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Asymptotic dynamics of nonlinear coupled suspension bridge equations

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ABSTRACT

In this paper we study the long-term dynamics of a doubly nonlinear abstract system which involves a single differential operator to different powers. For a special choice of the nonlinear terms, the system describes the motion of a suspension bridge where the road bed and the main cable are modeled as a nonlinear beam and a vibrating string, respectively, and their coupling is carried out by nonlinear springs. The set of stationary solutions turns out to be nonempty and bounded. As the external loads vanish, the null solution of the system is proved to be exponentially stable provided that the axial load does not exceed some critical value. Finally, we prove the existence of a bounded global attractor of optimal regularity in connection with an arbitrary axial load and quite general nonlinear terms.

1. Introduction

An increasing attention was paid in recent years to the analysis of nonlinear vibrations of suspension bridges, and many papers have been published in this field (see, for instance, [1,14] and references therein). Apart from a lot of works which deal with approximations and numerical simulations, some of them are devoted to scrutinize the periodic or longtime global dynamics by analytical methods [6,8,9,22,26].

It is worth noting that rigorous results concerning the global dynamics of a suspension bridge system are very difficult to achieve, especially if the coupling between the road bed and the suspension main cable is taken into account [2,7,18,21]. Most of this literature focus on mathematical models describing such a dynamics as the nonlinear coupling between a beam (the road bed) and a string (the main cable), according to the pioneer model introduced by Lazer and McKenna [19,20].

Taking into account the midplane stretching of the beam, which is due to its elongation when both ends are hinged, a doubly nonlinear system comes out. The former nonlinearity arises by modeling the flexible supporting stays as nonlinear elastic springs, according to Lazer and McKenna. The latter is a geometric nonlinearity which appears in the bending equation of the road-bed by accounting for its extension when an axial load is applied (see, for instance, [13,25]).

We assume the ratios between the length of the bridge and other dimensions to be very small, which entails that the torsional motion can be ignored, so that the road bed can be simply modeled as a vibrating one-dimensional beam. In addition, by neglecting the influence of the towers and side parts of the bridge, such a beam can be assumed to have simply supported ends (see Fig. 1).

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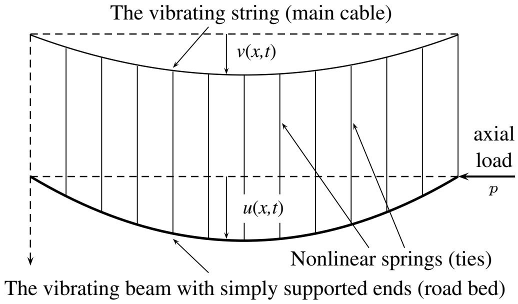

Fig. 1. Road bed hanged by the main cable.

The resulting dynamical system consists in the following coupled equations

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + a\partial_t u + \left(p - M(\|\partial_x u\|^2_{L^2(0,1)})\right)\partial_{xx}u + F(u-v, \partial_t(u-v)) = f, \\ \partial_{tt}v - \partial_{xx}v + b\partial_t v - F(u-v, \partial_t(u-v)) = g, \end{cases} \tag{1.1}$$

- $u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ represents the downward deflection of the beam mid-line (of unitary length) in the vertical plane with respect to his reference configuration.
- $v = v(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ measures the vertical displacement of the string.
- f and g are the external loads, which are assumed to be time-independent.
- $F(u-v, \partial_t(u-v))$ represents the nonlinear response of the stays connecting the beam and the string: each stay has a rest reference length which corresponds to $u(x) = v(x) = 0$. Since the action exerted by the suspension stays is mutual, a different sign appears in front of the term $F(u-v, \partial_t(u-v))$ within the two motion equations.
- $M(\|\partial_x u\|^2_{L^2(0,1)})$ takes into account the geometric nonlinearity of the beam bending due to its elongation.
- p is a parameter which regards the axial force acting at one end of the beam: it is negative when the beam is axially stretched, positive when compressed.
- $a, b > 0$ represent the viscous damping constants due to external resistant forces which linearly depend on the velocity.

For simplicity, the flexural rigidity of the structure and the coefficient of tensile strength of the cable are chosen equals to one. Some special choices of F and M will be discussed in the sequel.

The unknown fields u and v are required to satisfy the following initial conditions

$$\begin{cases} u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x), \\ v(x,0) = v_0(x), \qquad \partial_t v(x,0) = v_1(x). \end{cases} \tag{1.2}$$

Concerning the boundary conditions, the beam is considered with both pinned ends, the string has fixed ends. Accordingly, we choose

$$\begin{cases} u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0, \\ v(0,t) = v(1,t) = 0. \end{cases} \tag{1.3}$$

1.1. Constitutive assumptions

For further convenience, we list here all assumptions on the functions M and F which will be employed throughout the paper.

Assumptions on M

M_1- $M \in \mathcal{C}^1([0,\infty))$
M_2- $M(0) = 0$
M_3- $M'(s) \geq 0$ (non decreasing property)
M_4- $\lim_{s\to\infty} M(s) = M_\infty \in (|p|, \infty]$.

As a consequence of assumption M_4 and the continuity of M, for some positive constants k, m and γ we have (see [12])

$$(M(s) - p)s \geq -k, \tag{1.4}$$

$$\mathcal{M}(s) \geq \gamma s - m, \quad \gamma > |p|, \tag{1.5}$$

where

$$\mathcal{M}(s) = \int_0^s M(y)dy.$$

In addition, it is easy to check that

$$\mathcal{M}(s) \leq sM(s). \tag{1.6}$$

Examples

Usually, $M(s) = s$ in bending problems (see [8]). Nevertheless, M_1–M_4 are satisfied by a large class of increasing functions, either convex or concave. For instance

- $M(s) = s^\theta,\ \theta > 0$;
- $M(s) = \ln(1 + s)$.

Assumptions on F

A minimal set of assumptions on F is given by

F_1- $F(0, 0) = 0$
F_2- (Lipschitz condition) $\forall s_1, s_2, q_1, q_2 \in \mathbb{R}, \exists c_0,\ c_1 > 0$ such that

$$|F(s_1, q_1) - F(s_2, q_2)| \le c_0|s_1 - s_2| + c_1|q_1 - q_2|$$

F_3- (monotonicity property) $\forall s_1, s_2, q_1, q_2 \in \mathbb{R}$

$$[F(s_1, q) - F(s_2, q)](s_1 - s_2) \ge 0,$$
$$[F(s, q_1) - F(s, q_2)](q_1 - q_2) \ge 0.$$

In order to obtain uniform estimates, we further assume:

H_1- (decomposition) $\forall s, q \in \mathbb{R},\ \exists G_0, G, \Gamma_0, \Gamma$ such that

$$\begin{aligned} &F(s, q)q = G(s)q + G_0(s, q) \quad \text{with } G_0(s, q) \ge 0, \\ &F(s, q)s = \Gamma_0(s) + \Gamma(s)q \quad \text{with } \Gamma_0(s) \ge 0 \end{aligned} \tag{1.7}$$

H_2- $G(0) = 0, \Gamma(0) = 0$
H_3- (Lipschitz condition) $\forall s_1, s_2 \in \mathbb{R},\ \exists c_G, c_\Gamma > 0$ such that

$$|G(s_1) - G(s_2)| \le c_G|s_1 - s_2|,$$
$$|\Gamma(s_1) - \Gamma(s_2)| \le c_\Gamma|s_1 - s_2|$$

H_4- (monotonicity property) $\forall s_1, s_2 \in \mathbb{R}$

$$[G(s_1) - G(s_2)](s_1 - s_2) \ge 0,$$
$$[\Gamma(s_1) - \Gamma(s_2)](s_1 - s_2) \ge 0$$

H_5- $\forall s \in \mathbb{R}$

$$\Gamma_0(s) \ge 2\int_0^s G(\sigma)\, d\sigma \ge 0. \tag{1.8}$$

Following these assumptions, we can easily check that

$$\int_0^s \Gamma(\eta)\, d\eta \ge 0, \qquad \int_0^s G(\sigma)\, d\sigma \ge 0, \tag{1.9}$$

$$F(0, q)q \ge 0, \qquad F(s, 0)s \ge 0. \tag{1.10}$$

Examples

Some examples of function F satisfying F_1–F_3 and H_1–H_5 are as follows

- $F_L(s, q) = k^2 s + hq$, $h \ge 0$. In such a case

$$G(s) = k^2 s, \qquad G_0(q) = hq^2$$
$$\Gamma(s) = hs, \qquad \Gamma_0(s) = k^2 s^2.$$

- $F_{OS}(s, q) = k^2 s^+ + h1_s^+ q$, where $h \ge 0$ and

$$s^+ = \begin{cases} s & \text{if } s \ge 0 \\ 0 & \text{if } s < 0 \end{cases}, \qquad 1_s^+ = \frac{s^+}{s} = \begin{cases} 1 & \text{if } s \ge 0 \\ 0 & \text{if } s < 0. \end{cases}$$

Accordingly,

$$G(s) = k^2 s^+, \qquad G_0(s, q) = h1_s^+ q^2$$
$$\Gamma(s) = h1_s^+ s = hs^+, \qquad \Gamma_0(s) = k^2 (s^+)^2.$$

For both examples we recover the particular relation

$$\Gamma_0(s) = G(s)s.$$

When $h = 0$, it is apparent that functions F_L and F_{OS}, respectively, provide simple models of Linear and One-Sided springs with stiffness k^2 (see [6,8]). On the other hand, the dissipative terms appearing when $h > 0$ model some damping effect inside the springs. In the One-Sided case, this damping works only if the spring is stretched, obviously.

1.2. Outline of the paper

In Section 2 we formulate an abstract version of the problem, so proving that its solutions are bounded in the energy norm and that the set of the steady states is nonempty and bounded. Section 3 is focused on well-posedness results and the abstract system is shown to generate a strongly continuous semigroup (or dynamical system) on the energy space. We also prove the exponential decay of the energy norm when $f = g = 0$, provided that the parameter p is smaller than a critical value. In Section 4 we present the main result, namely the *existence of the regular global attractor* for a general value of p. This is done by exploiting the existence of a Lyapunov functional and without require any assumption on the strength of the dissipation terms. The asymptotic compactness property is shown by using a particular decomposition of the semigroup, as devised in [15], but a final bootstrap argument is needed here, unlike [15]. The Appendix at the end of the paper is devoted to recall some technical lemmas which are required to obtain main results.

2. Preliminary results

2.1. Functional setting

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let $A : \mathcal{D}(A) \Subset H \to H$ be a strictly positive selfadjoint operator. For any $r \in \mathbb{R}$, the scale of Hilbert spaces generated by the powers of A is introduced as follows

$$H^r = \mathcal{D}(A^{r/4}), \qquad \langle u, v\rangle_r = \langle A^{r/4}u, A^{r/4}v\rangle, \qquad \|u\|_r = \|A^{r/4}u\|.$$

When $r = 0$, the index r is omitted. The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality product between H^r and its dual space H^{-r}. In particular, we have the compact embeddings $H^{r+1} \Subset H^r$, along with the generalized Poincaré inequalities

$$\lambda_1\|w\|_r^4 \le \|w\|_{r+1}^4, \quad \forall w \in H^{r+1}, \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. From (2.1) it follows

$$\lambda\left(\|u\|_r^2 + \|v\|_r^2\right) \le \|u\|_{r+2}^2 + \|v\|_{r+1}^2, \tag{2.2}$$

where $\lambda = \min\{\lambda_1, \sqrt{\lambda_1}\}$, for all $u \in H^{r+2}$, $v \in H^{r+1}$. We denote by e_n and λ_n, with $n = 1, 2, \ldots$, the eigenfunctions and the strictly positive sequence of the distinct eigenvalues of A, respectively.

In order to establish more general results, we recast problem (1.1)–(1.3) into an abstract setting. Indeed, if we define the product Hilbert spaces

$$\mathcal{H}^r = H^{r+2} \times H^r \times H^{r+1} \times H^r,$$

then, for all $p \in \mathbb{R}$, the related abstract Cauchy problem is given by

$$\begin{cases} \partial_{tt}u + Au + a\partial_t u - (p - M(\|u\|_1^2))A^{1/2}u + F(u-v, \partial_t(u-v)) = f, \\ \partial_{tt}v + A^{1/2}v + b\partial_t v - F(u-v, \partial_t(u-v)) = g, \\ u(0) = u_0, \qquad \partial_t u(0) = u_1, \\ v(0) = v_0, \qquad \partial_t v(0) = v_1. \end{cases} \tag{2.3}$$

Problem (1.1)–(1.3) is just a particular case, which is obtained by setting $H = L^2(0, 1)$ and $A = \partial_{xxxx}$ with boundary conditions (1.3). The resulting operator is strictly positive, selfadjoint, with compact inverse. In addition, its domain is

$$\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0, 1) : w(0) = w(1) = \partial_{xx}w(0) = \partial_{xx}w(1) = 0\},$$

its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$, and $e_n = \sin n\pi x$ are the corresponding eigenfunctions.

Notation 1. Henceforth, a solution to problem (2.3) will be denoted by $\sigma : \mathbb{R}^+ \to \mathcal{H}$,

$$\sigma(t) = (u(t), \partial_t u(t), v(t), \partial_t v(t)),$$

and $z = (u_0, u_1, v_0, v_1)$ summarizes its initial data. Unless otherwise indicated, initial data of the problem are assumed to belong to a ball of radius R in $\mathcal{H}$, namely $\|z\|_{\mathcal{H}} \le R$. Henceforth, c will denote a *generic* positive constant which possibly (but implicitly) depends on the structural constants of the problem. In addition, $Q : \mathbb{R}_0^+ \to \mathbb{R}^+$ will denote a *generic* increasing monotone function which explicitly depends only on R, but implicitly also depends on the structural constants of the problem. The actual expressions of c and Q may change, even within the same line of a given equation.

2.2. The energy norm

The *total energy* of a solution σ is defined as

$$E(\sigma) = \mathcal{E}(\sigma) + \mathcal{M}(\|u\|_1^2) - p\|u\|_1^2 + 2\mathcal{G}(u - v) + m,$$

where m is the positive constant appearing into (1.5), $\mathcal{E}$ represents the *energy norm* of $\sigma(t)$ in $\mathcal{H}$, namely

$$\mathcal{E}(\sigma(t)) = \|\sigma(t)\|_{\mathcal{H}}^2 = \|u(t)\|_2^2 + \|\partial_t u(t)\|^2 + \|v(t)\|_1^2 + \|\partial_t v(t)\|^2$$

and

$$\mathcal{G}(w) = \left\langle \int_0^w G(\eta) d\eta, 1 \right\rangle.$$

The positivity of E is easily achieved by observing that (1.8) implies $\mathcal{G}(w) \geq 0$ and inequality (1.5) assures

$$\mathcal{M}(\|u(t)\|_1^2) - p\|u(t)\|_1^2 + m \geq (\gamma - p)\|u(t)\|_1^2.$$

In addition the following *energy identity* holds

$$\frac{1}{2}\frac{d}{dt}E(\sigma(t)) + a\|\partial_t u(t)\|^2 + b\|\partial_t v(t)\|^2 + \langle G_0(u(t) - v(t), \partial_t(u(t) - v(t))), 1\rangle = \langle \partial_t u(t), f\rangle + \langle \partial_t v(t), g\rangle. \tag{2.4}$$

Lemma 1 (*Energy Norm Boundedness*)**.** *Let $f \in H^{-2}$ and $g \in H^{-1}$. For all $t > 0$ and initial data $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} \leq R$, we have*

$$\mathcal{E}(\sigma(t)) \leq Q(R).$$

Proof. We introduce the functional $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ given by

$$\mathcal{L}(z) = E(z) - 2\langle u_0, f\rangle - 2\langle v_0, g\rangle, \tag{2.5}$$

where $z = (u_0, u_1, v_0, v_1) \in \mathcal{H}$. Along any solution $\sigma(t)$ to (2.3), the time function

$$\mathcal{L}(\sigma(t)) = E(\sigma(t)) - 2\langle u(t), f\rangle - 2\langle v(t), g\rangle$$

is non increasing. Actually, from the energy identity (2.4) it follows

$$\frac{d}{dt}\mathcal{L}(\sigma(t)) = -2a\|\partial_t u(t)\|^2 - 2b\|\partial_t v(t)\|^2 - \langle G_0(u(t) - v(t), \partial_t(u(t) - v(t))), 1\rangle \leq 0, \tag{2.6}$$

which ensures that

$$\mathcal{L}(\sigma(t)) \leq \mathcal{L}(z) \leq Q(R),$$

for all $z \in \mathcal{H}$, with $\|z\|_{\mathcal{H}} \leq R$. Since $E \geq \mathcal{E}$, from (2.5) we obtain the estimate

$$\mathcal{L} \geq \mathcal{E} - 2\langle u, f\rangle - 2\langle v, g\rangle \geq \frac{1}{2}\mathcal{E} - 2\left(\|f\|_{-2}^2 + \|g\|_{-1}^2\right),$$

which finally gets

$$\mathcal{E}(\sigma(t)) \leq 2\mathcal{L}(\sigma(t)) + 4\left(\|f\|_{-2}^2 + \|g\|_{-1}^2\right) \leq Q(R). \quad \square$$

2.3. The steady states

The set $\mathcal{S}$ of stationary solutions is made of vectors $z_0 = (u, 0, v, 0) \in \mathcal{H}$, such that $(u, v) \in H^2 \times H^1$ is a weak solution to the system

$$\begin{cases} Au - (p - M(\|u\|_1^2))A^{1/2}u + F(u - v, 0) = f, \\ A^{1/2}v - F(u - v, 0) = g. \end{cases} \tag{2.7}$$

It is worth mentioning here the following lemma,

Lemma 2 (*See [6, Lemma 4.5, p. 13]*)**.** *Let $p < \sqrt{\lambda_1}$ and $B_p = A - pA^{1/2}$. Then*

$$\langle B_p u, u\rangle \geq C_p \|u\|_2^2,$$

where

$$C_p = \begin{cases} 1 & \text{if } p \leq 0, \\ \left(1 - \frac{p}{\sqrt{\lambda_1}}\right) & \text{if } 0 < p < \sqrt{\lambda_1}. \end{cases} \tag{2.8}$$

As well as in proving the exponential stability of the system when the external sources vanish, this lemma is crucial to establish the following

Theorem 3. *Let $f \in H^{-2}$ and $g \in H$. Under the assumptions on M and F, the set $\mathcal{S}$ is nonempty and bounded in $\mathcal{H}$.*

Proof. As apparent, when F_1 holds, problem (2.7) with $f = g = 0$ admits the null solution $u = v = 0$. In the general case, the operator $B_p + M(\|u\|_1^2)A^{1/2}$ is (uniformly) strong elliptic for all $p \in \mathbb{R}$ and, by virtue of Lemma 2, it is coercive on H^2 provided that $p < \sqrt{\lambda_1}$. As a consequence, problem (2.7) has a solution which is unique when $p < \sqrt{\lambda_1}$, so that the set $\mathcal{S}$ is nonempty.

In order to prove the boundedness of $\mathcal{S}$, we start from the *steady-state energy identity*

$$\|u\|_2^2 + \|v\|_1^2 + \big(M(\|u\|_1^2) - p\big)\,\|u\|_1^2 + \langle F(u - v, 0), u - v\rangle = \langle u, f\rangle + \langle v, g\rangle\,, \tag{2.9}$$

which follows from multiplying $(2.7)_1$ by u and $(2.7)_2$ by v in H, then adding the results. Taking into account relations (1.4) and $(1.10)_2$, for all $(u, v) \in \mathcal{H}$ we conclude

$$\|u\|_2^2 + \|v\|_1^2 \le \|f\|_{-2}^2 + \|g\|_{-1}^2 + 2k,$$

so that $\mathcal{S}$ is bounded in $\mathcal{H}$. □

If we assume that $(u, v) \in H^2 \times H^2$ and $g \in H$, then by letting $w = u - v$ the system (2.7) leads to the single equation

$$Aw + \Lambda_p(w) + q_p(\|u\|_1)\,F(w, 0) = h_p(\|u\|_1), \tag{2.10}$$

where

$$b_p(\|u\|_1) = M(\|u\|_1^2) - p, \qquad q_p(\|u\|_1) = 1 + b_p(\|u\|_1), \qquad h_p(\|u\|_1) = f - A^{1/2}g - b_p(\|u\|_1)g,$$

and

$$\Lambda_p(w) = b_p(\|u\|_1)A^{1/2}w + A^{1/2}[F(w, 0)].$$

Each weak solution $w \in H^2$ to the elliptic equation (2.10) generates a weak solution (u, v) to the original system. Indeed, for any given w, we recover the original fields as follows,

$$v = A^{-1/2}[F(w, 0) + g], \qquad u = w + A^{-1/2}[F(w, 0) + g]. \tag{2.11}$$

Eq. (2.10) can be explicitly solved provided that F is linear (see, for instance, [7]).

3. The dynamical system

3.1. Well-posedness

We will prove here that the abstract system (2.3) defines a strongly continuous semigroup of operators on $\mathcal{H}$. First, by virtue of the interpolation inequality, namely $\|\theta\|_1^2 \le \|\theta\|, \|\theta\|_2,\ \theta \in H^2$, we can infer that

Lemma 4 (*See [15]*)**.** *For any $p \in \mathbb{R}$, there exist $\alpha > 0$ large enough and $\beta = \beta(p, \alpha) > 1$ such that*

$$\frac{1}{2}\|\theta\|_2^2 \le \|\theta\|_2^2 - p\|\theta\|_1^2 + \alpha\|\theta\|^2 \le \beta\|\theta\|_2^2. \tag{3.1}$$

This lemma is needed to prove the following well-posedness result

Proposition 1. *Let $f \in H^{-2}$ and $g \in H^{-1}$. For all initial data $z \in \mathcal{H}$, the abstract Cauchy problem (2.3) admits a unique solution*

$$\sigma = (u, \partial_t u, v, \partial_t v) \in \mathcal{C}(0, T; \mathcal{H}),$$

which continuously depends on the initial data.

Proof. We omit the proof of the existence result, which is straightforward. In particular, one can apply a standard Faedo–Galerkin approximation procedure (see, for instance [4,5,18,21]), together with a slight generalization of the usual Gronwall lemma. Indeed, the uniform-in-time estimates needed to obtain the global existence are exactly the same as in Lemma 1. On the contrary, the uniqueness result deserves a detailed discussion.

Let σ_1 and σ_2 be two weak solutions, both of which solve the abstract problem (2.3) on the time interval $(0, T)$ with initial data z. By virtue of Lemma 1, both solutions fulfill the bounds

$$\mathcal{E}(\sigma_i(t)) \le Q(R), \qquad \|z\|_{\mathcal{H}} \le R, \quad i = 1, 2.$$

Let $\omega = \sigma_1 - \sigma_2$ denotes their difference, with $\omega = (\theta, \partial_t\theta, \nu, \partial_t\nu)$ and

$$\theta = u_1 - u_2, \qquad \nu = v_1 - v_2.$$

Then, the unknown field ω solves the following homogeneous abstract problem

$$\begin{cases} \partial_{tt}\theta + A\theta + a\partial_t\theta - pA^{1/2}\theta + N(u_1) - N(u_2) + F(u_1 - v_1, \partial_t(u_1 - v_1)) - F(u_2 - v_2, \partial_t(u_2 - v_2)) = 0, \\ \partial_{tt}\nu + A^{1/2}\nu + b\partial_t\nu - F(u_1 - v_1, \partial_t(u_1 - v_1)) + F(u_2 - v_2, \partial_t(u_2 - v_2)) = 0, \\ \theta(x, 0) = \partial_t\theta(x, 0) = \nu(x, 0) = \partial_t\nu(x, 0) = 0, \end{cases} \tag{3.2}$$

where $N(u) = M(\|u\|_1^2)A^{1/2}u$. By letting $\chi(\rho) = \rho u_1 + (1-\rho)u_2$, $\rho \in [0, 1]$, we have $\chi(1) = u_1$, $\chi(0) = u_2$, $\chi'(\rho) = u_1 - u_2 = \theta$ and

$$\|\chi(\rho)\|_r \le \|u_1\|_r + \|u_2\|_r, \quad r = 0, 1, 2.$$

This allows the following representation,

$$N(u_1) - N(u_2) = 2B(u_1, u_2, \theta) + D(u_1, u_2)A^{1/2}\theta,$$

where

$$\begin{aligned} B(u_1, u_2, \theta) &= \int_0^1 M'\left(\|\chi(\rho)\|_1^2\right)\langle\chi(\rho), \theta\rangle_1 A^{1/2}\chi(\rho)\, d\rho, \\ D(u_1, u_2) &= \int_0^1 M\left(\|\chi(\rho)\|_1^2\right) d\rho. \end{aligned} \tag{3.3}$$

By virtue of Lemma 1 and assumption M_3, we obtain the following estimates,

$$\begin{aligned} \left|D(u_1, u_2)\left\langle A^{1/2}\theta, \partial_t\theta\right\rangle\right| &\le 2M\left(\left[\|u_1\|_1^2 + \|u_2\|_1^2\right]\right)\|\theta\|_2\|\partial_t\theta\| \le Q(R)\,\mathcal{E}(\omega), \\ |\langle B(u_1, u_2, \theta), \partial_t\theta\rangle| &= \left|\int_0^1 M'\left(\|\chi(\rho)\|_1^2\right)\langle\chi(\rho), \theta\rangle_1\left\langle A^{1/2}\chi(\rho), \partial_t\theta\right\rangle d\rho\right| \\ &\le Q(R)\,\|\theta\|_2\|\partial_t\theta\| \le Q(R)\,\mathcal{E}(\omega), \end{aligned}$$

which in turn imply

$$|\langle N(u_1) - N(u_2), \partial_t\theta\rangle| \le Q(R)\,\mathcal{E}(\omega).$$

On the other hand, by virtue of the Lipschitz condition F_2 we easily obtain

$$\begin{aligned} &|\langle F(u_1 - v_1, \partial_t(u_1 - v_1)) - F(u_2 - v_2, \partial_t(u_2 - v_2)), \partial_t\theta - \partial_t\nu\rangle| \\ &\quad \le \hat{c}\,(\|\theta\| + \|\nu\| + \|\partial_t\theta\| + \|\partial_t\nu\|)(\|\partial_t\theta\| + \|\partial_t\nu\|) \le \bar{c}\,\mathcal{E}(\omega). \end{aligned}$$

Now, we multiply $(3.2)_1$ by $\partial_t\theta$ and $(3.2)_2$ by $\partial_t\nu$ in H. After adding the resulting equations and taking into account previous estimates we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\mathcal{E}(\omega(t)) - p\,\|\theta(t)\|_1^2\right) \le Q(R)\mathcal{E}(\omega(t)). \tag{3.4}$$

Then, adding and subtracting the term $\alpha\,\langle\theta, \theta_t\rangle$, with $\alpha > 0$, we have

$$\frac{d}{dt}\mathcal{I}(\omega(t)) \le Q(R)\mathcal{E}(\omega(t)), \tag{3.5}$$

where

$$\mathcal{I}(\omega) = \mathcal{E}(\omega) - p\,\|\theta\|_1^2 + \alpha\,\|\theta\|^2.$$

Provided that α is large enough, according to Lemma 4 we have

$$\frac{1}{2}\,\mathcal{E}(\omega) \le \mathcal{I}(\omega) \le \beta\,\mathcal{E}(\omega), \tag{3.6}$$

where β is chosen as in (3.1). Then from (3.5) it follows that

$$\frac{d}{dt}\mathcal{I}(\omega(t)) \le Q(R)\mathcal{I}(\omega(t)).$$

Now, since $\omega(0) = (0, 0, 0, 0)$ and $\mathcal{I}(\omega(0)) = 0$, an application of the Gronwall lemma leads to $\mathcal{I}(\omega(t)) = 0$ for all $t > 0$. Finally, by virtue of (3.6), $\mathcal{E}(\omega(t)) = \|\omega(t)\|_{\mathcal{H}}^2 = 0$, so that $\sigma_1(t) = \sigma_2(t)$ and the uniqueness follows.

The same strategy leads to the continuous dependence of the solution on the initial data in $\mathcal{H}$. Indeed, if σ_1 and σ_2 are two solutions corresponding to initial data z_1 and z_2, respectively, then (3.4) holds with $\omega(0) = z_1 - z_2$ and implies

$$\|\sigma_1(t) - \sigma_2(t)\|^2_{\mathcal{H}} \le e^{Q(R)T} \ \|z_1 - z_2\|^2_{\mathcal{H}}, \quad \forall t \in (0, T). \quad \square$$

As a consequence of Proposition 1, the abstract system (2.3) generates a strongly continuous semigroup (or *dynamical system*) $S(t)$ on $\mathcal{H}$. That is to say, for a given initial data $z \in \mathcal{H}$, the function $\sigma(t) = S(t)z$, $t \ge 0$, is the unique weak solution to (2.3) and its related energy norm is given by $\mathcal{E}(t) = \|S(t)z\|^2_{\mathcal{H}}$.

3.2. The exponential decay

By virtue of Lemma 2, when $p < \sqrt{\lambda_1}$ we are able to prove the uniform decay of the special semigroup $S_0(t)$, which is generated by the system (2.3) under the restrictive assumptions $f = g = 0$.

Theorem 5. *Let $z \in \mathcal{H}$, $\|z\|_{\mathcal{H}} \le R$. Provided that $p < \sqrt{\lambda_1}$, the solutions $S_0(t)z$ to (2.3) with $f = g = 0$ decay exponentially, i.e.*

$$\mathcal{E}(t) \ \le \ Q(R)\, e^{-ct},$$

where c is a suitable positive constant and $\mathcal{E}(t) = \|S_0(t)z\|^2_{\mathcal{H}}$.

Proof. We introduce the functional

$$\Phi = E + \varepsilon \langle u, \partial_t u\rangle + \varepsilon \langle v, \partial_t v\rangle + \varepsilon \mathcal{N}(u - v) - m, \tag{3.7}$$

where

$$\mathcal{N}(w) = \left\langle \int_0^w \Gamma(\eta) d\eta, 1 \right\rangle$$

is positive by virtue of (1.9).

The first step is to prove the equivalence between $\mathcal{E}$ and Φ, that is

$$\frac{C_p}{2}\, \mathcal{E} \le \Phi \le Q(R)\, \mathcal{E}. \tag{3.8}$$

Taking into account (2.8), the constant

$$\mu = \min\{\lambda C_p, 1\}$$

is positive provided that $p < \sqrt{\lambda_1}$. Hence, by virtue of Lemma 2 and (2.2), the choice $\varepsilon \le \mu$ in (3.7) provides the following lower bound (see [8]),

$$\Phi \ge \frac{1}{2}\left(\|\partial_t u\|^2 + \|\partial_t v\|^2\right) + \frac{C_p}{2}\left(\|u\|_2^2 + \|v\|_1^2\right) \ge \frac{C_p}{2}\, \mathcal{E}.$$

On the other hand, by applying the Young inequality and using (2.2) we get the following inequality

$$\Phi \le \left(2 + \frac{1}{2\lambda} + \frac{\varepsilon^2}{2} - \frac{p}{\lambda}\right) \mathcal{E} + \mathcal{M}(\|u\|_1^2) + 2\mathcal{G}(u - v) + \varepsilon \mathcal{N}(u - v).$$

From (1.6) and Lemma 1, we obtain

$$\mathcal{M}(\|u\|_1^2) \ \le M(\|u\|_1^2)\|u\|_1^2 \le Q(R)\mathcal{E}.$$

In addition, by the Lipschitz condition H_3, we have

$$\mathcal{G}(u - v) \le c_G \|u - v\|^2 \le Q(R)\mathcal{E},$$
$$\mathcal{N}(u - v) \le c_\Gamma \|u - v\|^2 \le Q(R)\mathcal{E}.$$

Then, by collecting all previous results, we can write the upper estimate of Φ as

$$\Phi \le \left(2 + \frac{1}{2\lambda} + \frac{\varepsilon^2}{2} + Q(R) + \varepsilon Q(R)\right) \mathcal{E} = Q(R)\, \mathcal{E}.$$

In order to prove the exponential decay of Φ, we start from the identity

$$\begin{aligned}\frac{d}{dt}\,\Phi &+ \varepsilon\Phi + 2a\,\|\partial_t u\|^2 + 2b\,\|\partial_t v\|^2 + 2\,\langle G_0, 1\rangle - 2\varepsilon\left(\|\partial_t u\|^2 + \|\partial_t v\|^2\right) + \varepsilon\,\langle a\partial_t u,\ u\rangle \\ &+ \varepsilon\,\langle b\partial_t v,\ v\rangle - \varepsilon^2\left(\langle \partial_t u,\ u\rangle + \langle \partial_t v,\ v\rangle\right) + \varepsilon\,\langle \Gamma_0(u - v),\ 1\rangle \\ &+ \varepsilon M(\|u(t)\|_1^2)\|u(t)\|_1^2 - \varepsilon \mathcal{M}(\|u(t)\|_1^2) - 2\varepsilon\mathcal{G}(u - v) - \varepsilon^2 \mathcal{N}(u - v) = 0.\end{aligned}$$

Exploiting the Young inequality along with (1.6), (1.8), (2.1), (3.8) and the Lipschitz condition H_3 on Γ, we have

$$\frac{d}{dt}\,\Phi + \varepsilon\Phi + (\gamma_0 - \varepsilon)\left(\|\partial_t u\|^2 + \|\partial_t v\|^2\right) \leq \frac{\varepsilon^2\,(\varkappa - \varepsilon)}{4\lambda}\left(\|u\|_2^2 + \|v\|_1^2\right) \leq \frac{\varepsilon^2\varkappa}{2\lambda C_p}\Phi, \tag{3.9}$$

provided that

$$\varepsilon \leq \min\left\{\lambda C_p,\, 1,\, \gamma_0,\, \frac{2\lambda C_p}{\varkappa}\right\}, \tag{3.10}$$

where $\gamma_0 = \min\{a, b\}$, $\varkappa = \gamma_1 + c_\Gamma$ and $\gamma_1 = \max\{a, b\}$. From (3.9) it follows

$$\frac{d}{dt}\Phi + \frac{\varepsilon}{2\lambda C_p}\left[2\lambda C_p - \varepsilon\varkappa\right]\Phi \leq 0.$$

Letting $c = \varepsilon\left[2\lambda C_p - \varepsilon\varkappa\right]/2\lambda C_p$, which is positive because of (3.10), we conclude

$$\frac{C_p}{2}\,\mathcal{E}(t) \leq \Phi\,(t) \leq \Phi\,(0)\;e^{-c\,t} \leq Q(R)\,e^{-c\,t}. \quad \square$$

4. The global attractor

In this section, we prove the existence of a regular global attractor by exploiting a suitable Lyapunov functional and taking advantage from some technical lemmas which will be presented into the Appendix.

We begin to exhibit a Lyapunov functional $\mathcal{L}$ for the semigroup $S(t)$, that is a function $\mathcal{L} \in C(\mathcal{H}, \mathbb{R})$ satisfying the following conditions

(i) $\mathcal{L}(z) \to \infty$ if and only if $\|z\|_{\mathcal{H}} \to \infty$;
(ii) $\mathcal{L}(S(t)z)$ is non increasing for any $z \in \mathcal{H}$;
(iii) $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t > 0$ implies that $z \in \mathcal{S}$.

Proposition 2. *Let $f \in H^{-2}$ and $g \in H^{-1}$. The functional $\mathcal{L}$ defined in (2.5) is a Lyapunov functional for $S(t)$.*

Proof. By the continuity of $\mathcal{L}$ and by means of the estimates

$$\frac{1}{2}E(z) - c \leq \mathcal{L}(z) \leq \frac{3}{2}E(z) + c, \quad \forall z \in \mathcal{H},$$

assertion (i) can be easily proved.

The non increasing monotonicity of $\mathcal{L}$ along the trajectories departing from z, namely (ii), has been yet shown in the proof of Lemma 1.

Finally, if $\mathcal{L}$ is constant in time, from (2.6) we have $\partial_t u(t) = \partial_t v(t) = 0$ for all $t > 0$, which implies that $u(t) = u_0$ and $v(t) = v_0$ are constants and satisfy (2.7). Hence, $z = S(t)z = (u_0, 0, v_0, 0)$ for all $t > 0$, and then $z \in \mathcal{S}$. $\square$

As apparent, the existence of a Lyapunov functional ensures that $E(t)$ is bounded for all $t > 0$. In particular, bounded sets of initial data have bounded orbits. In addition, the set $\mathcal{S}$ of all stationary solutions,

$$\mathcal{S} = \{z \in \mathcal{H} : S(t)z = z,\ \forall t > 0\}$$

is nonempty and bounded in $\mathcal{H}$ (see Theorem 3). All these facts allow us to prove the existence of the global attractor by showing a suitable (really, exponential) asymptotic compactness property of the semigroup. Such a property will be obtained exploiting a particular decomposition of $S(t)$ devised in [15] and then applying a general result (see, for instance, [10]), tailored to our particular case. The existence of a bounded absorbing set in $\mathcal{H}$ can be obtained as a byproduct.

4.1. The global attractor

The main result of the paper is the following

Theorem 6. *Let $f \in H$ and $g \in H^1$. The semigroup $S(t)$ acting on $\mathcal{H}$ possesses a connected global attractor $\mathcal{A} \subset \mathcal{H}$. In particular, $\mathcal{A}$ is bounded in $\mathcal{H}^2$, so that its regularity is optimal.*

By standard arguments of the theory of dynamical systems (cf. [3,17,24]), this result can be established by exploiting the following lemma (cf. [15, Lemma 4.3]).

Lemma 7. *Assume that $S(t)$ fulfills the asymptotic compactness property. Namely, for every $R > 0$, there exist a function $\psi_R : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at infinity and a bounded set $\mathcal{K}_R \subset \mathcal{H}^2$ such that the semigroup $S(t)$ can be split into the sum $L(t) + K(t)$, where the one-parameter operators $L(t)$ and $K(t)$ fulfill*

$$\|L(t)z\|_{\mathcal{H}} \le \psi_R(t), \qquad K(t)z \in \mathcal{K}_R,$$

whenever $\|z\|_{\mathcal{H}} \le R$ and $t \ge 0$. Then, $S(t)$ possesses a connected global attractor $\mathcal{A}$. Moreover, $\mathcal{A} \subset \mathcal{K}_{R_0}$ for some $R_0 > 0$.

Theorem 6 is proved if $S(t)$ satisfies all assumptions required by this lemma. According to the scheme first devised in [15], we decompose the solution $S(t)z$ into the sum

$$S(t)z = L(t)z + K(t)z,$$

where

$$L(t)z = (\eta(t), \partial_t\eta(t), \xi(t), \partial_t\xi(t)) \quad \text{and} \quad K(t)z = (\vartheta(t), \partial_t\vartheta(t), \zeta(t), \partial_t\zeta(t))$$

respectively solve the systems

$$\begin{cases} \partial_{tt}\eta + A\eta + a\partial_t\eta - \Big(p - M(\|u\|_1^2)\Big)A^{1/2}\eta + \alpha\eta = 0, \\ \partial_{tt}\xi + A^{1/2}\xi + b\partial_t\xi = 0, \\ (\eta(0), \partial_t\eta(0), \xi(0), \partial_t\xi(0)) = z, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \partial_{tt}\vartheta + A\vartheta + a\partial_t\vartheta - \big(p - M\left(\|u\|_1^2\right)\big)A^{1/2}\vartheta - \alpha\eta + F(u - v, \partial_t(u - v)) = f, \\ \partial_{tt}\zeta + A^{1/2}\zeta + b\partial_t\zeta - F(u - v, \partial_t(u - v)) = g, \\ (\vartheta(0), \partial_t\vartheta(0), \zeta(0), \partial_t\zeta(0)) = 0, \end{cases} \tag{4.2}$$

where $\alpha > 0$ is large enough so that (3.1) holds.

The asymptotic compactness property of the semigroup $S(t)$ is shown thereafter by virtue of the following two subsections. In the former, we prove the exponential decay of $L(t)$. In the latter, we prove the asymptotic smoothing property of $K(t)$ and then we achieve the boundedness into $\mathcal{H}^2$ of the set

$$\{K(t)z : \|z\|_{\mathcal{H}} \le R,\ t \ge 0\}$$

by virtue of a bootstrap argument. Accordingly, on account of the compact embedding $\mathcal{H}^2 \Subset \mathcal{H}$, the existence of $\mathcal{K}_R$ follows.

In the sequel of this section, $R > 0$ is fixed and $\|z\|_{\mathcal{H}} \le R$.

4.2. The exponential decay of $L(t)$

As well as in Section 3.2, the exponential decay of $L(t)$ can be achieved by means of a suitable (generalized) uniform Gronwall lemma. In this case, however, some control on the dissipation of the original semigroup, and hence Lemma 11 (see Appendix), is needed because of the nonlinear (in u) term which appears into (4.1).

Now we start by proving that the first condition of Lemma 7 holds true.

Lemma 8. *Let $\|z\|_{\mathcal{H}} \le R$. There is $\varpi = \varpi(R) > 0$ such that*

$$\|L(t)z\|_{\mathcal{H}} \le Q(R)e^{-\varpi t}.$$

Proof. Let $\mathcal{E}_0(t)$ be the energy norm of the one-parameter operator $L(t)$, namely

$$\mathcal{E}_0(t) = \mathcal{E}_0(L(t)z) = \|L(t)z\|_{\mathcal{H}}^2 = \|\eta(t)\|_2^2 + \|\partial_t\eta(t)\|^2 + \|\xi(t)\|_1^2 + \|\partial_t\xi(t)\|^2.$$

We introduce the functional

$$\Phi_0(t) = \Phi_0(L(t)z, u(t)),$$

where $u(t)$ is the first component of $S(t)z$ and

$$\Phi_0(L(t)z, u(t)) = \mathcal{E}_0(L(t)z) - p\|\eta(t)\|_1^2 + \alpha\|\eta(t)\|^2 + M(\|u(t)\|_1^2)\|\eta(t)\|_1^2 + \varepsilon\langle\partial_t\eta(t), \eta(t)\rangle + \varepsilon\langle\partial_t\xi(t), \xi(t)\rangle.$$

In light of Lemma 1 and (3.1), we obtain the following lower and upper bounds for Φ_0,

$$\frac{1}{4}\mathcal{E}_0 \le \Phi_0 \le Q(R)\mathcal{E}_0. \tag{4.3}$$

The time-derivative of Φ_0 along a solution to system (4.1) reads

$$\frac{d\Phi_0}{dt} = -\varepsilon\Phi_0 - 2a\,\|\partial_t\eta\|^2 - 2b\,\|\partial_t\xi\|^2 + 2\left\langle\partial_t u, A^{1/2}u\right\rangle M'\left(\|u\|_1^2\right)\|\eta\|_1^2$$
$$+ 2\varepsilon\left(\|\partial_t\eta\|^2 + \|\partial_t\xi\|^2\right) - \varepsilon\left(\langle a\,\partial_t\eta, \eta\rangle + \langle b\,\partial_t\xi, \xi\rangle\right) + \varepsilon^2\left(\langle\,\partial_t\eta, \eta\rangle + \langle\,\partial_t\xi, \xi\rangle\right).$$

Since $M \in C^1([0,\infty))$, from Lemma 1 we get the bound $M'\left(\|u\|_1^2\right) \le C$. Hence

$$\frac{d\Phi_0}{dt} + \varepsilon\Phi_0 + 2(\gamma_0 - \varepsilon)\left(\|\partial_t\eta\|^2 + \|\partial_t\xi\|^2\right) \le 2\left\langle\partial_t u, A^{1/2}u\right\rangle C\ \|\eta\|_1^2 - \varepsilon\left(\langle a\,\partial_t\eta, \eta\rangle + \langle b\,\partial_t\xi, \xi\rangle\right)$$
$$+ \varepsilon^2\left(\langle\,\partial_t\eta, \eta\rangle + \langle\,\partial_t\xi, \xi\rangle\right)$$
$$\le 2\left\langle\partial_t u, A^{1/2}u\right\rangle C\ \|\eta\|_1^2 - \varepsilon(\gamma_0 - \varepsilon)\left(\langle\,\partial_t\eta, \eta\rangle - \langle\,\partial_t\xi, \xi\rangle\right),$$

where the r.h.s. can be estimated in terms of $\mathcal{E}_0$ as follows

$$2\left\langle\partial_t u, A^{1/2}u\right\rangle C\ \|\eta\|_1^2 - \varepsilon(\gamma_0 - \varepsilon)\left(\langle\,\partial_t\eta, \eta\rangle - \langle\,\partial_t\xi, \xi\rangle\right)$$
$$\le C\,\frac{1}{\varepsilon^2\sqrt{\lambda_1}}\,\|\partial_t u\|^2\,\|\eta\|_2^2 + C\frac{\varepsilon^2}{\sqrt{\lambda_1}}\left\|A^{1/2}u\right\|^2\|\eta\|_2^2 + (\gamma_0 - \varepsilon)\left(\|\partial_t\eta\|^2 + \|\partial_t\xi\|^2\right) + \varepsilon^2(\gamma_0 - \varepsilon)\left(\|\eta\|^2 + \|\xi\|^2\right)$$
$$\le \frac{1}{\varepsilon^2}Q(R)\ \|\partial_t u\|\ \mathcal{E}_0 + \varepsilon^2\,Q(R)\,\mathcal{E}_0 + (\gamma_0 - \varepsilon)\left(\|\partial_t\eta\|^2 + \|\partial_t\xi\|^2\right) + \frac{\varepsilon^2(\gamma_0 - \varepsilon)}{\lambda}\mathcal{E}_0.$$

Choosing ε as small as needed in order that $\varepsilon Q(R) + \frac{\varepsilon(\gamma_0 - \varepsilon)}{\lambda} < \frac{1}{8}$, we obtain

$$\frac{d}{dt}\Phi_0 + \varepsilon\Phi_0 \le \frac{1}{\varepsilon^2}Q(R)\ \|\partial_t u\|\ \mathcal{E}_0 + \frac{\varepsilon}{8}\mathcal{E}_0$$

and finally, since $\frac{1}{4}\mathcal{E}_0 \le \Phi_0$, we infer

$$\frac{d}{dt}\Phi_0 + \frac{\varepsilon}{2}\Phi_0 \le \frac{1}{\varepsilon^2}Q(R)\ \|\partial_t u\|\ \Phi_0.$$

The exponential decay of Φ_0 is entailed by exploiting Lemma 11 with $\epsilon = \varepsilon^3/Q(R)$ and then applying the generalized Gronwall Lemma 12 (see Appendix). Finally, from (4.3) the desired decay of $\mathcal{E}_0$ follows. □

4.3. *The smoothing property of* $K(t)$

On account of the compact embedding $\mathcal{H}^2 \Subset \mathcal{H}$, the next result provides the existence of a compact set $\mathcal{K}_R$ which contains all orbits $K(t)z$, $\|z\|_{\mathcal{H}} \le R$, $t \ge 0$.

Lemma 9. *Let $z \in \mathcal{H}$, $\|z\|_{\mathcal{H}} \le R$. Then, there exists $K_R \ge 0$ such that the estimate*

$$\|K(t)z\|_{\mathcal{H}^2} \le K_R \tag{4.4}$$

holds for every $t \ge 0$.

Unfortunately, a direct proof of this lemma seems out of reach. This is due to the extremely weak smoothing of solutions performed by the nonlinear damped wave equation $(4.2)_2$. So, we have to take advantage from a *bootstrap argument*, by paralleling some results devised in [10,16]. Precisely, taking advantage of the control in Lemma 13, we prove the smoothing property of $K(t)$, which in turn entails the desired proof of Lemma 9 by repeated applications.

In the next lemma we prove the smoothing property of $K(t)$, which means the boundedness of $K(t)z$ in a space, $\mathcal{H}_{\delta+1/4}$, which is more regular than the space $\mathcal{H}_\delta$ where the initial data z are bounded. Its proof strictly follows the arguments of Lemma 13 with $K(t)$ in place of $S(t)$, but working with the exponent $\delta + 1/4$ rather than τ.

Lemma 10 (*Smoothing Property*)**.** *Let $\delta \in [0, \frac{7}{4}]$ be fixed, and assume that $\|z\|_{\mathcal{H}^\delta} \le R$, for some $R > 0$. Then there exists $K_R^\delta \ge 0$ such that*

$$\|K(t)z\|_{\mathcal{H}^{\delta+1/4}} \le K_R^\delta, \quad \forall\, t \ge 0.$$

Proof. Hereafter, some estimates are performed in a formal way. Nevertheless, they can be rigorously justified within a suitable (but cumbersome) approximation scheme.

After introducing the energy of order $\delta + 1/4$ of $K(t)$, namely

$$\mathcal{E}_\delta^K = \|\vartheta\|_{\delta+9/4}^2 + \|\zeta\|_{\delta+5/4}^2 + \|\partial_t\vartheta\|_{\delta+1/4}^2 + \|\partial_t\zeta\|_{\delta+1/4}^2,$$

for any $\varepsilon \in (0, 1)$ we define the functional

$$J_\delta^K = \mathcal{E}_\delta^K + F_\delta^K + 2\,\varepsilon\,\langle \partial_t \vartheta, \vartheta\rangle_{\delta+1/4} + 2\,\varepsilon\,\langle \partial_t \zeta, \zeta\rangle_{\delta+1/4},$$

where

$$F_\delta^K = \left(M\left(\|u(t)\|_1^2\right) - p\right)\|\vartheta\|_{\delta+5/4}^2 - 2\,\langle f, \vartheta\rangle_{\delta+1/4} - 2\,\langle g, \zeta\rangle_{\delta+1/4}.$$

Choosing ε small enough, the following bounds hold true

$$\frac{1}{2}\mathcal{E}_\delta^K - Q(R) \le J_\delta^K \le 2\mathcal{E}_\delta^K + Q(R). \tag{4.5}$$

As in the proof of Lemma 13 (see Appendix), we estimate the time derivative of J_δ^K along a solution $K(t)z$. To this end we first compute

$$\frac{d}{dt}\left(\mathcal{E}_\delta^K + F_\delta^K\right) = -2a\,\|\partial_t\vartheta\|_{\delta+1/4}^2 - 2b\,\|\partial_t\zeta\|_{\delta+1/4}^2 + 2\alpha\,\langle\partial_t\vartheta, \eta\rangle_{\delta+1/4}$$
$$+ 2M'(\|u\|_1^2)\ \|\vartheta\|_{\delta+5/4}^2\,\langle u, \partial_t u\rangle_1 - 2\,\langle F(u-v, \partial_t(u-v)), \partial_t(\vartheta-\zeta)\rangle_{\delta+1/4}.$$

By virtue of the interpolation inequality

$$\left|\langle\phi, \psi\rangle_\ell\right| \le \|\phi\|_{\ell+1}\,\|\psi\|_{\ell-1}, \tag{4.6}$$

taking into account (2.1) and Lemma 1, we easily obtain

$$M'(\|u\|_1^2)\,\|\vartheta\|_{\delta+5/4}^2\ \langle u, \partial_t u\rangle_1 \le \tfrac{c}{\sqrt{\lambda_1}}\,\|\vartheta\|_{\delta+9/4}^2\,\|u\|_2\,\|\partial_t u\| \le Q(R)\,\|\partial_t u\|\,\mathcal{E}_\delta^K.$$

Moreover, letting $\tau = \delta + 1/4$, from assumption F_2 it follows

$$\begin{aligned}\left|\langle F\,(u-v, \partial_t(u-v))\,, \partial_t\,(\vartheta-\zeta)\rangle_{\delta+1/4}\right| &\le c_0\,\|u-v\|_{\delta+1/4}\,\|\partial_t(\vartheta-\zeta)\|_{\delta+1/4}\\ &\quad + c_1\,\|\partial_t(u-v)\|_{\delta+1/4}\,\|\partial_t(\vartheta-\zeta)\|_{\delta+1/4}\\ &\le Q(R)\sqrt{\mathcal{E}_\delta^K} + c\,h(t)\sqrt{\mathcal{E}_\delta^K},\end{aligned}$$

where $h(t) = \|\partial_t u(t)\|_\tau + \|\partial_t v(t)\|_\tau$. Finally, we get

$$\langle\partial_t\vartheta, \eta\rangle_{\delta+1/4} \le \frac{\alpha}{\varepsilon}\,\|\eta\|_{\delta+1/4}^2 + \frac{\varepsilon}{4\alpha}\,\|\partial_t\vartheta\|_{\delta+1/4}^2 \le \frac{c_\delta}{\varepsilon}\,\|\eta\|_2^2 + \frac{\varepsilon}{4\alpha}\mathcal{E}_\delta^K \le Q_\varepsilon(R) + \frac{\varepsilon}{4\alpha}\mathcal{E}_\delta^K.$$

In addition, a straightforward calculation leads to

$$\begin{aligned}\frac{d}{dt}\,\langle\partial_t\vartheta, \vartheta\rangle_{\delta+1/4} + \frac{d}{dt}\,\langle\partial_t\zeta, \zeta\rangle_{\delta+1/4} &= \|\partial_t\vartheta\|_{\delta+1/4}^2 + \|\partial_t\zeta\|_{\delta+1/4}^2 - \|\vartheta\|_{\delta+9/4}^2 - \|\zeta\|_{\delta+5/4}^2\\ &\quad - \langle a\partial_t\vartheta, \vartheta\rangle_{\delta+1/4} - \langle b\partial_t\zeta, \zeta\rangle_{\delta+1/4} + \langle\vartheta, \alpha\eta\rangle_{\delta+1/4} + \langle\vartheta, f\rangle_{\delta+1/4}\\ &\quad + \langle\zeta, g\rangle_{\delta+1/4} - \langle F(u-v, \partial_t(u-v)), \vartheta-\zeta\rangle_{\delta+1/4}\\ &\quad + \left[p - M\left(\|u\|_1^2\right)\right]\|\vartheta\|_{\delta+5/4}^2.\end{aligned}$$

From assumption F_2 and interpolation inequality (4.6), we have

$$\begin{aligned}\left|\langle F\,(u-v, \partial_t(u-v))\,, \vartheta-\zeta\rangle_{\delta+1/4}\right| &\le \|F(u-v, \partial_t(u-v))\|_{\delta-3/4}\,\|\vartheta-\zeta\|_{\delta+5/4}\\ &\le \|\vartheta-\zeta\|_{\delta+5/4}\left(c_0\,\|u-v\|_{\delta-3/4} + c_1\,\|\partial_t(u-v)\|_{\delta-3/4}\right)\\ &\le Q(R)\sqrt{\mathcal{E}_\delta^K} + c_\delta\,h(t)\sqrt{\mathcal{E}_\delta^K},\end{aligned}$$

hence

$$\begin{aligned}\frac{d}{dt}\,\langle\partial_t\vartheta, \vartheta\rangle_{\delta+1/4} + \frac{d}{dt}\,\langle\partial_t\zeta, \zeta\rangle_{\delta+1/4} &\le (1+\gamma_1)\left(\|\partial_t\vartheta\|_{\delta+1/4}^2 + \|\partial_t\zeta\|_{\delta+1/4}^2\right) - \frac{1}{2}\,\|\vartheta\|_{\delta+9/4}^2\\ &\quad - \frac{1}{2}\,\|\zeta\|_{\delta+5/4}^2 + Q_\varepsilon(R) + Q(R)\sqrt{\mathcal{E}_\delta^K} + c_\delta\,h(t)\sqrt{\mathcal{E}_\delta^K}.\end{aligned}$$

Collecting all these inequalities we end up with

$$\begin{aligned}&\frac{d}{dt}J_\delta^K + 2\left[\gamma_0 - \varepsilon(1+\gamma_1)\right]\left(\|\partial_t\vartheta\|_{\delta+1/4}^2 + \|\partial_t\zeta\|_{\delta+1/4}^2\right) + \varepsilon\left(\|\vartheta\|_{\delta+9/4}^2 + \|\zeta\|_{\delta+5/4}^2\right) - \frac{\varepsilon}{2}\mathcal{E}_\delta^K\\ &\quad \le Q_\varepsilon(R)\left[1 + h\,\mathcal{E}_\delta^K\right] + (Q_\varepsilon(R) + \bar{c}_\delta h)\sqrt{\mathcal{E}_\delta^K}.\end{aligned}$$

Choosing now $\varepsilon = \varepsilon_0 > 0$ small enough, in light of (4.5) this turns into

$$\frac{d}{dt}J_\delta^K + \frac{\varepsilon_0}{2}J_\delta^K \le Q(R)\big[1 + h + hJ_\delta^K\big] + (Q(R) + \bar{c}_\delta\, h)\sqrt{J_\delta^K}.$$

Notice that from Lemmas 11 and 13 (see Appendix) we have

$$Q(R)\int_s^t h(y)\,dy \le \varepsilon_0(t-s) + Q_{\varepsilon_0}(R).$$

Once again, from the generalized Gronwall Lemma 12 we get the thesis. □

The following *bootstrap* argument entails the desired result.

Proof of Lemma 9. In order to prove (4.4), we repeatedly apply Lemma 10 with $\delta = 0, 1/4, 1/2, \ldots$ and at the final stage, $\delta = 7/4$, the result follows by letting $K_R = K_R^{7/4}$. □

Appendix. Technical lemmas

Three technical results are needed to prove the existence of the global attractor of the previous section. The first lemma provides some integral control on the dissipation of the original semigroup. By virtue of the second lemma, we first obtain the boundedness of the semigroup $S(t)$ in a more regular space and, as a consequence, an integral control of the total dissipation in the same space. The last lemma is a generalized version of the Gronwall lemma.

Lemma 11 (*Integral Control of Dissipation*)**.** *For any* $\epsilon > 0$

$$\int_s^t (\|\partial_t u\,(y)\| + \|\partial_t v\,(y)\|)\,dy \le \epsilon(t-s) + Q_\epsilon(R),$$

for every $t \ge s \ge 0$.

Proof. After integrating (2.6) over (s, t) and taking Lemma 1 into account, we obtain

$$\mathcal{L}(S(t)z) + 2\int_s^t \left(a\,\|\partial_t u\,(y)\|^2 + b\,\|\partial_t v\,(y)\|^2\right) dy = \mathcal{L}(S(s)z) \le \mathcal{L}(z) \le Q(R),$$

from which it follows

$$\int_s^t \left(a\,\|\partial_t u\,(y)\|^2 + b\,\|\partial_t v\,(y)\|^2\right) dy \le \frac{1}{2}\left[\mathcal{L}(z) - \mathcal{L}(S(t)z)\right] \le Q(R).$$

Thanks to the Hölder inequality, this bound yields

$$\int_s^t (\|\partial_t u\,(y)\| + \|\partial_t v\,(y)\|)\,dy \le Q(R)\sqrt{t-s} \le \epsilon(t-s) + Q_\epsilon(R),$$

for any $\epsilon > 0$. □

Lemma 12 (*See [10, Lemma 2.1] and [11, Lemma 3.7]*)**.** *Let Φ be a nonnegative absolutely continuous function on $[0, \infty)$ which satisfies, for some $\varepsilon > 0$ and $0 \le \sigma < 1$, the differential inequality*

$$\frac{d}{dt}\Phi + \varepsilon\Phi \le h_1\Phi + h_2\Phi^\sigma + h_3$$

where

$$\int_s^t h_1(\tau)\,d\tau \le \varepsilon(t-s) + c$$

for all $s \in [0, t]$ and

$$\sup \int_t^{t+1} h_i(\tau)\,d\tau < m_i, \quad i = 2, 3$$

for some constants $c, m_i > 0$. Then there exist $C_1, C_2 \ge 0$ such that

$$\Phi(t) \le \frac{C_1}{1-\sigma}e^{-\varepsilon t} + C_2.$$

Moreover, if $m_3 = 0$ (that is, if $h_3 = 0$), it follows that $C_2 = 0$.

Lemma 13 (*See [11,23]*)**.** *Let* $\tau \in [0, 2]$ *be given and let*

$$z = (u_0, u_1, v_0, v_1) \in \mathcal{H}^\tau, \quad \mathcal{H}^\tau = H^{2+\tau} \times H^\tau \times H^{1+\tau} \times H^\tau,$$

such that $\|z\|_{\mathcal{H}^\tau} \le R$ *for some* $R > 0$*. Then,*

$$\|S(t)z\|_{\mathcal{H}^\tau} \le Q(R), \quad \forall t \ge 0. \tag{A.1}$$

Moreover, for all $\epsilon > 0$ *and for every* $t \ge s \ge 0$

$$\int_s^t (\|\partial_t u(y)\|_\tau + \|\partial_t v(y)\|_\tau)\, dy \le \epsilon(t-s) + Q_\epsilon(R). \tag{A.2}$$

Proof. When $\tau = 0$ the result is already known (see Lemmas 1 and 11). Therefore, hereafter we consider $\tau \in (0, 2]$. We define the functional

$$J_\tau = \mathcal{E}_\tau + F_\tau + 2\,\varepsilon\,\langle \partial_t u, u\rangle_\tau + 2\,\varepsilon\,\langle \partial_t v, v\rangle_\tau + 2\varepsilon \mathcal{N}_\tau(u-v),$$

where $\varepsilon > 0$,

$$\mathcal{E}_\tau = \|u\|^2_{2+\tau} + \|v\|^2_{1+\tau} + \|\partial_t u\|^2_\tau + \|\partial_t v\|^2_\tau\,,$$
$$F_\tau = 2\mathcal{G}_\tau(u-v) - 2\,\langle f, u\rangle_\tau - 2\,\langle g, v\rangle_\tau - \left(p - M\left(\|u\|_1^2\right)\right)\|u\|^2_{1+\tau}$$

and

$$\mathcal{N}_\tau(w) = \left\langle \int_0^w \Gamma(\eta)d\eta,\, 1\right\rangle_\tau, \qquad \mathcal{G}_\tau(w) = \left\langle \int_0^w G(\eta)d\eta,\, 1\right\rangle_\tau.$$

By evaluating these functionals along a solution $S(t)z$, it is straightforward to check that

$$\frac{d}{dt}(\mathcal{E}_\tau + F_\tau) = -2\left(a\,\|\partial_t u\|^2_\tau + b\,\|\partial_t v\|^2_\tau\right) + 2M'\left(\|u\|_1^2\right)\,\|u\|^2_{1+\tau}\,\langle u, \partial_t u\rangle_1 - 2\,\langle G_0, 1\rangle_\tau\,. \tag{A.3}$$

We first observe that $\|z\|_{\mathcal{H}^\tau} \le R$ implies $\|z\|_{\mathcal{H}} \le Q(R)$. Then, by virtue of the interpolation inequality (4.6) we obtain the following estimates of all terms on the r.h.s. of (A.3). First, taking advantage of (2.1) and Lemma 1, it follows

$$\|u\|^2_{1+\tau}\, M'\left(\|u\|_1^2\right)\langle u, \partial_t u\rangle_1 \le \frac{C}{\sqrt{\lambda_1}}\,\|u\|^2_{2+\tau}\,\|u\|_2\,\|\partial_t u\| \le Q(R)\,\|\partial_t u\|\;\mathcal{E}_\tau.$$

Then it follows

$$\frac{d}{dt}(\mathcal{E}_\tau + F_\tau) \le -2\gamma_0\left(\|\partial_t u\|^2_\tau + \|\partial_t v\|^2_\tau\right) + Q(R)\,\|\partial_t u\|\,\mathcal{E}_\tau.$$

Finally, taking into account that

$$\frac{d}{dt}\langle \partial_t u, u\rangle_\tau + \frac{d}{dt}\langle \partial_t v, v\rangle_\tau + \frac{d}{dt}\mathcal{N}_\tau(u-v) \le (1+\gamma_1)\left(\|\partial_t u\|^2_\tau + \|\partial_t v\|^2_\tau\right) - \frac{1}{2}\left(\|u\|^2_{2+\tau} + \|v\|^2_{1+\tau}\right) + Q(R)$$

this inequality holds for all $\varepsilon > 0$

$$\frac{d}{dt}J_\tau + 2[\gamma_0 - \varepsilon(1+\gamma_1)]\left(\|\partial_t u\|^2_\tau + \|\partial_t v\|^2_\tau\right) + \varepsilon\left(\|u\|^2_{2+\tau} + \|v\|^2_{1+\tau}\right) \le Q(R)\,\|\partial_t u\|\,\mathcal{E}_\tau + \varepsilon\,Q(R). \tag{A.4}$$

Choosing ε small enough, the following bounds hold true

$$\frac{1}{2}\mathcal{E}_\tau - c \le J_\tau \le Q(R)\mathcal{E}_\tau + Q(R).$$

Therefore, applying Lemma 1 again, we end up with

$$\frac{d}{dt}J_\tau + \frac{\varepsilon_0}{Q(R)}J_\tau \le Q_{\varepsilon_0}(R)\,\|\partial_t u\|\,J_\tau + Q_{\varepsilon_0}(R),$$

where $\varepsilon_0 = \min\{\varepsilon, 2[\gamma_0 - \varepsilon(1+\gamma_1)]\}$. By virtue of Lemma 11, with $\epsilon = \varepsilon_0/Q_{\varepsilon_0}(R)$, and the generalized Gronwall Lemma 12, we get the first part of the thesis.

To obtain the integral control, we consider (A.4) with $\varepsilon < \frac{2\gamma_0 - 1}{1+\gamma_1}$. Now, the bound $\mathcal{E}_\tau \le Q(R)$ holds true by virtue of (A.1). Then, we are lead to

$$\frac{d}{dt}J_\tau + \|\partial_t u\|^2_\tau + \|\partial_t v\|^2_\tau \le Q(R)\,\|\partial_t u\| + \varepsilon\,Q(R).$$

An integration on (s, t), along with further applications of (A.1) and Lemma 11 leads to

$$\int_s^t \left(\|\partial_t u(y)\|_\tau^2 + \|\partial_t v(y)\|_\tau^2\right) dy \le \varepsilon\, Q(R)(t-s) + Q_\varepsilon(R),$$

from which it follows

$$\int_s^t (\|\partial_t u\,(y)\|_\tau + \|\partial_t v\,(y)\|_\tau)\, dy \le \sqrt{\varepsilon}\, Q(R)(t-s) + Q_\epsilon(R).$$

For all $\epsilon > 0$, (A.2) follows by properly choosing $\sqrt{\varepsilon}$, and the argument is completed. □

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