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A Note on Maximal Elements for Acyclic Binary Relations on Compact Topological Spaces[1]

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Abstract

I introduce the concept of a *weakly tc-upper semicontinuous* acyclic binary relation $\prec$ on a topological space (X, τ), which appears as slightly more general than other concepts of continuity which have been introduced in the literature in connection with the problem concerning the existence of maximal elements. By using such a notion, I show that if an acyclic binary relation $\prec$ on a compact topological space is weakly tc-upper semicontinuous, then there exists a maximal element relative to $\prec$. In this way I generalize existing results concerning the existence of maximal elements on compact topological spaces.

1 Introduction

A classical and very nice theorem of Bergstrom [3] states that an acyclic binary relation $\prec$ on a compact topological space (X, τ) has a maximal element provided that $\prec$ is *upper semicontinuous* (i.e., $L_{\prec}(x) = \{z \in X : z \prec x\}$ is an open subset of X for every $x \in X$).

Several authors presented generalizations of the aforementioned result of Bergstrom by using different suitable notions of semicontinuity such as *weak lower continuity* (see Campbell and Walker [5]), *transfer lower continuity* (see e.g. Mehta [7] and Subiza and Peris [8]) and *tc-upper semicontinuity* (see Alcantud [1, 2]).

More recently, Kukushkin [6] was concerned with the existence of maximal elements for an *interval order* on a compact metric space.

In this paper we present the notion of a *weakly tc-upper semicontinuous*

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acyclic binary relation, which generalizes the notion of a tc -upper semicontinuous acyclic binary relation.

We recall that an acyclic binary relation $\prec$ on a topological space (X, τ) is said to be tc-upper semicontinuous if the *transitive closure* $\prec\prec$ of $\prec$ is upper semicontinuous. The concept of weak tc-upper semicontinuity which is presented in this paper resembles the notion of weak upper semicontinuity which is found in Bosi and Herden [4] in connection with the existence of a linear and upper semicontinuous extension of a partial order.

We show that if an acyclic binary relation $\prec$ on a compact topological space is weakly tc-upper semicontinuous, then there exists a maximal element relative to $\prec$.

As a corollary of our main result, we prove that an acyclic binary relation $\prec$ on a compact topological space (X, τ) has a maximal element provided that $\prec$ is tc-upper semicontinuous (see Alcantud [1, Proposition 2]).

2 Notation and definitions

We first recall that a binary relation $\prec$ on a nonempty set X is said to be *acyclic* if the following condition holds for every integer $n \geq 2$ and for all $x_1, ..., x_n \in X$:

$$(x_1 \prec x_2) \wedge (x_2 \prec x_3) \wedge ... \wedge (x_{n-1} \prec x_n) \Rightarrow x_1 \neq x_n.$$

Denote by $\prec\prec$ the *transitive closure* associated to an acyclic binary relation $\prec$ on a set X (namely, for every $x, y \in X$, $x \prec\prec y$ if and only if there exists an integer $n \geq 2$ and $x_1, ..., x_n \in X$ such that $x = x_1 \prec x_2 \prec ... \prec x_{n-1} \prec x_n = y$).

We say that a subset D of X is $\prec\prec$*-decreasing* if the following condition holds for every $w, z \in X$:

$$(w \prec\prec z) \text{ and } (z \in D) \Rightarrow w \in D.$$

An acyclic binary relation $\prec$ on a topological space (X, τ) is said to be *tc-upper semicontinuous* (see Alcantud [1]) if $L_{\prec\prec}(x) = \{z \in X : z \prec\prec x\}$ is an open subset of X for every $x \in X$.

Let us now present the most important definition in this paper.

Definition 2.1 If $\prec$ is an acyclic binary relation on a set X and τ is a topology on X, then we say that $\prec$ is *weakly tc-upper semicontinuous* if we may associate to every pair $(x, y) \in X \times X$ such that $x \prec y$ a subset O_{xy} of X so that the following conditions hold:

(i) O_{xy} is open;

(ii) O_{xy} is $\prec\!\prec$-decreasing;

(iii) $x \in O_{xy},\ y \notin O_{xy}$;

(iv) $O_{xy} \subsetneqq O_{zw}$ for every $x, y, z, w \in X$ such that $x \prec y$, $z \prec w$, $y \in O_{zw}$.

The reader may recall that an acyclic binary relation $\prec$ on a topological space (X, τ) is said to be *weakly upper semicontinuous* (see e.g. Alcantud [2]) if we may associate to every pair $(x, y) \in X \times X$ such that $x \prec y$ a neighborhood O_{xy} of x such that $y \prec z$ is false for all $z \in O_{xy}$. It is not hard to check that weak tc-upper semicontinuity implies weak upper semicontinuity. Indeed, if O_{xy} is $\prec\!\prec$-decreasing, then $y \prec z \in O_{xy}$ implies $y \in O_{xy}$ and this would contradict condition (iii) in Definition 2.1.

We recall that a real-valued function u on a set X is a *weak utility* for an acyclic binary relation $\prec$ on X if the following condition holds for every $x, y \in X$:

$$x \prec y \Rightarrow u(x) < u(y).$$

is an *almost weak utility* for an acyclic binary relation $\prec$ on X if the following condition holds for every $x, y \in X$:

$$x \prec\!\prec y \Rightarrow u(x) \leq u(y).$$

In the following proposition we show that the existence of an upper semicontinuous weak utility implies weak tc-upper semicontinuity.

Proposition 2.2 *Let $\prec$ be an acyclic binary relation on a topological space (X, τ). If there exists an upper semicontinuous weak utility u for $\prec$, then $\prec$ is weakly tc-upper semicontinuous.*

Proof. Let u be an upper semicontinuous weak utility for an acyclic binary relation $\prec$ on a topological space (X, τ). Define $O_{xy} = u^{-1}(]-\infty, u(y)[)$ for every $x, y \in X$ such that $x \prec y$. It is almost immediate to verify that the family $\{O_{xy} : x \prec y,\ x, y \in X\}$ satisfies conditions (i), (ii) and (iii) in Definition 2.1. In order to show that also condition (iv) in Definition 2.1 holds, consider $x, y, z, w \in X$ such that $x \prec y$, $z \prec w$, $u(y) < u(w)$ ($\Leftrightarrow y \in O_{zw}$). If a is any element of X such that $u(a) < u(y)$ ($\Leftrightarrow a \in O_{xy}$), then we have that $u(a) < u(y) < u(w)$ implies that $u(a) < u(w)$ ($\Leftrightarrow a \in O_{zw}$). If we further observe that $y \in O_{zw} \setminus O_{xy}$, then we immediately realize that $O_{xy} \subsetneqq O_{zw}$. $\square$

The concept of weak tc-upper semicontinuity generalizes the concept of tc-upper semicontinuity. Indeed the following proposition holds.

Proposition 2.3 *Let $\prec$ be an acyclic binary relation on a topological space (X, τ). If $\prec$ is tc-upper semicontinuous, then $\prec$ is also weakly tc-upper semicontinuous.*

Proof. If $\prec$ is tc-upper semicontinuous then we can define $O_{xy} = L_{\prec\prec}(y)$ for every $x, y \in X$ such that $x \prec y$. It is immediate to verify that the family $\{O_{xy} : x \prec y, x, y \in X\}$ satisfies conditions (i), (ii) and (iii) in Definition 2.1. In order to show that also condition (iv) in Definition 2.1 holds, consider $x, y, z, w \in X$ such that $x \prec y$, $z \prec w$, $y \prec\prec w$ ($\Leftrightarrow y \in O_{zw}$). If a is any element of X such that $a \prec\prec y$ ($\Leftrightarrow a \in O_{xy}$), then we have that $a \prec\prec y \prec\prec w$ implies that $a \prec\prec w$ ($\Leftrightarrow a \in O_{zw}$). Since it is clear that $y \in O_{zw} \setminus O_{xy}$, we have that actually $O_{xy} \subsetneqq O_{zw}$. □

3 The result

Let us now present the main result in this paper, which guarantees the existence of a maximal element for a weak tc-upper semicontinuous acyclic binary relation on a compact topological space.

Theorem 3.1 *Let $\prec$ be a weakly tc-upper semicontinuous acyclic binary relation on a compact topological space (X, τ). Then there exists a maximal element relative to $\prec$.*

Proof. Let $\prec$ be a weakly tc-upper semicontinuous acyclic binary relation on a compact topological space (X, τ). Assume by contraposition that there exists no maximal element relative to $\prec$. Since $\prec$ is weakly tc-upper semicontinuous we have that for every $x \in X$ there exist some element $y(x) \in X$ such that $x \prec y(x)$ and an open $\prec\prec$-decreasing subset $O_{xy(x)}$ of X such that $x \in O_{xy(x)}$, $y(x) \notin O_{xy(x)}$. It is clear that $\mathcal{O} = \{O_{xy(x)} : x \in X\}$ is an open covering of X. Since (X, τ) is compact there exist elements $x_1, ..., x_n$ of X ($n \in I\!N \setminus \{0, 1\}$) such that $\{O_{x_h y(x_h)} : h = 1, ..., n\}$ is also a covering of X ($x_h \prec y(x_h)$ for $h \in \{1, ..., n\}$). Without loss of generality, let us assume that $y(x_{h-1}) \in O_{x_h y(x_h)}$ for $h = 2, ..., n$ (observe that $y(x_{h-1}) \notin O_{x_{h-1} y(x_{h-1})}$). By condition (iv) in the definition of a weakly tc-upper semicontinuous acyclic binary relation (see Definition 2.1), we have that $O_{x_{h-1} y(x_{h-1})} \subsetneqq O_{x_h y(x_h)}$ for $h \in 2, ..., n$. Clearly, it should be $y(x_n) \in O_{x_{h^*} y(x_{h^*})} \setminus O_{x_n y(x_n)}$ for some $h^* \in \{2, ..., n-1\}$. But this is contradictory from the considerations above, since $y(x_n) \notin O_{x_n y(x_n)}$. So the proof is complete. □

Remark 3.2 It is clear from the proof of Theorem 3.1 and from the definition of weak tc-upper semicontinuity that Theorem 3.1 remains true if instead of requiring compactness of the topological space (X, τ) we only require *upper-compactness* of the topological related space $(X, \tau, \prec)$. We recall that if $\prec$ is an acyclic binary relation a topological space (X, τ), then the topological related space $(X, \tau, \prec)$ is said to be *upper-compact* if every open covering of X by $\prec\prec$-decreasing sets admits a finite subcovering. (see e.g, Alcantud [1]).

As an application of Theorem 3.1, Proposition 2.3 and Remark 3.2 we can easily obtain Proposition 2 in Alcantud [2] as a corollary.

Corollary 3.3 *Let $\prec$ be a tc-upper semicontinuous acyclic binary relation on a topological space (X, τ) and assume that $(X, \tau, \prec)$ is upper-compact. Then there exists a maximal element relative to $\prec$.*

Proof. Since $\prec$ is tc-upper semicontinuous, we have that $\prec$ is also weakly tc-upper semicontinuous by Proposition 2.3. Therefore Remark 3.2 and the proof of Theorem 3.1 guarantee the existence of a maximal element relative to $\prec$ since we have assumed upper-compactness of the topological related space $(X, \tau, \prec)$. □

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