Sampling complete designs

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Abstract

Suppose Γ' to be a subgraph of a graph Γ. We define a *sampling* of a Γ-design $\mathfrak{B} = (V, B)$ into a Γ'-design $\mathfrak{B}' = (V, B')$ as a surjective map $\xi : B \to B'$ mapping each block of B into one of its subgraphs. A sampling will be called *regular* when the number of preimages of each block of B' under ξ is a constant. This new concept is closely related with the classical notion of *embedding*, which has been extensively studied, for many classes of graphs, by several authors; see, for example, the survey [29]. Actually, a sampling ξ might induce several embeddings of the design $\mathfrak{B}'$ into $\mathfrak{B}$, although the converse is not true in general. In the present paper we study in more detail the behaviour of samplings of Γ–complete designs of order n into Γ'–complete designs of the same order and show how the natural necessary condition for the existence of a regular sampling is actually sufficient. We also provide some explicit constructions of samplings, as well as propose further generalizations.

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1 Introduction

Denote by K_n the complete graph on n vertices, and assume $\Gamma \leq K_n$ to be a subgraph of K_n. Write ${}^{\lambda}K_n$ for the multigraph obtained from K_n by repeating each of its edges exactly λ times. A $({}^{\lambda}K_n, \Gamma)$*–design* is a set $\mathfrak{B}$ of graphs, called *blocks*, isomorphic to Γ and partitioning the edges of ${}^{\lambda}K_n$. Recall that an automorphism of $\mathfrak{B}$ is a permutation of the vertices of K_n leaving $\mathfrak{B}$ invariant. These designs are a topic of current investigation. In particular, for those with $\lambda = 1$ and endowed with a rich automorphism group, several constructions and existence results are known; see, for instance, [2, 3, 28]. In the present paper, we generalize the classical concept of (v, k)–complete design, see [6, 23], to the context of graph decompositions.

Definition 1.1. Suppose $\Gamma \leq K_n$. By a (K_n, Γ)*–complete design* we mean the set $K_n(\Gamma)$ consisting of all subgraphs of K_n isomorphic to Γ.

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Clearly, a (K_v, K_k)–complete design is just a (v, k)–complete design.

The main topic of this paper is the new concept of *sampling* of designs; we propose the following definition.

Definition 1.2. Given a $({}^{\lambda}K_n, \Gamma)$–design $\mathfrak{B}$ and a $({}^{\mu}K_v, \Gamma')$–design $\mathfrak{B}'$ with $\Gamma' \leq \Gamma$, a $\mathfrak{B}'$*–sampling* of $\mathfrak{B}$ is a surjective function $\xi : \mathfrak{B} \to \mathfrak{B}'$ such that $\xi(B) \leq B$, for any $B \in \mathfrak{B}$.

When $n = v$ and $\lambda = \mu = 1$, we shall speak simply of Γ'–samplings of $\mathfrak{B}$. In general, to prove the existence of samplings of arbitrary designs is an interesting yet difficult problem. Here, we shall be concerned with samplings of complete designs.

Our notion of sampling is closely related to that of embedding of designs. We recall that an *embedding* of $\mathfrak{B}'$ into $\mathfrak{B}$, see [29], is a function $\psi : \mathfrak{B}' \to \mathfrak{B}$ such that for any $G \in \mathfrak{B}'$,

$$G \leq \psi(G).$$

In recent years, embeddings have been extensively investigated and several results have been obtained for various classes of graphs; see for instance, [7, 10, 17, 21, 25, 26, 24, 27, 30, 31]. An injective embedding is called *strict*. Note that given a sampling $\xi : K_n(\Gamma) \to K_n(\Gamma')$, there is always a strict embedding $\psi : K_n(\Gamma') \to K_n(\Gamma)$ such that $\xi\psi$ is the identity on $K_n(\Gamma')$. However, the reverse, namely that any strict embedding induces a sampling, is not true, unless the embedding is taken to be bijective.

Samplings are also related to nestings of cycle systems; see Section 2 for some details.

We also propose this new definition for samplings between complete designs.

Definition 1.3. A *regular* Γ'*–sampling* of the complete design $K_n(\Gamma)$ is a sampling $\xi : K_n(\Gamma) \to K_n(\Gamma')$ such that the number of preimages of any $G \in K_n(\Gamma')$ is a constant $\lambda > 0$. Such a sampling is said to have *redundancy* λ.

Observe that constructing a regular Γ'–sampling of redundancy λ is the same as to extract from every block of $K_n(\Gamma)$ a subgraph isomorphic to Γ' in such a way as to cover the set $K_n(\Gamma')$ exactly λ times.

We propose an analogous definition for embeddings.

Definition 1.4. A λ*–fold regular embedding* of $K_n(\Gamma')$ in $K_n(\Gamma)$ is an embedding $\psi : K_n(\Gamma') \to K_n(\Gamma)$ such that any $G \in K_n(\Gamma)$ has λ preimages under ψ.

The main result of the present paper is contained in Theorem 4.3: it shows that the natural necessary conditions for the existence of regular Γ'–samplings and λ–fold regular embeddings for $K_n(\Gamma)$ are also sufficient. Theorem 4.6 shows that when the hypotheses of Theorem 4.3 are not fulfilled, there might still exist, in some cases, samplings with some regularity.

In Section 5, we shall provide some direct constructions of regular samplings, using suitable automorphism groups, and focusing our attention to the case in which both Γ and Γ' are complete graphs. We will also propose a generalisation of the notion of (K_n, Γ)–complete design to arbitrary graphs.

2 Samplings and nestings

In this section we use standard notations of graph theory; see [14]. In particular, by C_m we mean the cycle of length m, whereas S_{m+1} denotes the star with m rays and $m+1$ vertices; with W_{m+1} we write the wheel with $m+1$ vertices, that is the graph obtained from a cycle C_m, by adding a further vertex adjacent to all the preexisting ones; this vertex is the *centre* of the wheel.

An *m–cycle system $\mathfrak{C}$ of order n* is just a (K_n, C_m)–design; see [4]. A *nesting* of $\mathfrak{C}$ is a function $f : \mathfrak{C} \to V(K_n)$ such that

$$\mathfrak{S} = \{\{x, f(C)\} \,|\, C \in \mathfrak{C}, x \in V(C)\}$$

is a (K_n, S_{m+1})–design.

Nestings of cycle systems have been extensively studied; see [9, 11, 18, 19, 20, 32, 33]. Observe that, by construction, a nesting of $\mathfrak{C}$ always determines a bijection $g : \mathfrak{C} \to \mathfrak{S}$. Consider the set

$$\mathfrak{W} = \{C \cup g(C) \,|\, C \in \mathfrak{C}\}.$$

All elements of $\mathfrak{W}$ are wheels W_{m+1}. It is immediate to see that $\mathfrak{W}$ is a $({}^2K_n, W_{m+1})$–design. Since both $C_m \le W_{m+1}$ and $S_{m+1} \le W_{m+1}$, there exist at least two samplings, $\xi_1 : \mathfrak{W} \to \mathfrak{C}$ and $\xi_2 : \mathfrak{W} \to \mathfrak{S}$. Actually, in this case, it is possible to reconstruct the nesting f from just ξ_2. Proceed as follows: for any $W \in \mathfrak{W}$, let $\zeta(W) = W \setminus \xi_2(W)$. Clearly $\zeta(W)$ is always an m–cycle. Since $\mathfrak{W}$ is a $({}^2K_n, W_{m+1})$–design and $\mathfrak{S}$ is a (K_n, S_{m+1})–design, the set

$$\mathfrak{C} = \{\zeta(W) \,|\, W \in \mathfrak{W}\}$$

is an m–cycle system of order n. The function $f : \mathfrak{C} \to V(K_n)$ which sends any cycle $C = \zeta(W)$ into the centre of the wheel W is uniquely defined and is a nesting. We observe that, for $m > 3$ there is only one cycle, say C, in W_{m+1} such that $W_{m+1} \setminus C$ is a star, even if the wheel W_{m+1} actually contains $m+1$ distinct cycles C_m. Hence, a sampling $\xi_1 : \mathfrak{W} \to \mathfrak{C}$, is not, in general, associated with a sampling $\xi_2 : \mathfrak{W} \to \mathfrak{S}$; thus, it might not induce a nesting.

3 Preliminaries on graph and matching theory

Here we recall some known results about matchings of bipartite graphs; for further references, see [22, 34].

By a graph we shall always mean a finite unordered graph $\Gamma = (V, E)$ without loops, having vertex set V and edge set E.

Recall that a graph $\Gamma = (V, E)$ is *bipartite* if V can be partitioned into two sets Γ_1, Γ_2 such that every edge of Γ has one vertex in Γ_1 and one in Γ_2. The *degree* of $v \in V$ in Γ is the number $\deg_\Gamma(v)$ of edges of Γ containing v. A bipartite graph Γ with vertex set $\Gamma_1 \cup \Gamma_2$ is *(d, e)–regular* if each vertex in Γ_1 has degree d while each vertex in Γ_2 has degree e. If $d = e$, we say that Γ is *regular of degree d*.

A *matching* in a graph Γ is a set of edges of Γ, no two of which are adjacent. A *perfect matching* (or 1–*factor*) of Γ is a matching partitioning the vertex set $V(\Gamma)$. In a bipartite graph Γ with vertex set $\Gamma_1 \cup \Gamma_2$, a matching is *full* if it contains $\min(|\Gamma_1|, |\Gamma_2|)$ edges. Clearly, a full matching of Γ is perfect if, and only if, $|\Gamma_1| = |\Gamma_2|$.

The following lemma shall be used throughout the paper.

Lemma 3.1 ([1], pag. 397, Corollary 8.13)**.** *Any bipartite (d, e)–regular graph Γ possesses a full matching.*

An *edge colouring* of Γ is a function $w : E \to \mathbb{N}$ such that for any adjacent edges, say e_1, e_2,

$$w(e_1) \neq w(e_2).$$

An *n–edge colouring* of Γ is an edge colouring using exactly n colours. The *chromatic index* $\chi'(\Gamma)$ is the minimum n such that Γ has an n–edge colouring.

Theorem 3.2 (König Line Colouring Theorem, [15, 16])**.** *For any bipartite graph Γ,*

$$\chi'(\Gamma) = \max_{v \in \Gamma} \deg_\Gamma(v).$$

4 Embeddings and samplings of (K_n, Γ)–complete designs

Throughout this section, let $\Gamma' \leq \Gamma$ be two subgraphs of K_n.

By ${}^{\lambda}K_n(\Gamma)$ we will denote the multiset obtained from $K_n(\Gamma)$ by repeating each of its elements exactly λ times.

Lemma 4.1. *Let b_1 be the number of blocks of $K_n(\Gamma)$ and b_2 be that of $K_n(\Gamma')$ and let $m = \mathrm{lcm}(b_1, b_2)$. Then there is a bijective embedding*

$$\psi : {}^{(m/b_2)}K_n(\Gamma') \to {}^{(m/b_1)}K_n(\Gamma).$$

Proof. Introduce the bipartite graph Δ with vertex set $V = K_n(\Gamma) \cup K_n(\Gamma')$ and $x, y \in V$ are adjacent if, and only if, $x \neq y$ and either $x \leq y$ or $y \leq x$.

In the first step of the proof, we verify that Δ is (d, e)–regular, for some $d, e \in \mathbb{N}$. As the automorphism group of K_n is $\mathrm{Aut}(K_n) \simeq S_n$, we have that $\mathrm{Aut}(K_n)$ is transitive on both $K_n(\Gamma)$ and $K_n(\Gamma')$. We now argue by way of contradiction. Suppose that there are $\Gamma_1, \Gamma_2 \in K_n(\Gamma)$ such that

$$d_1 = \deg \Gamma_1 < \deg \Gamma_2 = d_2.$$

Then, there exists $\sigma \in \mathrm{Aut}(K_n)$ such that $\sigma(\Gamma_2) = \Gamma_1$. In particular, the image under σ of the d_2 subgraphs of Γ_2 isomorphic to Γ' consists of d_2 subgraphs of Γ_1, all isomorphic to Γ'. However, we supposed the number of subgraphs of Γ_1 isomorphic to Γ' to be $d_1 < d_2$; this yields a contradiction.

Likewise, suppose we have two graphs $\Gamma'_1, \Gamma'_2 \in K_n(\Gamma')$ with

$$e_1 = \deg \Gamma'_1 < \deg \Gamma'_2 = e_2.$$

As $\mathrm{Aut}(K_n)$ is transitive on $K_n(\Gamma')$, there is $\sigma \in \mathrm{Aut}(K_n)$ with $\sigma(\Gamma'_2) = \Gamma'_1$. This permutation σ, in particular, sends the e_2 graphs isomorphic to Γ containing Γ'_2 into e_2 distinct graphs containing Γ'_1 isomorphic to Γ. This yields a contradiction, since $e_1 < e_2$.

Let now Δ' be the graph obtained from Δ by replicating (m/b_1)–times $K_n(\Gamma)$ and (m/b_2)–times $K_n(\Gamma')$. By construction, Δ' is a $(dm/b_2, em/b_1)$–regular bipartite graph. By Lemma 3.1, Δ' admits a full matching M. Furthermore, since both parts of Δ' have the same cardinality, M is perfect.

For any $x \in K_n(\Gamma')$, define $\psi(x) = y$ where $(x, y) \in M$. This provides an embedding, as required. □

Remark 4.2. Since ψ in Lemma 4.1 is bijective, the function

$$\xi = \psi^{-1} : {}^{(m/b_1)}K_n(\Gamma) \to {}^{(m/b_2)}K_n(\Gamma')$$

is a sampling.

Now we are ready to prove our main result.

Theorem 4.3. *The complete design $K_n(\Gamma)$ admits a regular Γ'–sampling if, and only if, there is $\lambda \in \mathbb{N}$ such that*

$$|K_n(\Gamma)| = \lambda |K_n(\Gamma')|. \tag{1}$$

The redundancy of any such sampling is λ.

Conversely, there is a λ–fold regular embedding of $K_n(\Gamma')$ in $K_n(\Gamma)$ if, and only if

$$|K_n(\Gamma')| = \lambda |K_n(\Gamma)|.$$

Proof. Clearly, Condition (1) is necessary for the existence of a regular Γ'–sampling. By Remark 4.2, there is a bijective sampling

$$\vartheta : K_n(\Gamma) \to {}^{\lambda}K_n(\Gamma').$$

Each $y \in K_n(\Gamma')$ appears exactly λ times in ${}^{\lambda}K_n(\Gamma')$. As such, y is the image of λ elements of $K_n(\Gamma)$ under ϑ. Thus, ϑ induces a regular Γ'–sampling $\xi : K_n(\Gamma) \to K_n(\Gamma')$ with redundancy λ.

The second part of the theorem is proved in an analogous way, using the bijective embedding

$$\psi : K_n(\Gamma') \to {}^{\lambda}K_n(\Gamma)$$

provided by Lemma 4.1. □

We have to point out that the proof of previous theorem is not constructive. In order to determine actual samplings we may, in general, need to use some of the algorithms for matchings in graphs, like the ones in [22], applied to the graph Δ' of Lemma 4.1. Also, in Section 5, we will construct explicitly regular samplings for some complete designs.

When $|K_n(\Gamma')|$ is not a divisor of $|K_n(\Gamma)|$, the previous theorem fails. Under the assumption

$$|K_n(\Gamma)| = \lambda |K_n(\Gamma')| + r,$$

with $0 < r < |K_n(\Gamma')|$, we may investigate the existence of sampling functions which are "as regular as possible", namely in which the number of preimages of any given element is either λ or $\lambda + 1$. These samplings shall be called $(\lambda, \lambda + 1)$*–semiregular.*

We start with the following lemma.

Lemma 4.4. *Suppose*

$$\lambda = \left\lfloor \frac{|K_n(\Gamma)|}{|K_n(\Gamma')|} \right\rfloor ;$$

then, there is a strict embedding $\xi : {}^{\lambda}K_n(\Gamma') \to K_n(\Gamma)$.

Proof. Argue as in the first part of the proof of Lemma 4.1 and introduce the (d, e)–regular bipartite graph Δ. Let now Δ' be the graph obtained from Δ by replicating λ–times $K_n(\Gamma')$. As Δ' is a $(\lambda d, e)$–regular bipartite graph, it admits by Lemma 3.1 a full matching M of size $\lambda |K_n(\Gamma')|$; in particular, each vertex in ${}^{\lambda}K_n(\Gamma')$ is matched to exactly one vertex of $K_n(\Gamma)$. □

By collapsing the multiset ${}^{\lambda}K_n(\Gamma')$ in the proof of the previous lemma, we have that M associates λ elements of $K_n(\Gamma)$ to each element of $K_n(\Gamma')$. This leads to the following corollary.

Corollary 4.5. *Suppose*

$$\lambda = \left\lfloor \frac{|K_n(\Gamma)|}{|K_n(\Gamma')|} \right\rfloor ;$$

then, there exists a sampling $\xi : K_n(\Gamma) \to K_n(\Gamma')$ *such that any* $g \in K_n(\Gamma')$ *has* at least λ *preimages.*

It can be seen directly from Corollary 4.5 that when

$$|K_n(\Gamma)| = \lambda |K_n(\Gamma')| + 1,$$

there always exists a $(\lambda, \lambda + 1)$–semiregular sampling. In general, however, further hypotheses on the nature of the embedding of $K_n(\Gamma')$ into $K_n(\Gamma)$ are required. We prove a result for the case $\lambda = 1$.

Theorem 4.6. *Let* $\Gamma' \leq \Gamma \leq K_n$ *with*

$$|K_n(\Gamma)| - |K_n(\Gamma')| = r < |K_n(\Gamma')|. \tag{2}$$

Suppose

$$|K_n(\Gamma)| > \frac{er^2}{e + r - 1}, \tag{3}$$

where $e \in \mathbb{N}$ *is the number of elements of* $K_n(\Gamma)$ *containing* Γ'*. Then, there is a* $(1, 2)$*–semiregular sampling* $\xi : K_n(\Gamma) \to K_n(\Gamma')$*.*

Proof. Argue as in the proof of Lemma 4.1, and construct a (d,e)–regular bipartite graph Δ. As

$$d|K_n(\Gamma)| = e|K_n(\Gamma')|, \tag{4}$$

by (2) we have

$$(e-d)|K_n(\Gamma')| = dr > 0.$$

Thus, $e-1 \geq d$. Also, by (2) and (4), we have $er = (e-d)|K_n(\Gamma)|$, hence using (3) it results

$$d > \frac{r-1}{r}(e-1).$$

Determine now, using Lemma 3.1, a full matching of Δ; this provides values for the function ξ on a subset T of $K_n(\Gamma)$ with $T = |K_n(\Gamma')|$. Let now $R = K_n(\Gamma)\backslash T$ be the set of the vertices of $K_n(\Gamma)$ which have not been already matched, and consider the bipartite graph $\Delta' = (V', E')$ obtained from Δ by keeping just the vertices in $R \cup K_n(\Gamma')$. This graph, in general, is not regular; however, $\deg_{\Delta'}(v) = d$ if $v \in R$ and $\deg_{\Delta'}(v) < e$ if $v \in K_n(\Gamma')$. By Theorem 3.2, $\chi'(\Delta') \leq e-1$. Let $w : E' \to L \subset \mathbb{N}$ be a colouring of Δ' with $|L| \leq e-1$. Since $R = \Delta' \cap K_n(\Gamma)$ contains exactly r vertices and each of these is incident with at least $(e-1)(r-1)/r+1$ differently coloured edges, there is at least an $\ell \in L$ such that each vertex of R is incident with an edge of colour ℓ. Thus, the set

$$C_\ell = \{x \in E' : w(x) = \ell\}$$

is a full matching for Δ'. Now, for any $r \in R$, define $\xi(r) = s$, where $(r,s) \in C_\ell$. This completes the proof. □

Example 4.7. An interesting case for Theorem 4.6 occurs when d is taken to be as large as possible, namely $d = e-1$. Since, in this case, by (2) and (4)

$$r = \frac{1}{e}|K_n(\Gamma)|,$$

condition (3) is always satisfied. We provide now an actual example where this happens. Suppose n to be even and let $\Gamma = K_{n/2}$ and $\Gamma' = K_{n/2-1}$. Each element of $K_n(\Gamma')$ is contained in $e = n/2+1$ elements of $K_n(\Gamma)$, while Γ contains $d = n/2$ elements of $K_n(\Gamma')$.

5 Examples and applications

As said above, in this section we will show direct constructions of samplings for some complete designs. A convenient approach is to consider a suitable automorphism group of the designs which is also compatible with the sampling we wish to find, in the sense of the following definition.

Definition 5.1. Let $\mathfrak{B}$ be a (K_n,Γ)–design and let $\mathfrak{B}'$ be a (K_n,Γ')-design, with $\Gamma' \leq \Gamma$, and suppose $\xi : \mathfrak{B} \to \mathfrak{B}'$ to be a sampling. An *automorphism* α of ξ is an automorphism of $\mathfrak{B}'$ and $\mathfrak{B}$ such that for any $B \in \mathfrak{B}$,

$$\xi(\alpha(B)) = \alpha(\xi(B)).$$

Observe that an analogous definition is possible also for embeddings; see, for instance [5, Theorem 3.2] where a $2-(p,3,1)$ design is cyclically embedded into a cyclic $2-(4p,4,1)$ design.
Let n be an integer. From now on, we shall mean by $\mathbb{Z}_n^{\square}$ the group of all the invertible elements in $\mathbb{Z}_n$ which are squares.

Example 5.2. Fix n. Let $\Gamma = C_k \leq K_n$ be a cycle on k vertices, and write $\Gamma' = P_h \leq C_k$ for a path in C_k with h vertices. We have

$$|K_n(C_k)| = \binom{n}{k}\frac{(k-1)!}{2}; \qquad |K_n(P_h)| = \binom{n}{h}\frac{h!}{2}.$$

By Theorem 4.3, $K_n(C_k)$ admits a regular P_h–sampling, if, and only if,

$$\lambda = \frac{|K_n(C_k)|}{|K_n(P_h)|} = \frac{(n-h)!}{(n-k)!k} \in \mathbb{N}.$$

In general, even with the existence of a group action compatible with the structures involved, it is not easy to actually determine a sampling.

We write a full example just for $n=7$, $k=4$ and $h=3$; here, $\lambda = 1$. Identify the vertices of K_7 with the elements of $\mathbb{Z}_7$. First, we wish to find a group G which is

1. transitive on K_7;
2. acts in a semiregular way on $K_7(C_4)$ and, possibly, on $K_7(P_3)$.

Such a group G, if it exists, it is necessarily isomorphic to $H/\operatorname{Stab}_H(C)$ for some $H \leq S_7$ and any $C \in K_7(C_4)$. Thus, we need first to determine all $H \leq S_7$ normalising at least $\operatorname{Stab}_H(C_4)$. A direct computation shows that, up to conjugacy, there are just two classes of such subgroups:

(a) one consisting of cyclic groups of order 7 isomorphic to $(\mathbb{Z}_7,+)$,

(b) the other containing groups of order 21 isomorphic to $G = \mathbb{Z}_7^{\square} \ltimes \mathbb{Z}_7$, where, for $(\alpha,\beta),(\alpha',\beta') \in G$,

$$(\alpha,\beta)(\alpha',\beta') = (\alpha\alpha', \beta + \alpha\beta').$$

The action of G on $V(K_7) = \mathbb{Z}_7$ is given by

$$(\alpha,\beta)(x) = \alpha x + \beta.$$

It might be checked that the action induced by both these groups on $K_7(C_4)$ and $K_7(P_3)$ is semiregular.

We now describe 2 different samplings.

First consider the group $(\mathbb{Z}_7,+)$. In this case, a complete system of representatives for $K_7(P_3)$ is given by the paths

012 013 014 015 016 023 024 025 026 034
035 036 045 046 056.

Recall that two triples abc and def represent the same path in $K_7(P_3)$ if, and only if, either $abc = def$ or $abc = fed$. Suitable representatives for the orbits of $K_7(C_4)$ turn out to be

$$\begin{array}{ccccc} \underline{012}6 & \underline{013}6 & \underline{014}6 & \underline{015}2 & \underline{016}3 \\ \underline{023}1 & \underline{024}5 & \underline{025}3 & \underline{026}1 & \underline{034}6 \\ \underline{035}1 & \underline{036}2 & \underline{045}1 & \underline{046}1 & \underline{056}2 \end{array}$$

The subpath selected by the sampling ξ is given, for each representative, by the underlined elements, which have to be taken in the order they actually appear. As above, observe that the elements of $K_7(C_4)$ are not sets; in particular, two quadruples $abcd$ and $a'b'c'd'$ represent the same graph if, and only if, they belong to the same orbit in the vector space $\mathbb{Z}_7^4$ under the action of D_8, the dihedral group of order 8.

If the group $\mathbb{Z}_7^\square \ltimes \mathbb{Z}_7$ is chosen for the construction, then a complete system of representatives for $K_7(P_3)$ is just

$$012 \quad 013 \quad 014 \quad 015 \quad 035.$$

A related system of representatives for $K_7(C_4)$ is determined as

$$\underline{012}6 \quad \underline{013}6 \quad \underline{014}6 \quad \underline{015}4 \quad \underline{035}1.$$

The image of 0126, 0136, 0146 and 0351 in the sampling ζ induced by these representatives is the same as that in the sampling ξ described above; however, it is not possible to choose $\zeta(0152) = 015$ again, since 0152 now belongs to the orbit of 0146; thus, its sample is uniquely determined by $\zeta(0146) = 014$ and it must be $\zeta(0152) = 025$. This shows that G is an automorphism group of ζ, but not of ξ.

If we take both the graphs Γ and Γ' to be complete, we may apply Theorem 4.3 to the study of embeddings and samplings of complete designs in the classical sense.

We remark that the problem of determining a regular K_1–sampling with redundancy 1 of $K_n(K_k)$ is exactly that of determining a system of distinct representatives for the k–subsets of a set of cardinality n; see [12, 13].

For any finite set S of cardinality n, denote by $\binom{S}{k}$ the set of all subsets of S with k elements.

Corollary 5.3. *There exists a regular k'–sampling*

$$\xi : \binom{S}{k} \to \binom{S}{k'}$$

if, and only if, there is $\lambda \in \mathbb{N}$ such that

$$\binom{n}{k} = \lambda \binom{n}{k'}.$$

The redundancy of this sampling is λ.

Corollary 5.4. *Let S be a finite set with $|S| = n$. Suppose $k \leq \lfloor n/2 \rfloor$. Then, there exists a bijective sampling*

$$\xi : \binom{S}{n-k} \to \binom{S}{k}.$$

Remark 5.5. Corollary 5.3 guarantees the existence of a regular k'–sampling of $\binom{S}{k}$ with redundancy λ every time the necessary condition

$$\binom{n}{k} = \lambda \binom{n}{k'}$$

holds; yet our proof is non–constructive.

In at least some cases, however, it might possible to write at least some sampling functions ξ in a more direct way.

The main idea, as before, is to describe both $\binom{S}{k}$ and $\binom{S}{k'}$ as union of orbits under the action of a suitable group G, acting on S, and determine systems of representatives T and U such that:

1. $T \subseteq \binom{S}{k}$, $U \subseteq \binom{S}{k'}$;
2. G is semiregular on $\binom{S}{k}$;
3. any element of $u \in U$ is a sample of $\lambda / |\operatorname{Stab}_G(u)|$ elements of T;
4. for any $\sigma \in G$ and $t \in T$,
$$\sigma(\xi(t)) = \xi(\sigma(t)).$$

Example 5.6. Suppose $k = 3$. We are looking for a regular 2–sampling of $\binom{S}{3}$; thus, $\lambda = (n-2)/3$. By Corollary 5.3 such a regular 2–sampling ξ exists if, and only if, $n \equiv 2 \pmod 3$. Observe that when n is even, λ is also even.

In order to explicitly find ξ, consider the natural action of the cyclic group $(\mathbb{Z}_n, +)$ on the set $S = \{0, 1, \ldots, n-1\}$: for any $\eta \in \mathbb{Z}_n$ and $s \in S$, let

$$\eta(s) = s + \eta \pmod n.$$

Fix $v = \lfloor n/2 \rfloor$. It is easy to show that a complete system of representatives for the orbits of $\mathbb{Z}_n$ on $\binom{S}{3}$ is given by either $T = T_1 \cup T_2 \cup T_3 \cup T_4^1$ for n odd, or $T = T_1 \cup T_2 \cup T_3 \cup T_4^2$ for n even, where

$$\begin{array}{rcl} T_1 & = & \{\{0, i, i+t\} \mid i = 1, \ldots, \lambda+1; t = 1, \ldots, \lambda\} \\ T_2 & = & \{\{0, j, u\} \mid j = \lambda+2, \ldots, v-1; u = 1, \ldots, j-\lambda-1\} \\ T_3 & = & \{\{0, \ell, m\} \mid \ell = \lambda+2, \ldots, v-1; m = \ell+1, \ldots, 2\lambda+1\} \\ T_4^1 & = & \{\{0, v, u\} \mid u = 1, \ldots, v-\lambda-1, v+1, \ldots, 2\lambda+1\} \\ T_4^2 & = & \{\{0, v, p\} \mid p = 1, \ldots, \lambda/2\}. \end{array} \quad (5)$$

A set of representatives for the orbits of $\mathbb{Z}_n$ on $\binom{S}{2}$ is just $U = \{\{0, x\} : 1 \leq x \leq v\}$. All orbits of $\mathbb{Z}_n$ on $\binom{S}{3}$ have length n; thus $\mathbb{Z}_n$ acts semiregularly on $\binom{S}{3}$.

When n is odd, it is also true that $\mathrm{Stab}_{\mathbb{Z}_n}(y) = \{0\}$ for any $y \in U$. However, when n is even, $\mathrm{Stab}_{\mathbb{Z}_n}(y) = \{0\}$ for $y \neq \{0, v\}$ but $\mathrm{Stab}_{\mathbb{Z}_n}(y) = \{0, v\}$ when $y = \{0, v\}$. We now introduce a function $\widehat{\xi} : T \to U$ such that each $y \in U$, $y \neq \{0, v\}$ has λ preimages in T, while $\{0, v\}$ for v even has $\lambda/2$ preimages. Indeed, for each element $\{0, \ell, m\}$ in T_i, where ℓ and m are to be taken in the same order as they appear in (5), let

$$\widehat{\xi}(\{0, \ell, m\}) = \{0, \ell\}.$$

The group $\mathbb{Z}_n$ is semiregular on $\binom{S}{3}$ and T is a set of representatives for its orbits. Hence, for any $\{a, b, c\} \in \binom{S}{3}$ there are unique $\{0, \ell, m\} \in T$ and $\eta \in \mathbb{Z}_n$ such that $\eta(\{0, \ell, m\}) = \{a, b, c\}$. Thus, the definition

$$\xi(\{a, b, c\}) = \xi(\eta(\{0, \ell, m\})) = \eta(\widehat{\xi}(\{0, \ell, m\}) = \eta(\{0, \ell\}) = \{\eta, \ell + \eta\}$$

is well posed. We claim that ξ is a regular 2–sampling of $\binom{S}{3}$. Since for any $\{s, t\} \in \binom{S}{2}$ there exists $\eta \in \mathbb{Z}_n$ such that $\{s, t\} = \eta(\{0, \ell\})$, in order to show that ξ is a sampling we just need to prove that the number of preimages in $\binom{S}{3}$ of $\{0, \ell\}$ under $\widehat{\xi}$ is exactly λ. Observe that for n odd or $\ell \neq v$, the only preimages of $\{0, \ell\}$ are elements of T; thus, we have to analyse the following cases:

1. for $1 < \ell \leq \lambda + 1$, the set $\{0, \ell\}$ has λ preimages in the class T_1 and none in T_2, T_3, T_4^1 or T_4^2 and we are done;
2. for $\lambda + 2 \leq \ell < v$, the set $\{0, \ell\}$ has $\ell - \lambda - 1$ preimages in T_2 and $2\lambda + 1 - \ell$ preimages in T_3, for a total of λ;
3. if $\ell = v$ and n is odd, then $\{0, \ell\}$ has λ preimages in T_4^1.

In n is even and $\ell = v$, then each orbit of an element of T_4^2 contains two preimages of $\{0, v\}$; since $|T_4^2| = \lambda/2$, we get that also in this case the total number of preimages is λ. It follows that ξ is, as requested, a regular 2–sampling for $\binom{S}{3}$.

We now show how this construction might be used for some small cases:

1. for $n = 14$ we have $\lambda = 4$, $v = 7$. The set T is as follows:

T_1:	$\underline{01}2$	$\underline{01}3$	$\underline{01}4$	$\underline{01}5$	$\underline{02}3$	$\underline{02}4$	$\underline{02}5$	$\underline{02}6$	$\underline{03}4$	$\underline{03}5$
	$\underline{03}6$	$\underline{03}7$	$\underline{04}5$	$\underline{04}6$	$\underline{04}7$	$\underline{04}8$	$\underline{05}6$	$\underline{05}7$	$\underline{05}8$	$\underline{05}9$
T_2:	$\underline{06}7$	$\underline{06}8$	$\underline{06}9$							
T_3:	$\underline{06}1$									
T_4^2:	$\underline{07}1$	$\underline{07}2$								

The underlined elements in the preceding table are the image under $\widehat{\xi}$ of the corresponding set.

2. for $n = 17$ we have $\lambda = 5$, $v = 8$. We describe T and $\widehat{\xi}$ as before. In the following table, by a and b we respectively mean 10 and 11.

$T_1:$	$\underline{01}2$	$\underline{01}3$	$\underline{01}4$	$\underline{01}5$	$\underline{01}6$	$\underline{02}3$	$\underline{02}4$	$\underline{02}5$	$\underline{02}6$	$\underline{02}7$
	$\underline{03}4$	$\underline{03}5$	$\underline{03}6$	$\underline{03}7$	$\underline{03}8$	$\underline{04}5$	$\underline{04}6$	$\underline{04}7$	$\underline{04}8$	$\underline{04}9$
	$\underline{05}6$	$\underline{05}7$	$\underline{05}8$	$\underline{05}9$	$\underline{05}a$	$\underline{06}7$	$\underline{06}8$	$\underline{06}9$	$\underline{06}a$	$\underline{06}b$
$T_2:$	$\underline{07}8$	$\underline{07}9$	$\underline{07}a$	$\underline{07}b$						
$T_3:$	$\underline{07}1$									
$T_4^1:$	$\underline{08}1$	$\underline{08}2$	$\underline{08}9$	$\underline{08}a$	$\underline{08}b$					

Example 5.7. We wish to determine a regular 3–sampling ξ of $\binom{S}{4}$. In this case, $k = 4, \lambda = 5$. To construct ξ it is convenient to describe $\binom{S}{4}$ as union of orbits of a fairly large semiregular group. We may consider the group

$$G = \mathbb{Z}_n^{\square} \ltimes \mathbb{Z}_n.$$

For n a prime with $n \equiv 11 \pmod{12}$, simple, but tedious, computations show that the action of G is semiregular on both $\binom{S}{3}$ and $\binom{S}{4}$. The smallest interesting case occurs for $n = 23$. Here, a system of representatives for the orbits of $\binom{S}{3}$ is

$$U = \{\ 0,1,2 \quad 0,1,3 \quad 0,1,4 \quad 0,1,5 \quad 0,1,7 \quad 0,1,9 \quad 0,1,13\ \}.$$

By a computer assisted search with [8], we determined the following system compatible with U of representatives for the orbits of $\binom{S}{4}$:

$\underline{0,1,2}, 5$	$\underline{0,1,2}, 7$	$\underline{0,1,2}, 10$	$\underline{0,1,2}, 11$	$\underline{0,1,2}, 14$
$\underline{0,1,3}, 7$	$\underline{0,1,3}, 15$	$\underline{0,1,3}, 19$	$\underline{0,1,3}, 21$	$\underline{0,1,3}, 22$
$\underline{0,1,4}, 5$	$\underline{0,1,4}, 7$	$\underline{0,1,4}, 11$	$\underline{0,1,4}, 15$	$\underline{0,1,4}, 17$
$\underline{0,1,5}, 6$	$\underline{0,1,5}, 14$	$\underline{0,1,5}, 15$	$\underline{0,1,5}, 20$	$\underline{0,1,5}, 22$
$\underline{0,1,7}, 5$	$\underline{0,1,7}, 9$	$\underline{0,1,7}, 10$	$\underline{0,1,7}, 21$	$\underline{0,1,7}, 22$
$\underline{0,1,9}, 2$	$\underline{0,1,9}, 5$	$\underline{0,1,9}, 8$	$\underline{0,1,9}, 16$	$\underline{0,1,9}, 20$
$\underline{0,1,13}, 3$	$\underline{0,1,13}, 5$	$\underline{0,1,13}, 7$	$\underline{0,1,13}, 9$	$\underline{0,1,13}, 12.$

The sampling map $\widehat{\xi}$ is defined as in the previous example.

We remark that, instead of the group G, we might also have considered the action of the cyclic group $\mathbb{Z}_{23}$. However, had this been the case, we would have needed to write 77 distinct representatives for the orbits of $\binom{S}{3}$ and 385 compatible representatives for the orbits of $\binom{S}{4}$.

Remark 5.8. If $\binom{S}{k}$ admits a regular k_1–sampling ξ_1 with redundancy λ_1 and $\binom{S}{k_1}$ admits, in turn, a regular k_2–sampling ξ_2 with redundancy λ_2, then $\xi = \xi_2\xi_1$ is a regular k_2–sampling of $\binom{S}{k}$ with redundancy $\lambda_1\lambda_2$, since any $x_2 \in \binom{S}{k_2}$ is a sample of λ_2 elements of $\binom{S}{k_1}$, while, on the other hand, any $x_1 \in \binom{S}{k_1}$ is a sample of λ_1 elements of $\binom{S}{k}$. However, it has to be stressed that not all k_2–samplings of $\binom{S}{k}$ arise in this way.

Example 5.9. For $n = 11$, the necessary condition for the existence of a regular 2–sampling of $\binom{S}{3}$ as well as that for the existence of a regular 3–sampling of $\binom{S}{4}$ are simultaneously fulfilled. In particular, it is possible to construct a regular 2–sampling ξ to $\binom{S}{4}$ with redundancy $\lambda = 6$ as composition of a regular 2–sampling ξ_1 of $\binom{S}{3}$ and a regular 3–sampling ξ_2 of $\binom{S}{4}$. We consider the action of the group G introduced in Example 5.7. As before, G is semiregular on $\binom{S}{4}$ and $\binom{S}{3}$. Furthermore, a direct computation shows that G is regular on $\binom{S}{2}$. We provide suitable systems of representatives for, respectively, $\binom{S}{2}$, $\binom{S}{3}$ and $\binom{S}{4}$:

U_2: 01
U_3: $\underline{01}2 \quad \underline{01}3 \quad \underline{01}5$
U_4: $\underline{012}3 \quad \underline{012}4 \quad \underline{013}5 \quad \underline{013}7 \quad \underline{015}4 \quad \underline{015}8$.

The samplings arise, as in the previous examples, from $\widehat{\xi_1} : U_3 \to U_2$ and $\widehat{\xi_2} : U_4 \to U_3$. It is immediate to see that $\widehat{\xi} = \widehat{\xi_1}\widehat{\xi_2}$ is a regular 2–sampling of $\binom{S}{4}$. On the other hand, it is possible to define a regular 2–sampling of $\binom{S}{4}$ also from the starter set

U_4: $\underline{01}23 \quad \underline{01}24 \quad \underline{01}25 \quad \underline{01}26 \quad \underline{01}28 \quad \underline{01}34$.

However, it is not possible to extract a regular 3–sampling (with redundancy 2) from U_4.

The notion of (K_n, Γ)–complete design can be further generalized to that of a (Δ, Γ)–complete design, where Δ is an arbitrary graph. As before, given $\Gamma' \leq \Gamma \leq \Delta$, we might want to investigate the existence of a regular Γ'–sampling of $\Delta(\Gamma)$ or, conversely, an embedding of $\Delta(\Gamma')$ into $\Delta(\Gamma)$. However, in this general case, Lemma 4.1 fails, since it is not possible to guarantee that $\mathrm{Aut}(\Delta)$ acts transitively on the blocks of $\Delta(\Gamma)$ and $\Delta(\Gamma')$; hence it is not possible to get an analogue of Theorem 4.3. The following example contains a case in which a sampling might be shown to exist (and at least one of these samplings can be determined in an independent way).

Example 5.10. Let q be any prime power, and consider the projective space $PG(3, q)$. Let $\hat{\Delta}$ be the bipartite point–line incidence graph of this geometry. Let $\widehat{\Gamma}$ be the point–line incidence graph of $PG(2, q)$, seen as a plane embedded into $PG(3, q)$. Clearly $\widehat{\Gamma} < \widehat{\Delta}$. Define two new graphs Δ and Γ by replacing in $\widehat{\Delta}$ and $\widehat{\Gamma}$ each vertex v corresponding to a point of $PG(3, q)$ with a triangle $vv'v''$. Let Γ' be a triangle C_3. We observe that $\Delta(\Gamma')$ corresponds to the set of the points of $PG(3, q)$. Since there are as many points in $PG(3, q)$ as planes, we have $|\Delta(\Gamma')| = |\Delta(\Gamma)|$. Furthermore, the full automorphism group of Δ contains a subgroup isomorphic to $PGL(3, q)$, which acts in a transitive way on $\Delta(\Gamma)$ and also $\Delta(\Gamma')$. Thus, we may apply the same techniques as in the proof of Lemma 4.1, as to obtain a perfect matching of Δ, as in Theorem 4.3. This gives a sampling that associates to every plane π of $PG(3, q)$ a point $p \in \pi$.

There are several possible matchings of this kind; one of these is given by the action of a symplectic polarity σ of $PG(3, q)$.

6 Conclusion and open problems

In this paper, the notion of sampling has been introduced. We have shown in Theorem 4.3 that the obvious necessary numerical condition for the existence of a regular sampling $\xi : K_n(\Gamma) \to K_n(\Gamma')$ is also sufficient. In particular, it has been proved that the existence of such a sampling is independent from the actual nature of the graphs involved.

The case of semiregular samplings is more complicated. More in detail, Theorem 4.6 provides a sufficient condition for the existence of a $(1, 2)$-semiregular sampling. It is worth noticing that here some further hypotheses on the relationship between Γ and Γ', apart from $\Gamma' \leq \Gamma$, have been used. More in general, it appears it would be most interesting to determine suitable classes of graphs for which it is possible to guarantee the existence of $(\lambda, \lambda+1)$-semiregular samplings with $\lambda > 1$.

A different kind of generalisation can be obtained by considering samplings between designs on complete graphs of different order, namely from $K_n(\Gamma)$ to $K_{n'}(\Gamma')$, where $K_{n'} \leq K_n$ and $\Gamma' \leq \Gamma$. It is immediate to see that in this case a numerical condition cannot be sufficient by itself. For instance it is always necessary to suppose that any graph Γ'' obtained by removing $n - n'$ vertices from Γ contains at least one subgraph isomorphic to Γ'. In this case, a promising approach could be to investigate existence of samplings when Γ and Γ' are choosen in well characterised families of graphs.

References

[1] Aigner, M., “Combinatorial Theory”, Classics in Mathematics, *Springer-Verlag* (1997).

[2] Bosák, J., “Decompositions of graphs”, Mathematics and its Applications (1990), *Kluwer Academic Publishers Group.*

[3] Bryant, D., El-Zanati, S., *Graph Decompositions*, CRC Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz, CRC Press (2006), 477–486.

[4] Bryant, D., Rodger, C., *Cycle Decompositions*, CRC Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz Editors, CRC Press (2006), 373–382.

[5] Buratti, M., *Cyclic designs with block size 4 and related optimal optical orthogonal codes*, Des. Codes Cryptogr. **26** (2002), 111–125.

[6] Cameron, P.J., “Parallelisms of Complete Designs”, Cambridge University Press (1976).

[7] Colbourn, C. J., Ling, A. C. H., Quattrocchi, G., *Minimum embedding of P_3-designs into $(K_4 - e)$-designs*, J. Combin. Des. **11** (2003), 352–366.

[8] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, (http://www.gap-system.org).

[9] Gionfriddo, L., Lindner, C. C., *Nesting kite and 4-cycle systems*, Australas. J. Combin. **33** (2005), 247–254

[10] Gionfriddo, M., Quattrocchi, G., *Embedding balanced P_3-designs into (balanced) P_4-designs*, Discrete Math. **308** (2008), 155–160.

[11] Granville, A., Moisiadis, A., Rees, R., *Nested Steiner n-cycle systems and perpendicular arrays*, J. Combin. Math. Combin. Comput. **3** (1988), 163–167.

[12] Hall, M. Jr., "Combinatorial Theory", Wiley Classics Library. *John Wiley & Sons, Inc.* (1998).

[13] Hall, P., *On representatives of subsets*, J. London Math. Soc. **10** (1935), pp. 26–30.

[14] Harary, F., "Graph Theory", Addison–Wesley Publishing Co. (1969).

[15] Kőnig, D., *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. **77** (1916), 453–465.

[16] Kőnig, D., *Gráfok és alkalmazásuk a determinánsok és a halmazok elméletére*, Math. Termész. Ért. **34** (1916), 104–119.

[17] Lindner, C. C., Quattrocchi, G., Rodger, C. A., *Embedding Steiner triple systems in hexagon triple systems*, Discrete Math. **309** (2009), 487–490.

[18] Lindner, C. C., Rodger, C. A., Stinson, D. R., *Nesting of cycle systems of odd length*, Combinatorial designs—a tribute to Haim Hanani. Discrete Math. **77** (1989), 191–203.

[19] Lindner, C. C., Rodger, C. A., Stinson, D. R., *Nestings of directed cycle systems*, Ars Combin. **32** (1991), 153–159.

[20] Lindner, C. C., Stinson, D. R., *Nesting of cycle systems of even length*, J. Combin. Math. Combin. Comput. **8** (1990), 147–157.

[21] Ling, A. C. H., Colbourn, C. J., Quattrocchi, G., *Minimum embeddings of Steiner triple systems into $(K_4 - e)$-designs. II*, Discrete Math. **309** (2009), 400–411.

[22] Lovász, L., Plummer, M.D., "Matching Theory", Annals of Discrete Mathematics **29** (1986).

[23] Mathon, R., Rosa, A., $2-(v,k,\lambda)$ *Designs of Small Order*, CRC Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz Editors, CRC Press (2006), 25–58.

[24] Meszka, M., Rosa, A., *Embedding Steiner triple systems into Steiner systems* $S(2,4,v)$, Discrete Math. **274** (2004), 199–212.

[25] Milici, S., Quattrocchi, G., Shen, H., *Embeddings of simple maximum packings of triples with* λ *even*, Discrete Math. **145** (1995), 191–200.

[26] Milici, S., Quattrocchi, G., *Embedding handcuffed designs with block size* 2 *or* 3 *in* 4*-cycle systems*, Combinatorics (Assisi, 1996). Discrete Math. **208/209** (1999), 443–449

[27] Milici, S., *Minimum embedding of* P_3*-designs into* $\mathrm{TS}(v,\lambda)$, Discrete Math. **308** (2008), 331–338.

[28] Pasotti, A., "Graph Decompositions with a sharply vertex transitive automorphism group", Ph.D. Thesis, Università degli Studi di Milano Bicocca (2006).

[29] Quattrocchi, G., *Embedding* G_1*–designs into* G_2*–designs, a short survey*, Rend. Sem. Mat. Messina Ser. II **8** (2001), 129–143.

[30] Quattrocchi, G., *Embedding path designs in 4-cycle systems*, Combinatorics '98 (Palermo). Discrete Math. **255** (2002), 349–356.

[31] Quattrocchi, G., *Embedding handcuffed designs in* D*-designs, where* D *is the triangle with attached edge*, Discrete Math. **261** (2003), 413–434.

[32] Rodger, C. A., Stinson, D. R., *Nesting directed cycle systems of even length*, European J. Combin. **13** (1992), 213–218.

[33] Stinson, D. R., *The construction of nested cycle systems*, Coding theory and design theory, Part II, 362–367, IMA Vol. Math. Appl., **21**, Springer, New York, 1990.

[34] Tutte W.T., "Graph Theory", Cambridge University Press (2001).