and (4.1). Then there exists $\gamma_0 \ge 1$ such that for all $\gamma \ge \gamma_0$ and for all $f_{v\,,\gamma} \in H^1_{tan\,,\gamma}(Q^+_T)$ vanishing in the past, namely for $t < 0$, problem (2.29) has a unique solution $(U_\gamma, \mathcal{H}_\gamma, \varphi_\gamma)$ such that $(q_\gamma, W_\gamma, \mathcal{H}_\gamma, \varphi_\gamma) \in \dot H^1_\gamma(Q^+_T) \times H^1_{tan\,,\gamma}(Q^+_T) \times H^1_\gamma(Q^-_T) \times H^1_\gamma(\omega_T)$ with the trace $(q_\gamma, u_{1,\gamma}, h_{1,\gamma}, \mathcal{H}_\gamma)|_{\omega_T} \in H^{1/2}_\gamma(\omega_T)$ and obeys the a priori estimate

$$\gamma\Big(\|W_\gamma\|^2_{H^1_{tan\,,\gamma}(Q^+_T)} + \|\nabla q_\gamma\|^2_{L^2(Q^+_T)} + \|\mathcal{H}_\gamma\|^2_{H^1_\gamma(Q^-_T)} + \|(q_\gamma, u_{1,\gamma}, h_{1,\gamma}, \mathcal{H}_\gamma)|_{\omega_T}\|^2_{H^{1/2}_\gamma(\omega_T)}\Big) + \gamma^2\|\varphi_\gamma\|^2_{H^1_\gamma(\omega_T)} \le \frac{C}{\gamma}\|f_{v,\gamma}\|^2_{H^1_{tan\,,\gamma}(Q^+_T)}\,, \tag{8.1}$$

where $C = C(K, T, \delta) > 0$ is a constant independent of the data f_v and the parameter γ.

Proof. For every $\varepsilon > 0$, such that $0 < \varepsilon < \varepsilon_0$, $\gamma \ge \gamma_0$ (where ε_0 and γ_0 are given by Lemma 7.3) and $f_{v,\gamma} \in H^1_{tan,\gamma}(Q^+_T)$, let $(q^\varepsilon_\gamma, u^\varepsilon_\gamma, h^\varepsilon_\gamma, \mathcal{W}^\varepsilon_\gamma, \varphi^\varepsilon_\gamma) \in \dot H^1_\gamma(Q^+_T) \times H^1_{tan,\,\gamma}(Q^+_T) \times H^1_\gamma(Q^-_T) \times H^{3/2}_\gamma(\omega_T)$ be the unique solution to the ε-regularized problem (5.1), (5.3), (5.7) with data $\widetilde F_\gamma = (0, \partial_1\widehat\Phi_1 \widehat J^T f_{v,\gamma}, 0)$.

The a priori estimate (7.1) yields that $\{\nabla q^\varepsilon_\gamma, u^\varepsilon_\gamma, h^\varepsilon_\gamma, \mathcal{W}^\varepsilon_\gamma, \varphi^\varepsilon_\gamma\}_{0<\varepsilon<\varepsilon_0}$ and $\{(q^\varepsilon_\gamma, u^\varepsilon_{1,\gamma}, h^\varepsilon_{1,\gamma}, \mathcal{W}^\varepsilon_\gamma)|_{\omega_T}\}_{0<\varepsilon\le\varepsilon_0}$ are bounded sequences respectively in $L^2(Q^+_T) \times H^1_{tan,\,\gamma}(Q^+_T) \times H^1_\gamma(Q^-_T) \times H^1_\gamma(\omega_T)$ and $H^{1/2}_\gamma(\omega_T)$. Thus, one can pass to the weak limit as $\varepsilon \to 0$, up to subsequences.

In particular,

$$(\nabla q^\varepsilon_\gamma, u^\varepsilon_\gamma, h^\varepsilon_\gamma, \mathcal{W}^\varepsilon_\gamma, \varphi^\varepsilon_\gamma) \rightharpoonup (r_\gamma, u_\gamma, h_\gamma, \mathcal{W}_\gamma, \varphi_\gamma)$$

in $L^2(Q^+_T) \times H^1_{tan,\gamma}(Q^+_T) \times H^1_\gamma(Q^-_T) \times H^1_\gamma(\omega_T)$, with

$$r_\gamma = \nabla q_\gamma, \quad q_\gamma \in \dot H^1_\gamma(Q^+_T), \quad (u_\gamma, h_\gamma) \in H^1_{tan,\,\gamma}(Q^+_T),$$

$$\mathcal{W}_\gamma = (\mathfrak{H}_\gamma, \mathfrak{E}_\gamma) \in H^1_\gamma(Q^-_T), \quad \varphi_\gamma \in H^1_\gamma(\omega_T).$$

We also define $v_\gamma = \widehat J u_\gamma$ and, similarly, we define H_γ, $\mathcal{H}_\gamma$, E_γ through h_γ, $\mathfrak{H}_\gamma$, $\mathfrak{E}_\gamma$.

Firstly, we pass to the limit as $\varepsilon \to 0$ in (5.1a), restated for $q'^{\varepsilon} = \varepsilon q^{\varepsilon}$. Using that q'^{ε} is bounded in $H^1_{tan,\gamma}(Q^+_T)$ from estimate (7.1), we get that the limit u of $\{u^{\varepsilon}\}$ satisfies

$$\operatorname{div} u = 0 \quad \text{in } Q^+_T\,. \tag{8.2}$$

Secondly, passing to the limit as $\varepsilon \to 0$ in the other equations of problem (5.1), (5.3), (5.7) we get that $(v, H, \mathcal{H}, \varphi)$ solves the original problem (2.30), and one has that $E = \mathfrak{E} = 0$.

Passing to the limit as $\varepsilon \to 0$ in the a priori estimate (7.1), we get estimate (8.1) of $(\nabla q, v, H, \mathcal{H}, \varphi)$ (recall that $v = \widehat{J}u$, $H = \widehat{J}h$). This estimate gives the uniqueness of the solution. □

9. Current-vorticity-type linearized system: proof of Theorem 4.1. Just as in (6.13), from (2.30a) and (2.31) we can estimate the normal derivatives of the normal components of the velocity and the plasma magnetic field through conormal derivatives and the source term:

$$\|\partial_1(u_{1,\gamma}, h_{1,\gamma})\|^2_{L^2(Q^+)} \leq C\left\{\|W_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} + \|q_\gamma\|^2_{\dot{H}^1_{tan}(Q^+)} + \|f_{v,\gamma}\|^2_{L^2(Q^+)}\right\},$$

with

$$\|u\|^2_{\dot{H}^1_{tan}(Q^+)} = \sum_{|\alpha|=1} \|\partial^\alpha_{tan} u\|^2_{L^2(Q^+)}.$$

Then, it follows from Lemma 8.1 that there exist $\partial_1 u_{1,\gamma} \in L^2(Q^+_T)$ and $\partial_1 h_{1,\gamma} \in L^2(Q^+_T)$ obeying the estimate

$$\|\partial_1(u_{1,\gamma}, h_{1,\gamma})\|_{L^2(Q^+)} \leq \frac{C}{\gamma}\|f_{v,\gamma}\|_{H^1_{tan,\gamma}(Q^+_T)}. \tag{9.1}$$

To prove the existence of missing normal derivatives of W_γ (and obtain estimates for them) we use arguments similar to those in [12] for incompressible current-vortex sheets. That is, we write down a current-vorticity-type linearized system which is a linear symmetric hyperbolic system for the linearized vorticity $\xi_\gamma = \nabla \times \mathfrak{U}_\gamma$ and current $z_\gamma = \nabla \times \mathfrak{B}_\gamma$, where

$$\mathfrak{U}_\gamma = e^{-\gamma t}\mathfrak{U}, \quad \mathfrak{U} = (v_1\partial_1\widehat{\Phi}_1, v_{\hat{\tau}_2}, v_{\hat{\tau}_3}), \quad v_{\hat{\tau}_i} = v_1\partial_i\widehat{\Psi} + v_i, \quad i = 2,3,$$

$$\mathfrak{B}_\gamma = e^{-\gamma t}\mathfrak{B}, \quad \mathfrak{B} = (H_1\partial_1\widehat{\Phi}_1, H_{\hat{\tau}_2}, H_{\hat{\tau}_3}), \quad H_{\hat{\tau}_i} = H_1\partial_i\widehat{\Psi} + H_i, \quad i = 2,3.$$

For obtaining this system we rewrite equations (2.30a) and (2.30b) as

$$\begin{aligned} &\partial_t\mathfrak{U}_\gamma + \gamma\mathfrak{U}_\gamma + \frac{1}{\partial_1\widehat{\Phi}_1}\left\{(\hat{w},\nabla)\mathfrak{U}_\gamma - (\hat{h},\nabla)\mathfrak{B}_\gamma\right\} + \nabla q_\gamma + \text{l.o.t.}^{(1)} = \tilde{f}_{v,\gamma}\,,\\ &\partial_t\mathfrak{B}_\gamma + \gamma\mathfrak{B}_\gamma + \frac{1}{\partial_1\widehat{\Phi}_1}\left\{(\hat{w},\nabla)\mathfrak{B}_\gamma - (\hat{h},\nabla)\mathfrak{U}_\gamma\right\} + \text{l.o.t.}^{(2)} = 0 \qquad \text{in } Q^+_T\,, \end{aligned} \tag{9.2}$$

where

$$\tilde{f}_{v,\gamma} = e^{-\gamma t}\tilde{f}_v, \quad \tilde{f}_v = (f_1\partial_1\widehat{\Phi}_1, f_{\hat{\tau}_2}, f_{\hat{\tau}_3}), \quad f_{\hat{\tau}_i} = f_1\partial_i\widehat{\Psi} + f_i, \quad i = 2,3,$$

and $\text{l.o.t.}^{(k)}$ $(k = 1,2)$ represent lower-order terms whose exact forms have no meaning (they are linear combinations of components of W_γ).

Applying the curl operator to (9.2) and using the fact that $\partial_1 u_{1,\gamma}$ and $\partial_1 h_{1,\gamma}$ can be expressed through conormal derivatives, we obtain the current-vorticity-type system

$$\partial_t Z_\gamma + \gamma Z_\gamma + \frac{1}{\partial_1\widehat{\Phi}_1}\begin{pmatrix} \hat{w}\cdot\nabla & -\hat{h}\cdot\nabla \\ -\hat{h}\cdot\nabla & \hat{w}\cdot\nabla \end{pmatrix} Z_\gamma + \mathfrak{C}(\widehat{W},\widehat{\Psi})Z_\gamma = \mathfrak{F} \qquad \text{in } Q_T^+\,, \tag{9.3}$$

where

$$Z_\gamma = \begin{pmatrix} \xi_\gamma \\ z_\gamma \end{pmatrix}, \qquad \mathfrak{F} = \begin{pmatrix} \nabla\times\tilde{f}_{v,\gamma} + \text{l.o.t.}_{(1)} \\ \text{l.o.t.}_{(2)} \end{pmatrix},$$

the coefficients of the matrix $\mathfrak{C} = \mathfrak{C}(\widehat{W},\widehat{\Psi})$ are of no interest and $\text{l.o.t.}_{(k)}$ $(k = 1,2)$ represent linear combinations of components of W_γ and its conormal derivatives. Clearly, (9.3) is a symmetric hyperbolic system for the vector Z_γ, provided the right-hand side $\mathfrak{F}$ is given. It is worth noting that, in view of Lemma 8.1, $\mathfrak{F}\in L^2(Q_T^+)$. Since $\hat{w}_1|_{\omega_T} = \hat{h}_1|_{\omega_T} = 0$ (see (2.4c), (2.5)), the linear symmetric hyperbolic system (9.3) does not need any boundary conditions on ω_T. Thanks to classical results the Cauchy problem for this system has a unique strong solution $Z_\gamma\in L^2(Q_T^+)$ obeying the estimate

$$\|Z_\gamma\|^2 \le \frac{C}{\gamma^2}\|\mathfrak{F}\|^2_{L^2(Q_T^+)} \le \frac{C}{\gamma^2}\left\{\|W_\gamma\|^2_{H^1_{tan,\gamma}(Q^+)} + \|f_{v,\gamma}\|^2_{H^1_\gamma(Q^+)}\right\}. \tag{9.4}$$

Since the components of W can be expressed through u_1, h_1 and the components of $\mathfrak{U}$ and $\mathfrak{B}$, we can express $\partial_1 W_\gamma$ through $\partial_1 u_{1,\gamma}$, $\partial_1 h_{1,\gamma}$, ξ_γ and z_γ. Hence, there exists $\partial_1 W_\gamma\in L^2(Q_T^+)$. Moreover, estimates (8.1), (9.1) and (9.4) imply the a priori estimate (4.2). This completes the proof of Theorem 4.1.

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