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Steady states analysis and exponential stability of an extensible thermoelastic system

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Abstract

In this work we consider a nonlinear model for the vibrations of a thermoelastic beam with fixed ends resting on an elastic foundation. The behavior of the related dissipative system accounts for both the midplane stretching of the beam and the Fourier heat conduction. The nonlinear term enters the motion equation, only, while the dissipation is entirely contributed by the heat equation. Under stationary axial load and uniform external temperature the problem uncouples and the bending equilibria of the beam satisfy a semilinear equation. For a general axial load p, the existence of a finite/infinite set of steady states is proved and buckling occurrence is discussed. Finally, long-term dynamics of solutions and exponential stability of the straight position are scrutinized.

Keywords: `Thermoelastic system, hinged beam, steady state solutions, nonlinear buckling, exponential stability.`

1. Introduction.

Our goal is to scrutinize here the following evolution system

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta + \left(p - \|\partial_x u\|^2_{L^2(0,1)}\right)\partial_{xx}u = f - k\,u, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \end{cases} \tag{1.1}$$

in the variables $u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, accounting for the vertical deflection of the beam with respect to its reference configuration, and $\theta = \theta(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$, accounting for the variation of temperature with respect to its reference value. The real function $f = f(x)$ is the (given) lateral load distribution, $-ku$ represents the lateral action effected by the elastic foundation, and $g = g(x)$ is the thermal source. The real constant p represents the axial force acting in the reference configuration: it is negative when the beam is stretched, positive when compressed.

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Letting $x \in [0,1]$, the initial condition reads

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x). \tag{1.2}$$

A Dirichlet boundary condition is assumed for the temperature variation,

$$\theta(0,t) = \theta(1,t) = 0\,, \qquad t \in [0,T]\,,$$

and both ends of the beam are assumed to be hinged,

$$u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0\,, \qquad t \in [0,T]\,. \tag{1.3}$$

The solutions to problem (1.1)-(1.3) describe the mechanical and thermal evolution (in the transversal direction) of an hinged extensible thermoelastic beam of unitary natural length resting on an adiabatic elastic foundation. The geometric nonlinearity which is involved accounts for midplane stretching due to the elongation of the beam (see [4] for a rigorous derivation).

The static counterpart of (1.1)-(1.3) reduces to an uncoupled system, and stationary solutions $(\tilde{u}(x), \tilde{\theta}(x))$ can be obtained by solving separately

$$\begin{cases} u'''' + \left(p - \int_0^1 |u'(\xi,\cdot)|^2 \mathrm{d}\xi\right) u'' + k\,u = f + g, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.4}$$

$$\begin{cases} \theta'' = -g, \\ \theta(0) = \theta(1) = 0. \end{cases} \tag{1.5}$$

The investigation of the solutions to (1.4), in dependence on p, represents a classical *nonlinear buckling problem* in the structural mechanics literature. In the case $k = 0$, a careful analysis of the corresponding buckled stationary states was performed in [2] for all values of p in the presence of a source $f + g$ with a general shape. When $f = g = 0$, see also [8,9]. For every $k > 0$ and vanishing sources exact solutions to (1.4) can be found in [1].

In recent years, an increasing attention was payed on the analysis of vibrations and post-buckling dynamics of nonlinear beam models, especially in connection with industrial applications of micromachined beams [3] and microbridges [7]. Unfortunately, most of the paper in the literature deal with the isothermal case, only. Unlike purely mechanical devices, in (1.1) the critical parameter p depends on the thermal expansion, besides on the given axial load, so that the buckling behavior is affected by the mean axial temperature even in the static case (see [4]). Then, under suitable heating conditions, buckling can occur in a thermoelastic beam even if the axial displacements of the ends vanish (thermal buckling). This is a relevant

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phenomenon in a lot of beam-like metallic structures exposed to the sun heating, such as edge rails, for instance.

The global dynamics of (1.1) with $k = 0$ has been addressed in [5], where the existence of the global attractor and its optimal regularity is obtained. A related problem, dealing with purely mechanical vibrations of an extensible viscoelastic beam, has been studied in [6]. In all these papers the exponential decay of the energy for the homogeneous problem is provided when the axial load p does not exceed p_c, the *Euler buckling load*, so that the null solution is both unique and exponentially stable. On the contrary, as $p > p_c$ the straight position loses stability and the beam buckles. In this case, a finite number of buckled solutions occurs and the global (exponential) attractor coincides with the unstable trajectories connecting them.

At a first sight, the case $k > 0$ looks like a slight modification of previously scrutinized models where k vanishes. This is partially true, but similarities are confined to the static case. Indeed, the restoring elastic force $-ku$ opposes the buckling phenomenon and increases the critical value p_c, which turns out to be a *piecewise* linear function of k. In particular, when f and g vanish, the null solution is unique provided that $p \leq p_c(k)$ (see Theorem 3.2). On the contrary, the (unique) null solution to (1.1) is not exponentially stable for all values $p < p_c(k)$, as it occurs when $k = 0$. Quite surprisingly, for large values of k, let say $k > k_0$, the energy decays with a sub-exponential rate when $\bar{p}(k) < p < p_c(k)$ (see Theorem 4.1), and $\bar{p}(k) = p_c(k)$ only if $0 \leq k \leq \lambda_1$ (λ_1 is the first eigenvalue of ∂_{xxxx}). A similar result has been recently obtained for purely mechanical vibrations of an extensible elastic beam resting on a viscoelastic foundation [1].

Finally, we remark that our analysis is carried over an abstract version (independent of the space dimension) of the original problem, so it could be extended to scrutinize stability even in a thermoelastic plate model.

2. The abstract problem

We will consider an abstract version of problem (1.1). To this aim, let H_0 be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $A : \mathcal{D}(A) \Subset H_0 \to H_0$ be a strictly positive selfadjoint operator. We denote by λ_n, with $n = 1, 2, \ldots$, the strictly positive (possibly finite) increasing sequence of the distinct eigenvalues of A. For $\ell \in \mathbb{R}$, we introduce the scale of Hilbert spaces

$$H_\ell = \mathcal{D}(A^{\ell/4}), \qquad \langle u, v \rangle_\ell = \langle A^{\ell/4} u, A^{\ell/4} v \rangle, \qquad \|u\|_\ell = \|A^{\ell/4} u\|.$$

Then, $H_{\ell+1} \Subset H_\ell$ and the following scale of Poincaré inequalities holds

$$\sqrt{\lambda_1}\, \|u\|_\ell^2 \leq \|u\|_{\ell+1}^2, \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Finally, we define the product Hilbert spaces

$$\mathcal{H}_\ell = H_{\ell+2} \times H_\ell \times H_\ell.$$

For $p \in \mathbb{R}$, $k > 0$ and $f, g \in H_0$, we investigate the evolution system on $\mathcal{H}_0$ in the unknowns $u(t) : [0, \infty) \to H_2$ and $\partial_t u(t)$, $\theta(t) : [0, \infty) \to H_0$

$$\begin{cases} \partial_{tt} u + Au - A^{1/2}\theta - \big(p - \|u\|_1^2\big) A^{1/2} u + ku = f, \\ \partial_t \theta + A^{1/2}\theta + A^{1/2}\partial_t u = g, \end{cases} \tag{2.2}$$

with initial conditions

$$(u(0), \partial_t u(0), \theta(0)) = (u_0, u_1, \theta_0) = z \in \mathcal{H}_0. \tag{2.3}$$

In the sequel $z(t) = (u(t), \partial_t u(t), \theta(t))$ will denote the solution. System (2.2) generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{H}_0$. For any initial data $z \in \mathcal{H}_0$, $z(t) = S(t)z$ is the unique weak solution to (2.2), with related energy given by

$$\mathcal{E}(t) = \frac{1}{2}\|S(t)z\|_{\mathcal{H}_0}^2 = \frac{1}{2}\big(\|u(t)\|_2^2 + \|\partial_t u(t)\|^2 + \|\theta(t)\|^2\big). \tag{2.4}$$

Besides, $S(t)$ continuously depends on the initial data. We omit the proof, which can be demonstrated either by means of a Galerkin procedure, or with a standard fixed point method. In both cases, it is crucial to have uniform estimates on any finite time-interval.

Remark 2.1. Problem (1.1)–(1.3) is just a particular case of the abstract system (2.2), obtained by setting $H_0 = L^2(0,1)$ and $A = \partial_{xxxx}$ with the boundary condition (1.3). Nonetheless, the abstract result applies to more general situations, including, for instance, thermoelastic plates.

The differential operator ∂_{xxxx} acting on $L^2(0,1)$ is strictly positive selfadjoint with compact inverse. Its domain is

$$\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0,1) : w(0) = w(1) = w''(0) = w''(1) = 0\}$$

and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. Thus, $\lambda_1 = \pi^4$ is the smallest eigenvalue. Besides, the peculiar relation $(\partial_{xxxx})^{1/2} = -\partial_{xx}$ holds true, with $\mathcal{D}(-\partial_{xx}) = H^2(0,1) \cap H_0^1(0,1)$.

3. Stationary solutions

Assuming $f, g \in H_0$, in the sequel we characterize the set $\mathcal{S}$ of all stationary solutions $(\tilde{u}, \tilde{\theta})$ to the problem

$$\begin{cases} Au - \big(p - \|u\|_1^2\big) A^{1/2} u + ku = f + g, \\ A^{1/2}\theta = g\,, \end{cases} \tag{3.1}$$

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which is the abstract version of (1.4)-(1.5).

Steady temperature distributions. First, we easily obtain the following

Theorem 3.1. *For any $g \in H_0$ the steady heat equation $(3.1)_2$ has a unique solution $\tilde{\theta} \in \mathcal{D}(A^{1/2})$ that vanishes when $g = 0$. In the particular case (1.5)*

$$\tilde{\theta}(x) = \int_0^x \left[G - \int_0^\xi g(\eta)\, \mathrm{d}\eta \right] \mathrm{d}\xi\,, \qquad G = \int_0^1 \int_0^\xi g(\zeta)\, \mathrm{d}\zeta \mathrm{d}\xi\,.$$

Steady deflections. Letting $B = A^{-1/2}(A + kI)$ we can rewrite $(3.1)_1$ as

$$Bu - \big(p - \|u\|_1^2\big)u = h, \tag{3.2}$$

where $h \in \mathcal{D}(A^{1/2})$ solves the problem $A^{1/2}h = f + g$. Because of our assumptions on A, B is a strictly positive selfadjoint operator for every fixed $k > 0$. We denote by μ_n, with $n = 1, 2, \ldots$, the strictly positive sequence of the distinct eigenvalues of B, which are given by

$$\mu_n = \sqrt{\lambda_n} + k/\sqrt{\lambda_n}. \tag{3.3}$$

Moreover, A and B have the same eigenvectors and $\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B)$. Let $\mathcal{E}_n$ the eigenspace corresponding to μ_n (λ_n), with finite orthogonal dimension $\dim(\mathcal{E}_n) = d_n$. For every n, let $e_{n,i}$, with $i \in \{1, \ldots, d_n\}$, be an orthonormal basis of $\mathcal{E}_n$. In particular, the equalities

$$A^q e_{n,i} = \lambda_n^q e_{n,i}\,, \qquad B^q e_{n,i} = \mu_n^q e_{n,i}\,,$$

hold for every $q \in \mathbb{R}$. We introduce the subset of the natural numbers (depending on the given value of p in (3.2))

$$\mathbb{S}_p = \big\{ n : p - \mu_n > 0 \big\}.$$

and we assume that $|\mathbb{S}_p|$ (the cardinality of $\mathbb{S}_p$) is finite for every $p \in \mathbb{R}$.

Remark 3.1. Due to the structure of the equation (3.2), if $h \in H_0$, then $\tilde{u} \in H^2$ and so it is a solution in the strong sense. Moreover, if $h \in \mathcal{D}(A^{1/2})$ then $\tilde{u} \in \mathcal{D}(A)$ and solves $(3.1)_1$ in the strong sense with $f \in H_0$. Such regular solutions represents the stationary deflection states of the beam.

Our aim is to analyze the multiplicity of solutions to (3.2). In particular, we will show that there is always at least one solution, and at most a finite number of solutions, whenever the eigenvalues not exceeding p are simple. First, we scrutinize the homogeneous case. For the abstract problem, we can parallel the proofs of Th. 4.1 and Remark 4.2 in [2].

Theorem 3.2. *Let $h = 0$ and $n_* = |\mathbb{S}_p|$. Given $k > 0$, if there exists an eigenvalue μ_n of B which is not simple for some $n \in \mathbb{S}_p$, then (3.2) has infinitely many solutions. Otherwise, it has exactly $2n_\star + 1$ solutions:*

$$\tilde{u}_0 = 0 \quad \text{and} \quad \tilde{u}_n^{\pm} = C_n^{\pm} e_{n,1}, \qquad \text{for every } n \in \mathbb{S}_p\,,$$

where

$$C_n^{\pm} = \pm\sqrt{(p-\mu_n)/\sqrt{\lambda_n}}\,.$$

Proof. If $n \in \mathbb{S}_p$ and $d_n > 1$, then any $u \in \mathcal{E}_n$ satisfying $\|u\|_1^2 = p - \mu_n$ is a solution to (3.2). Clearly, there are infinitely many such u, given by

$$\tilde{u} = \sum_i u_i e_{i,n}, \qquad u_i \in \mathbb{R},$$

such that

$$\sum_i u_i^2 = \frac{1}{\sqrt{\lambda_n}}(p - \mu_n).$$

Assume then that μ_n is simple whenever $n \in \mathbb{S}_p$. Obviously, $\tilde{u} = 0$ is a solution. Let us look for a nontrivial solution $\tilde{u}$. Setting

$$\nu = -p + \|\tilde{u}\|_1^2, \tag{3.4}$$

such $\tilde{u}$ solves $A\tilde{u} + \nu\tilde{u} = 0$. Hence, $\nu = -\mu_n$, $\tilde{u} = Ce_{n,1}$, for some $C \neq 0$. In particular, $\|\tilde{u}\|_1^2 = C^2\sqrt{\lambda_n}$. The value C is determined by (3.4), which yields the relation

$$C^2\sqrt{\lambda_n} = p - \mu_n.$$

Therefore, we have nontrivial solutions if and only if $n \in \mathbb{S}_p$. Namely, there are exactly $2n_\star$ nontrivial solutions, explicitly computed. □

The homogeneous case of the physical model (1.4) has been scrutinized in Th. 2.2 of [1]. There, the eigenvalues of B are proved to be

$$\mu_n = n^2\pi^2 + k/n^2\pi^2, \qquad n \in \mathbb{N},\ k > 0,$$

which are all simple and increasingly ordered provided that $k < 4\pi^4$. The corresponding eigenvectors are $e_n(x) = \sqrt{2}\sin n\pi x$ with $d_n = 1$. On the contrary, $\mu_n = \mu_m$, $n \neq m$, when $k = n^2m^2\pi^4$ (resonant values) and

$$\min_{n\in\mathbb{N}} \mu_n = \mu_{n_k}, \qquad n_k \in \mathbb{N} : (n_k - 1)^2 n_k^2 \leq k/\pi^4 < n_k^2(n_k + 1)^2\,.$$

Moreover, $\mathbb{S}_p = \left\{n \in \mathbb{N} : n^4\pi^2 + k/\pi^2 < pn^2\right\}$. In the sequel we denote

$$p_c(k) = \min_{n\in\mathbb{N}} \mu_n, \qquad \mathcal{R} = \{\rho : \rho = n^2m^2\pi^4, n, m \in \mathbb{N}, m < n\}\,.$$

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When $k \in \mathcal{R}$, there exists at least an eigenvalue of B which is not simple. Let μ_m be the smallest one. According to Th. 3.2, if $m \in \mathcal{S}_p$ system (1.4) with $h = 0$ has infinitely many solutions. Otherwise, it has at most $2n_\star + 1$ solutions: when $p \le p_c(k)$ there is only the null (straight) solution. If $p > p_c(k)$, then besides the null solution there are also $2n_\star$ buckled solutions,

$$\tilde{u}_n^{\pm}(x) \;=\; \pm \frac{\sqrt{2}}{n\,\pi} \sqrt{p - \frac{k}{n^2\pi^2} - n^2\pi^2}\, \sin n\pi x\,, \qquad n = 1, 2, \dots, n_\star.$$

Remark 3.2. Assuming $k = 0$, we recover the results of [8,9].

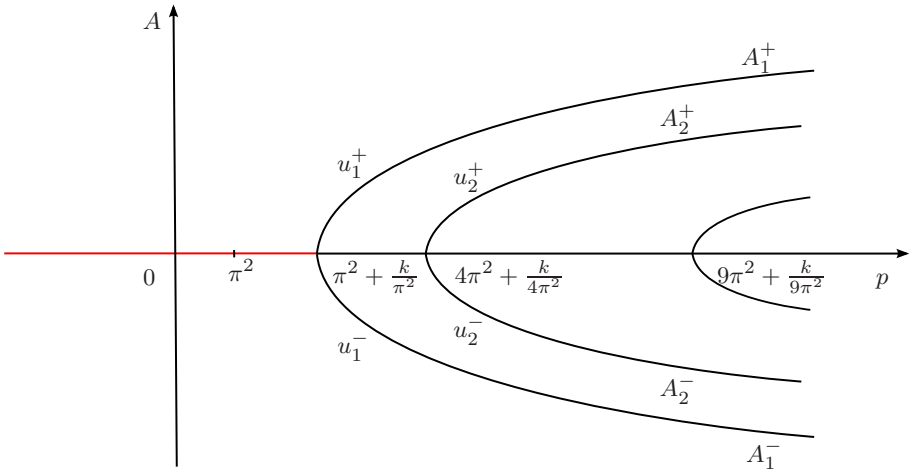

Fig. 1. The static response of the beam when $k < 4\pi^4$.

In the nonhomogeneous case, the picture is more complicated, and the shape of $h = f + g$ plays a crucial role (see for instance [2]). In essence, if there is an eigenvalue exceeding p, whose multiplicity is greater than one, then infinite solutions may appear, unless the projection of the external source $h = f + g$ on the relative eigenspace is not zero.

In order to prove this behavior, we take advantage of the following

Definition 3.1. Let $h \in H_{-1}$. A function $\tilde{u} \in H_1$ is a solution to (3.2) if

$$\langle B^{1/2}\tilde{u}, B^{1/2}w\rangle - \big(p - \|\tilde{u}\|_1^2\big)\langle \tilde{u}, w\rangle = \langle B^{-1/2}h, B^{1/2}w\rangle,$$

for every $w \in H_1$. In fact, $H_1 = \mathcal{D}(A^{1/4}) \subset \mathcal{D}(B^{1/2})$.

Let us set $h_{n,i} = \langle B^{-1/2}h, B^{1/2}e_{n,i}\rangle$. Besides, $h_{n,i} \neq 0$ for some n and some i, otherwise $h = 0$. We define

$$Q_j = \sum_{n\neq j,\, i} \frac{\sqrt{\lambda_n}\, h_{n,i}^2}{(\mu_n - \mu_j)^2}, \qquad j \in \mathbb{N}.$$

Theorem 3.3. *Let $h \neq 0$ and $k > 0$. Along with $n_\star = |\mathbb{S}_p|$, we define*

$$j_\star = \big|\{j \in \mathbb{N} : p - \mu_j > 0,\ \ Q_j < p - \mu_j,\ \ \mathbb{P}_j h = 0\}\big|,$$
$$j_\star^0 = \big|\{j \in \mathbb{N} : p - \mu_j > 0,\ \ Q_j = p - \mu_j,\ \ \mathbb{P}_j h = 0\}\big|,$$

where $\mathbb{P}_n$ is the projection of H_{-1} onto $\mathcal{E}_n$. Then, (3.2) has infinitely many solutions if and only if the conditions

$$d_j > 1\,, \qquad \mathbb{P}_j h = 0\,, \qquad Q_j < p - \mu_j$$

simultaneously hold for some j. Otherwise, (3.2) has $m_\star$ solutions, with

$$1 \leq m_\star \leq 2n_\star + 2j_\star + j_\star^0 + 1.$$

Proof. The proof is carried out by paralleling Th. 5.1 of [2]. First, we set

$$\nu = -p + \|\tilde{u}\|_1^2, \tag{3.5}$$

which, since $\tilde{u} = 0$ is not a solution anymore, yields the constraint

$$p + \nu > 0. \tag{3.6}$$

Writing $\tilde{u} = \sum_{n,i} u_{n,i} e_{n,i}$, with $u_{n,i} = \langle \tilde{u}, e_{n,i} \rangle$, we have

$$\|\tilde{u}\|_1^2 = \sum_{n,i} \sqrt{\lambda_n} u_{n,i}^2.$$

Thus, (3.5) turns into

$$\nu = -p + \sum_{n,i} \sqrt{\lambda_n} u_{n,i}^2. \tag{3.7}$$

Projecting (3.2) on the orthonormal basis, we obtain, for every n, i,

$$(\mu_n + \nu) u_{n,i} = h_{n,i}. \tag{3.8}$$

The solution $\tilde{u}$ is known once we determine all the coefficients $u_{n,i}$.

• We begin to look for solutions $\tilde{u}$ for which $\nu \neq -\mu_n$, for all n. In that case the coefficients $u_{n,i}$ are uniquely determined by (3.8) as

$$u_{n,i} = \frac{h_{n,i}}{\mu_n + \nu}. \tag{3.9}$$

Setting

$$\Phi(\nu) = \sum_{n,i} \frac{\sqrt{\lambda_n} h_{n,i}^2}{(\mu_n + \nu)^2},$$

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we plug (3.9) into (3.7). Recalling (3.6), we realize at once that the admissible values of ν are the solutions to the equation

$$\Lambda(\nu) = -p - \nu + \Phi(\nu) = 0, \tag{3.10}$$

in $D = (-p, +\infty) \setminus \{-\mu_n\}$. The set D is the union (empty if $n_\star = 0$) of $n_\star$ bounded open interval I_n and of the open interval $I_0 = (\alpha, +\infty)$, where

$$\alpha = \begin{cases} \inf\limits_{n\in\mathcal{S}} -\mu_n & \text{if } n_\star > 0, \\ -p & \text{if } n_\star = 0. \end{cases}$$

For every $\nu \in D$, we have

$$\Lambda''(\nu) = \Phi''(\nu) = 6 \sum_{n,i} \frac{\sqrt{\lambda_n} h_{n,i}^2}{(\mu_n + \nu)^4} > 0.$$

Thus, Λ is strictly convex on each $I_n \subset D$, $n \in \{1, ..., n_\star\}$. Hence, the equation $\Lambda(\mu) = 0$ can have at most two solutions on each I_n. In the unbounded interval I_0, the function Λ is strictly decreasing. Moreover, since $\Phi(\infty) = 0$, then $\lim_{\nu\to+\infty} \Lambda(\nu) = -\infty$, and

$$\lim_{\nu\to\alpha^+} \Lambda(\nu) = \begin{cases} +\infty & \text{if } n_\star > 0, \\ \Phi(-p) > 0 & \text{if } n_\star = 0. \end{cases}$$

So, we conclude that there is exactly one solution in I_0. Summarizing, the equation $\Lambda(\nu) = 0$, and then (3.2), has at least one solution and at most $2n_\star + 1$ solutions with the property that $\nu \neq -\mu_n$. Indeed, for every $\nu \in D$ such that $\Lambda(\nu) = 0$, the vector $\tilde{u}$ with Fourier coefficients given by (3.9) belongs to H^1. By virtue of (3.3), this is guaranteed by the convergence of the series $\Phi(\nu)$, since ν cannot accumulate $-\mu_n$ and the assumption $h \in H_{-1}$ translates into the summability conditions

$$\sum_{n,i} \frac{\sqrt{\lambda_n}}{\mu_n^2} h_{n,i}^2 \leq \sum_{n,i} \frac{1}{\mu_n} h_{n,i}^2 \leq \sum_{n,i} \frac{1}{\sqrt{\lambda_n}} h_{n,i}^2 < \infty.$$

• Next, we look for solutions $\tilde{u}$ such that $\nu = -\mu_j$, for some given j. We preliminarily observe that, due to (3.6), if $p - \mu_j \notin (0, +\infty)$, no such solutions exist. In the other case, for $n \neq j$, the values $u_{n,i}$ are fixed by (3.9) with $\nu = -\mu_j$. We are left to determine the values $u_{j,i}$. But (3.7) now reads

$$\mu_j \sum_i u_{j,i}^2 + Q_j = p - \mu_j.$$

Therefore, we have no solutions whenever $Q_j > p - \mu_j$. Assume then that $Q_j \leq p - \mu_j$. From (3.8), we have no solutions unless $h_{j,i} = \mathbb{P}_i h = 0$ for all i. In which case, we have only the trivial solution ($u_{j,i} = 0$ for all i) if $Q_j = p - \mu_j$. On the other hand, if $Q_j < p - \mu_j$, we have two solutions provided that $d_j = 1$, corresponding to

$$u^{\pm}_{j,1} = \pm\sqrt{(p - \mu_j - Q_j)/\sqrt{\lambda_j}}\,,$$

and infinitely many solutions if $d_j > 1$. □

4. Exponential Stability

Recalling Theorems 3.1 and 3.2, when $f = g = 0$ the set $\mathcal{S}_0$ of stationary solutions to (1.1) reduces to the singleton of the null pair provided that

$$p \leq p_c(k) = \min_{n\in\mathbb{N}} \mu_n, \qquad k > 0. \tag{4.1}$$

In spite of this, we shall prove that the energy $\mathcal{E}(t)$, given by (2.4), does not decay exponentially for all values of p and k which satisfy (4.1), but in a smaller range. For futher purposes, we define

$$\bar{p}(k) = \begin{cases} 2\sqrt{k}, & k > \lambda_1 \\ \\ p_c(k), & 0 < k \leq \lambda_1. \end{cases}$$

Lemma 4.1 (see [1], Lemma 4.1). *Let $p \in \mathbb{R}$, $k > 0$ and denote*

$$Lu \;=\; Au - p\,A^{1/2}u + k\,u\;.$$

Provided that $p < \bar{p}(k)$, there exists a positive function $\nu = \nu(p,k)$ such that

$$\langle Lu\,,\,u\rangle = \|u\|_2^2 - p\,\|u\|_1^2 + k\,\|u\|^2 \geq \nu\,\|u\|_2^2\,.$$

For all $p \in \mathbb{R}$ and $k > 0$ the positive functional

$$\mathcal{L}(t) = \mathcal{E}(t) + \frac{k}{2}\|u(t)\|^2 + \frac{1}{4}\big(\|u(t)\|_1^2 - p\big)^2$$

is a Lyapunov functional for $S(t)$. It is an easy matter to show (see [5])

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) + \|\theta(t)\|_1^2 = 0 \tag{4.3}$$

and this ensures a stability result relative to the energy norm. Indeed,

$$\mathcal{L}(S(t)z) \leq \mathcal{L}(z) = \frac{1}{2}\|z\|^2_{\mathcal{H}_0} + \frac{k}{2}\|u_0\|^2 + \frac{1}{4}\big(\|u_0\|_1^2 - p\big)^2.$$

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Then, given $R > 0$, if we consider initial data $z \in \mathcal{H}_0$ such that $\|z\|_{\mathcal{H}_0} \leq R$, the previous inequality provides

$$\mathcal{E}(t) \leq \mathcal{L}(t) \leq C, \tag{4.4}$$

where C depends (increasingly) only on R, besides on the structural quantities p and k. We are now in a position to prove the following result.

Theorem 4.1. *When $f = g = 0$, the solutions to (2.2) decay exponentially,*

$$\mathcal{E}(t) \;\leq\; \mathcal{E}(0)\, e^{-ct}$$

for some $c > 0$, provided that $p < \bar{p}(k)$.

Proof. Let Φ be defined as

$$\Phi(t) = \mathcal{L}(t) - \frac{1}{4}\, p^2 + \frac{\varepsilon}{2}\, \langle \partial_t u(t)\,,\, u(t) \rangle + \varepsilon \langle \partial_t u(t), \theta(t) \rangle_{-1}.$$

In view of applying Lemma 4.1, we note that

$$\Phi = \frac{1}{2}\left(\langle Lu\,,\, u \rangle + \|\theta\|^2 + \|\partial_t u\|^2 \right) + \frac{1}{4}\, \|u\|_1^4 + \frac{\varepsilon}{2}\, \langle \partial_t u, u \rangle + \varepsilon \langle \partial_t u, \theta \rangle_{-1}.$$

Then, Φ is equivalent to $\mathcal{E}$. Indeed, from (4.4) we have

$$\|u\|_1^4 \leq C \|u\|_1^2 \leq C\mathcal{E},$$

and by virtue of Lemma 4.1 we can easily obtain

$$c\,\mathcal{E} \leq \Phi \leq C\mathcal{E},$$

for some $c = c(p, k)$, provided that ε is small enough and $p < \bar{p}(k)$. So, in the sequel it is enough to prove the exponential decay of Φ. To this aim, exploiting (2.2) with $f = g = 0$, we have the equality

$$\begin{aligned} \frac{d}{dt}\Phi + \varepsilon\Phi + \frac{\varepsilon}{4}\|u\|_1^4 + \|\theta\|_1^2 = \frac{3\varepsilon}{2}\|\theta\|^2 + \frac{\varepsilon}{2}\Big[& 2(p - \|u\|_1^2)\langle u, \theta \rangle \\ & - \langle u, \theta \rangle_1 - 2\langle \partial_t u, \theta \rangle - 2k\langle u, \theta \rangle_{-1} + 2\varepsilon \langle \partial_t u, \theta \rangle_{-1} + \varepsilon \langle \partial_t u, u \rangle \Big]. \end{aligned} \tag{4.6}$$

Exploiting the inequalities

$$\begin{aligned} \varepsilon \langle \partial_t u, u \rangle &\leq \nu \|\partial_t u\|^2 + \frac{\varepsilon^2}{4\nu\lambda_1}\|u\|_2^2 \\ 2\varepsilon \langle \partial_t u, \theta \rangle_{-1} &\leq \nu \|\partial_t u\|^2 + \frac{\varepsilon^2}{\nu\lambda_1}\|\theta\|^2 \\ 2\langle \partial_t u, \theta \rangle &\leq \sqrt{\nu}\|\partial_t u\|^2 + \frac{1}{\sqrt{\nu}}\|\theta\|^2 \\ \langle u, \theta \rangle_\ell \leq \|u\|_2 \|\theta\|_{2(\ell-1)} &\leq \sqrt{\nu}\|u\|_2^2 + \frac{\lambda_1^{\ell-1}}{4\sqrt{\nu}}\|\theta\|^2, \quad \ell = -1, 0, 1, \end{aligned}$$

and choosing $\nu = \varepsilon$, from (4.4) and (4.6) it follows

$$\frac{d}{dt}\Phi + \varepsilon\Phi + \|\theta\|_1^2 \leq \frac{1}{2}\left(\frac{\varepsilon^2}{\lambda_1} + 3\varepsilon + \frac{C_1\sqrt{\varepsilon}}{4\lambda_1^2}\right)\|\theta\|^2 + \frac{\varepsilon}{2}\left[(2\varepsilon + \sqrt{\varepsilon})\|\partial_t u\|^2 + \left(\frac{\varepsilon}{4\lambda_1} + C_2\sqrt{\varepsilon}\right)\|u\|_2^2\right],$$

where $C_1 = 5\lambda_1^2 + 2k + 2|p|\lambda_1 + 4C\sqrt{\lambda_1}$ and $C_2 = 2|p| + 1 + 2k + 4C/\sqrt{\lambda_1}$. Finally, applying the inequality

$$\frac{1}{2}A(\varepsilon)\|\theta\|^2 - \|\theta\|_1^2 \leq -\Big(1 - \frac{A(\varepsilon)}{2\sqrt{\lambda_1}}\Big)\|\theta\|_1^2$$

and choosing ε as small as needed, we obtain

$$\frac{d}{dt}\Phi + \varepsilon\,\Phi + \frac{1}{2}\,\|\theta\|_1^2 \leq \frac{\varepsilon}{2}\,c\,\mathcal{E} \leq \frac{\varepsilon}{2}\,\Phi$$

which in turn implies by the Gronwall Lemma

$$\Phi(t) \leq \Phi(0)\,e^{-\frac{\varepsilon}{2}t}.$$

□

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