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Variational Principles for Electromagnetic Systems with Memory (*).

ELENA VUK

Abstract. – *«Saddle point» or «mini-max» principles are set up for first order integrodifferential systems arising in the hereditary electromagnetic theory. Variational formulations are obtained directly in terms of electric and magnetic field vectors. In particular, these principles are shown to hold as a consequence of thermodynamic restrictions, when the quasi-static case and the linear evolution problem are considered.*

1. – Introduction.

Variational formulations in the electromagnetic theory has been investigated in many papers (see [1, 3, 10, 18, 19]). Concerning evolution problems, these methods often use time-convolution bilinear forms and lead to a stationary principle, not necessarily to a minimum one. They become extremum principles for the Laplace transform of the functionals and are still valid in the transform, but not in the original domain.

A new technique to obtain stationary and minimum principles for linear evolution equations was earlier introduced by Reiss [13] and a specific application to electromagnetism has been developed by Reiss & Haug [14]. This method rests upon the introduction of bilinear forms of convolution type with a suitable weight function involving Laplace transformation with respect to time, in such a way that the functionals could be transformed back to the time domain, while preserving their extremum characteristic.

This result has been extended in [11] to electromagnetic materials whose dissipativity is due to the memory in the constitutive relations. There, minimum principles are obtained by transforming Maxwell's equations system into a second order integrodifferential equation in terms of the electric field.

We notice that Reiss' approach, unfortunately, can be applied to initial boundary value problems, only. Neverthless, as observed in [7], this method can be modified for the treatement of quasi-static problems. Actually, the appropriate class of convolution type function-

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Indirizzo dell'A.: Dipartimento di Matematica, Università degli Studi di Brescia, Via Valotti 9, 25133 Brescia, Italy; e-mail: vuk@bsing.ing.unibs.it

als is taken in terms of bilinear forms with a weight function involving the Fourier transform.

Recently, using a different technique, extremum principles are formulated for time-harmonic and quasi-static problems for lossy systems (subject to impedance-like boundary conditions) in terms of both electric and magnetic field [8]. Actually, if dissipation is involved, variational expressions achieve the extremum property. Moreover, extremum principles (minimum or «mini-max»), if avalaible allow numerical approximations to be applied and may be regarded as a contribution to optimization theory [16].

In this work we formulate some «mini-max» principles for materials subject to conservative and hereditary impedance-like dissipative boundary conditions, which describe the behaviour of a well (but not perfectly) conducting surface. In this framework, we exhibit variational expressions concerning the quasi-static and evolution problems in terms of E and H, such that the stationary property yields both Maxwell equations and mixed boundary conditions.

After introducing proper bilinear forms and making use of an appropriate decomposition of the electromagnetic field, we show that every (unique) solution of first order Maxwell equations can be characterized as a «saddle point» of the corresponding functional.

2. – Setting of the problem.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$ and let $t \in \mathbb{R}$ be the time. In absence of free charge density, the evolution of the electromagnetic field in the space-time domain $Q = \Omega \times \mathbb{R}^+$ is ruled by the well-known Maxwell equations

$$\begin{cases} \nabla \times E(x, t) + \dfrac{\partial}{\partial t} B(x, t) = I_i(x, t) \\ \nabla \cdot B(x, t) = 0 \\ \nabla \times H(x, t) - \dfrac{\partial}{\partial t} D(x, t) = J(x, t) + J_i(x, t) \\ \nabla \cdot D(x, t) = 0\,. \end{cases} \tag{2.1}$$

Vectors fields B, E, H, D and J represent the magnetic flux density, the electric field, the magnetic intensity, the electric displacement and the induced electric current density, respectively. I_i and J_i denote the forced magnetic and electric current densities and are known vector functions on Q.

Assuming the free charge density to be zero we have as a consequence

$$\nabla \cdot (J(x, t) + J_i(x, t)) = 0\,.$$

In spite of this, later on we take into account this constraint by introducing suitable spaces for the field E and for the forced current densities.

For later convenience, for any function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we denote by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\tau) \exp(-i\omega\tau)\, d\tau\,, \qquad \omega \in \mathbb{R}$$

its Fourier transform. If f is a causal function, i.e. $f(s) = 0$, $s < 0$, then

$$\hat{f}(\omega) = f_c(\omega) - if_s(\omega)\,, \qquad \bar{\hat{f}}(\omega) = f_c(\omega) + if_s(\omega)\,,$$

where f_c, f_s are the half-range cosine and sine Fourier transforms of f and «$\bar{\ }$» stands for the complex conjugation. Similarly, for any $z \in \mathbb{C}^+ = \{z = s + i\omega \in \mathbb{C},\ s \geq 0\}$, we denote by

$$\tilde{f}(z) = \int_0^{\infty} f(\tau) \exp(-z\tau)\, d\tau$$

the Laplace transform of f and

$$\tilde{f}_c(\omega) = \int_0^{\infty} f(\tau) \exp(-s\tau) \cos\omega\tau\, d\tau\,, \qquad \tilde{f}_s(\omega) = \int_0^{\infty} f(\tau) \exp(-s\tau) \sin\omega\tau\, d\tau$$

represent its cosine and sine Laplace transforms. In particular, we denote by

$$\tilde{f}(s) = \int_0^{\infty} f(\tau) \exp(-s\tau)\, d\tau\,, \qquad s \in \mathbb{R}^+$$

the real Laplace transform of f.

Now, we consider a conducting material modelled by the following constitutive equations

$$\begin{cases} D(x, t) = \varepsilon(x)\, E(x, t) \\ B(x, t) = \mu(x)\, H(x, t) \\ J(x, t) = \sigma(x)\, E(x, t) \end{cases} \tag{2.2}$$

where ε, μ, σ are symmetric, positive definite second-order tensors called *permittivity, permeability* and *conductivity*, respectively.

Regarding to (2.1) we will scrutinize different boundary conditions. So, the boundary $\partial\Omega$ is taken to consist of two disjoint subsets, namely $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, with $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$; $\partial\Omega_1$ consists of a perfect conductor, namely

$$n(x) \times E(x, t) = 0\,, \qquad \forall x \in \partial\Omega_1\,, \tag{2.3}$$

where n is the unit outnormal, whereas $\partial\Omega_2$ is a good (non perfect) conductor ruled by the hereditary impedance-like constitutive relation proposed by Fabrizio and Morro [5]

$$(2.4) \qquad E_\tau(x,t) = \beta_0(x)\, H(x,t) \times n(x) + \int_0^\infty \beta(x,s)\, H^t(x,s) \times n(x)\, ds\,, \qquad x \in \partial\Omega_2$$

where E_τ is the tangential component of E, β_0 and β are symmetric second-order tensors acting at the tangent bundle τ_x of $\partial\Omega_2$ in x, with (1)

$$\beta_0(x) > 0\,, \quad \beta(x,\cdot) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)\,, \quad \forall x \in \Omega\,.$$

In (2.4) H^t represents the *history* of the magnetic field up to time t, namely

$$H^t(x,s) = H(x, t-s)\,, \quad x \in \Omega,\ s \in \mathbb{R}^+\,.$$

The last equation (2.4) turns out to be an appropriate generalization to time-dependent fields of the relation proposed by Graffi and Schelkunoff [9, 15] for time-harmonic fields

$$(2.5) \qquad E_\tau(x,\omega) = \lambda(x,\omega)\, H(x,\omega) \times n(x)\,.$$

Here the *surface impedance* λ is given by

$$\lambda(x,\omega) = \lambda'(x,\omega) + i\lambda''(x,\omega)$$

where λ' and λ'' are called *surface resistance* and *reactance*, respectively. Actually, if time-harmonic fields are considered, namely

$$E(x,t) = E(x,\omega)\, e^{i\omega t}, \qquad H(x,t) = H(x,\omega)\, e^{i\omega t}$$

then (2.4) reduces to (2.5) and

$$\lambda(x,\omega) = \beta_0(x) + \widehat{\beta}(x,\omega)\,.$$

The local dissipativity of the boundary medium $\partial\Omega_2$ follows from the inequality

$$(2.6) \qquad \int_0^d E_\tau(x,t) \times H(x,t) \cdot n(x)\, dt > 0\,, \qquad x \in \partial\Omega_2$$

which must hold for any non trivial cycle of duration d. This condition implies the following thermodynamic restriction:

$$(2.7) \qquad \beta_0(x) + \beta_c(x,\omega) > 0\,, \quad \forall x \in \partial\Omega_2,\ \forall\omega \in \mathbb{R}^+\,.$$

In the sequel, whenever no ambiguity arises, the dependence on x is understood and not written.

It is worth noting that if there exist two real functions a, b such that $\beta_0 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$,

(1) For any tensor A, the notation $A > 0$ ($A \geqslant 0$) means that A is positive definite (semi-definite) on the space of the symmetric tensors Sym.

then β_0 commutes with the operator «$n\times$», namely

$$n\times(\beta_0 v)=\beta_0(n\times v)\,,\qquad \forall v\in\tau_x .$$

Of course, if the tensor β satisfies a similar assumption, then condition (2.4) can also be expressed in the form

$$n\times E(t)=-\beta_0 n\times n\times H(t)-\int_0^\infty \beta(s)\, n\times n\times H^t(s)\, ds\,. \tag{2.8}$$

With the same arguments presented in [2] it is possible to invert the boundary condition (2.8). This problem is connected to a Wiener-Hopf integral equation [17], whose solution is achieved by the crucial thermodynamic request $\beta_0+\widehat{\beta}(\omega)\neq 0$, $\forall\omega\in\mathbb{R}$.

LEMMA 2.1. – *If β satisfies* (2.7) *and $\beta_0>0$, then it is possible to invert the boundary relation* (2.8), *i.e. there exists a symmetric, causal, tensor valued function α such that*

$$n\times H(t)=\alpha_0 n\times n\times E(t)+\int_0^\infty \alpha(s)\, n\times n\times E^t(s)\, ds \tag{2.9}$$

where

$$\alpha_0=\frac{1}{\beta_0}\,,\qquad \widehat{\alpha}(\omega)=-\frac{\widehat{\beta}(\omega)}{\beta_0[\beta_0+\widehat{\beta}(\omega)]}\,,\qquad \omega\in\mathbb{R}\,. \tag{2.10}$$

As a consequence of previous thermodynamic restrictions on β and β_0, it holds

$$\alpha_0>0\,,\qquad \alpha_0+\alpha_c(\omega)>0\,,\qquad \forall\omega\in\mathbb{R}^+ . \tag{2.11}$$

Besides, in force of condition (2.7), we prove the following

LEMMA 2.2. – *If $\beta_0>0$ and $\beta_0+\beta_c(\omega)>0$ for any $\omega\neq 0$, then*

$$\beta_0+\tilde{\beta}(s)>0\,,\qquad s>0\,. \tag{2.12}$$

PROOF. – By definition of real Laplace transform and Plancherel's theorem we have:

$$\beta_0+\tilde{\beta}(s)=\beta_0+\frac{2}{\pi}\int_0^\infty (e^{-s\tau})_c(\omega)\,\beta_c(\omega)\, d\omega$$

$$=\beta_0+\frac{2}{\pi}\int_0^\infty \frac{s}{s^2+\omega^2}\,\beta_c(\omega)\, d\omega\,.$$

Taking into account the identities

$$f(\tau) = \frac{2}{\pi} \int_0^\infty f_c(\omega) \cos \omega\tau \, d\omega , \qquad f(0) = \frac{2}{\pi} \int_0^\infty f_c(\omega) \, d\omega$$

and choosing $f(\tau) = \exp(-s\tau)$, we obtain

$$f(0) = 1 = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2 + \omega^2} \, d\omega$$

so that

$$\beta_0 + \tilde{\beta}(s) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2 + \omega^2} [\beta_0 + \beta_c(\omega)] \, d\omega > 0 .$$

Therefore, in view of (2.12), we achieve

$$\alpha_0 + \tilde{\alpha}(s) > 0 , \quad s > 0 . \tag{2.13}$$

In order to give an accurate formulation to the problem, we introduce suitable functional spaces. Let

$$\mathcal{D}(\Omega) = \left\{ u \in L^2(\Omega), \int_\Omega u(x) \cdot \nabla\phi(x) \, dx = 0 \ \ \forall \phi \in C_0^\infty(\Omega) \right\}.$$

The space $\mathcal{D}(\Omega)$ is a closed subspace of $L^2(\Omega)$ and can be defined also as the closure of the set of the solenoidal vectors $u \in C_0^\infty(\Omega)$.

$$\mathcal{D}_\mu(\Omega) = \left\{ H \in L^2(\Omega), \int_\Omega \mu(x) \, H(x) \cdot \nabla\phi(x) \, dx = 0 \ \ \forall \phi \in C_0^\infty(\Omega) \right\},$$

$$\mathcal{D}_\varepsilon(\Omega) = \left\{ E \in L^2(\Omega), \int_\Omega \varepsilon(x) \, E(x) \cdot \nabla\phi(x) \, dx = 0 \ \ \forall \phi \in C_0^\infty(\Omega) \right\},$$

$$\mathcal{D}_\sigma(\Omega) = \left\{ E \in L^2(\Omega), \int_\Omega \sigma(x) \, E(x) \cdot \nabla\phi(x) \, dx = 0 \ \ \forall \phi \in C_0^\infty(\Omega) \right\},$$

$$\mathcal{D}_{\varepsilon,\sigma}(\Omega) = \left\{ E \in L^2(\Omega), \int_\Omega [\varepsilon(x) \, E(x) \cdot \nabla\phi(x) + \sigma(x) \, E(x) \cdot \nabla\psi(x)] \, dx = 0 \ \ \forall \phi, \psi \in C_0^\infty(\Omega) \right\},$$

$$\mathcal{H}(\Omega) = \{ (E, H) \in \mathcal{D}_{\varepsilon,\sigma}(\Omega) \times \mathcal{D}_\mu(\Omega) : \nabla E \in L^2(\Omega), \nabla H \in L^2(\Omega) \},$$

$$\mathfrak{J}_l(Q) = L^2_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{H}(\Omega)) \cap H^1_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{D}_{\varepsilon,\sigma}(\Omega) \times \mathcal{D}_\mu(\Omega)),$$

$$\mathfrak{J}(Q) = L^2(\mathbb{R}^+, \mathcal{H}(\Omega)) \cap H^1(\mathbb{R}^+, \mathcal{D}_{\varepsilon,\sigma}(\Omega) \times \mathcal{D}_\mu(\Omega)).$$

3. – A saddle-point principle for a quasi-static problem.

In this section we give a mini-max principle for conducting materials subject to conservative and dissipative boundary conditions. First we observe that Reiss' technique, properly modified, can be also applied to quasi-static problems by introducing fitting bilinear forms in terms of full-range Fourier transforms.

Moving along the lines of [7], let Y be a real function belonging to $L^1(\mathbb{R}^+)$ and Y_e be its even extension, i.e.:

$$Y_e(s) = \begin{cases} Y(s), & s \in \mathbb{R}^+ \\ Y(-s), & s \in \mathbb{R}^- . \end{cases}$$

We define the function y on $\mathbb{R}$ as

$$y(t) = \widehat{Y}_e(t) = \int_{-\infty}^{\infty} Y_e(s) \exp(-ist)\, ds = 2\int_0^{\infty} Y(s) \cos st\, ds . \tag{3.1}$$

It is a real, even, absolutely continuous function which vanishes as t tends to infinite. For any pair (p, q) of vector- or tensor-valued functions on $\Omega \times \mathbb{R}$ or $\partial\Omega \times \mathbb{R}$ we introduce the following bilinear forms

$$\langle p, q\rangle_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t+\tau)\int_{\Omega} p(x, t)\cdot q(x, \tau)\, dx\, dt\, d\tau , \tag{3.2}$$

$$[p, q]_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t+\tau)\int_{\partial\Omega} p(x, t)\cdot q(x, \tau)\, da\, dt\, d\tau . \tag{3.3}$$

Substituting (3.1) into (3.2) it follows that:

$$\begin{aligned} \langle p, q\rangle_y &= \int_{-\infty}^{\infty} Y_e(\omega)\int_{\Omega} \widehat{p}(x, \omega)\cdot\widehat{q}(x, \omega)\, dx\, d\omega \\ &= \int_{-\infty}^{\infty} Y_e(\omega)\, \mathrm{Re}\,(\widehat{p}(\omega), \widehat{q}(\omega))_{\Omega}\, d\omega \\ &= 2\int_0^{\infty} Y(\omega)[(p_c(\omega), q_c(\omega))_{\Omega} - (p_s(\omega), q_s(\omega))_{\Omega}]\, d\omega \end{aligned} \tag{3.4}$$

where $\mathrm{Re}\, z$ denotes the real part of the complex number z and $(\cdot, \cdot)_{\Omega}$, $(\cdot, \cdot)_{\partial\Omega}$ represent the inner products of L^2. The bilinear form (3.4) is well-defined on $L^1(\mathbb{R}, L^2(\Omega))$ whenever

$Y \in L^1(\mathbb{R}^+)$ as well as on $L^2(\mathbb{R}, L^2(\Omega))$ whenever $Y \in C_0^\infty(\mathbb{R}^+)$. Analogously, we have:

$$[p, q]_y = 2 \int_0^\infty Y(\omega)[(p_c(\omega), q_c(\omega))_{\partial\Omega} - (p_s(\omega), q_s(\omega))_{\partial\Omega}]\, d\omega . \tag{3.5}$$

It is easy to check that $\langle\cdot, \cdot\rangle_y$ and $[\cdot, \cdot]_y$ are symmetric. Letting

$$p * q(t) = \int_{-\infty}^{\infty} p(s)\, q(t-s)\, ds$$

we observe that

$$\langle p * q, r\rangle_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y(t+\tau)\left(\int_{-\infty}^{\infty} p(\tau - \sigma)\, q(\sigma)\, d\sigma,\, r(t)\right)_{\Omega} d\tau\, dt$$

$$= \int_{-\infty}^{\infty} Y_e(\omega)\, \mathrm{Re}\, (\hat{p}(\omega)\, \hat{q}(\omega), \hat{r}(\omega))_{\Omega}\, d\omega$$

and

$$[p * q, r]_y = \int_{-\infty}^{\infty} Y_e(\omega)\, \mathrm{Re}\, (\hat{p}(\omega)\, \hat{q}(\omega), \hat{r}(\omega))_{\partial\Omega}\, d\omega .$$

Now, we consider the quasi-static approximation of system (2.1), by assuming $\frac{\partial}{\partial t} D = 0$, $\frac{\partial}{\partial t} B = 0$

$$\begin{cases} \nabla \times E(x, t) = I_i(x, t) \\ \nabla \times H(x, t) = J(x, t) + J_i(x, t) . \end{cases} \tag{3.6}$$

The solenoidal condition on the electric current, namely $\nabla \cdot J = \nabla \cdot (\sigma E) = 0$ is inherent into the quasi-static approximation. Actually, it follows taking the divergence of the second equation of (3.6) and recalling $J_i \in \mathcal{D}(\Omega)$.

We associate the mixed boundary conditions:

$$n(x) \times E(x, t) = 0 , \quad x \in \partial\Omega_1 \tag{3.7$_1$}$$

$$\begin{cases} n(x) \times E(x, t) = -[\beta_0(x)\delta(x) + \beta(x, \cdot)] * n(x) \times n(x) \times H(x, t) \\ n(x) \times H(x, t) = [\alpha_0(x)\delta(x) + \alpha(x, \cdot)] * n(x) \times n(x) \times E(x, t) \end{cases}, \quad x \in \partial\Omega_2 \tag{3.7$_2$}$$

where δ is the Dirac distribution and $*$ denotes the convolution.

Let «$\nabla \times$» denote the *curl* differential operator and «$n \times$» the skew tensor Γ_n related to the unit outnormal n, namely:

$$\Gamma_n = n \times = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

so that $\Gamma_n v = n \times v$, $\Gamma_n^2 v = n \times n \times v$. If we introduce the operators:

$$R = \begin{bmatrix} 0 & \nabla \times \\ \nabla \times & 0 \end{bmatrix}; \quad N = \begin{bmatrix} 0 & n \times \\ n \times & 0 \end{bmatrix}; \quad P = \begin{bmatrix} 0 & -n \times \\ n \times & 0 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0 & -(\beta_0 \delta + \beta) * \Gamma_n \\ (\alpha_0 \delta + \alpha) * \Gamma_n & 0 \end{bmatrix}$$

a straightforward calculation proves the following properties:

$$N^T = -N; \quad P^T = P; \quad N\Lambda = -\Lambda^T N,$$

where the superscript T stands for the transposition. Then, letting

$$D = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}; \quad U = \begin{bmatrix} E \\ H \end{bmatrix}; \quad F = \begin{bmatrix} J_i \\ I_i \end{bmatrix},$$

system (3.6), (3.7) takes the form

$$(3.8) \qquad (\mathcal{P}_1) \begin{cases} RU = DU + F & \text{in } \Omega \times \mathbb{R} \\ NU = -PU & \text{on } \partial\Omega_1 \times \mathbb{R}. \\ NU = N\Lambda * U & \text{on } \partial\Omega_2 \times \mathbb{R} \end{cases}$$

DEFINITION 3.1. – A function $U = (E, H)$ is called a strong solution of the quasi-static problem $\mathcal{P}_1$ with source term F belonging to $L^2(\mathbb{R}, \mathcal{D}(\Omega) \times \mathcal{D}(\Omega))$ if U belongs to $L^2(\mathbb{R}, \mathcal{H}(\Omega))$ and satisfies almost everywhere (3.8).

As to the solvability of (3.8), we recall the existence and uniqueness results proved by Nibbi [12] under more general conditions.

THEOREM 3.1. – *Under thermodynamic restrictions* (2.7), (2.11), *for any positive function* $Y \in C_0^\infty(\mathbb{R}^+)$, U_f *is a strong solution of problem* $\mathcal{P}_1$ *if and only if* U_f *is a saddle-point on* $L^2(\mathbb{R}, \mathcal{H}(\Omega))$ *of the functional*

$$\mathcal{L}_y(U) = \frac{1}{2}\langle U, RU\rangle_y - \frac{1}{2}\langle U, DU\rangle_y - \langle U, F\rangle_y - \frac{1}{4}[U, PU]_y - \frac{1}{4}[NU, \Lambda U]_y.$$

PROOF. – Let $U_f \in L^2(\mathbb{R}, \mathcal{H}(\Omega))$ be a strong solution of $\mathcal{P}_1$ with $F \in L^2(\mathbb{R}, \mathcal{D}(\Omega) \times \mathcal{D}(\Omega))$. For all $\omega \in \mathbb{R}$, the Fourier transform $\widehat{U}_f(\omega) \in \mathcal{H}(\Omega)$ satisfies almost everywhere the problem

$$\tag{3.9} \begin{cases} R\widehat{U} = D\widehat{U} + \widehat{F} & \text{in } \Omega \times \mathbb{R} \\ N\widehat{U} = -P\widehat{U} & \text{on } \partial\Omega_1 \times \mathbb{R}\,. \\ N\widehat{U} = N\widehat{\Lambda}\widehat{U} & \text{on } \partial\Omega_2 \times \mathbb{R} \end{cases}$$

Because $\widehat{U} = U_c - iU_s$, the pair (U_c, U_s) must satisfy a.e. the coupled systems

$$(a) \quad \begin{cases} RU_c = DU_c + F_c \\ NU_c = -PU_c \\ NU_c = N\Lambda_c U_c - N\Lambda_s U_s \end{cases} \quad ; \quad (b) \quad \begin{cases} RU_s = DU_s + F_s \\ NU_s = -PU_s \\ NU_s = N\Lambda_c U_s + N\Lambda_s U_c \end{cases}$$

where

$$\begin{aligned} N\widehat{\Lambda} &= N\Lambda_c - iN\Lambda_s \\ &= \begin{bmatrix} (\alpha_0 + \alpha_c)\Gamma_n^2 & 0 \\ 0 & -(\beta_0 + \beta_c)\Gamma_n^2 \end{bmatrix} - i \begin{bmatrix} \alpha_s \Gamma_n^2 & 0 \\ 0 & -\beta_s \Gamma_n^2 \end{bmatrix}. \end{aligned}$$

Using (3.4)-(3.5), we obtain:

$$\tag{3.10} \begin{aligned} \mathcal{L}_y(U) = \int_0^\infty Y(\omega) \Big[& (U_c, RU_c)_\Omega - (U_s, RU_s)_\Omega - (U_c, DU_c)_\Omega + (U_s, DU_s)_\Omega - \\ & -2(U_c, F_c)_\Omega + 2(U_s, F_s)_\Omega - \frac{1}{2}(U_c, PU_c)_{\partial\Omega_1} + \frac{1}{2}(U_s, PU_s)_{\partial\Omega_1} + \\ & + \frac{1}{2}(U_c, N\Lambda_c U_c - N\Lambda_s U_s)_{\partial\Omega_2} - \frac{1}{2}(U_s, N\Lambda_s U_c + N\Lambda_c U_s)_{\partial\Omega_2} \Big] d\omega \end{aligned}$$

where the dependence on ω is understood and all quantities considered are real valued. Taking into account that for any pair of real vectors U, V, the following properties hold

$$\tag{Q1} (U, NV)_{\partial\Omega} = -(NU, V)_{\partial\Omega}$$

$$\tag{Q2} (U, \Lambda V)_{\partial\Omega} = (\Lambda^T U, V)_{\partial\Omega}$$

$$\tag{Q3} (NU, \Lambda V)_{\partial\Omega} = (\Lambda U, NV)_{\partial\Omega}$$

$$\tag{Q4} (U, RV)_\Omega = (V, RU)_\Omega + (U, NV)_{\partial\Omega}$$

the first variation of $\mathcal{L}_y$ can be expressed by

$$\delta\mathcal{L}_y(U) = \int_0^\infty Y(\omega)[2(\delta U_c, (R-D)U_c - F_c)_\Omega - 2(\delta U_s, (R-D)\, U_s - F_s)_\Omega -$$
$$-(\delta U_c, (N+P)U_c)_{\partial\Omega_1} + (\delta U_s, (N+P)\, U_s)_{\partial\Omega_1} - (\delta U_c,\, NU_c - N\Lambda_c U_c + N\Lambda_s U_s)_{\partial\Omega_2} +$$
$$+(\delta U_s,\, NU_s - N\Lambda_s U_c - N\Lambda_c U_s)_{\partial\Omega_2}]\, d\omega$$

and it vanishes provided that, for any $Y \in C_0^\infty(\mathbb{R}^+)$, $\widehat{U}_f$ satisfies (3.9).

The second variation of $\mathcal{L}_y$ at $\widehat{U}_f$ is given by

$$\delta^2\mathcal{L}_y(\widehat{U}_f) = \int_0^\infty Y(\omega)[2(\delta U_c, (R-D)\,\delta U_c)_\Omega - 2(\delta U_s, (R-D)\,\delta U_s)_\Omega -$$
$$-(\delta U_c,\, P\delta U_c)_{\partial\Omega_1} + (\delta U_s,\, P\delta U_s)_{\partial\Omega_1} + (\delta U_c,\, N\Lambda_c\delta U_c)_{\partial\Omega_2} - (\delta U_s,\, N\Lambda_c\delta U_s)_{\partial\Omega_2} -$$
$$-2(\delta U_c,\, N\Lambda_s\delta U_s)_{\partial\Omega_2}]\, d\omega\, .$$

Letting $W = L^2(\mathbb{R}, \mathcal{H}(\Omega))$, we can represent the space W as

$$W = W_a \oplus W_b$$

so that, for any $U \in W$, its Fourier transform $\widehat{U} \in W$ and can be written as $\widehat{U} = U_a + U_b$ where $U_a \in W_a$, $U_b \in W_b$ are as follows

$$U_a = \begin{bmatrix} E_c \\ -iH_s \end{bmatrix}, \quad U_b = \begin{bmatrix} -iE_s \\ H_c \end{bmatrix}.$$

As a consequence

$$\delta U_a = \begin{bmatrix} \delta E_c \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ \delta H_s \end{bmatrix}, \quad \delta U_b = \begin{bmatrix} 0 \\ \delta H_c \end{bmatrix} - i \begin{bmatrix} \delta E_s \\ 0 \end{bmatrix}.$$

Then, if $\delta E_s = \delta H_c = 0$, we obtain

$$\delta^2\mathcal{L}_y(U_a) = -\int_0^\infty Y(\omega)[2(\delta E_c,\, \sigma\delta E_c)_\Omega + (n\times\delta E_c, (\alpha_0+\alpha_c)\, n\times\delta E_c)_{\partial\Omega_2} + \tag{3.11}$$
$$+(n\times\delta H_s, (\beta_0+\beta_c)\, n\times\delta H_s)_{\partial\Omega_2}]\, d\omega < 0\, ,$$

while, if $\delta E_c = \delta H_s = 0$, we have

$$\delta^2\mathcal{L}_y(U_b) = \int_0^\infty Y(\omega)[2(\delta E_s,\, \sigma\delta E_s)_\Omega + (n\times\delta E_s, (\alpha_0+\alpha_c)\, n\times\delta E_s)_{\partial\Omega_2} + \tag{3.12}$$
$$+(n\times\delta H_c, (\beta_0+\beta_c)\, n\times\delta H_c)_{\partial\Omega_2}]\, d\omega > 0\, .$$

Finally, using the bijectivity of the Fourier transform from L^2 into itself, we deduce that

$U_f \in W$ is a strong solution of $\mathcal{P}_1$ if and only if U_f is a saddle-point of $\mathcal{L}_y$ on W, for every positive $Y \in C_0^\infty(\mathbb{R}^+)$. The saddle-point exists and is unique in W under thermodynamic restrictions (2.7), (2.11) by virtue of [12].

REMARK 3.1. – The previous theorem is still valid for dielectric materials (i.e. when the conductivity $\sigma = 0$), provided that the whole boundary $\partial\Omega$ satisfies the dissipative conditions $(3.7)_2$ only.

4. – Saddle-point principles for an initial-value problem.

In this section we are dealing with two variational formulations related to the evolution of materials described by the constitutive equations (2.2), and subject to mixed boundary conditions (3.7). The first is set up through the Laplace transform; other, by means of Reiss' method, through a specific bilinear form with a weight function.

In the space-time domain Q, the Maxwell equations become

$$\left\{\begin{array}{l} \nabla \times E(x,t) + \mu(x) \dfrac{\partial}{\partial t} H(x,t) = I_i(x,t) \\[2ex] \nabla \times H(x,t) - \varepsilon(x) \dfrac{\partial}{\partial t} E(x,t) = \sigma(x)\, E(x,t) + J_i(x,t) \end{array}\right. \tag{4.1}$$

with initial data

$$\left\{\begin{array}{lll} E(x,0) = E_0(x), & H(x,0) = H_0(x) & x \in \Omega \\ E_\tau(x,t) = 0\,, & H_\tau(x,t) = 0 \quad t < 0\,, & x \in \partial\Omega_2 \end{array}\right. . \tag{4.2}$$

Taking advantage of previous differential operators and setting

$$C = \begin{bmatrix} -\varepsilon & 0 \\ 0 & \mu \end{bmatrix}; \quad U(x,0) = U_0(x); \quad U_\tau(x,t) = 0, \; t < 0$$

system (4.1) becomes

$$(\mathcal{P}_2) \quad \left\{\begin{array}{ll} RU(t) + C \dfrac{\partial}{\partial t} U(t) = DU(t) + F(t) & \text{in } \Omega \times \mathbb{R}^+ \\ NU(t) = -PU(t) & \text{on } \partial\Omega_1 \times \mathbb{R}^+ \\ NU(t) = N\Lambda(t) * U(t) & \text{on } \partial\Omega_2 \times \mathbb{R}^+ \\ U(0) = U_0 & \text{in } \Omega \\ U_\tau(t) = 0 & \text{on } \partial\Omega_2 \times \mathbb{R}^- \end{array}\right. . \tag{4.3}$$

DEFINITION 4.1. – A function $U = (E, H) \in \mathfrak{J}_l(Q)$ is called a strong solution to the evolution problem $\mathcal{P}_2$, with initial data $U_0 \in \mathcal{D}_\varepsilon(\Omega) \times \mathcal{D}_\mu(\Omega)$ and source $F \in L^2_{\text{loc}}(\mathbb{R}^+, \mathcal{D}(\Omega) \times$

$\times \mathcal{D}(\Omega))$ if U satisfies (4.3) almost everywhere and

$$\lim_{t\to 0^+} \|U(x,t) - U_0(x)\|_{L^2} = 0 .$$

As to the solvability of (4.3), we recall the existence and uniqueness results proved in [5, 6].

The application of the Laplace transform to problem $\mathcal{P}_2$ leads

$$\begin{cases} R\tilde{U} + M\tilde{U} = G & \text{in } \Omega \\ N\tilde{U} = -P\tilde{U} & \text{on } \partial\Omega_1 \\ N\tilde{U} = N\tilde{\Lambda}\tilde{U} & \text{on } \partial\Omega_2 \end{cases} \tag{4.4}$$

where

$$M = zC - D = \begin{bmatrix} -(z\varepsilon + \sigma) & 0 \\ 0 & z\mu \end{bmatrix}; \quad G = \tilde{F} - CU_0 = \begin{bmatrix} \tilde{J}_i + \mu H_0 \\ \tilde{I}_i - \varepsilon E_0 \end{bmatrix}.$$

The solution $\tilde{U}_f(z)$ of (4.4) is obviously a stationary point of the functional

$$\mathcal{G}_z(U) = \frac{1}{2}(U, RU)_\Omega + \frac{1}{2}(U, MU)_\Omega - (U, G)_\Omega - \frac{1}{4}(U, PU)_{\partial\Omega_1} - \frac{1}{4}(NU, \tilde{\Lambda} U)_{\partial\Omega_2}. \tag{4.5}$$

On the other hand, by setting $\Upsilon_z(U) = \mathcal{G}_z(\tilde{U}(z))$, the functional (4.5) may be considered as parameterized by the complex number z. Therefore the solution $U_f \in \mathcal{J}_l(Q)$ to problem $\mathcal{P}_2$ is a stationary point for $\Upsilon_z(U)$, $\forall z \in \mathbb{C}^{++} = \{z \in \mathbb{C};\ \mathrm{Re}\, z > 0\}$.

THEOREM 4.1. – *Let* $\mathcal{F} = \{U \in \mathcal{J}(Q) : \tilde{U} \in \mathcal{H}(\Omega),\ \forall z \in \mathbb{C}^{++}\}$. *Under the thermodynamic restrictions* (2.12), (2.13), $U_f \in \mathcal{F}$ *is a strong solution of problem* $\mathcal{P}_2$ *if and only if* U_f *is a saddle-point of the functional* Υ_s *on* $\mathcal{F}$, *for any* $s > 0$.

PROOF. – a) If $U_f \in \mathcal{F}$ is a solution of $\mathcal{P}_2$, then $\tilde{U}_f(z)$ satisfies (4.4) and is a stationary point for $\mathcal{G}_z$, $\forall s > 0$.

Letting z be real, i.e. $z = s > 0$, the second variation of $\mathcal{G}_s$ is given by

$$\begin{aligned} \delta^2 \mathcal{G}_s = & \int_\Omega [\delta\tilde{E}\cdot\nabla \times \delta\tilde{H} + \delta\tilde{H}\cdot\nabla\times\delta\tilde{E} - \delta\tilde{E}\cdot(s\varepsilon + \sigma)\,\delta\tilde{E} + \delta\tilde{H}\cdot s\mu\delta\tilde{H}]\,dx \\ & - \int_{\partial\Omega_1} n \times \delta\tilde{E}\cdot\delta\tilde{H}\,da \\ & + \frac{1}{2}\int_{\partial\Omega_2} [(n\times\delta\tilde{H})\cdot(\beta_0 + \tilde{\beta})(n\times\delta\tilde{H}) - (n\times\delta\tilde{E})\cdot(\alpha_0 + \tilde{\alpha})(n\times\delta\tilde{E})]\,da . \end{aligned} \tag{4.6}$$

If $\delta\widetilde{H}=0$, then

$$\delta^2 \mathcal{G}_{s,\widetilde{E}} = -\int_\Omega \delta\widetilde{E}\cdot(s\varepsilon+\sigma)\,\delta\widetilde{E}\,dx - \frac{1}{2}\int_{\partial\Omega_2}(n\times\delta\widetilde{E})\cdot(\alpha_0+\tilde{\alpha})(n\times\delta\widetilde{E})\,da<0 \tag{4.7}$$

while, if $\delta\widetilde{E}=0$,

$$\delta^2 \mathcal{G}_{s,\widetilde{H}} = \int_\Omega \delta\widetilde{H}\cdot s\mu\delta\widetilde{H}\,dx + \frac{1}{2}\int_{\partial\Omega_2}(n\times\delta\widetilde{H})\cdot(\beta_0+\tilde{\beta})(n\times\delta\widetilde{H})\,da>0\,. \tag{4.8}$$

b) As a consequence of thermodynamic restrictions and definition of $\Upsilon_z(U)$, $\widetilde{U}_f(s)$ is a saddle-point for $\mathcal{G}_s$, i.e. $\delta\mathcal{G}_s=0$, $\forall s>0$ and conditions (4.7), (4.8) hold. Then, for any $s>0$, $\widetilde{U}_f(s)$ satisfies (4.4) almost everywhere and this is the unique solution in force of (2.12), (2.13). In addition, by virtue of (2.12)-(2.13), the complex operator $L_z=R+M(z)$ is strongly elliptic $\forall z\in\mathbb{C}^{++}$ [5, 6] and system (4.4) has a unique solution $V(z)\in\mathcal{H}(\Omega)$ which is analytic in z on $\mathbb{C}^{++}$ and equals to $\widetilde{U}_f$ on the real positive half-line. Then, V and $\widetilde{U}_f$ must coincide on the whole $\mathbb{C}^{++}$ [4]. Uniqueness of the Laplace transform implies that U_f is the unique solution of (4.3) in $\mathcal{F}$.

With regard to the second extremum principle, it is convenient to introduce further functional spaces, namely

$$L_L^2(\mathbb{R}^+,\mathcal{V})=\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^+,\mathcal{V}):\tilde{u}(z)\in\mathcal{V},\ z\in\mathbb{C}^{++}\}$$

with $\mathcal{V}$ a suitable Hilbert space contained in $L^2(\Omega)$, and

$$H_L^1(\mathbb{R}^+,L^2(\Omega))=\{u\in H^1_{\mathrm{loc}}(\mathbb{R}^+,L^2(\Omega)):\tilde{u}(z)\in L^2(\Omega),\ z\in\mathbb{C}^{++}\}\,,$$

$$\mathcal{S}(Q)=L_L^2(\mathbb{R}^+,\mathcal{H}(\Omega))\cap H_L^1(\mathbb{R}^+,L^2(\Omega))\}\,.$$

Consider a non negative function Y belonging to $C(\mathbb{R}^+)$ such that

$$y(t)=\int_0^\infty Y(s)\exp(-ist)\,ds \tag{4.9}$$

exists for any $t\in\mathbb{R}^+$ and define the bilinear forms $(\cdot,\cdot)_y$ and $((\cdot,\cdot))_y$ on $L_L^2(\mathbb{R}^+,L^2(\Omega))$ and $L_L^2(\mathbb{R}^+,L^2(\partial\Omega))$, respectively, as

$$\begin{aligned}(p,q)_y&=\int_0^\infty\int_0^\infty y(t+\tau)\int_\Omega p(x,t)\cdot q(x,\tau)\,dx\,dt\,d\tau\\&=\int_0^\infty Y(s)(\tilde{p}(s),\tilde{q}(s))_\Omega\,ds\end{aligned}$$

$$\begin{aligned}((p,q))_y&=\int_0^\infty\int_0^\infty y(t+\tau)\int_{\partial\Omega} p(x,t)\cdot q(x,\tau)\,da\,dt\,d\tau\\&=\int_0^\infty Y(s)(\tilde{p}(s),\tilde{q}(s))_{\partial\Omega}\,ds\,.\end{aligned}$$

They are both positive definite if Y does not vanish identically, and well-defined on L_L^2 when $Y \in C_0^\infty(\mathbb{R}^+)$.

THEOREM 4.2. – *Under thermodynamic restrictions* (2.12), (2.13), $U_f \in \mathcal{S}(Q)$ *is a strong solution to problem* $\mathcal{P}_2$ *with source* $F \in L_L^2(\mathbb{R}^+, \mathcal{D}(\Omega) \times \mathcal{D}(\Omega))$ *if and only if, for every* $Y \in C_0^\infty(\mathbb{R}^+)$, U_f *is a saddle-point on* $\mathcal{S}(Q)$ *of the functional*

$$\mathcal{F}_y(U) = \frac{1}{2}(U, RU)_y + \frac{1}{2}\left(\frac{\partial}{\partial t}U, CU\right)_y - \frac{1}{2}(U, DU)_y - (U, F)_y - \tag{4.10}$$

$$- \frac{1}{4}((U, PU))_y - \frac{1}{4}((NU, \Lambda U))_y + \frac{1}{2}\int_0^\infty y(t)\int_\Omega CU \cdot [U(0) - 2U_0]\, dx\, dt\,.$$

PROOF. – It is easy to check that the first variation of $\mathcal{F}_y$ vanishes if $\tilde{U}_f$ satisfies (4.4), for every $Y \in C_0^\infty(\mathbb{R}^+)$. The second variation of the functional (4.10) at $\tilde{U}_f$ is given by

$$\delta^2 \mathcal{F}_y(\tilde{U}_f) = \int_0^\infty Y(s)[(\delta\tilde{U}, R\delta\tilde{U})_\Omega + (\delta\tilde{U}, M(s)\delta\tilde{U})_\Omega -$$

$$- \frac{1}{2}(\delta\tilde{U}, P\delta\tilde{U})_{\partial\Omega_1} + \frac{1}{2}(\delta\tilde{U}, N\tilde{\Lambda}\delta\tilde{U})\,\partial\Omega_2]\, ds$$

so that if $\delta\tilde{H} = 0$ yields

$$\delta^2 \mathcal{F}_{y,\tilde{E}} = -\int_0^\infty Y(s)\left[\int_\Omega \delta\tilde{E}\cdot(s\varepsilon + \sigma)\,\delta\tilde{E}\, dx + \frac{1}{2}\int_{\partial\Omega_2}(n \times \delta\tilde{E})\cdot(\alpha_0 + \tilde{\alpha})(n \times \delta\tilde{E})\, da\right] ds < 0$$

while, if $\delta\tilde{E} = 0$, it follows

$$\delta^2 \mathcal{F}_{y,\tilde{H}} = \int_0^\infty Y(s)\left[\int_\Omega \delta\tilde{H}\cdot s\mu\delta\tilde{H}\, dx + \frac{1}{2}\int_{\partial\Omega_2}(n \times \delta\tilde{H})\cdot(\beta_0 + \tilde{\beta})(n \times \delta\tilde{H})\, da\right] ds > 0\,.$$

The viceversa parallels the procedure of Theorem 4.1.

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