Unitals in $\mathrm{PG}(2, q^2)$ with a large 2-point stabiliser

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Abstract

Let $\mathcal{U}$ be a unital embedded in the Desarguesian projective plane $\mathrm{PG}(2, q^2)$. Write M for the subgroup of $\mathrm{PGL}(3, q^2)$ which preserves $\mathcal{U}$. We show that $\mathcal{U}$ is classical if and only if $\mathcal{U}$ has two distinct points P, Q for which the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$.

1 Introduction

In the Desarguesian projective plane $\mathrm{PG}(2, q^2)$, a *unital* is defined to be a set of $q^3 + 1$ points containing either 1 or $q+1$ points from each line of $\mathrm{PG}(2, q^2)$. Observe that each unital has a unique 1-secant at each of its points. The idea of a unital arises from the combinatorial properties of the non-degenerate unitary polarity π of $\mathrm{PG}(2, q^2)$. The set of absolute points of π is indeed a unital, called the *classical* or *Hermitian* unital. Therefore, the projective group preserving the classical unital is isomorphic to $\mathrm{PGU}(3, q)$ and acts on its points as $\mathrm{PGU}(3, q)$ in its natural 2-transitive permutation representation. Using the classification of subgroups of $\mathrm{PGL}(3, q^2)$, Hoffer [14] proved that a unital is classical if and only if is preserved by a collineation group isomorphic to $\mathrm{PSU}(3, q^2)$. Hoffer's characterisation has been the starting point for several investigations of unitals in terms of the structure of their automorphism group, see [3, 6, 4, 5, 8, 9, 10, 11, 12, 15, 16]; see also the survey [2, Appendix B]. In $\mathrm{PG}(2, q^2)$ with q odd, L.M. Abatangelo [1] proved that a Buekenhout–Metz unital with a cyclic 2–point stabiliser of order $q^2 - 1$ is necessarily classical. In their talk at Combinatorics 2010, G. Donati e N. Durante have conjectured that Abatangelo's characterisation holds true for any unital in $\mathrm{PG}(2, q^2)$. In this note, we provide a proof of this conjecture.

Our notation and terminology are standard, see [2], and [13]. We shall assume $q > 2$, since all unitals in $\mathrm{PG}(2, 4)$ are classical.

2 Some technical lemmas

Let M be the subgroup of $\mathrm{PGL}(3, q^2)$ which preserves a unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$. A 2*-point stabiliser of* $\mathcal{U}$ is a subgroup of M which fixes two distinct points of $\mathcal{U}$.

Lemma 2.1. *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$ with a 2–point stabiliser G of order $q^2 - 1$. Then, G is cyclic, and there exists a projective frame in $\mathrm{PG}(2, q^2)$ such that G is generated by a projectivity with matrix representation*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where λ is a primitive element of $\mathrm{GF}(q^2)$ and μ is a primitive element of $\mathrm{GF}(q)$.

Proof. Let O, Y_∞ be two distinct points of $\mathcal{U}$ such that the stabiliser $G = M_{O,Y_\infty}$ has order q^2-1. Choose a projective frame in $\mathrm{PG}(2,q^2)$ so that $O=(0,0,1)$, $Y_\infty=(0,1,0)$ and the 1-secants of $\mathcal{U}$ at those points are respectively $\ell_X : X_2 = 0$ and $\ell_\infty : X_3 = 0$. Write $X_\infty = (1,0,0)$ for the common point of ℓ_X and ℓ_∞. Observe that G fixes the vertices of the triangle $OX_\infty Y_\infty$. Therefore, G consists of projectivities with diagonal matrix representation. Now let $h \in G$ be a projectivity that fixes a further point $P \in \ell_X$ apart from O, X_∞. Then, h fixes ℓ_X point-wise; that is, h is a perspectivity with axis ℓ_X. Since h also fixes Y_∞, the centre of h must be Y_∞. Take any point $R \in \ell_X$ with $R \neq O, X_\infty$. Obviously, h preserves the line $r = Y_\infty R$; hence, it also preserves $r \cap \mathcal{U}$. Since $r \cap \mathcal{U}$ comprises q points other than R, the subgroup H generated by h has a permutation representation of degree q in which no non-trivial permutation fixes a point. As $q = p^r$ for a prime p, this implies that p divides $|H|$. On the other hand, h is taken from a group of order q^2-1. Thus, h must be the trivial element in G. Therefore, G has a faithful action on ℓ_X as a 2-point stabiliser of $\mathrm{PG}(1,q^2)$. This proves that G is cyclic. Furthermore, a generator g of G has a matrix representation

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with λ a primitive element of $\mathrm{GF}(q^2)$.

As G preserves the set $\Delta = \mathcal{U} \cap OY_\infty$, it also induces a permutation group $\bar{G}$ on Δ. Since any projectivity fixing three points of OY_∞ must fix OY_∞ point-wise, $\bar{G}$ is semiregular on Δ. Therefore, $|\bar{G}|$ divides $q-1$. Now let F be the subgroup of G fixing Δ point-wise. Then, F is a perspectivity group with centre X_∞ and axis $\ell_Y : X_1 = 0$. Take any point $R \in \ell_Y$ such that the line $r = RX_\infty$ is a $(q+1)$-secant of $\mathcal{U}$. Then, $r \cap \mathcal{U}$ is disjoint from ℓ_Y. Hence, F has a permutation representation on $r \cap \mathcal{U}$ in which no non-trivial permutation fixes a point. Thus, $|F|$ divides $q+1$. Since $|G| = q^2-1$, we have $|\bar{G}| \leq q-1$ and $|G| = |\bar{G}||F|$. This implies $|\bar{G}| = q-1$ and $|F| = q+1$. From the former condition, μ must be a primitive element of $\mathrm{GF}(q)$. □

Lemma 2.2. *In $\mathrm{PG}(2,q^2)$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two non-degenerate Hermitian curves which have the same tangent at a common point P. Denote by $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ the intersection multiplicity of $\mathcal{H}_1$ and $\mathcal{H}_2$ at P Then,*

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = q+1. \tag{1}$$

Proof. Since, up to projectivities, there is a unique class of Hermitian curves in $\mathrm{PG}(2,q^2)$, we may assume $\mathcal{H}_1$ to have equation $-X_1^{q+1} + X_2^q X_3 + X_2 X_3^q = 0$. Furthermore, as the projectivity group $\mathrm{PGU}(3,q)$ preserving $\mathcal{H}_1$ acts transitively on the points of $\mathcal{H}_1$ in $\mathrm{PG}(2,q^2)$, we may also suppose $P = (0,0,1)$. Within this setting, the tangent r of $\mathcal{H}_1$ at P coincides with the line $X_2 = 0$. As no term X_1^j with $0 < j \leq q$ occurs in the equation of $\mathcal{H}_1$, the intersection multiplicity $I(P, \mathcal{H}_1 \cap r)$ is equal to $q+1$.

The equation of the other Hermitian curve $\mathcal{H}_2$ might be written as

$$F(X_1, X_2, X_3) = a_0\, X_3^q X_2 + a_1 X_3^{q-1} G_1(X_1, X_2) + \ldots + a_q G_q(X_1, X_2) = 0,$$

where $a_0 \neq 0$ and $\deg G_i(X_1, X_2) = i+1$. Since the tangent of $\mathcal{H}_2$ at P has no other common point with $\mathcal{H}_2$, even over the algebraic closure of $\mathrm{GF}(q^2)$, no terms X_1^j with $0 < j \leq q$ can occur in the polynomials $G_i(X_1, X_2)$. In other words, $I(P, \mathcal{H}_2 \cap r) = q+1$.

A primitive representation of the unique branch of $\mathcal{H}_1$ centred at P has components

$$x(t) = t,\ y(t) = ct^i + \ldots,$$

where i is a positive integer and $y(t) \in \mathrm{GF}(q^2)[[t]]$, that is, $y(t)$ stands for a formal power series with coefficients in $\mathrm{GF}(q^2)$.

From $I(P, \mathcal{H}_1 \cap r) = q + 1$,

$$y(t)^q + y(t) - t^{q+1} = 0,$$

whence $y(t) = t^{q+1} + H(t)$, where $H(t)$ is a formal power series of order at least $q + 2$. That is, the exponent j in the leading term ct^j of $H(t)$ is larger than $q + 1$.

It is now possible to compute the intersection multiplicity $I(P, \mathcal{H}_1 \cap \mathcal{H}_2)$ using [13, Theorem 4.36]:

$$I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = \mathrm{ord}_t\, F(t, y(t), 1) = \mathrm{ord}_t\, (a_0 t^{q+1} + G(t)),$$

with $G(t) \in \mathrm{GF}(q^2)[[t]]$ of order at least $q + 2$. From this, the assertion follows. □

Lemma 2.3. *In* $\mathrm{PG}(2, q^2)$*, let* $\mathcal{H}$ *be a non-degenerate Hermitian curve and let* $\mathcal{C}$ *be a Hermitian cone whose centre does not lie on* $\mathcal{H}$*. Assume that there exist two points* $P_i \in \mathcal{H} \cap \mathcal{C}$*, with* $i = 1, 2$*, such that the tangent line of* $\mathcal{H}$ *at* P_i *is a linear component of* $\mathcal{C}$*. Then*

$$I(P_1, \mathcal{H} \cap \mathcal{C}) = q + 1. \tag{2}$$

Proof. We use the same setting as in the proof of Lemma 2.2 with $P = P_1$. Since the action of $\mathrm{PGU}(3, q)$ is 2-transitive on the points of $\mathcal{H}$, we may also suppose that $P_2 = (0, 1, 0)$. Then the centre of $\mathcal{C}$ is the point $X_\infty = (1, 0, 0)$, and $\mathcal{C}$ has equation $c^q X_2^q X_3 + c X_2 X_3^q = 0$ with $c \neq 0$. Therefore,

$$I(P, \mathcal{H} \cap \mathcal{C}) = \mathrm{ord}_t\, (c^q y(t)^q + c y(t)) = \mathrm{ord}_t\, (c^q t^{q+1} + K(t))$$

with $K(t) \in \mathrm{GF}(q^2)[[t]]$ of order at least $q + 2$, whence the assertion follows. □

3 Main result

Theorem 3.1. *In* $\mathrm{PG}(2, q^2)$*, let* $\mathcal{U}$ *be a unital and write* M *for the group of projectivities which preserves* $\mathcal{U}$*. If* $\mathcal{U}$ *has two distinct points* P, Q *such that the stabiliser* $G = M_{P,Q}$ *has order* $q^2 - 1$*, then* $\mathcal{U}$ *is classical.*

The main idea of the proof is to build up a projective plane of order q using, for the definition of points, non-trivial G-orbits in the affine plane $\mathrm{AG}(2, q^2)$ which arise from $\mathrm{PG}(2, q^2)$ by removing the line $\ell_\infty : X_3 = 0$ with all its points. For this purpose, take $\mathcal{U}$ and G as in Lemma 2.1, and define an incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ as follows.

1. Points are all non-trivial G-orbits in $\mathrm{AG}(2, q^2)$.

2. Lines are ℓ_Y, and the non-degenerate Hermitian curves of equation

$$\mathcal{H}_b : \; -X_1^{q+1} + b X_3 X_2^q + b^q X_3^q X_2 = 0, \tag{3}$$

with b ranging over $\mathrm{GF}(q^2)^*$, together with the Hermitian cones of equation

$$\mathcal{C}_c : \; c^q X_2^q X_3 + c X_2 X_3^q = 0, \tag{4}$$

with c ranging over a representative system of cosets of $(\mathrm{GF}(q), *)$ in $(\mathrm{GF}(q^2), *)$.

3. Incidence is the natural inclusion.

Lemma 3.2. *The incidence structure* $\Pi = (\mathcal{P}, \mathcal{L})$ *is a projective plane of order* q.

Proof. In $\mathrm{AG}(2, q^2)$, the group G has $q^2 + q + 1$ non-trivial orbits, namely its q^2 orbits disjoint from ℓ_Y, each of length $q^2 - 1$, and its $q + 1$ orbits on ℓ_Y, these of length $q - 1$. Therefore, the total number of points in $\mathcal{P}$ is equal to $q^2 + q + 1$. By construction of Π, the number of lines in $\mathcal{L}$ is also $q^2 + q + 1$. Incidence is well defined as G preserves ℓ_Y and each Hermitian curve and cone representing lines of $\mathcal{L}$.

Now we count the points incident with a line in Π. Each G-orbit on ℓ_Y distinct from O and Y_∞ has length $q - 1$. Hence there are exactly $q + 1$ such G-orbits; in terms of Π, the line represented by ℓ_Y is incident with $q + 1$ points. A Hermitian curve $\mathcal{H}_b$ of Equation (3) has q^3 points in $\mathrm{AG}(2, q^2)$ and meets ℓ_Y in a G-orbit, while it contains no point apart from O of line ℓ_X. As $q^3 - q = q(q^2 - 1)$, the line represented by $\mathcal{H}_b$ is incident with $q + 1$ points in $\mathcal{P}$. Finally, a Hermitian cone $\mathcal{C}_c$ of Equation (4) has q^3 points in $\mathrm{AG}(2, q^2)$ and contains q points from ℓ_Y. One of these q points is O, the other $q - 1$ forming a non-trivial G-orbit. The remaining $q^3 - q$ points of $\mathcal{C}_c$ are partitioned into q distinct G-orbits. Hence, the line represented by $\mathcal{C}_c$ is also incident with $q + 1$ points. This shows that each line in Π is incident with exactly $q + 1$ points.

Therefore, it is enough to show that any two distinct lines of $\mathcal{L}$ have exactly one common point. Obviously, this is true when one of these lines is represented by ℓ_Y. Furthermore, the point of $\mathcal{P}$ represented by ℓ_X is incident with each line of $\mathcal{L}$ represented by a Hermitian cone of equation (4). We are led to investigate the case where one of the lines of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}_b$ of equation (4), and the other line of $\mathcal{L}$ is represented by a Hermitian curve $\mathcal{H}$ which is either another Hermitian curve $\mathcal{H}_d$ of the same type of Equation (3), or a Hermitian cone $\mathcal{C}_c$ of Equation (4).

Clearly, both O and Y_∞ are common points of $\mathcal{H}_b$ and $\mathcal{H}$. From Kestenband's classification [17], see also [2, Theorem 6.7], $\mathcal{H}_b \cap \mathcal{H}$ cannot consist of exactly two points. Therefore, there exists another point, say $P \in \mathcal{H}_b \cap \mathcal{H}$. Since ℓ_X and ℓ_∞ are 1-secants of $\mathcal{H}_b$ at the points O and Y_∞, respectively, either P is on ℓ_Y or P lies outside the fundamental triangle. In the latter case, the G-orbit Δ_1 of P has size $q^2 - 1$ and represents a point in $\mathcal{P}$. Assume that $\mathcal{H}_b \cap \mathcal{H}$ contains a further point $Q \notin \Delta_1$ which does not belong to ℓ_Y and denote by Δ_2 its G-orbit. Then,

$$|\mathcal{H}_b \cap \mathcal{H}| \geq |\Delta_1| + |\Delta_2| = 2(q^2 - 1) + 2 = 2q^2.$$

However, from Bézout's theorem, see [13, Theorem 3.14],

$$|\mathcal{H}_b \cap \mathcal{H}| \leq (q + 1)^2.$$

Therefore, $Q \in \ell_Y$, and its G-orbit Δ_3 has length $q - 1$. Hence, $\mathcal{H}_b$ and $\mathcal{H}$ share $q + 1$ points on ℓ_Y. If $\mathcal{H} = \mathcal{H}_d$ is a Hermitian curve of Equation (3), each of these $q + 1$ points is the tangency point of a common inflection tangent with multiplicity $q + 1$ of the Hermitian curves $\mathcal{H}_b$ and $\mathcal{H}$. Write $R_1, \ldots R_{q+1}$ for these points. Then, by (1) the intersection multiplicity is $I(R_i, \mathcal{H}_b \cap \mathcal{H}_d) = q + 1$. This holds true also when $\mathcal{H}$ is a Hermitian cone $\mathcal{C}_c$ of Equation (4); see Lemma 2.3. Therefore, in any case,

$$\sum_{i=1}^{q+1} I(R_i, \mathcal{H}_b \cap \mathcal{H}) = (q + 1)^2.$$

From Bézout's theorem, $\mathcal{H}_b \cap \mathcal{H} = \{R_1, \ldots R_{q+1}\}$. Therefore, $\mathcal{H}_b \cap \mathcal{H} = \Delta_3 \cup \{O, Y_\infty\}$. This shows that if $Q \notin \ell_Y$, the lines represented by $\mathcal{H}_b$ and $\mathcal{H}$ have exactly one point in common. The above argument can also be adapted to prove this assertion in the case where $Q \in \ell_Y$. Therefore, any two distinct lines of $\mathcal{L}$ have exactly one common point. $\square$

Proof of Theorem 3.1. Construct a projective plane Π as in Lemma 3.2. Since $\mathcal{U} \setminus \{O, Y_\infty\}$ is the union of G-orbits, $\mathcal{U}$ represents a set Γ of $q+1$ points in Π. From [7], $N \equiv 1 \pmod{p}$ where N is the number of common points of $\mathcal{U}$ with any Hermitian curve $\mathcal{H}_b$. In terms of Π, Γ contains some point from every line Λ in $\mathcal{L}$ represented by a Hermitian curve of Equation (3). Actually, this holds true when the line Λ in $\mathcal{L}$ is represented by a Hermitian cone $\mathcal{C}$ of Equation (4). To prove it, observe that $\mathcal{C}$ contains a line r distinct from both lines ℓ_X and ℓ_∞. Then, $r \cap \mathcal{U}$ is not empty and contains neither O nor Y_∞. If P is point in $r \cap \mathcal{U}$, then the G-orbit of P represents a common point of Γ and Λ. Since the line in $\mathcal{L}$ represented by ℓ_Y meets Γ, it turns out that Γ contains some point from every line in $\mathcal{L}$.

Therefore, Γ itself is a line in $\mathcal{L}$. Note that $\mathcal{U}$ contains no line. In terms of $\mathrm{PG}(2, q^2)$, this yields that $\mathcal{U}$ coincides with a Hermitian curve of Equation (3). In particular, $\mathcal{U}$ is a classical unital. $\square$

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