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# Unitals in $PG(2, q^2)$ with a large 2-point stabiliser

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ABSTRACT

Let $\mathcal{U}$ be a unital embedded in the Desarguesian projective plane $PG(2, q^2)$. Write $M$ for the subgroup of $PGL(3, q^2)$ which preserves $\mathcal{U}$. We show that $\mathcal{U}$ is classical if and only if $\mathcal{U}$ has two distinct points $P, Q$ for which the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$.



## 1. Introduction

In the Desarguesian projective plane $PG(2, q^2)$, a *unital* is defined to be a set of $q^3 + 1$ points containing either 1 or $q + 1$ points from each line of $PG(2, q^2)$. Observe that each unital has a unique 1-secant at each of its points. The idea of a unital arises from the combinatorial properties of the non-degenerate unitary polarity $\pi$ of $PG(2, q^2)$. The set of absolute points of $\pi$ is indeed a unital, called the *classical* or *Hermitian* unital. Therefore, the projective group preserving the classical unital is isomorphic to $PGU(3, q)$ and acts on its points as $PGU(3, q)$ in its natural 2-transitive permutation representation. Using the classification of subgroups of $PGL(3, q^2)$, Hoffer [14] proved that a unital is classical if and only if it is preserved by a collineation group isomorphic to $PSU(3, q^2)$. Hoffer's characterisation has been the starting point for several investigations of unitals in terms of the structure of their automorphism group, see [3,6,4,5,8,9,11,10,12,15,16]; see also the survey [2, Appendix B]. In $PG(2, q^2)$ with $q$ odd, Abatangelo [1] proved that a Buekenhout–Metz unital with a cyclic 2-point stabiliser of order $q^2 - 1$ is necessarily classical. In their talk at Combinatorics 2010, G. Donati e N. Durante have conjectured that Abatangelo's characterisation holds true for any unital in $PG(2, q^2)$. In this note, we provide a proof of this conjecture.

Our notation and terminology are standard, see [2,13]. We shall assume $q > 2$, since all unitals in $PG(2, 4)$ are classical.

## 2. Some technical lemmas

Let $M$ be the subgroup of $PGL(3, q^2)$ which preserves a unital $\mathcal{U}$ in $PG(2, q^2)$. A *2-point stabiliser of* $\mathcal{U}$ is a subgroup of $M$ which fixes two distinct points of $\mathcal{U}$.

**Lemma 2.1.** *Let $\mathcal{U}$ be a unital in $PG(2, q^2)$ with a 2-point stabiliser $G$ of order $q^2 - 1$. Then, $G$ is cyclic, and there exists a projective frame in $PG(2, q^2)$ such that $G$ is generated by a projectivity with matrix representation*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

*where $\lambda$ is a primitive element of $GF(q^2)$ and $\mu$ is a primitive element of $GF(q)$.*

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**Proof.** Let $O$, $Y_\infty$ be two distinct points of $\mathcal{U}$ such that the stabiliser $G = M_{O,Y_\infty}$ has order $q^2-1$. Choose a projective frame in $\mathrm{PG}(2,q^2)$ so that $O=(0,0,1)$, $Y_\infty=(0,1,0)$ and the 1-secants of $\mathcal{U}$ at those points are respectively $\ell_X : X_2 = 0$ and $\ell_\infty : X_3 = 0$. Write $X_\infty = (1,0,0)$ for the common point of $\ell_X$ and $\ell_\infty$. Observe that $G$ fixes the vertices of the triangle $OX_\infty Y_\infty$. Therefore, $G$ consists of projectivities with diagonal matrix representation. Now let $h \in G$ be a projectivity that fixes a further point $P \in \ell_X$ apart from $O, X_\infty$. Then, $h$ fixes $\ell_X$ point-wise; that is, $h$ is a perspectivity with axis $\ell_X$. Since $h$ also fixes $Y_\infty$, the centre of $h$ must be $Y_\infty$. Take any point $R \in \ell_X$ with $R \neq O, X_\infty$. Obviously, $h$ preserves the line $r = Y_\infty R$; hence, it also preserves $r \cap \mathcal{U}$. Since $r \cap \mathcal{U}$ comprises $q$ points other than $R$, the subgroup $H$ generated by $h$ has a permutation representation of degree $q$ in which no non-trivial permutation fixes a point. As $q = p^r$ for a prime $p$, this implies that $p$ divides $|H|$. On the other hand, $h$ is taken from a group of order $q^2-1$. Thus, $h$ must be the trivial element in $G$. Therefore, $G$ has a faithful action on $\ell_X$ as a 2-point stabiliser of $\mathrm{PG}(1,q^2)$. This proves that $G$ is cyclic. Furthermore, a generator $g$ of $G$ has a matrix representation

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\lambda$ a primitive element of $\mathrm{GF}(q^2)$.

As $G$ preserves the set $\Delta = \mathcal{U} \cap OY_\infty$, it also induces a permutation group $\bar{G}$ on $\Delta$. Since any projectivity fixing three points of $OY_\infty$ must fix $OY_\infty$ point-wise, $\bar{G}$ is semiregular on $\Delta$. Therefore, $|\bar{G}|$ divides $q-1$. Now let $F$ be the subgroup of $G$ fixing $\Delta$ point-wise. Then, $F$ is a perspectivity group with centre $X_\infty$ and axis $\ell_Y : X_1 = 0$. Take any point $R \in \ell_Y$ such that the line $r = RX_\infty$ is a $(q+1)$-secant of $\mathcal{U}$. Then, $r \cap \mathcal{U}$ is disjoint from $\ell_Y$. Hence, $F$ has a permutation representation on $r \cap \mathcal{U}$ in which no non-trivial permutation fixes a point. Thus, $|F|$ divides $q+1$. Since $|G| = q^2-1$, we have $|\bar{G}| \le q-1$ and $|G| = |\bar{G}||F|$. This implies $|\bar{G}| = q-1$ and $|F| = q+1$. From the former condition, $\mu$ must be a primitive element of $\mathrm{GF}(q)$. □

**Lemma 2.2.** *In $\mathrm{PG}(2,q^2)$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two non-degenerate Hermitian curves which have the same tangent at a common point $P$. Denote by $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ the intersection multiplicity of $\mathcal{H}_1$ and $\mathcal{H}_2$ at $P$. Then,*

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = q+1. \tag{1}$$

**Proof.** Since up to projectivities there is a unique class of Hermitian curves in $\mathrm{PG}(2,q^2)$, we may assume $\mathcal{H}_1$ to have equation $-X_1^{q+1} + X_2^q X_3 + X_2 X_3^q = 0$. Furthermore, as the projectivity group $\mathrm{PGU}(3,q)$ preserving $\mathcal{H}_1$ acts transitively on the points of $\mathcal{H}_1$ in $\mathrm{PG}(2,q^2)$, we may also suppose $P = (0,0,1)$. Within this setting, the tangent $r$ of $\mathcal{H}_1$ at $P$ coincides with the line $X_2 = 0$. As no term $X_1^j$ with $0 < j \le q$ occurs in the equation of $\mathcal{H}_1$, the intersection multiplicity $I(P, \mathcal{H}_1 \cap r)$ is equal to $q+1$.

The equation of the other Hermitian curve $\mathcal{H}_2$ might be written as

$$F(X_1,X_2,X_3) = a_0 X_3^q X_2 + a_1 X_3^{q-1} G_1(X_1,X_2) + \cdots + a_q G_q(X_1,X_2) = 0,$$

where $a_0 \neq 0$ and $\deg G_i(X_1,X_2) = i+1$. Since the tangent of $\mathcal{H}_2$ at $P$ has no other common point with $\mathcal{H}_2$, even over the algebraic closure of $\mathrm{GF}(q^2)$, no terms $X_1^j$ with $0 < j \le q$ can occur in the polynomials $G_i(X_1,X_2)$. In other words, $I(P, \mathcal{H}_2 \cap r) = q+1$.

A primitive representation of the unique branch of $\mathcal{H}_1$ centred at $P$ has components

$$x(t) = t, \qquad y(t) = ct^i + \cdots,$$

where $i$ is a positive integer and $y(t) \in \mathrm{GF}(q^2)[[t]]$, that is, $y(t)$ stands for a formal power series with coefficients in $\mathrm{GF}(q^2)$.

From $I(P, \mathcal{H}_1 \cap r) = q+1$,

$$y(t)^q + y(t) - t^{q+1} = 0,$$

whence $y(t) = t^{q+1} + H(t)$, where $H(t)$ is a formal power series of order at least $q+2$. That is, the exponent $j$ in the leading term $ct^j$ of $H(t)$ is larger than $q+1$.

It is now possible to compute the intersection multiplicity $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ using [13, Theorem 4.36]:

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = \mathrm{ord}_t F(t, y(t), 1) = \mathrm{ord}_t (a_0 t^{q+1} + G(t)),$$

with $G(t) \in \mathrm{GF}(q^2)[[t]]$ of order at least $q+2$. From this, the assertion follows. □

**Lemma 2.3.** *In $\mathrm{PG}(2,q^2)$, let $\mathcal{H}$ be a non-degenerate Hermitian curve and let $\mathcal{C}$ be a Hermitian cone whose centre does not lie on $\mathcal{H}$. Assume that there exist two points $P_i \in \mathcal{H} \cap \mathcal{C}$, with $i = 1, 2$, such that the tangent line of $\mathcal{H}$ at $P_i$ is a linear component of $\mathcal{C}$. Then*

$$I(P_1, \mathcal{H} \cap \mathcal{C}) = q+1. \tag{2}$$

**Proof.** We use the same setting as in the proof of Lemma 2.2 with $P = P_1$. Since the action of $\mathrm{PGU}(3, q)$ is 2-transitive on the points of $\mathcal{H}$, we may also suppose that $P_2 = (0, 1, 0)$. Then the centre of $\mathcal{C}$ is the point $X_\infty = (1, 0, 0)$, and $\mathcal{C}$ has equation $c^q X_2^q X_3 + cX_2X_3^q = 0$ with $c \neq 0$. Therefore,

$$I(P, \mathcal{H} \cap \mathcal{C}) = \mathrm{ord}_t(c^q y(t)^q + cy(t)) = \mathrm{ord}_t(c^q t^{q+1} + K(t))$$

with $K(t) \in \mathrm{GF}(q^2)[[t]]$ of order at least $q + 2$, whence the assertion follows. □

## 3. Main result

**Theorem 3.1.** *In $\mathrm{PG}(2, q^2)$, let $\mathcal{U}$ be a unital and write $M$ for the group of projectivities which preserves $\mathcal{U}$. If $\mathcal{U}$ has two distinct points $P, Q$ such that the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$, then $\mathcal{U}$ is classical.*

The main idea of the proof is to build up a projective plane of order $q$ using, for the definition of points, non-trivial $G$-orbits in the affine plane $\mathrm{AG}(2, q^2)$ which arise from $\mathrm{PG}(2, q^2)$ by removing the line $\ell_\infty : X_3 = 0$ with all its points. For this purpose, take $\mathcal{U}$ and $G$ as in Lemma 2.1, with $\mu = \lambda^{q+1}$, and define an incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ as follows.

1. Points are all non-trivial $G$-orbits in $\mathrm{AG}(2, q^2)$.
2. Lines are $\ell_Y$, and the non-degenerate Hermitian curves of equation

$$\mathcal{H}_b : -X_1^{q+1} + bX_3X_2^q + b^qX_3^qX_2 = 0, \quad (3)$$

   with $b$ ranging over $\mathrm{GF}(q^2)^*$, together with the Hermitian cones of equation

$$\mathcal{C}_c : c^qX_2^qX_3 + cX_2X_3^q = 0, \quad (4)$$

   with $c$ ranging over a representative system of cosets of $(\mathrm{GF}(q), *)$ in $(\mathrm{GF}(q^2), *)$.
3. Incidence is the natural inclusion.

**Lemma 3.2.** *The incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ is a projective plane of order $q$.*

**Proof.** In $\mathrm{AG}(2, q^2)$, the group $G$ has $q^2 + q + 1$ non-trivial orbits, namely its $q^2$ orbits disjoint from $\ell_Y$, each of length $q^2 - 1$, and its $q + 1$ orbits on $\ell_Y$, these of length $q - 1$. Therefore, the total number of points in $\mathcal{P}$ is equal to $q^2 + q + 1$. By construction of $\Pi$, the number of lines in $\mathcal{L}$ is also $q^2 + q + 1$. Incidence is well defined as $G$ preserves $\ell_Y$ and each Hermitian curve and cone representing lines of $\mathcal{L}$.

Now we count the points incident with a line in $\Pi$. Each $G$-orbit on $\ell_Y$ distinct from $O$ and $Y_\infty$ has length $q - 1$. Hence there are exactly $q + 1$ such $G$-orbits; in terms of $\Pi$, the line represented by $\ell_Y$ is incident with $q + 1$ points. A Hermitian curve $\mathcal{H}_b$ of Eq. (3) has $q^3$ points in $\mathrm{AG}(2, q^2)$ and meets $\ell_Y$ in a $G$-orbit, while it contains no points apart from $O$ of line $\ell_X$. As $q^3 - q = q(q^2 - 1)$, the line represented by $\mathcal{H}_b$ is incident with $q + 1$ points in $\mathcal{P}$. Finally, a Hermitian cone $\mathcal{C}_c$ of Eq. (4) has $q^3$ points in $\mathrm{AG}(2, q^2)$ and contains $q$ points from $\ell_Y$. One of these $q$ points is $O$, the other $q - 1$ forming a non-trivial $G$-orbit. The remaining $q^3 - q$ points of $\mathcal{C}_c$ are partitioned into $q$ distinct $G$-orbits. Hence, the line represented by $\mathcal{C}_c$ is also incident with $q + 1$ points. This shows that each line in $\Pi$ is incident with exactly $q + 1$ points.

Therefore, it is enough to show that any two distinct lines of $\mathcal{L}$ have exactly one common point. Obviously this is true when one of these lines is represented by $\ell_Y$. Furthermore, the point of $\mathcal{P}$ represented by $\ell_X$ is incident with each line of $\mathcal{L}$ represented by a Hermitian cone of Eq. (4). We are led to investigate the case where one of the lines of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}_b$ of Eq. (4), and the other line of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}$ which is either another Hermitian curve $\mathcal{H}_d$ of the same type of Eq. (3), or a Hermitian cone $\mathcal{C}_c$ of Eq. (4).

Clearly, both $O$ and $Y_\infty$ are common points of $\mathcal{H}_b$ and $\mathcal{H}$. From Kestenband's classification [17], see also [2, Theorem 6.7], $\mathcal{H}_b \cap \mathcal{H}$ cannot consist of exactly two points. Therefore, there exists another point, say $P \in \mathcal{H}_b \cap \mathcal{H}$. Since $\ell_X$ and $\ell_\infty$ are 1-secants of $\mathcal{H}_b$ at the points $O$ and $Y_\infty$, respectively, either $P$ is on $\ell_Y$ or $P$ lies outside the fundamental triangle. In the latter case, the $G$-orbit $\Delta_1$ of $P$ has size $q^2 - 1$ and represents a point in $\mathcal{P}$. Assume that $\mathcal{H}_b \cap \mathcal{H}$ contains a further point $Q \notin \Delta_1$ which does not belong to $\ell_Y$ and denote by $\Delta_2$ its $G$-orbit. Then,

$$|\mathcal{H}_b \cap \mathcal{H}| \geq |\Delta_1| + |\Delta_2| = 2(q^2 - 1) + 2 = 2q^2.$$

However, from Bézout's theorem, see [13, Theorem 3.14],

$$|\mathcal{H}_b \cap \mathcal{H}| \leq (q + 1)^2.$$

Therefore, $Q \in \ell_Y$, and its $G$-orbit $\Delta_3$ has length $q - 1$. Hence, $\mathcal{H}_b$ and $\mathcal{H}$ share $q + 1$ points on $\ell_Y$. If $\mathcal{H} = \mathcal{H}_d$ is a Hermitian curve of Eq. (3), each of these $q + 1$ points is the tangency point of a common inflection tangent with multiplicity $q + 1$ of the Hermitian curves $\mathcal{H}_b$ and $\mathcal{H}$. Write $R_1, \ldots, R_{q+1}$ for these points. Then, by (1) the intersection multiplicity is $I(R_i, \mathcal{H}_b \cap \mathcal{H}_d) = q + 1$. This holds true also when $\mathcal{H}$ is a Hermitian cone $\mathcal{C}_c$ of Eq. (4); see Lemma 2.3. Therefore, in any case,

$$\sum_{i=1}^{q+1} I(R_i, \mathcal{H}_b \cap \mathcal{H}) = (q + 1)^2.$$

From Bézout's theorem, $\mathcal{H}_b \cap \mathcal{H} = \{R_1, \ldots, R_{q+1}\}$. Therefore, $\mathcal{H}_b \cap \mathcal{H} = \Delta_3 \cup \{O, Y_\infty\}$. This shows that if $Q \notin \ell_Y$, the lines represented by $\mathcal{H}_b$ and $\mathcal{H}$ have exactly one point in common. The above argument can also be adapted to prove this assertion in the case where $Q \in \ell_Y$. Therefore, any two distinct lines of $\mathcal{L}$ have exactly one common point. □

**Proof of Theorem 3.1.** Assume first $\mu = \lambda^{q+1}$. Construct a projective plane $\Pi$ as in Lemma 3.2. Since $\mathcal{U} \setminus \{O, Y_\infty\}$ is the union of $G$-orbits, $\mathcal{U}$ represents a set $\Gamma$ of $q+1$ points in $\Pi$. From [7], $N \equiv 1 (\mathrm{mod}\ p)$ where $N$ is the number of common points of $\mathcal{U}$ with any Hermitian curve $\mathcal{H}_b$. In terms of $\Pi$, $\Gamma$ contains some point from every line $\Lambda$ in $\mathcal{L}$ represented by a Hermitian curve of Eq. (3). Actually, this holds true when the line $\Lambda$ in $\mathcal{L}$ is represented by a Hermitian cone $\mathcal{C}$ of Eq. (4). To prove it, observe that $\mathcal{C}$ contains a line $r$ distinct from both lines $\ell_X$ and $\ell_\infty$. Then $r \cap \mathcal{U}$ is not empty and contains neither $O$ nor $Y_\infty$. If $P$ is a point in $r \cap \mathcal{U}$, then the $G$-orbit of $P$ represents a common point of $\Gamma$ and $\Lambda$. Since the line in $\mathcal{L}$ represented by $\ell_Y$ meets $\Gamma$, it turns out that $\Gamma$ contains some point from every line in $\mathcal{L}$.

Therefore, $\Gamma$ itself is a line in $\mathcal{L}$. Note that $\mathcal{U}$ contains no line. In terms of $PG(2, q^2)$, this yields that $\mathcal{U}$ coincides with a Hermitian curve of Eq. (3). In particular, $\mathcal{U}$ is a classical unital.

To investigate the case $\mu \neq \lambda^{q+1}$, we still work in the above plane $\Pi$. By a straightforward computation, the projectivity $g$ given in Lemma 2.1 induces a non-trivial collineation on $\Pi$. Also, $g$ preserves every Hermitian cone of Eq. (4) and the common line $\ell_X$ of these Hermitian cones. In terms of $\Pi$, $\bar{g}$ is a perspectivity with centre at the point represented by $\ell_X$. Since $g$ also preserves the line $\ell_Y$, the axis of $\bar{g}$ is $\ell_Y$, regarded as a line in $\Pi$. Therefore, every point of $\Pi$ lying on $\ell_Y$ is fixed by $g$. Consequently, $\bar{g}^{q-1}$ is the identity collineation. As $g$ has order $q^2-1$, this yields that $g^{q+1}$ preserves every Hermitian curve of Eq. (3). Thus, $\mu^{q+1} = (\lambda^{q+1})^{q+1}$, whence $\mu = -\lambda^{q+1}$. In particular, $p \neq 2$.

Consider now the $q+1$ non-trivial $G$-orbits in $\mathcal{U}$ with $G = \langle g \rangle$. For any point $P \in \Pi$, let $n_P$ the number of the non-trivial $G$-orbits in $\mathcal{U}$ intersecting the set $\rho(P)$ representing $P$ in $PG(2, q^2)$. Then $n_P = 1$ when $\rho(P)$ is the unique $G$-orbit in $\mathcal{U}$ which lies on $\ell_Y$. Otherwise, $0 \leq n_P \leq 2$, with $n_P = 2$ if and only if $\rho(P)$ is not a $G$-orbit but the union of two $H$-orbits with $H = \langle g^2 \rangle$.

Let $\Gamma$ be the multiset in $\Pi$ consisting of all points with $n_P > 0$ and define the weight $\nu_P$ of $P$ to be either 1 or 2, according as $n_P = 2$ or $n_P = 1$. Then, $\sum_{P \in \Gamma} \nu_P = 2q + 2$. We show that $\Gamma$ is a 2-fold blocking multiset of $\Pi$. For this purpose, let $\mathcal{H}$ be either a Hermitian curve of Eq. (3) or a Hermitian cone of Eq. (4). Write $m$ for the number of common points of $\mathcal{H}_b$ and $\mathcal{U}$, different from $O$ and $Y_\infty$; thus, the total number of common points is $N = m + 2$. As $N \equiv 1 (\mathrm{mod}\ p)$, we have $m \geq 1$. Take $P \in \mathcal{H} \cap \mathcal{U}$. If $\nu_P = 2$, then the line representing $\mathcal{H}$ meets $\Gamma$ in a point with weight 2. If $\nu_P = 1$, then the $H$-orbit of $P$ has size $(q^2-1)/2$ and lies on both $\mathcal{H}$ and $\mathcal{U}$. Since $(q^2-1)/2 + 2 \not\equiv 1 (\mathrm{mod}\ p)$, $\mathcal{H}$ and $\mathcal{U}$ must share a further point $Q$ other than $O$ and $Y_\infty$. Therefore, the points $P'$ and $Q'$ of $\Pi$ which represent the subsets containing $P$ and $Q$ are distinct. This shows that $\Gamma$ meets the line represented by $\mathcal{H}$ in two distinct points. Therefore, $\Gamma$ is a 2-fold blocking multiset.

Since $\Gamma$ has at least one point with weight 2, this yields that $\Gamma$ comprises all points of a line, each with weight 2. Hence, $\mathcal{U}$ coincides with either a Hermitian curve or a Hermitian cone. On the other hand, $\mathcal{U}$ is definitely not a Hermitian cone. As $\mu \neq \lambda^{q+1}$, $\mathcal{U}$ is neither a Hermitian curve; therefore, this case cannot actually occur. □

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