Intersections of the Hermitian surface with irreducible quadrics in $\mathrm{PG}(3, q^2)$, q odd

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Abstract

In $\mathrm{PG}(3, q^2)$, with q odd, we determine the possible intersection sizes of a Hermitian surface $\mathcal{H}$ and an irreducible quadric $\mathcal{Q}$ having the same tangent plane at a common point $P \in \mathcal{Q} \cap \mathcal{H}$.

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1. Introduction

The study of the intersection of hypersurfaces in projective spaces is a deep classical problem in algebraic geometry; see [13] for the general theory. Many of the usual properties over algebraically closed fields, e.g. Bézout's theorem, fail to fully hold in the finite case. There are several results on the combinatorial characterization of the intersection pattern of curves and surfaces as there is a close relationship between the size of the intersection of two varieties and the weight distribution of some functional linear codes; see [18]. The problem, when both surfaces are of the same degree has been widely investigated; see for instance [3, 9, 4] for the case of quadrics or [14, 7, 5] for that of Hermitian surfaces.

When one of the variety is Hermitian and the other is a quadric the problem appears to be more difficult to tackle. For instance, the possible intersection patterns between Hermitian curves and conics have been studied in [6], whereas the possible intersections between a conic and a curve of a non-classical Buekenhout-Metz unital in $\mathrm{PG}(2, q^2)$ have been determined in [1].

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In a recent series of papers, the largest values for the spectrum of the intersection of Hermitian and quadratic varieties have been investigated, as it is related to the minimum distance of certain functional codes. We refer to [8, 11, 10, 15, 12] for codes defined on a Hermitian variety by quadratic forms and [2] for the converse, i.e. codes defined by Hermitian forms on a quadric. In both cases, it appears that the maximum cardinality should be attained when one of the varieties splits in the union of hyperplanes, even if this, in the case of [2], is, for the time being, still an open conjecture.

The setting of the present paper is slightly different. Here we restrict ourselves to dimension 3 and assume the characteristic to be odd. We aim to provide the spectrum of all possible intersection numbers between a Hermitian surface and an irreducible quadric, under the further assumption that they share a common tangent plane. Our main result is contained in the following Theorem 1.1.

Theorem 1.1. *In* $\mathrm{PG}(3, q^2)$*, with q odd, let $\mathcal{H}$ and $\mathcal{Q}$ be respectively a Hermitian surface and an irreducible quadric having the same tangent plane π at a common nonsingular point P. Then, the possible intersection sizes for $\mathcal{H} \cap \mathcal{Q}$ in* $\mathrm{PG}(3, q^2)$ *are as follows.*

- *For $\mathcal{Q}$ elliptic:*

$$q^3-q^2+1, q^3-q^2+q+1, q^3-q+1, q^3+1, q^3+q+1, q^3+q^2-q+1, q^3+q^2+1.$$

- *For $\mathcal{Q}$ a 1–degenerate cone and $q > 3$,*

$$q^3-q^2+1, q^3-q^2+q+1, q^3-q+1, q^3+1, q^3+q+1,$$
$$q^3+q^2-q+1, q^3+q^2+1, q^3+2q^2-q+1.$$

- *For $\mathcal{Q}$ hyperbolic:*

$$q^2+1, q^3-q^2+1, q^3-q^2+q+1, q^3-q+1, q^3+1, q^3+q+1, q^3+q^2-q+1, q^3+q^2+1,$$
$$q^3+2q^2-q+1, q^3+2q^2+1, q^3+3q^2-q+1, 2q^3+q^2+1.$$

The proof of this theorem is contained in Section 2.

In Section 3 we characterize the geometric configurations corresponding respectively to the minimum and maximum cardinality allowable: in both cases, detailed in Theorems 3.1 and 3.2, the quadric $\mathcal{Q}$ is hyperbolic and in permutable position with the Hermitian surface $\mathcal{H}$.

2. Proof of Theorem 1.1

Fix a projective frame in $\mathrm{PG}(3,q^2)$ with homogeneous coordinates (J,X,Y,Z), and consider the affine space $\mathrm{AG}(3,q^2)$ whose infinite hyperplane Σ_∞ has equation $J=0$. Then, $\mathrm{AG}(3,q^2)$ has affine coordinates (x,y,z), where $x=X/J$, $y=Y/J$ and $z=Z/J$.

Since all non-degenerate Hermitian surfaces are projectively equivalent, we can assume without loss of generality $\mathcal{H}$ to have affine equation

$$z^q+z=x^{q+1}+y^{q+1}. \tag{1}$$

The unitary group $\mathrm{PGU}(4,q)$ is transitive on the points of $\mathcal{H}$; see [19]. Thus, we can also suppose $P\in\mathcal{H}\cap\mathcal{Q}$ to be $P=P_\infty(0,0,0,1)$; the common tangent plane π between $\mathcal{Q}$ and $\mathcal{H}$ at P is then Σ_∞. Under the aforementioned assumptions, the equation of the irreducible quadric $\mathcal{Q}$ is of the form

$$z=ax^2+by^2+cxy+dx+ey+f, \tag{2}$$

where $(a,b,c)\neq(0,0,0)$ and $\Delta=4ab-c^2\neq 0$ when $\mathcal{Q}$ is non-singular.

Write $\mathcal{C}_\infty:=\mathcal{Q}\cap\mathcal{H}\cap\Sigma_\infty$. When $\mathcal{Q}$ is elliptic, that is Δ is a nonsquare in $\mathrm{GF}(q^2)$, the point P_∞ is, clearly, its only point at infinity of the intersection; that is $\mathcal{C}_\infty=\{P_\infty\}$. The structure of $\mathcal{C}_\infty$ when $\mathcal{Q}$ is hyperbolic or a 1–degenerate cone, is detailed by the following lemma.

Lemma 2.1. *If $\mathcal{Q}$ is a 1–degenerate cone, then $\mathcal{C}_\infty$ consists of either one or q^2+1 points on a line. When $\mathcal{Q}$ is a hyperbolic quadric, then $\mathcal{C}_\infty$ consists of either one, or q^2+1 or $2q^2+1$ points. All cases may actually occur.*

Proof. As both $\mathcal{H}\cap\Sigma_\infty$ and $\mathcal{Q}\cap\Sigma_\infty$ split in lines through P_∞, it is straightforward to see that the only possibilities for $\mathcal{C}_\infty$ are the following: when $\mathcal{Q}$ is a 1–degenerate cone $\mathcal{C}_\infty$ is either a point or one line, whereas, when $\mathcal{Q}$ is a hyperbolic quadric, $\mathcal{C}_\infty$ consists of either a point or one or two lines. Actually, the set $\mathcal{C}_\infty$ is determined by the following equations

$$\begin{cases} x^{q+1}+y^{q+1}=0\\ ax^2+by^2+cxy=0. \end{cases} \tag{3}$$

Suppose $\mathcal{Q}$ to be a 1–degenerate cone. Under our assumptions, when $c=0$ we can assume either $a=0$ or $b=0$. In both cases $\mathcal{C}_\infty$ is the point P_∞.

If $c\neq 0$, then the vertex of $\mathcal{Q}$ is the point $V=(0,-c,2a,e-dc)$. As P_∞ is not the vertex of $\mathcal{Q}$, the conic $\mathcal{C}_\infty$ consists of a line ℓ if and only if $V\in\mathcal{H}$. This is the same as to require $c^{q+1}+4a^{q+1}=0$, that is $||c||=-4||a||$, where $||x||$ denotes the norm of $x\in\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$.

Suppose now that $\mathcal{Q}$ is a hyperbolic quadric. We consider the intersection of $\mathcal{C}_\infty$ with a line ℓ of Σ_∞ not through P_∞. When $a = 0$, let $\ell : x = 1$; then from (3) we get

$$\begin{cases} 1 + y^{q+1} = 0 \\ y(by + c) = 0. \end{cases} \tag{4}$$

This system admits solution if and only if $b \neq 0$ and $||cb^{-1}|| = -1$. In this case, clearly, its solution is unique and $\mathcal{C}_\infty$ is just a line. An analogous argument applies when $b = 0$ and $a \neq 0$.

Suppose now $a, b \neq 0$. We can take $\ell : y = 1$. Then, from System (3) we obtain

$$\begin{cases} x^{q+1} + 1 = 0 \\ x^2 + \frac{c}{a}x + \frac{b}{a} = 0. \end{cases} \tag{5}$$

Set $s^2 = c^2 - 4ab \neq 0$. The solutions of the second equation in (5) are

$$x_{1/2} = \frac{-c \pm s}{2a}.$$

If $c = 0$, then (5) has either 2 or 0 solutions, according as $||s|| = -4||a||$ or not. Consequently, $\mathcal{C}_\infty$ consists of either $2q^2 + 1$ points or just one point, namely P_∞.

If $c \neq 0$ and $||s - c|| = ||s + c|| = -4||b||$, then $x_1^{q+1} = x_2^{q+1} = -1$; thus, $\mathcal{C}_\infty$ contains $2q^2 + 1$ points.

If $c \neq 0$, $||s - c|| \neq ||s + c||$ and either $||s + c|| = -4||b||$ or $||s - c|| = -4||b||$, then $\mathcal{C}_\infty$ consists of $q^2 + 1$ points. In all of the remaining cases, $\mathcal{C}_\infty = \{P_\infty\}$. $\square$

Lemma 2.2. *Under the assumptions of Theorem 1.1, the possible sizes of* $(\mathcal{H} \cap \mathcal{Q}) \setminus \pi$ *are either*

$$q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2$$

when $\mathcal{Q}$ *is elliptic or a* 1*–degenerate cone or*

$$q^2, q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2, 2q^3 - q^2$$

when $\mathcal{Q}$ *is hyperbolic.*

Proof. We use the same setup as at the beginning of this section. Thus, we need to compute the number of common points between $\mathcal{H}$ and $\mathcal{Q}$ in $\mathrm{AG}(3, q^2)$; hence, we have to study the following system of equations

$$\begin{cases} z^q + z = x^{q+1} + y^{q+1} \\ z = ax^2 + by^2 + cxy + dx + ey + f. \end{cases} \tag{6}$$

Once the value of z is recovered from the second equation and substituted in the first, we obtain

$$\begin{aligned} &a^q x^{2q} + b^q y^{2q} + c^q x^q y^q + d^q x^q + e^q y^q + f^q + ax^2 + by^2 \\ &+cxy + dx + ey + f = x^{q+1} + y^{q+1}. \end{aligned} \tag{7}$$

We now need to determine the number of solutions of (7) as a, b, c, d, e, f vary in $\mathrm{GF}(q^2)$. To this purpose, choose a primitive element β of $\mathrm{GF}(q^2)$. As q is odd, $\varepsilon = \beta^{(q+1)/2} \in \mathrm{GF}(q^2)$ and $\varepsilon^q = -\varepsilon$; furthermore, ε^2 is a primitive element of $\mathrm{GF}(q)$. Since $\varepsilon \notin \mathrm{GF}(q)$, it is immediate to see that $\{1, \varepsilon\}$ is a basis of $\mathrm{GF}(q^2)$, regarded as a vector space over $\mathrm{GF}(q)$. We shall consequently write each $x \in \mathrm{GF}(q^2)$ as a $\mathrm{GF}(q)$–linear combination $x = x_0 + x_1\varepsilon$ with $x_0, x_1 \in \mathrm{GF}(q)$.

By regarding $\mathrm{GF}(q^2)$ as a 2–dimensional vector space over $\mathrm{GF}(q)$, (7) can be rewritten as

$$\begin{aligned} &(2a_0 - 1)x_0^2 + (2a_0 + 1)\varepsilon^2 x_1^2 + 4\varepsilon^2 a_1 x_0 x_1 + (2b_0 - 1)y_0^2 + \varepsilon^2(2b_0 + 1)y_1^2 + \\ &4\varepsilon^2 b_1 y_0 y_1 + 2c_0 x_0 y_0 + 2\varepsilon^2 c_0 x_1 y_1 + 2\varepsilon^2 c_1 x_0 y_1 + 2\varepsilon^2 c_1 x_1 y_0 + \\ &2d_0 x_0 + 2\varepsilon^2 d_1 x_1 + 2e_0 y_0 + 2\varepsilon^2 e_1 y_1 + 2f_0 = 0. \end{aligned} \tag{8}$$

It is thus possible to consider the solutions (x_0, x_1, y_0, y_1) of (8) as the affine points of the (possibly degenerate) quadric hypersurface Ξ of $\mathrm{PG}(4, q)$ associated to the symmetric 5×5 matrix

$$A = \begin{pmatrix} (2a_0 - 1) & 2\varepsilon^2 a_1 & c_0 & \varepsilon^2 c_1 & d_0 \\ 2\varepsilon^2 a_1 & (2a_0 + 1)\varepsilon^2 & \varepsilon^2 c_1 & \varepsilon^2 c_0 & \varepsilon^2 d_1 \\ c_0 & \varepsilon^2 c_1 & (2b_0 - 1) & 2\varepsilon^2 b_1 & e_0 \\ \varepsilon^2 c_1 & \varepsilon^2 c_0 & 2\varepsilon^2 b_1 & (2b_0 + 1)\varepsilon^2 & \varepsilon^2 e_1 \\ d_0 & \varepsilon^2 d_1 & e_0 & \varepsilon^2 e_1 & 2f_0 \end{pmatrix}.$$

The above argument shows that the number of affine points of Ξ equals the number of points in $\mathrm{AG}(3, q^2)$ which lie in $\mathcal{H} \cap \mathcal{Q}$; using the results of [17] it is possible to actually count these points. To this purpose, first determine the number points at infinity of Ξ. These points are those of the quadric Ξ_∞ of $\mathrm{PG}(3, q)$ associated to the symmetric 4×4 block matrix

$$A_\infty = \begin{pmatrix} \mathfrak{A} & \mathfrak{C} \\ \mathfrak{C} & \mathfrak{B} \end{pmatrix},$$

where

$$\mathfrak{A} := \begin{pmatrix} 2a_0 - 1 & 2\varepsilon^2 a_1 \\ 2\varepsilon^2 a_1 & (2a_0 + 1)\varepsilon^2 \end{pmatrix}, \quad \mathfrak{B} := \begin{pmatrix} 2b_0 - 1 & 2\varepsilon^2 b_1 \\ 2\varepsilon^2 b_1 & (2b_0 + 1)\varepsilon^2 \end{pmatrix},$$

$$\mathfrak{C} := \begin{pmatrix} c_0 & \varepsilon^2 c_1 \\ \varepsilon^2 c_1 & \varepsilon^2 c_0 \end{pmatrix}.$$

Denote by (t, u, v, w) homogeneous coordinates for this $\mathrm{PG}(3, q)$. Observe that

$$\det A_\infty = (c^2 - 4ab)^{q+1} - 4a^{q+1} - 4b^{q+1} - 2c^{q+1} + 1. \tag{9}$$

We first show that $\operatorname{rank} A_\infty \geq 2$. Actually, if it were $\operatorname{rank} A_\infty = 1$, then

$$\frac{1}{\varepsilon^2} \det \mathfrak{C} = c_0^2 - \varepsilon^2 c_1^2 = 0. \tag{10}$$

As ε^2 is a non-square in $\mathrm{GF}(q)$, Condition (10) is equivalent to $c_0 = c_1 = 0$. Thus, A_∞ would be of the form

$$A_\infty = \begin{pmatrix} 2a_0 - 1 & 2\varepsilon^2 a_1 & 0 & 0 \\ 2\varepsilon^2 a_1 & (2a_0+1)\varepsilon^2 & 0 & 0 \\ 0 & 0 & 2b_0 - 1 & 2\varepsilon^2 b_1 \\ 0 & 0 & 2\varepsilon^2 b_1 & (2b_0+1)\varepsilon^2 \end{pmatrix}.$$

Denote by R_1, R_2, R_3, R_4 the rows of A_∞. For $\operatorname{rank} A_\infty = 1$ and R_1 not null, there should be α_i such that $R_i = \alpha_i R_1$ for $2 \leq i \leq 4$. As the last two entries of R_1 are 0, $\alpha_3 = \alpha_4 = 0$ and the matrix $\mathfrak{B}$ would be null. This gives $b_0 = 2^{-1} = -2^{-1}$, a contradiction as the characteristic of $\mathrm{GF}(q)$ is odd. The case in which R_1 is null, but a different row is not, is entirely analogous.

Our next step is to prove that when $\operatorname{rank} A_\infty = 2$, then $\mathcal{Q}$ is hyperbolic. Recall that a quadric $\mathcal{Q}$ of the form (2) is hyperbolic if and only if $4ab - c^2$ is a non-zero square in $\mathrm{GF}(q^2)$. We distinguish two cases.

- Suppose $c = 0$; since q is odd, neither $\mathfrak{A}$ nor $\mathfrak{B}$ can be null matrices. Furthermore, $\operatorname{rank} A_\infty = 2$ implies $(a, b) \neq (0, 0)$ and $\det \mathfrak{A} = \det \mathfrak{B} = 0$. Hence, $a^{q+1} = b^{q+1} = 2^{-2}$ and $(ab^{-1})^{q+1} = 1$. In particular, $ab^{-1} \in \langle \varepsilon^{q-1} \rangle$ is a square in $\mathrm{GF}(q^2)$. Consequently, $4ab \neq 0$ is also a square in $\mathrm{GF}(q^2)$. As $c^2 - 4ab \neq 0$, the quadric $\mathcal{Q}$ is non-singular and, thus, hyperbolic.

- Assume $c \neq 0$; then, $\det \mathfrak{C} \neq 0$. Denote by A^{ij} the 3×3 minor of A_∞ obtained by deleting its i–th row and j–th column. Clearly, as $\operatorname{rank} A_\infty = 2$, we have $\det A^{ij} = 0$ for any $1 \leq i, j \leq 4$. In particular, the following system of equations holds:

$$\begin{cases} \frac{1}{4\varepsilon^4} \det A^{41} - \frac{1}{4\varepsilon^4} \det A^{32} = (a_0 - b_0)c_1 + (b_1 - a_1)c_0 = 0 \\ \frac{1}{4\varepsilon^4} \det A^{31} - \frac{1}{4\varepsilon^2} \det A^{42} = (a_1 + b_1)\varepsilon^2 c_1 - (a_0 + b_0)c_0 = 0. \end{cases} \tag{11}$$

  As $\varepsilon^2 c_1^2 - c_0^2 = -||c|| \neq 0$, System (11) has exactly one solution in a_0, a_1, namely

$$a_0 = \frac{(-b_0 c_1 + 2b_1 c_0)c_1 \varepsilon^2 - b_0 c_0^2}{c^{q+1}}, \ a_1 = \frac{(b_1 c_1 \varepsilon^2 - 2b_0 c_0)c_1 + b_1 c_0^2}{c^{q+1}}. \tag{12}$$

Replacing these values in A^{ij} we get $\det A^{ij} = 0$ if and only if $c^{q+1} + 4b^{q+1} = 1$. Thus,

$$4ab - c^2 = -\frac{c}{c^q}\left(c^{q+1} + 4b^{q+1}\right) = \frac{-1}{c^{q-1}}.$$

Since q is odd, $4ab - c^2 \neq 0$ is a square of $\mathrm{GF}(q^2)$ and $\mathcal{Q}$ is hyperbolic.

Now we determine the number N of affine points of Ξ; $N = |\Xi| - |\Xi_\infty|$. Exactly one of the following cases (C1)–(C8) happens.

(C1) $\det A \neq 0$, $\det A_\infty \neq 0$ and $\det A_\infty$ is a square.

In this case, Ξ is a parabolic quadric, Ξ_∞ is a hyperbolic quadric and

$$N = (q+1)(q^2+1) - (q+1)^2 = q^3 - q$$

(C2) $\det A \neq 0$, $\det A_\infty \neq 0$ and $\det A_\infty$ is a nonsquare.

Here the quadric Ξ_∞ is elliptic and

$$N = (q+1)(q^2+1) - (q^2+1) = q^3 + q$$

(C3) $\det A = 0$, $\det A_\infty \neq 0$ and $\det A_\infty$ is a square.

We have that Ξ is a cone projecting a hyperbolic quadric of $\mathrm{PG}(3, q)$; thus

$$N = q(q+1)^2 + 1 - (q+1)^2 = q^3 + q^2 - q.$$

(C4) $\det A = 0$, $\det A_\infty \neq 0$ and $\det A_\infty$ is a nonsquare.

In this case Ξ is a cone projecting an elliptic quadric of $\mathrm{PG}(3, q)$ and thus

$$N = q(q^2+1) + 1 - (q^2+1) = q^3 - q^2 + q$$

(C5) $\operatorname{rank} A = 4$, $\operatorname{rank} A_\infty = 3$.

We get that Ξ_∞ is a cone comprising the join of a point to a conic. Thus, either

$$N = q(q+1)^2 + 1 - [q(q+1)+1] = q^3 + q^2$$

or

$$N = q(q^2+1) + 1 - [q(q+1)+1] = q^3 - q^2,$$

according as Ξ is a cone projecting a hyperbolic or an elliptic quadric.

(C6) $\operatorname{rank} A = \operatorname{rank} A_\infty = 3$.

Here Ξ is the join of a line to a conic; so

$$N = q^3 + q^2 + q + 1 - (q^2 + q + 1) = q^3$$

(C7) rank $A = 3$, rank $A_\infty = 2$.

We get that either

$$N = q^3 + q^2 + q + 1 - (2q^2 + q + 1) = q^3 - q^2$$

or

$$N = q^3 + q^2 + q + 1 - q - 1 = q^3 + q^2,$$

according as Ξ_∞ is a pair of planes or a line.

(C8) rank $A =$ rank $A_\infty = 2$.

In this case either Ξ is a pair of solids and Ξ_∞ is a pair of planes or Ξ is a plane and Ξ_∞ is a line. Thus we get either

$$N = 2q^3 + q^2 + q + 1 - (2q^2 + q + 1) = 2q^3 - q^2$$

or

$$N = q^2 + q + 1 - (q + 1) = q^2.$$

Observe cases (C7) and (C8) are possible only when $\mathcal{Q}$ is hyperbolic. $\square$

We now need to provide compatibility conditions between the set of the affine points of $\mathcal{H} \cap \mathcal{Q}$ and $\mathcal{C}_\infty$, as to explicitly determine $|\mathcal{H} \cap \mathcal{Q}|$. In order to restrict the values of some of the parameters of (2) we use a geometric argument.

Lemma 2.3. *If $\mathcal{Q}$ is a hyperbolic quadric, we can assume without loss of generality:*

1. $b = 0$, *and* $||c|| \neq -||a||$ *if* $\mathcal{C}_\infty$ *is just the point* P_∞*;*
2. $b = 0$, $c = \beta^{q-1/2}a$ *if* $\mathcal{C}_\infty$ *is a line;*
3. $b = -\beta^{(q-1)}a$, $c = 0$ *if* $\mathcal{C}_\infty$ *is the union of two lines.*

When $\mathcal{Q}$ is a cone, we can assume without loss of generality:

1. $b = c = 0$ *if* $\mathcal{C}_\infty$ *is a point;*
2. $b = \beta^{q-1}a$, $c = 2\beta^{(q-1)/2}a$ *if* $\mathcal{C}_\infty$ *is a line.*

Proof. The stabilizer $\mathcal{G}$ of P_∞ in $\mathrm{PGU}(4, q)$ acts on the points of Σ_∞ as the automorphism group of a degenerate Hermitian curve. As such, it has three orbits on the points of Σ_∞, namely, the common tangency point P_∞, the points of $\Sigma_\infty \cap \mathcal{H}$ different from P_∞ and those in $\Sigma_\infty \setminus \mathcal{H}$; see [19, §35 page 47]. Consequently, its action on the lines through P_∞ is the same as that of $\mathrm{PGU}(2, q)$ on the points of $\mathrm{PG}(1, q^2)$: it affords two orbits, say Λ_1 and Λ_2 where Λ_1 consists of the totally isotropic lines of $\mathcal{H}$ through P_∞ while Λ_2 contains the remaining q^2-q lines of Σ_∞ through P_∞. Recall that $\mathcal{G}$ is doubly transitive on Λ_1 and the stabilizer of any $m\in\Lambda_1$ is transitive on Λ_2. Let $\mathcal{Q}_\infty=\mathcal{Q}\cap\Sigma_\infty$. If $\mathcal{Q}$ is hyperbolic and $\mathcal{C}_\infty=\{P_\infty\}$ we can assume $\mathcal{Q}_\infty$ to be the union of the line $\ell: x=0$ and another line, say $ax+cy=0$ with $||a||\neq -||c||$. Thus, $b=0$. Otherwise, up to a suitable element $\sigma\in\mathcal{G}$, we can always take $\mathcal{Q}_\infty$ as the union of any two lines in $\{\ell,m,n\}$ with

$$\ell\colon\ x=0,\qquad m\colon\ x-\beta^{(q-1)/2}y=0,\qquad n\colon\ x+\beta^{(q-1)/2}y=0.$$

Actually, when $\mathcal{C}_\infty$ contains one line we take $\mathcal{Q}_\infty=\ell\, m$, while if $\mathcal{C}_\infty$ is the union of two lines we have $\mathcal{Q}_\infty=mn$. When $\mathcal{Q}$ is a cone, we get either $\mathcal{Q}_\infty=\ell^2$ or $\mathcal{Q}_\infty=n^2$. The lemma follows. $\square$

Lemma 2.4. *Suppose* $\operatorname{rank} A_\infty=2$ *and that* $\mathcal{C}_\infty$ *is the union of two lines. Then,* $|\Xi_\infty|=2q^2+q+1$.

Proof. By Lemma 2.3, assume $c=0$ and $b=-\beta^{q-1}a$. Observe that with this choice $b_0\neq\pm\frac{1}{2}$. We distinguish two cases.

1. If $a=-\frac{1}{2}$, then $b=\frac{1}{2}\beta^{q-1}$. Let M be the nonsingular matrix

$$M=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(2b_0+1)\varepsilon^2&0\\0&0&-2b_1\varepsilon^2&1\end{pmatrix};$$

a direct computation shows that

$$M^TA_\infty M=\begin{pmatrix}-2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&(2b_0+1)\varepsilon^2\end{pmatrix}.$$

Here,

$$2b_0+1=b^q+b+1=\frac{1}{2}\left(\frac{\beta^{q-1}+1}{\beta^{(q-1)/2}}\right)^2.$$

Hence Ξ_∞ is projectively equivalent to

$$-2t^2+(2b_0+1)\varepsilon^2w^2=0$$

and it is the union of two distinct planes if and only if $2(2b_0+1)\varepsilon^2$ is a square of $\mathrm{GF}(q)$.

As $(\beta^{(q+1)/2})^q = -\beta^{(q+1)/2}$, we have $\left(\frac{\beta^{q-1}+1}{\beta^{(q-1)/2}}\right)^q = -\frac{\beta^{q-1}+1}{\beta^{(q-1)/2}}$. Thus, $\frac{\beta^{q-1}+1}{\beta^{(q-1)/2}}$ is not an element of $\mathrm{GF}(q)$. It follows that

$$2(2b_0+1)\varepsilon^2 = \left(\frac{\beta^{q-1}+1}{\beta^{(q-1)/2}}\right)^2 \varepsilon^2$$

is the product of two non-squares and hence it is a square. The quadric Ξ_∞ is reducible in the union of two distinct planes and our lemma follows.

2. Consider now the case $2a_0 + 1 \neq 0$. Take as M the non-singular matrix

$$M = \begin{pmatrix} (2a_0+1)\varepsilon^2 & 0 & 0 & 0 \\ -2a_1\varepsilon^2 & 1 & 0 & 0 \\ 0 & 0 & (2b_0+1)\varepsilon^2 & 0 \\ 0 & 0 & -2b_1\varepsilon^2 & 1 \end{pmatrix}.$$

A straightforward computation proves that

$$M^T A_\infty M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (2a_0+1)\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (2b_0+1)\varepsilon^2 \end{pmatrix}.$$

In particular, the quadric Ξ_∞ defined by A_∞ is projectively equivalent to

$$(2a_0+1)u^2 + (2b_0+1)w^2 = 0.$$

From $4a^{q+1} = 1 = 4b^{q+1}$ and $b = -a\beta^{q-1}$ we get $a = \frac{\beta^{s(q-1)}}{2}$ and $b = -\frac{\beta^{(s+1)(q-1)}}{2}$ for some value of s. As $2a_0+1 = a^q+a+1$ and $2b_0+1 = b^q+b+1$,

$$2a_0+1 = \frac{1}{2}\left(\frac{\beta^{s(q-1)}+1}{\beta^{s(q-1)/2}}\right)^2, \qquad 2b_0+1 = -\frac{1}{2}\left(\frac{\beta^{(s+1)(q-1)}-1}{\beta^{(s+1)(q-1)/2}}\right)^2.$$

If s is odd, then $\left(\frac{\beta^{(s+1)(q-1)}-1}{\beta^{(s+1)(q-1)/2}}\right)$ and $\left(\frac{\beta^{s(q-1)}+1}{\beta^{s(q-1)/2}}\right)$ are in $\mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ while for s even they are elements of $\mathrm{GF}(q)$.
Thus

$$-(2a_0+1)(2b_0+1) = \frac{1}{4}\left(\frac{\beta^{(s+1)(q-1)}-1}{\beta^{(s+1)(q-1)/2}}\right)^2 \left(\frac{\beta^{(s+1)(q-1)}-1}{\beta^{(s+1)(q-1)/2}}\right)^2$$

is a square and the quadric is reducible in the union of two planes.

$\square$

Lemma 2.5. *Suppose $\mathcal{Q}$ to be a hyperbolic quadric with* $\operatorname{rank} A_\infty = 2$ *and that* $\mathcal{C}_\infty$ *is just a point. Then,* $|\Xi_\infty| = q+1$.

Proof. By Lemma 2.3, we may assume $b = 0$. The equations in (12) give now $a = 0$ and $||c|| = 1$. Take

$$M = \begin{pmatrix} 1 & 0 & c_0 & c_1\varepsilon^2 \\ 0 & 1 & -c_1 & -c_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$M^T A_\infty M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \varepsilon^2 & 0 & 0 \\ 0 & 0 & c_0^2 - c_1^2\varepsilon^2 - 1 & 0 \\ 0 & 0 & 0 & c_1^2\varepsilon^4 + (1-c_0^2)\varepsilon^2 \end{pmatrix}$$

As $||c|| = c_0^2 - \varepsilon^2 c_1^2 = 1$, the quadric Ξ_∞ defined by A_∞ is equivalent to $\varepsilon^2 u^2 - t^2 = 0$. As $\varepsilon \notin \mathrm{GF}(q)$, this is the union of two conjugate planes and it consists of just $q+1$ points. □

When $\mathcal{Q}$ is elliptic, clearly $\mathcal{C}_\infty = \{P_\infty\}$. The possible sizes for the affine part of $\mathcal{H} \cap \mathcal{Q}$ correspond to cases (C1)–(C6) of Lemma 2.2, whence the theorem follows.

Consider now the case in which $\mathcal{Q}$ is a cone. Here $\mathcal{C}_\infty$ is either a point or one line; once more, the size of the affine part of $\mathcal{H} \cap \mathcal{Q}$ falls in one of cases (C1)–(C6) of Lemma 2.2. If $\mathcal{C}_\infty = \{P_\infty\}$, by Lemma 2.3, we can assume $b = c = 0$. Under this assumption for $q > 3$ all cases (C1)–(C6) of Lemma 2.2 may occur. For $q = 3$ the only square in GF(3) is 1 and $\det A_\infty = 1 - 4a^{q+1} = 1$ yields $a = 0$. Thus, cases (C1) and (C3) cannot occur.

When $\mathcal{C}_\infty$ consists of one line, then, by Lemma 2.3, we can assume $b = \beta^{(q-1)}a$ and $c = 2\beta^{(q-1)/2}a$. Consequently, $\det A_\infty = 1$ and only cases (C1) and (C3) may occur.

If $\mathcal{Q}$ is a hyperbolic quadric, then we have three possibilities for $\mathcal{C}_\infty$. When $\mathcal{C}_\infty$ consists of two lines, by Lemma 2.3, we can assume $b = -\beta^{(q-1)}a$, $c = 0$. Consequently, $\det A_\infty = (4a^{q+1} - 1)^2$. If $4a^{q+1} \neq 1$, then $\det A_\infty$ is a non-zero square in $\mathrm{GF}(q)$ and cases (C1) and (C3) in Lemma 2.2 may occur. If $4a^{q+1} = 1$, then $\operatorname{rank} A_\infty = 2$ and either (C7) or (C8) occurs. By Lemma 2.4 it follows that in these latter cases Ξ_∞ is the union of two planes. Thus, Case (C7) yields $|\mathcal{H} \cap \mathcal{Q} \cap \mathrm{AG}(3,q^2)| = q^3 - q^2$, while from Case (C8) we obtain $|\mathcal{H} \cap \mathcal{Q} \cap \mathrm{AG}(3,q^2)| = 2q^3 - q^2$.

Suppose now $\mathcal{C}_\infty$ to be just one line. By Lemma 2.3 we can assume $b = 0$, $c = \beta^{q-1/2}a$. This implies $\det A_\infty = (a^{q+1} - 1)^2$, which is a square in $\mathrm{GF}(q)$. When this is zero we have $\operatorname{rank} A_\infty = 3$. Thus only cases (C1), (C3), (C5) and (C6) in Lemma 2.2 might happen.

Finally, if $\mathcal{C}_\infty = \{P_\infty\}$, then all cases (C1)–(C8) might occur. When $\operatorname{rank} A_\infty = 2$, from Lemma 2.5 we have that Ξ_∞ is a line, thus from (C7) we get $|\mathcal{H} \cap \mathcal{Q} \cap \mathrm{AG}(3, q^2)| = q^3 + q^2$, whereas from (C8) $|\mathcal{H} \cap \mathcal{Q} \cap \mathrm{AG}(3, q^2)| = q^2$. Our Theorem 1.1 follows.

3. Extremal configurations

It is possible to characterize the configurations arising when the intersection size is either $q^2 + 1$ or $2q^3 + q^2 + 1$. These are respectively the minimum and the maximum yielded by Theorem 1.1. and they can happen only when $\mathcal{Q}$ is an hyperbolic quadric. Throughout this section we assume that the hypotheses of Theorem 1.1 hold, namely that $\mathcal{H}$ and $\mathcal{Q}$ share a tangent plane at some point P. We prove the following two theorems.

Theorem 3.1. *Suppose $|\mathcal{H} \cap \mathcal{Q}| = q^2 + 1$. Then, $\mathcal{Q}$ is a hyperbolic quadric and $\Omega = \mathcal{H} \cap \mathcal{Q}$ is an elliptic quadric contained in a subgeometry* $\mathrm{PG}(3, q)$ *embedded in* $\mathrm{PG}(3, q^2)$.

Proof. By Theorem 1.1, $\mathcal{Q}$ is hyperbolic. We first show that Ω must be an ovoid of $\mathcal{Q}$. Indeed, suppose there is a generator r of $\mathcal{Q}$ meeting $\mathcal{H}$ in more than 1 point. Then, $|r \cap \mathcal{H}| \geq q + 1$. On the other hand, any generator $\ell \neq r$ of $\mathcal{Q}$ belonging to the same regulus $\mathcal{R}$ as r necessarily meets $\mathcal{H}$ in at least one point. As there are q^2 such generators, we get $|\Omega| \geq q^2 + q + 1$ — a contradiction. In particular, by the above argument, any generator ℓ of $\mathcal{Q}$ through a point of Ω must be tangent to $\mathcal{H}$. Thus, at all points $P \in \Omega$ the tangent planes to $\mathcal{H}$ and to $\mathcal{Q}$ are the same. A direct counting argument shows that Ω contains a 4–simplex. Let ρ and θ be the polarities induced respectively by $\mathcal{H}$ and $\mathcal{Q}$, and denote by $\Psi = \rho\theta$ the collineation they induce. By [19, §83], ρ and θ commute. Thus, Ψ is an involution fixing pointwise Ω and, with respect to any fixed frame, it acts as the conjugation $X \to X^q$. It follows that Ω is contained in the Baer subgeometry $\mathrm{PG}(3, q)$ fixed by Ψ. Actually we see that this is the complete intersection of $\mathcal{Q}$ with this subgeometry. $\square$

Theorem 3.2. *Suppose $|\mathcal{H} \cap \mathcal{Q}| = 2q^3 + q^2 + 1$. Then, $\mathcal{Q}$ is a hyperbolic quadric in permutable position with $\mathcal{H}$. In particular, there is a quadric $\mathcal{Q}' \subseteq \mathcal{H} \cap \mathcal{Q}$, contained in a subgeometry* $\mathrm{PG}(3, q)$*, such that all points of $\mathcal{H} \cap \mathcal{Q}$ lie on (extended) generators of $\mathcal{Q}'$.*

Theorem 3.2 can be obtained as a consequence of the analysis contained in [8, §5.2.1], in light of [16, Lemma 19.3.1]. Here we present a direct argument.

Proof. By Theorem 1.1, $\mathcal{Q}$ is a hyperbolic quadric. Let now $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two reguli of $\mathcal{Q}$ and denote by Ω_i the set of lines of $\mathcal{R}_i$ which are also generators of

$\mathcal{H}$. Any line of $\mathcal{R}_i$ meets $\mathcal{H}$ in either $1, q+1$ or q^2+1 points; let r_1^i, r_2^i, r_3^i be the respective number of lines; in particular $r_3^i = |\Omega_i|$. Then,

$$r_1^i + (q+1)r_2^i + (q^2+1)(q^2+1-r_1^i-r_2^i) = 2q^3+q^2+1.$$

After some direct manipulations we get

$$q^2(q-1)^2 = q^2(r_1^i+r_2^i) - qr_2^i;$$

whence,

$$q((q-1)^2 - r_1^i) = r_2^i(q-1).$$

Thus, there are integers t and s such that $r_1^i = (q-1)s$ and $r_2^i = qt$; and $s+t = q-1$. By the above argument, there are at least $(q^2+1)-(q^2-q) = q+1$ generators of $\mathcal{Q}$ in each $\mathcal{R}_i$ which belong to Ω_i. Consider the set

$$\mathcal{Q}' = \{P_{\mathfrak{x}\mathfrak{y}} = \mathfrak{x} \cap \mathfrak{y} : \mathfrak{x} \in \Omega_1, \mathfrak{y} \in \Omega_2\}.$$

At any point $P_{\mathfrak{x}\mathfrak{y}} \in \mathcal{Q}'$, the tangent plane to $\mathcal{Q}$ and the tangent plane to $\mathcal{H}$ are the same. Furthermore, as $q+1 \geq 3$, there is at least a 4–simplex contained in $\mathcal{Q}'$. Thus, by [19, §83], the quadric $\mathcal{Q}$ and the Hermitian surface $\mathcal{H}$ are permutable, see also [16, §19.3], and, by [19, §75 page 135], $\mathcal{Q}'$ is a hyperbolic quadric contained in a subgeometry $\mathrm{PG}(3,q)$. $\square$

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