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A three-dimensional phase transition model in ferromagnetism: Existence and uniqueness

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ABSTRACT

We scrutinize both from the physical and the analytical viewpoint the equations ruling the paramagnetic-ferromagnetic phase transition in a rigid three-dimensional body. Starting from the order structure balance, we propose a non-isothermal phase-field model which is thermodynamically consistent and accounts for variations in space and time of all fields (the temperature θ, the magnetic field vector $\mathbf{H}$ and the magnetization vector $\mathbf{M}$). In particular, we are able to establish a well-posedness result for the resulting coupled system.

1. Introduction

Ferromagnetism is a typical phenomenon occurring in metals like iron, cobalt, nickel and many alloys containing these elements. The phenomenon appears when a small external magnetic field yields a large magnetization inside the material, due to the alignment of the spin magnetic moments. Below a particular value of the temperature θ_c, called Curie temperature, the spin magnetic moments stay aligned even if the external magnetic field is removed. On the contrary, when the temperature overcomes the critical value θ_c, the residual alignment disappears and the material reverts to the paramagnetic phase.

Usually, the passage from the paramagnetic to the ferromagnetic "state" is modeled as a second order phase transition (see, for instance [5,15]). Indeed, no latent heat is released or absorbed during the change at the Curie temperature. Within this approach, we suggest and scrutinize here a phase-field model that sets the phenomenon in the general framework of the Ginzburg–Landau theory of second-order phase transitions.

Unlike some previous phase-transition models in ferromagnetics (e.g. [5,11,15]), the point of view we will adopt here identifies the components of the magnetization $\mathbf{M}$ with the set of parameters able to characterize the "amount of order" of the internal structure of the material. Since the order parameter $\mathbf{M}$ is a vector, our model takes into account not only the number of the oriented spin, but also their direction. Accordingly, the phase-field $\mathbf{M}(x,t)$ is a vector-valued rather than a scalar-valued field.

We follow here the same procedure proposed in [2,6], but accounting for phase-transition phenomena in a different setting. That is to say, the kinetic equation governing the evolution of the phase-field $\mathbf{M}(x,t)$ is deduced from a local balance law concerning the organization of the structure at (x,t). Within this scheme, the relation between the magnetization $\mathbf{M}$ and the magnetic field $\mathbf{H}$ takes the form of a vector-valued Ginzburg–Landau equation, namely

$$\gamma\dot{\mathbf{M}} = \nu\Delta\mathbf{M} - \theta_c F'(\mathbf{M}) - \theta G'(\mathbf{M}) + \mathbf{H}, \tag{1.1}$$

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where γ, ν are positive constants, and F, G are scalar-valued (possibly anisotropic) functions. In particular, γ is responsible of the dissipation (as apparent from (6.3)) and, at the same time, it is in inverse proportion to the gyroscopic inertia, as we shall show later in this section. Exploiting the formal similarity of (1.1) with the much more well-known second-order phase transition model of superconductivity (see e.g. [1,18]), we assume that

(i) at saturation $|\mathbf{M}| = 1$;
(ii) F and G are even polynomial forms involving $\mathbf{M}$;
(iii) the sum $F + G$ has a global minimum at $\mathbf{M} = 0$.

The first item simply means that we identify $\mathbf{M}$ with the rescaled dimensionless field $\mathbf{M}/M_s$, where M_s is the saturation magnetization. According to conditions (ii) and (iii), in anisotropic ferromagnetic transitions we are allowed to choose

$$F(\mathbf{M}) = \frac{1}{4}(\mathbb{F}\mathbf{M} \otimes \mathbf{M}) \cdot \mathbf{M} \otimes \mathbf{M} - \frac{1}{2}\mathcal{F} \cdot (\mathbf{M} \otimes \mathbf{M}), \qquad G(\mathbf{M}) = \frac{1}{2}\mathcal{F} \cdot (\mathbf{M} \otimes \mathbf{M}),$$

where $\mathbb{F}$ and $\mathcal{F}$ are fourth-order and second-order positive definite tensors, respectively, and the dot denotes a scalar product between both vector and second-order tensors. A similar expression has been introduced in [9], where a dyadic order parameter is used in order to model the isotropic-nematic phase transition occurring in liquid crystals.

As we shall see later, the magnetic potential $W = \theta_c F + \theta G$ enters the free energy functional, and its critical points represent the magnetization (phase) equilibria at $\mathbf{H} = 0$.

When isotropic materials are concerned, as well as it is assumed throughout the paper, then $\mathbb{F}$ and $\mathcal{F}$ reduce (to within a constant factor) to the identity tensor of the corresponding order. Accordingly

$$F(\mathbf{M}) = \frac{1}{4}|\mathbf{M}|^4 - \frac{1}{2}|\mathbf{M}|^2, \qquad G(\mathbf{M}) = \frac{1}{2}|\mathbf{M}|^2 \tag{1.2}$$

so that

$$F'(\mathbf{M}) = \big(|\mathbf{M}|^2 - 1\big)\mathbf{M}, \qquad G'(\mathbf{M}) = \mathbf{M}. \tag{1.3}$$

Therefore (1.1) becomes

$$\gamma\dot{\mathbf{M}} = \nu\Delta\mathbf{M} - \theta_c|\mathbf{M}|^2\mathbf{M} - (\theta - \theta_c)\mathbf{M} + \mathbf{H}. \tag{1.4}$$

In order to complete the dynamical system, this equation needs to be coupled with a suitable local representation of the heat equation ruling the evolution of the absolute temperature field $\theta(x,t)$ and Maxwell's equations governing the changes of $\mathbf{H}(x,t)$.

Under assumptions (1.2), it is worth noting that

$$W = W\big(|\mathbf{M}|, \theta\big) = \frac{1}{4}\theta_c|\mathbf{M}|^4 + \frac{1}{2}(\theta - \theta_c)|\mathbf{M}|^2.$$

This expression of the magnetic potential, which leads to (1.4), traces back to Landau and coworkers (see [15, p. 137]) and has a global minimum at $\mathbf{M} = 0$ for all $\theta \geqslant \theta_c$, but, due to the vector character of $\mathbf{M}$, it has infinitely many minima at any fixed temperature $\theta < \theta_c$, that is all vectors $\mathbf{M}$ such that

$$|\mathbf{M}| = \frac{\theta_c - \theta}{\theta_c} < 1.$$

This occurs in isotropic ferromagnets since the magnetization vector $\mathbf{M}$ has no preferred (easy) direction at equilibrium. A typical such material is an amorphous (non-crystalline) ferromagnetic-metallic alloy. On the contrary, the general case accounting for anisotropy exhibits a finite set of minima. Although isotropy is a crude assumption, in that most of the ferromagnetic materials have a crystalline structure, here we confine our attention to Eq. (1.4). The more general (and realistic) anisotropic case will be addressed in a forthcoming work.

The main goal of this paper consists in describing the three-dimensional evolution of both thermodynamic and electromagnetic properties of the material as a whole. Allowing both spatial nonuniform behavior and diffusion, the resulting differential system provided by our model allows to account for temperature induced transitions from paramagnetic to ferromagnetic regime, and vice versa. In the following, we show the well-posedness of the related initial–boundary value problem, by proving both the existence and the uniqueness theorem for weak solutions in the same functional class.

To the best of our knowledge, this approach to the paramagnetic–ferromagnetic transition is a novelty. Indeed, most of the works in the literature scrutinize either the turning movement of $\mathbf{M}$ at saturation, that is when $|\mathbf{M}| = 1$ (see, for instance [12,15,17] and references therein), or the evolution of its component M along a fixed axis r, which occurs when both $\mathbf{M}$ and $\mathbf{H}$ point in the same direction r at all times $t > 0$ (see, for instance [13,19] and references therein). In spite of its general vector form, Eq. (1.1) works better in the latter occurrence, for instance when the material is uniaxial and the magnetic field $\mathbf{H}$ is aligned with the axis of easy magnetization r, namely $\mathbf{M} = Mr$ and $\mathbf{H} = Hr$. If this is the case,we can replace vectors by components. From (1.1), the evolution equation for the (scalar) magnetization field $M(x,t)$ takes then the form

$$\gamma\dot{M} = \nu\Delta M - \theta_c F'(M) - \theta G'(M) + H.$$

This is a widely accepted expression for the kinetics of M. The Weiss model, for instance, can be easily recovered from this equation by letting

$$F'(M) = -\frac{\beta}{\theta_c} M, \qquad G'(M) = \alpha \mathcal{L}^{-1}(M),$$

where $\mathcal{L}$ is the so-called Langevin function (see e.g. [13,19]). On the other hand, in the saturation occurrence, the model we suggest here looks very poor if compared with well-known theories on micromagnetics. As apparent, neither the Landau and Lifshitz [14] nor Gilbert [12] models can be recovered directly from (1.4). In fact, when $|\mathbf{M}| = 1$ it reads

$$\gamma \dot{\mathbf{M}} = \nu \Delta \mathbf{M} - \theta \mathbf{M} + \mathbf{H}. \tag{1.5}$$

Actually, this vector equation contains much more informations than the corresponding scalar one. Under quite restrictive assumptions, indeed, an equivalent evolution equation, which looks like the Gilbert one, can be derived. By applying $\tau \mathbf{M} \times$, $\tau > 0$, to (1.5) and then adding the result to (1.5) itself, after a division by $\tau\gamma$ we obtain

$$\frac{1}{\tau} \dot{\mathbf{M}} = -\frac{\theta}{\tau\gamma} \mathbf{M} + \frac{1}{\tau\gamma} (\mathbf{H} + \nu \Delta \mathbf{M}) + \frac{1}{\gamma} \mathbf{M} \times (\mathbf{H} + \nu \Delta \mathbf{M}) - \mathbf{M} \times \dot{\mathbf{M}}.$$

The parameter τ is dimensionless and some possible meaning will be proposed in a while. Now, assuming that $1/\gamma$ and $1/\tau$ are of the same order with respect to a small parameter ε, all terms containing the factor $1/\tau\gamma$ are negligible, so that we recover the approximate equation

$$\dot{\mathbf{M}} = \frac{\tau}{\gamma} \mathbf{M} \times (\mathbf{H} + \nu \Delta \mathbf{M}) - \tau \mathbf{M} \times \dot{\mathbf{M}}.$$

Now, if we compare it with the Gilbert equation (remember that $M_s = 1$)

$$\dot{\mathbf{M}} = \lambda \mathbf{M} \times \mathbf{H}_{\mathrm{eff}} - \alpha \mathbf{M} \times \dot{\mathbf{M}},$$

we can establish a strong similarity by letting $\mathbf{H}_{\mathrm{eff}} = \mathbf{H} + \nu \Delta \mathbf{M}$ (as usual in isotropic materials), $\lambda = \tau/\gamma$ and $\alpha = \tau$, which is the (dimensionless) Gilbert damping constant. In addition, by this comparison we may infer that

$$\gamma = \frac{\alpha}{1 + \alpha^2} \frac{1}{\delta},$$

where δ is the gyromagnetic ratio. As a consequence, γ is proportional to α and vanishes with it.

The plan of the paper is the following. In Section 2, we introduce the differential equations governing the evolution of the ferromagnetic material and prove the thermodynamic consistence of the model. Since the differential system involves a logarithmic non-linearity, in Section 3, we regularize the problem with the technique of the Yosida approximation, obtaining a family of differential problems (P_ε). Such a regularization allows to prove, in Section 4, existence of solutions to problem (P_ε) for every $\varepsilon > 0$. Then, by means of uniform estimates in ε, in Section 5 the solutions to (P_ε) are proved to converge to solutions of the original differential system as $\varepsilon \to 0$. Finally, uniqueness of solutions is proved in Section 6, by means of a continuous dependence result.

2. The model

Let us consider a rigid ferromagnetic conductor occupying a domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. Moreover we suppose that its mass density is constant and we let $\rho = 1$, for simplicity. Denoting by $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}$ the electric field, the magnetic field, the electric displacement and the magnetic induction, the behavior of the material is ruled by Maxwell's equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \qquad \nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}, \tag{2.1a}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = \rho_e, \tag{2.1b}$$

where $\mathbf{J}$ is the current density and ρ_e is the free charge density. Throughout the paper, we assume the electromagnetic isotropy of the material and the following constitutive equations

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H} + \mathbf{M}, \qquad \mathbf{J} = \sigma \mathbf{E}, \tag{2.2}$$

where ε, μ, σ are respectively the dielectric constant, the magnetic permeability and the conductivity, and $\mathbf{M}$ is the magnetization vector.

As well-known, in paramagnetic materials $\mathbf{M}$ is a function of $\mathbf{H}$. On the contrary, in a ferromagnetic body $\mathbf{M}$ is uniquely determined by $\mathbf{H}$ only for large values of this field, namely $|\mathbf{H}| > H_c$, where H_c is named *coercive field*. Here, our goal is to specify the law relating the variation of the magnetization $\mathbf{M}$ to the magnetic field $\mathbf{H}$, in such a way that the thermally-induced change between the paramagnetic and the ferromagnetic state is captured.

First, we observe that the order of the internal structure for a ferromagnetic material is characterized by both the position and the direction of the magnetic spins. Thus, we are allowed to define the internal structure order $\mathbf{K}^i(\mathcal{A})$ of a sub-body $\mathcal{A} \subset \Omega$ as the vector-valued measure whose representation is

$$\mathbf{K}^i(\mathcal{A}) = \int_{\mathcal{A}} \mathbf{k}\,dx,$$

where the vector $\mathbf{k}$, called *specific internal structure order*, accounts for the internal order of the spins determined by their orientation. On the other hand, we assume the external structure order vector $\mathbf{K}^e(\mathcal{A})$ to have the form

$$\mathbf{K}^e(\mathcal{A}) = \int_{\partial\mathcal{A}} \mathbf{P}\mathbf{n}_{\mathcal{A}}\,ds + \int_{\mathcal{A}} \boldsymbol{\sigma}\,dx,$$

where $\partial\mathcal{A}$ denotes the boundary of $\mathcal{A}$ and $\mathbf{n}_{\mathcal{A}}$ is its unit outward normal vector. Here $\mathbf{P}$ is a second order tensor such that $\mathbf{P}\mathbf{n}$ provides the specific flux of the structure order through the boundary and $\boldsymbol{\sigma}$ is the structure order supply. In other words, the tensor $\mathbf{P}(x,t)$ describes the distribution of the internal order in a neighborhood of x at the instant t, while $\boldsymbol{\sigma}$ represents a source of internal order inside the domain Ω. In the sequel we let $\boldsymbol{\sigma} = \mathbf{0}$.

The structure order balance on every sub-body $\mathcal{A} \subset \Omega$ states $\mathbf{K}^i(\mathcal{A}) = \mathbf{K}^e(\mathcal{A})$ and then

$$\int_{\mathcal{A}} \mathbf{k}\,dx = \int_{\partial\mathcal{A}} \mathbf{P}\mathbf{n}_{\mathcal{A}}\,ds,$$

from which we obtain a.e. in Ω the local equation

$$\mathbf{k} = \nabla \cdot \mathbf{P}. \tag{2.3}$$

Since the transition between paramagnetism and ferromagnetism is a typical second order phase transition, we suggest the following constitutive equations for $\mathbf{k}$ and $\mathbf{P}$

$$\mathbf{k} = \gamma\dot{\mathbf{M}} + \theta_c F'(\mathbf{M}) + \theta G'(\mathbf{M}) - \mathbf{H}, \tag{2.4a}$$

$$\mathbf{P} = \nu\nabla\mathbf{M}, \tag{2.4b}$$

where γ, ν are positive constants, $\theta > 0$ is the absolute temperature, and θ_c is the (critical) Curie temperature.

As suggested in the previous section, functions F, G are defined by (1.2). Accordingly (2.3) reduces to

$$\gamma\dot{\mathbf{M}} = \nu\Delta\mathbf{M} - \theta_c\big(|\mathbf{M}|^2 - 1\big)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}. \tag{2.5}$$

By following the general scheme proposed in [2,6], Eq. (2.3) has to be regarded as a field equation which is able to yield a power (and then an energy variation) accounting for the internal structure order of the material. Indeed, in view of (2.4b), by multiplying (2.3) by $\dot{\mathbf{M}}$, we obtain

$$\mathbf{k}\cdot\dot{\mathbf{M}} + \nu\nabla\mathbf{M}\cdot\nabla\dot{\mathbf{M}} = \nu\nabla\cdot\big(\nabla\mathbf{M}^T\dot{\mathbf{M}}\big),$$

where the superscript T denotes the transpose of a tensor and we have denoted with the same symbol $\cdot$ the scalar product between vectors and tensors. The latter is defined as $\mathbf{A}\cdot\mathbf{B} = \sum_{i,j} A_{ij}B_{ij}$.

The previous identity suggests to split the power into the internal and external contributions as follows

$$\mathcal{P}^i_{\mathbf{M}} = \mathbf{k}\cdot\dot{\mathbf{M}} + \nu\nabla\mathbf{M}\cdot\nabla\dot{\mathbf{M}},$$

$$\mathcal{P}^e_{\mathbf{M}} = \nu\nabla\cdot\big(\nabla\mathbf{M}^T\dot{\mathbf{M}}\big).$$

By means of (2.4a) and (1.3) the internal power takes the form

$$\mathcal{P}^i_{\mathbf{M}} = \gamma|\dot{\mathbf{M}}|^2 + \nu\nabla\mathbf{M}\cdot\nabla\dot{\mathbf{M}} + \big[\theta_c\big(|\mathbf{M}|^2 - 1\big) + \theta\big]\mathbf{M}\cdot\dot{\mathbf{M}} - \mathbf{H}\cdot\dot{\mathbf{M}}. \tag{2.6}$$

In order to provide a coherent model which is able to include both thermal and electromagnetic effects, it is essential to obtain the representation of the heat equation pertinent to this context. As known, the thermal balance law is expressed by the following equation

$$h = -\nabla\cdot\mathbf{q} + r, \tag{2.7}$$

where h is the rate at which heat is absorbed, $\mathbf{q}$ is the heat flux vector and r is the heat source.

By introducing the internal energy e, from the first law of thermodynamics, we deduce

$$\dot{e} = \mathcal{P}_{el} + \mathcal{P}^i_{\mathbf{M}} + h, \tag{2.8}$$

where $\mathcal{P}_{el}$ is the electromagnetic power defined as

$$\mathcal{P}_{el} = \dot{\mathbf{D}}\cdot\mathbf{E} + \dot{\mathbf{B}}\cdot\mathbf{H} + \mathbf{J}\cdot\mathbf{E} = \varepsilon\dot{\mathbf{E}}\cdot\mathbf{E} + \mu\dot{\mathbf{H}}\cdot\mathbf{H} + \dot{\mathbf{M}}\cdot\mathbf{H} + \sigma|\mathbf{E}|^2. \tag{2.9}$$

Now, we choose the following expression of the internal energy

$$e = c(\theta) + \frac{1}{2}\varepsilon|\mathbf{E}|^2 + \frac{1}{2}\mu|\mathbf{H}|^2 + \frac{\theta_c}{4}\big(|\mathbf{M}|^2 - 1\big)^2 + \frac{\nu}{2}|\nabla\mathbf{M}|^2. \tag{2.10}$$

It takes into account the purely thermal contribution, $c(\theta)$, the purely electromagnetic part, $(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2)/2$, and the energy interface density, $\nu|\nabla\mathbf{M}|^2/2$. Substitution into (2.8) yields

$$h = c'(\theta)\dot\theta - \sigma|\mathbf{E}|^2 - \gamma|\dot{\mathbf{M}}|^2 - \theta\mathbf{M}\cdot\dot{\mathbf{M}},$$

from which the evolution equation for the temperature can be obtained by means of (2.7) and the constitutive equation

$$\mathbf{q} = -k(\theta)\nabla\theta, \tag{2.11}$$

where $k(\theta) > 0$ is the thermal conductivity. Hence we deduce

$$c'(\theta)\dot\theta - \sigma|\mathbf{E}|^2 - \gamma|\dot{\mathbf{M}}|^2 - \theta\mathbf{M}\cdot\dot{\mathbf{M}} = \nabla\cdot\big(k(\theta)\nabla\theta\big) + r. \tag{2.12}$$

We are in a position to prove that our model is consistent with the second law of thermodynamics written in the Clausius–Duhem form, namely

$$\dot\eta \geqslant -\nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) + \frac{r}{\theta},$$

where η is the entropy density. Since the thermal balance law (2.7) yields

$$\theta\dot\eta \geqslant \frac{\mathbf{q}}{\theta}\cdot\nabla\theta + h,$$

after introducing the free energy $\psi = e - \theta\eta$, the previous inequality and (2.8) lead to

$$\dot\psi + \eta\dot\theta + \frac{\mathbf{q}}{\theta}\cdot\nabla\theta - \mathcal{P}_{el} - \mathcal{P}^i_{\mathbf{M}} \leqslant 0,$$

and substituting the expressions of the powers given by (2.6) and (2.9) we obtain

$$\dot\psi + \eta\dot\theta + \frac{\mathbf{q}}{\theta}\cdot\nabla\theta - \varepsilon\dot{\mathbf{E}}\cdot\mathbf{E} - \mu\dot{\mathbf{H}}\cdot\mathbf{H} - \sigma|\mathbf{E}|^2 - \gamma|\dot{\mathbf{M}}|^2 - \nu\nabla\mathbf{M}\cdot\nabla\dot{\mathbf{M}} - \big[\theta_c\big(|\mathbf{M}|^2 - 1\big) + \theta\big]\mathbf{M}\cdot\dot{\mathbf{M}} \leqslant 0. \tag{2.13}$$

This inequality enforces the following choices

$$\psi = \frac{\mu}{2}|\mathbf{H}|^2 + \frac{\varepsilon}{2}|\mathbf{E}|^2 + \frac{\nu}{2}|\nabla\mathbf{M}|^2 + \frac{\theta_c}{4}\big(|\mathbf{M}|^2 - 1\big)^2 + \frac{1}{2}\theta|\mathbf{M}|^2 + \alpha(\theta), \tag{2.14a}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} = -\frac{1}{2}|\mathbf{M}|^2 - \alpha'(\theta), \tag{2.14b}$$

which agree with (2.10) if α satisfies the relation

$$\alpha(\theta) - \theta\alpha'(\theta) = c(\theta).$$

Substitution of (2.11), (2.14a), (2.14b) into (2.13) provides the reduced inequality

$$-\frac{k(\theta)}{\theta}|\nabla\theta|^2 - \sigma|\mathbf{E}|^2 - \gamma|\dot{\mathbf{M}}|^2 \leqslant 0,$$

which holds along any process in view of the assumption $k(\theta) \geqslant 0$ and of the positiveness of the absolute temperature. Then the thermodynamical consistence of the model is proved.

In the wide mathematical literature related to phase transition phenomena, a great variety of assumptions about heat conductivity and specific heat is depicted (see, for instance, [2,3,5,7,10]). In this model, we suppose that heat conductivity and specific heat depend on the absolute temperature according the polynomial laws

$$k(\theta) = k_0 + k_1\theta, \qquad c(\theta) = c_1\theta + \frac{c_2}{2}\theta^2, \tag{2.15}$$

with $k_0, k_1, c_1, c_2 > 0$. In addition, we restrict our attention to processes for which the fields $\mathbf{E}$, $\dot{\mathbf{M}}$, $\nabla\theta$ are small enough so that the quadratic terms

$$-\sigma|\mathbf{E}|^2 - \gamma|\dot{\mathbf{M}}|^2 - k_1|\nabla\theta|^2$$

are negligible if compared to other contributions in (2.12). Within this approximation scheme, the energy balance reduces to

$$(c_1 + c_2\theta)\dot\theta - \theta\mathbf{M}\cdot\dot{\mathbf{M}} = (k_0 + k_1\theta)\Delta\theta + r.$$

After dividing by θ, we obtain

$$c_1\partial_t(\ln\theta) + c_2\dot{\theta} - \mathbf{M}\cdot\dot{\mathbf{M}} = \frac{k_0}{\theta}\Delta\theta + k_1\Delta\theta + \hat{r},$$

where $\hat{r} = r/\theta$. Therefore, by ignoring once more the term proportional to $|\nabla\theta|^2$, we obtain

$$c_1\partial_t(\ln\theta) + c_2\dot{\theta} - \mathbf{M}\cdot\dot{\mathbf{M}} = k_0\Delta(\ln\theta) + k_1\Delta\theta + \hat{r}. \tag{2.16}$$

A further simplification can be introduced if we neglect the displacement current $\varepsilon\dot{\mathbf{E}}$. This is a customary assumption in describing ferromagnetic phenomena. As a consequence, from (2.1a)–(2.2) we deduce

$$\mu\dot{\mathbf{H}} + \dot{\mathbf{M}} = -\frac{1}{\sigma}\nabla\times\nabla\times\mathbf{H}, \tag{2.17a}$$

$$\nabla\cdot(\mu\mathbf{H} + \mathbf{M}) = 0. \tag{2.17b}$$

3. The differential system

According to the previous section, the system governing the evolution of the ferromagnetic material reads

$$\gamma\dot{\mathbf{M}} = \nu\Delta\mathbf{M} - \theta_c\big(|\mathbf{M}|^2 - 1\big)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}, \tag{3.1a}$$

$$c_1\partial_t(\ln\theta) + c_2\dot{\theta} - \mathbf{M}\cdot\dot{\mathbf{M}} = k_0\Delta(\ln\theta) + k_1\Delta\theta + \hat{r}, \tag{3.1b}$$

$$\mu\dot{\mathbf{H}} + \dot{\mathbf{M}} = -\frac{1}{\sigma}\nabla\times\nabla\times\mathbf{H}, \tag{3.1c}$$

$$\nabla\cdot(\mu\mathbf{H} + \mathbf{M}) = 0. \tag{3.1d}$$

For the sake of simplicity, we assume here that $\hat{r}$ is a known function of x, t.

In order to prove existence and uniqueness results, Eqs. (3.1a)–(3.1d) have to be fulfilled with initial and boundary conditions. Concerning the boundary conditions of the magnetization, we assume

$$\nabla\mathbf{M}\mathbf{n}\big|_{\partial\Omega} = \mathbf{0}, \tag{3.2a}$$

which is a typical condition in phase transitions. The same Neumann boundary condition is assumed for the temperature, i.e.

$$\nabla\theta\cdot\mathbf{n}\big|_{\partial\Omega} = 0, \tag{3.2b}$$

and the magnetic field is required to satisfy

$$(\nabla\times\mathbf{H})\times\mathbf{n}\big|_{\partial\Omega} = 0. \tag{3.2c}$$

Finally, let the initial data

$$\mathbf{M}(x,0) = \mathbf{M}_0(x), \qquad \theta(x,0) = \theta_0(x), \qquad \mathbf{H}(x,0) = \mathbf{H}_0(x) \tag{3.3}$$

be given functions in Ω. We now recall the following result

Remark. (See [8].) In view of $(2.1\text{a})_1$ and (3.1d), if we impose the following constraints on the initial data

$$(\mu\mathbf{H}_0 + \mathbf{M}_0)\cdot\mathbf{n}\big|_{\partial\Omega} = 0, \qquad \nabla\cdot(\mu\mathbf{H}_0 + \mathbf{M}_0) = 0, \tag{3.4}$$

then at any subsequent $t > 0$

$$(\mu\mathbf{H} + \mathbf{M})\cdot\mathbf{n}\big|_{\partial\Omega} = 0, \tag{3.5}$$

$$\nabla\cdot(\mu\mathbf{H} + \mathbf{M}) = 0, \quad \text{a.e. in } \Omega. \tag{3.6}$$

Here we introduce some notation. For any Hilbert space X let $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$ denote the X-inner product and X-norm, respectively. In particular, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ stand for the inner product and norm in $L^2(\Omega)$. Moreover we let X' be the dual space of X.

Then the functional formulation of the problem is the following

Problem (P). To find a triplet $(\mathbf{M}, \theta, \mathbf{H})$ such that

$$\mathbf{M} \in L^2\big(0,T,H^2(\Omega)\big) \cap H^1\big(0,T,L^2(\Omega)\big),$$

$$\theta \in L^2\big(0,T,H^1(\Omega)\big), \quad \theta > 0, \qquad \ln\theta \in L^2\big(0,T,H^1(\Omega)\big),$$

$$c_1\ln\theta + c_2\theta \in H^1\big(0,T,H^1(\Omega)'\big),$$

$$\mathbf{H} \in L^2\big(0,T,H^1(\Omega)\big) \cap H^1\big(0,T,H^1(\Omega)'\big)$$

satisfying (3.2a)–(3.5) and

$$\gamma\dot{\mathbf{M}} - \nu\Delta\mathbf{M} + \big[\theta_c\big(|\mathbf{M}|^2 - 1\big) + \theta\big]\mathbf{M} - \mathbf{H} = 0, \quad \text{a.e. in } \Omega, \tag{3.7a}$$

$$\int_\Omega \big[c_1\partial_t(\ln\theta)\omega + c_2\dot{\theta}\omega - \mathbf{M}\cdot\dot{\mathbf{M}}\omega + k_0\nabla(\ln\theta)\cdot\nabla\omega + k_1\nabla\theta\cdot\nabla\omega - \hat{r}\omega\big]\,dx = 0, \tag{3.7b}$$

$$\int_\Omega \left[\mu\dot{\mathbf{H}}\cdot\mathbf{w} + \dot{\mathbf{M}}\cdot\mathbf{w} + \frac{1}{\sigma}(\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{w})\right]dx = 0, \tag{3.7c}$$

$$\mu\nabla\cdot\mathbf{H} + \nabla\cdot\mathbf{M} = 0, \quad \text{a.e. in } \Omega \tag{3.7d}$$

for any $\omega, \mathbf{w} \in H^1(\Omega)$, a.e. $t \in (0,T)$.

Well-posedness of problem (P) is ensured by the following theorems, whose proof will be performed in Sections 5 and 6.

Theorem 3.1. *Let $\hat{r} \in L^2(0,T,H^1(\Omega)')$ and $\mathbf{M}_0 \in H^1(\Omega)$, $\theta_0, \mathbf{H}_0 \in L^2(\Omega)$, such that* (3.4) *hold. For every $T > 0$, problem (P) admits a solution $(\mathbf{M},\theta,\mathbf{H})$ satisfying* (3.7a)–(3.7c).

Theorem 3.2. *Let $(\mathbf{M}_1,\theta_1,\mathbf{H}_1)$ and $(\mathbf{M}_2,\theta_2,\mathbf{H}_2)$ be two solutions of problem (P), with sources $\hat{r}_1, \hat{r}_2 \in L^2(0,T,H^1(\Omega)')$ and initial data $(\mathbf{M}_{01},\theta_{01},\mathbf{H}_{01})$, $(\mathbf{M}_{02},\theta_{02},\mathbf{H}_{02}) \in H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ satisfying* (3.4). *Then, for each $T > 0$, there exists a positive constant $C(T)$ such that*

$$\int_0^T \big(\|\mathbf{M}_1 - \mathbf{M}_2\|_{H^1}^2 + \|\ln\theta_1 - \ln\theta_2\|^2 + \|\theta_1 - \theta_2\|^2 + \|\mathbf{H}_1 - \mathbf{H}_2\|^2\big)\,d\tau$$

$$\leqslant C(T)\left[\|\mathbf{M}_{01} - \mathbf{M}_{02}\|_{H^1}^2 + \|\ln\theta_{01} - \ln\theta_{02}\|^2 + \|\theta_{01} - \theta_{02}\|^2 + \|\mathbf{H}_{01} - \mathbf{H}_{02}\|^2 + \int_0^T \|\hat{r}_1 - \hat{r}_2\|^2\,d\tau\right].$$

In particular, the solution of problem (P) is unique.

Existence of solutions to problem (P) is proved by introducing a suitable approximation of the logarithmic nonlinearities. More precisely, we denote by $\ln_\varepsilon$ the Yosida approximation of the logarithm function (see e.g. [4]) and consider the following problem:

Problem (P_ε). To find a triplet $(\mathbf{M}_\varepsilon,\theta_\varepsilon,\mathbf{H}_\varepsilon)$ such that

$$\mathbf{M}_\varepsilon \in L^2\big(0,T,H^2(\Omega)\big) \cap H^1\big(0,T,L^2(\Omega)\big),$$
$$\theta_\varepsilon \in L^2\big(0,T,H^1(\Omega)\big), \qquad \ln_\varepsilon\theta_\varepsilon \in L^2\big(0,T,H^1(\Omega)\big),$$
$$c_1\ln_\varepsilon\theta_\varepsilon + c_2\theta_\varepsilon \in H^1\big(0,T,H^1(\Omega)'\big),$$
$$\mathbf{H}_\varepsilon \in L^2\big(0,T,H^1(\Omega)\big) \cap H^1\big(0,T,H^1(\Omega)'\big),$$

satisfying (3.2a)–(3.2c) and (3.5), the equations

$$\gamma\dot{\mathbf{M}}_\varepsilon - \nu\Delta\mathbf{M}_\varepsilon + \big[\theta_c\big(|\mathbf{M}_\varepsilon|^2 - 1\big) + \theta\big]\mathbf{M}_\varepsilon - \mathbf{H}_\varepsilon = 0, \quad \text{a.e. in } \Omega, \tag{3.8a}$$

$$\int_\Omega \big[c_1\partial_t(\ln_\varepsilon\theta_\varepsilon)\omega + c_2\dot{\theta}_\varepsilon\omega - \mathbf{M}_\varepsilon\cdot\dot{\mathbf{M}}_\varepsilon\omega + k_0\nabla(\ln_\varepsilon\theta_\varepsilon)\cdot\nabla\omega + k_1\nabla\theta_\varepsilon\cdot\nabla\omega - \hat{r}\omega\big]\,dx = 0, \tag{3.8b}$$

$$\int_\Omega \left[\mu\dot{\mathbf{H}}_\varepsilon\cdot\mathbf{w} + \dot{\mathbf{M}}_\varepsilon\cdot\mathbf{w} + \frac{1}{\sigma}(\nabla\times\mathbf{H}_\varepsilon)\cdot(\nabla\times\mathbf{w})\right]dx = 0, \tag{3.8c}$$

$$\mu\nabla\cdot\mathbf{H}_\varepsilon + \nabla\cdot\mathbf{M}_\varepsilon = 0, \quad \text{a.e. in } \Omega \tag{3.8d}$$

for any $\omega, \mathbf{w} \in H^1(\Omega)$, a.e. $t \in (0,T)$, and the initial conditions

$$\mathbf{M}_\varepsilon(x,0) = \mathbf{M}_0(x), \qquad \theta_\varepsilon(x,0) = \theta_0(x), \qquad \mathbf{H}_\varepsilon(x,0) = \mathbf{H}_0(x), \tag{3.8e}$$

a.e. in Ω.

The advantage of this procedure stands in the regularizing properties of the Yosida approximation, which we recall here for convenience. The Yosida regularization of the logarithm function is defined as

$$\ln_\varepsilon \tau = \frac{\tau - \rho_\varepsilon(\tau)}{\varepsilon}, \quad \tau \in \mathbb{R},$$

where $\rho_\varepsilon(\tau)$ is the unique solution of equation

$$\rho_\varepsilon(\tau) + \varepsilon \ln \rho_\varepsilon(\tau) = \tau.$$

The function $\ln_\varepsilon$ is Lipschitz continuous with constant $1/\varepsilon$ [4, Proposition 2.6]. Moreover it is easy to check that $\ln_\varepsilon$ is C^∞ and it satisfies

$$0 < \ln'_\varepsilon \tau \leqslant \frac{2}{\varepsilon}, \quad \tau \in \mathbb{R}, \tag{3.9a}$$

$$|\ln_\varepsilon \tau| \leqslant |\ln \tau|, \quad \tau > 0. \tag{3.9b}$$

For later convenience, we introduce the function

$$I_\varepsilon(\tau) = \int_0^\tau s \ln'_\varepsilon(s)\, ds. \tag{3.10}$$

In view of (3.9a), I_ε satisfies

$$I_\varepsilon(\tau) \geqslant 0, \quad \tau \in \mathbb{R}.$$

4. Existence of solutions to (P_ε)

In order to establish the existence of solutions to problem (P_ε) for every $\varepsilon > 0$, we prove the following

Theorem 4.1. *Let $\hat r \in L^2(0, T, H^1(\Omega)')$ and $\mathbf{M}_0 \in H^1(\Omega)$, $\theta_0, \mathbf{H}_0 \in L^2(\Omega)$, such that* (3.4) *hold. For every $\varepsilon > 0$ and $T > 0$, problem (P_ε) admits a solution $(\mathbf{M}_\varepsilon, \theta_\varepsilon, \mathbf{H}_\varepsilon)$ satisfying* (3.8a)–(3.8e)*.*

Proof. The proof is based on the Galerkin procedure. For every fixed $n \in \mathbb{N}$, let us consider the increasing sequences V_n^1, V_n^2, V_n^3 of n-dimensional subspaces of $H^1(\Omega)$ such that each set $\bigcup_{n\in\mathbb{N}} V_n^k$, $k = 1, 2, 3$, is dense in $H^1(\Omega)$ and

$$V_n^1 \subset V^1 = \{\mathbf{v} \in H^2(\Omega),\ \nabla \mathbf{v}\mathbf{n}|_{\partial\Omega} = \mathbf{0}\},$$
$$V_n^2 \subset V^2 = \{\phi \in H^2(\Omega),\ \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$
$$V_n^3 \subset V^3 = \{\mathbf{u} \in H^2(\Omega),\ (\nabla \times \mathbf{v}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}\}.$$

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $\{\phi_1, \dots, \phi_n\}$, $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be the orthogonal bases of V_n^1, V_n^2, V_n^3, respectively, and define the approximated solution $(\mathbf{M}_n, \theta_n, \mathbf{H}_n) \in V_n^1 \times V_n^2 \times V_n^3$ as

$$\mathbf{M}_n(x,t) = \sum_{i=1}^n \alpha_{in}(t)\mathbf{v}_i(x),$$
$$\theta_n(x,t) = \sum_{i=1}^n \beta_{in}(t)\phi_i(x),$$
$$\mathbf{H}_n(x,t) = \sum_{i=1}^n \gamma_{in}(t)\mathbf{u}_i(x),$$

a.e. in $\Omega \times (0,T)$, where $\alpha_{in}, \beta_{in}, \gamma_{in}$ satisfy

$$\int_\Omega \{\gamma \dot{\mathbf{M}}_n \cdot \mathbf{v}_i + \nu \nabla \mathbf{M}_n \cdot \nabla \mathbf{v}_i + [\theta_c(|\mathbf{M}_n|^2 - 1) + \theta_n]\mathbf{M}_n \cdot \mathbf{v}_i - \mathbf{H}_n \cdot \mathbf{v}_i\}\, dx = 0, \tag{4.1a}$$

$$\int_\Omega [(c_1 \ln'_\varepsilon \theta_n + c_2)\dot\theta_n \phi_i - \mathbf{M}_n \cdot \dot{\mathbf{M}}_n \phi_i + (k_0 \ln'_\varepsilon \theta_n + k_1)\nabla\theta_n \cdot \nabla\phi_i - \hat r \phi_i]\, dx = 0, \tag{4.1b}$$

$$\int_\Omega \left(\mu \dot{\mathbf{H}}_n \cdot \mathbf{u}_i + \dot{\mathbf{M}}_n \cdot \mathbf{u}_i + \frac{1}{\sigma}\nabla \times \mathbf{H}_n \cdot \nabla \times \mathbf{u}_i\right) dx = 0, \tag{4.1c}$$

for any $i = 1, \dots, n$. Moreover, as $n \to \infty$ the initial data are supposed to verify

$$\mathbf{M}_n(\cdot,0) \to \mathbf{M}_0 \quad \text{in } H^1(\Omega),$$
$$\theta_n(\cdot,0) \to \theta_0 \quad \text{in } L^2(\Omega),$$
$$\mathbf{H}_n(\cdot,0) \to \mathbf{H}_0 \quad \text{in } L^2(\Omega).$$

Inequality (3.9a) leads to

$$c_1 \ln'_\varepsilon \theta_n + c_2 > c_2 > 0.$$

Hence (4.1a)–(4.1c) is a system of ordinary differential equations which can be put into normal form. Accordingly, there exists a unique local solution $(\mathbf{M}_n, \theta_n, \mathbf{H}_n)$ for every $n \in \mathbb{N}$.

We prove now that a suitable subsequence of $(\mathbf{M}_n, \theta_n, \mathbf{H}_n)$ approaches a solution $(\mathbf{M}_\varepsilon, \theta_\varepsilon, \mathbf{H}_\varepsilon)$ to problem (P_ε) as $n \to \infty$. To do this we perform some a priori uniform (with respect to n) estimates which allow the passage to the limit into (4.1a)–(4.1c) *via* a compactness argument. The same estimates prove that the maximal solution to problem (P_ε) is a global solution defined in the time interval $(0, T)$ for every $T > 0$.

First, let us multiply (4.1a), (4.1b), (4.1c) respectively by $\dot\alpha_{in}, \beta_{in}, \gamma_{in}$ and sum for $i = 1, \dots, n$ the resulting equations. We obtain

$$\frac{1}{2}\frac{d}{dt}\left(\nu\|\nabla\mathbf{M}_n\|^2 + \frac{\theta_c}{2}\|\mathbf{M}_n\|_4^4 + c_2\|\theta_n\|^2 + \mu\|\mathbf{H}_n\|^2\right) + \gamma\|\dot{\mathbf{M}}_n\|^2 + c_1\int_\Omega \dot\theta_n\theta_n \ln'_\varepsilon\theta_n\,dx + k_1\|\nabla\theta_n\|^2 + \frac{1}{\sigma}\|\nabla\times\mathbf{H}_n\|^2$$
$$+ k_0\int_\Omega \ln'_\varepsilon\theta_n|\nabla\theta_n|^2\,dx = \int_\Omega(\theta_c\mathbf{M}_n\cdot\dot{\mathbf{M}}_n + \hat r\theta_n)\,dx. \tag{4.2}$$

By recalling the definition (3.10), the first integral can be written as

$$\int_\Omega \dot\theta_n\theta_n\ln'_\varepsilon\theta_n\,dx = \frac{d}{dt}\int_\Omega I_\varepsilon(\theta_n)\,dx.$$

Moreover, by means of the Young inequality, the right-hand side of (4.2) can be estimated as

$$\int_\Omega(\theta_c\mathbf{M}_n\cdot\dot{\mathbf{M}}_n + \hat r\theta_n)\,dx \leqslant \frac{\gamma}{2}\|\dot{\mathbf{M}}_n\|^2 + \frac{\theta_c^2}{2\gamma}\|\mathbf{M}_n\|^2 + \frac{1}{2k_1}\|\hat r\|^2_{H^1(\Omega)'} + \frac{k_1}{2}\|\theta_n\|^2_{H^1}$$
$$\leqslant \frac{\gamma}{2}\|\dot{\mathbf{M}}_n\|^2 + \frac{k_1}{2}\|\nabla\theta_n\|^2 + C\big(1 + \|\mathbf{M}_n\|_4^4 + \|\hat r\|^2_{H^1(\Omega)'} + \|\theta_n\|^2\big), \tag{4.3}$$

where C is a suitable positive constant.

By substituting into (4.2) and using (3.9a), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\nu\|\nabla\mathbf{M}_n\|^2 + \frac{\theta_c}{2}\|\mathbf{M}_n\|_4^4 + c_2\|\theta_n\|^2 + \mu\|\mathbf{H}_n\|^2 + c_1\int_\Omega I_\varepsilon(\theta_n)\,dx\right) + \frac{\gamma}{2}\|\dot{\mathbf{M}}_n\|^2 + \frac{k_1}{2}\|\nabla\theta_n\|^2 + \frac{1}{\sigma}\|\nabla\times\mathbf{H}_n\|^2$$
$$\leqslant C\big(1 + \|\mathbf{M}_n\|_4^4 + \|\hat r\|^2_{H^1(\Omega)'} + \|\theta_n\|^2\big). \tag{4.4}$$

Therefore, Gronwall's inequality yields

$$\|\mathbf{M}_n\|^2_{H^1} + \|\theta_n\|^2 + \|\mathbf{H}_n\|^2 \leqslant C_0 \tag{4.5}$$

and

$$\int_0^t \big(\|\dot{\mathbf{M}}_n\|^2 + \|\nabla\theta_n\|^2 + \|\nabla\times\mathbf{H}_n\|^2\big)\,d\tau \leqslant C_0, \tag{4.6}$$

where C_0 is a positive constant depending on the norms $\|\mathbf{M}_0\|_{H^1}$, $\|\theta_0\|$, $\|\mathbf{H}_0\|$, $\|\hat r\|_{L^2(0,T,H^1(\Omega)')}$ and on ε, T. Moreover, from (4.1a)-(4.1c), we deduce

$$\int_0^t \|\Delta\mathbf{M}_n\|^2\,d\tau \leqslant C\int_0^t\big[\|\dot{\mathbf{M}}_n\|^2 + \|\mathbf{M}_n\|^2_{H^1}\big(\|\mathbf{M}_n\|^4_{H^1} + 1 + \|\theta_n\|^2_{H^1}\big) + \|\mathbf{H}_n\|^2\big]\,d\tau,$$
$$\int_0^t \big\|\partial_t(c_1\ln_\varepsilon\theta_n + c_2\theta_n)\big\|^2_{H^1(\Omega)'}\,d\tau \leqslant C_\varepsilon\int_0^t\big(\|\mathbf{M}_n\|^2_{H^1}\|\dot{\mathbf{M}}_n\|^2 + \|\nabla\theta_n\|^2 + \|\hat r\|^2_{H^1(\Omega)'}\big)\,d\tau,$$
$$\int_0^t \|\dot{\mathbf{H}}_n\|^2_{H^1(\Omega)'} \leqslant C\int_0^t\big(\|\dot{\mathbf{M}}_n\|^2 + \|\nabla\times\mathbf{H}_n\|^2\big)\,d\tau.$$

Therefore, in view of (4.5)–(4.6), we obtain the estimate

$$\int_0^t \left[\|\Delta\mathbf{M}_n\|^2 + \|\partial_t(c_1 \ln_\varepsilon \theta_n + c_2\theta_n)\|^2_{H^1(\Omega)'} + \|\dot{\mathbf{H}}_n\|^2_{H^1(\Omega)'}\right] d\tau \leqslant C_0. \tag{4.7}$$

Inequalities (4.5)–(4.7) ensure the existence of a subsequence, denoted also $(\mathbf{M}_n, \theta_n, \mathbf{H}_n)$ such that

$$\mathbf{M}_n \to \mathbf{M}_\varepsilon \quad \text{weakly in } L^2\big(0,t,H^2(\Omega)\big) \cap H^1\big(0,t,L^2(\Omega)\big), \tag{4.8a}$$

$$\theta_n \to \theta_\varepsilon \quad \text{weakly in } L^2\big(0,t,H^1(\Omega)\big), \tag{4.8b}$$

$$\mathbf{H}_n \to \mathbf{H}_\varepsilon \quad \text{weakly in } L^2\big(0,t,L^2(\Omega)\big) \cap H^1\big(0,t,H^1(\Omega)'\big), \tag{4.8c}$$

$$\nabla\times\mathbf{H}_n \to \nabla\times\mathbf{H}_\varepsilon \quad \text{weakly in } L^2\big(0,t,L^2(\Omega)\big). \tag{4.8d}$$

In particular, owing to (3.9a), (4.8b) yields

$$\ln_\varepsilon \theta_n \to \ell_\varepsilon \quad \text{weakly in } L^2\big(0,t,H^1(\Omega)\big),$$

which in view of (4.7) implies

$$c_1 \ln_\varepsilon \theta_n + c_2\theta_n \to c_1\ell_\varepsilon + c_2\theta_\varepsilon \quad \text{weakly in } L^2\big(0,t,H^1(\Omega)\big) \cap H^1\big(0,t,H^1(\Omega)'\big). \tag{4.9}$$

By means of a compactness argument, from (4.6), (4.7), we deduce

$$\mathbf{M}_n \to \mathbf{M}_\varepsilon \quad \text{strongly in } L^2\big(0,t,H^1(\Omega)\big) \cap C\big(0,t,L^2(\Omega)\big),$$

$$c_1 \ln_\varepsilon \theta_n + c_2\theta_n \to c_1\ell_\varepsilon + c_2\theta_\varepsilon \quad \text{strongly in } L^2\big(0,t,L^2(\Omega)\big).$$

Accordingly

$$\int_\Omega (c_1 \ln_\varepsilon \theta_n + c_2\theta_n)\theta_n\, dx \to \int_\Omega (c_1\ell_\varepsilon + c_2\theta_\varepsilon)\theta_\varepsilon\, dx,$$

which proves that (see [4, Proposition 2.5])

$$\ell_\varepsilon = \ln_\varepsilon \theta_\varepsilon.$$

Moreover (see [16, p. 12])

$$\big(\mathbf{M}_n^2 - 1\big)\mathbf{M}_n \to \big(\mathbf{M}_\varepsilon^2 - 1\big)\mathbf{M}_\varepsilon \quad \text{weakly in } L^2\big(0,T,L^2(\Omega)\big).$$

The previous convergences allow to take the limit as $n\to\infty$ into (4.1a)–(4.1c) and to prove that $(\mathbf{M}_\varepsilon, \theta_\varepsilon, \mathbf{H}_\varepsilon)$ satisfies (3.8a)–(3.8c).

By integrating (3.8c) over $(0,t)$ we obtain

$$\int_\Omega (\mu\mathbf{H}_\varepsilon + \mathbf{M}_\varepsilon)\cdot\mathbf{w}\, dx + \int_0^t \int_\Omega \nabla\times\mathbf{H}_\varepsilon \cdot \nabla\times\mathbf{w}\, dx\, d\tau = \int_\Omega (\mu\mathbf{H}_0 + \mathbf{M}_0)\cdot\mathbf{w}\, dx.$$

Letting $\mathbf{w} = \nabla\chi$, $\chi \in H^2(\Omega)$, an integration by parts yields

$$\int_{\partial\Omega} \chi(\mu\mathbf{H}_\varepsilon + \mathbf{M}_\varepsilon)\cdot\mathbf{n}\, ds - \int_\Omega \chi\nabla\cdot(\mu\mathbf{H}_\varepsilon + \mathbf{M}_\varepsilon)\, dx = \int_{\partial\Omega} \chi(\mu\mathbf{H}_0 + \mathbf{M}_0)\cdot\mathbf{n}\, ds - \int_\Omega \chi\nabla\cdot(\mu\mathbf{H}_0 + \mathbf{M}_0)\, dx.$$

In view of (3.4) and the arbitrariness of χ, we prove (3.8d) and (3.5).

Finally, relations (3.8d), (4.8c) and (4.8d) imply that

$$\mathbf{H}_n \to \mathbf{H}_\varepsilon \quad \text{weakly in } L^2\big(0,t,H^1(\Omega)\big) \cap H^1\big(0,t,H^1(\Omega)'\big).$$

Therefore $(\mathbf{M}_\varepsilon, \theta_\varepsilon, \mathbf{H}_\varepsilon)$ is a solution to problem (P_ε). □

5. Proof of Theorem 3.1

In this section we prove that a solution $(\mathbf{M}_\varepsilon, \theta_\varepsilon, \mathbf{H}_\varepsilon)$ of problem (P_ε) converges to a solution $(\mathbf{M}, \theta, \mathbf{H})$ of problem (P) as $\varepsilon \to 0$. In order to pass to the limit we need some uniform estimates in ε. To this aim, we test (3.8a) with $\dot{\mathbf{M}}_\varepsilon$, (3.8b) with θ_ε, (3.8c) with $\mathbf{H}_\varepsilon$. Adding the resulting equation we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\nu\|\nabla \mathbf{M}_\varepsilon\|^2 + c_1\int_\Omega I_\varepsilon(\theta_\varepsilon)\,dx + c_2\|\theta_\varepsilon\|^2 + \mu\|\mathbf{H}_\varepsilon\|^2 + \frac{\theta_c}{2}\|\mathbf{M}_\varepsilon\|_4^4\right]$$
$$+ \gamma\|\dot{\mathbf{M}}_\varepsilon\|^2 + k_1\|\nabla\theta_\varepsilon\|^2 + \frac{1}{\sigma}\|\nabla\times\mathbf{H}_\varepsilon\|^2 + k_0\int_\Omega \ln_\varepsilon'(\theta_\varepsilon)|\nabla\theta_\varepsilon|^2\,dx$$
$$= \int_\Omega (\theta_c\mathbf{M}_\varepsilon\cdot\dot{\mathbf{M}}_\varepsilon + \hat{r}\theta_\varepsilon)\,dx. \tag{5.1}$$

Using the same argument of the previous section (see Eq. (4.3)), but emphasizing the dependence on ε, we prove

$$\|\mathbf{M}_\varepsilon\|_{H^1}^2 + \|\theta_\varepsilon\|^2 + \|\mathbf{H}_\varepsilon\|^2 + \int_0^t \left(\|\dot{\mathbf{M}}_\varepsilon\|^2 + \|\nabla\theta_\varepsilon\|^2 + \|\nabla\times\mathbf{H}_\varepsilon\|^2\right)d\tau \leqslant C_0 + \int_\Omega \left|I_\varepsilon(\theta_0)\right|dx.$$

In order to obtain an estimate independent on ε, we consider the identity

$$I_\varepsilon(\tau) = \tau\ln_\varepsilon(\tau) - \int_0^\tau \ln_\varepsilon(s)\,ds, \quad \tau\in\mathbb{R}.$$

Hence (3.9b) leads to

$$\left|I_\varepsilon(\theta_0)\right| \leqslant \theta_0\ln_\varepsilon\theta_0 + \int_0^{\theta_0} |\ln_\varepsilon s|\,ds \leqslant 2\theta_0^2,$$

which implies

$$\int_\Omega \left|I_\varepsilon(\theta_0)\right|dx \leqslant 2\|\theta_0\|^2.$$

Thus

$$\|\mathbf{M}_\varepsilon\|_{H^1}^2 + \|\theta_\varepsilon\|^2 + \|\mathbf{H}_\varepsilon\|^2 + \int_0^t \left(\|\dot{\mathbf{M}}_\varepsilon\|^2 + \|\nabla\theta_\varepsilon\|^2 + \|\nabla\times\mathbf{H}_\varepsilon\|^2\right)d\tau \leqslant C_0. \tag{5.2}$$

Now let us test (3.8b) by $c_1\ln_\varepsilon(\theta_\varepsilon) + c_2\theta_\varepsilon$ and integrate over Ω. We deduce

$$\frac{1}{2}\frac{d}{dt}\|c_1\ln_\varepsilon\theta_\varepsilon + c_2\theta_\varepsilon\|^2 + k_0c_1\left\|\nabla(\ln_\varepsilon\theta_\varepsilon)\right\|^2 + k_1c_2\|\nabla\theta_\varepsilon\|^2$$
$$= \int_\Omega \left[(c_1\ln_\varepsilon\theta_\varepsilon + c_2\theta_\varepsilon)(\mathbf{M}_\varepsilon\cdot\dot{\mathbf{M}}_\varepsilon + \hat{r}) - (k_0c_2 + k_1c_1)\nabla(\ln_\varepsilon\theta_\varepsilon)\cdot\nabla\theta_\varepsilon\right]dx.$$

By means of Hölder's and Young's inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|c_1\ln_\varepsilon\theta_\varepsilon + c_2\theta_\varepsilon\|^2 + \frac{k_0c_1}{2}\left\|\nabla(\ln_\varepsilon\theta_\varepsilon)\right\|^2 + k_1c_2\|\nabla\theta_\varepsilon\|^2$$
$$\leqslant \lambda\left(\|c_1\ln_\varepsilon(\theta_\varepsilon) + c_2\theta_\varepsilon\|^2 + \|\mathbf{M}_\varepsilon\|_{H^1}^2\|\dot{\mathbf{M}}_\varepsilon\|^2 + \|\hat{r}\|_{H^1(\Omega)'}^2 + \|\nabla\theta_\varepsilon\|^2\right), \tag{5.3}$$

where λ is a positive constant.

Moreover, in view of (3.9b), we have

$$\|c_1\ln_\varepsilon\theta_0 + c_2\theta_0\| \leqslant c_1\|\ln\theta_0\| + c_2\|\theta_0\|.$$

By recalling (5.2), from Gronwall's inequality, we obtain

$$\|c_1\ln_\varepsilon\theta_\varepsilon + c_2\theta_\varepsilon\| \leqslant C_0.$$

Finally, from (5.3) we deduce

$$\int_0^t \left\|\nabla(\ln_\varepsilon\theta_\varepsilon)\right\|^2 dt \leqslant C_0$$

and comparison with (3.8b) yields

$$\int_0^t \|c_1\partial_t \ln_\varepsilon \theta_\varepsilon + c_2\dot{\theta}_\varepsilon\|^2_{H^1(\Omega)'}\,dt \leqslant C_0.$$

The a priori estimates proved in the previous section allow to pass to the limit as $\varepsilon \to 0$ and to obtain existence of a solution to problem (P). Thus, the proof of Theorem 3.1 is complete.

Remark. The a priori estimate (5.2) holds true even for the solutions $(\mathbf{M}, \theta, \mathbf{H})$ to problem (P). Accordingly $(\mathbf{M}, \theta, \mathbf{H})$ satisfies

$$\|\mathbf{M}\|^2_{H^1} + \|\theta\|^2 + \|\mathbf{H}\|^2 + \int_0^t \big(\|\dot{\mathbf{M}}\|^2 + \|\nabla\theta\|^2 + \|\nabla\times\mathbf{H}\|^2\big)\,d\tau \leqslant C_0. \tag{5.4}$$

6. Proof of Theorem 3.2

In order to prove uniqueness of solutions to problem (P), let $(\mathbf{M}_i, \theta_i, \mathbf{H}_i)$, $i = 1, 2$, be two solutions corresponding to the data $\hat{r}_i, \mathbf{M}_{0i}, \theta_{0i}, \mathbf{H}_{0i}$, $i = 1, 2$, satisfying (3.4), respectively. We introduce the differences

$$\mathbf{M} = \mathbf{M}_1 - \mathbf{M}_2, \qquad \theta = \theta_1 - \theta_2, \qquad \xi = \ln\theta_1 - \ln\theta_2, \qquad \mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2, \qquad \hat{r} = \hat{r}_1 - \hat{r}_2,$$

which solve the differential problem

$$\gamma\dot{\mathbf{M}} = \nu\Delta\mathbf{M} - \theta_c\big(|\mathbf{M}_1|^2 - 1\big)\mathbf{M}_1 + \theta_c\big(|\mathbf{M}_2|^2 - 1\big)\mathbf{M}_2 - \theta_1\mathbf{M}_1 + \theta_2\mathbf{M}_2 + \mathbf{H}, \tag{6.1a}$$

$$c_1\dot{\xi} + c_2\dot{\theta} = \mathbf{M}_1\cdot\dot{\mathbf{M}}_1 - \mathbf{M}_2\cdot\dot{\mathbf{M}}_2 + k_0\Delta\xi + k_1\Delta\theta + \hat{r}, \tag{6.1b}$$

$$\mu\dot{\mathbf{H}} + \dot{\mathbf{M}} = -\frac{1}{\sigma}\nabla\times\nabla\times\mathbf{H} \tag{6.1c}$$

with boundary conditions

$$\nabla\mathbf{M}\,\mathbf{n}\big|_{\partial\Omega} = \mathbf{0}, \qquad \nabla\theta\cdot\mathbf{n}\big|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{H})\times\mathbf{n}\big|_{\partial\Omega} = \mathbf{0}, \tag{6.1d}$$

and initial data

$$\mathbf{M}(x,0) = \mathbf{M}_{01}(x) - \mathbf{M}_{02}(x) = \mathbf{M}_0(x),$$
$$\theta(x,0) = \theta_{01}(x) - \theta_{02}(x) = \theta_0(x),$$
$$\mathbf{H}(x,0) = \mathbf{H}_{01}(x) - \mathbf{H}_{02}(x) = \mathbf{H}_0(x)$$

satisfying (3.4).

By integrating (6.1b) and (6.1c) over $(0, t)$, we obtain

$$c_1\xi + c_2\theta = \frac{1}{2}\big(|\mathbf{M}_1|^2 - |\mathbf{M}_2|^2\big) + \int_0^t [k_0\Delta\xi + k_1\Delta\theta + \hat{r}]\,d\tau + c_1\xi_0 + c_2\theta_0 - \frac{1}{2}\big(|\mathbf{M}_{01}|^2 - |\mathbf{M}_{02}|^2\big), \tag{6.2a}$$

$$\mu\mathbf{H} + \mathbf{M} + \frac{1}{\sigma}\int_0^t \nabla\times\nabla\times\mathbf{H}\,d\tau = \mu\mathbf{H}_0 + \mathbf{M}_0. \tag{6.2b}$$

Let us multiply (6.1a) by $\mathbf{M}$ and (6.2b) by $\mathbf{H}$. Adding the resulting equations and integrating over Ω, we deduce

$$\frac{1}{2}\frac{d}{dt}\left(\gamma\|\mathbf{M}\|^2 + \frac{1}{\sigma}\left\|\int_0^t \nabla\times\mathbf{H}\,d\tau\right\|^2\right) + \nu\|\nabla\mathbf{M}\|^2 + \mu\|\mathbf{H}\|^2 \leqslant \big(\mu\|\mathbf{H}_0\| + \|\mathbf{M}_0\|\big)\|\mathbf{H}\| + I_1, \tag{6.3}$$

where

$$I_1 = \int_\Omega \big[-\theta_c\big(|\mathbf{M}_1|^2 - 1\big)\mathbf{M}_1 + \theta_c\big(|\mathbf{M}_2|^2 - 1\big)\mathbf{M}_2 - \theta_1\mathbf{M}_1 + \theta_2\mathbf{M}_2\big]\cdot\mathbf{M}\,dx.$$

The terms in I_1 can be arranged as

$$I_1 = \int_\Omega \big\{\big[\theta_c\big(1 - |\mathbf{M}_1|^2\big) - \theta_1\big]\mathbf{M} - \theta_c\big[(\mathbf{M}_1 + \mathbf{M}_2)\cdot\mathbf{M}\big]\mathbf{M}_2 - \mathbf{M}_2\theta\big\}\cdot\mathbf{M}\,dx.$$

Hence, owing to (5.4), and the embedding inequality

$$\|\mathbf{w}\|_\infty \leqslant C\|\mathbf{w}\|_{H^2}, \quad \mathbf{w} \in H^2(\Omega),$$

we have

$$\begin{aligned} I_1 &\leqslant C\big[\big(1 + \|\theta_1\|_{H^1(\Omega)}\big)\|\mathbf{M}\|_{H^1} + \|\mathbf{M}_2\|_{H^2}\|\theta\|\big]\|\mathbf{M}\| \\ &\leqslant \delta\|\mathbf{M}\|^2_{H^1} + C\big(1 + \|\theta_1\|^2_{H^1} + \|\mathbf{M}_2\|^2_{H^2}\big)\|\mathbf{M}\|^2 + \delta\|\theta\|^2, \end{aligned} \tag{6.4}$$

for every $\delta > 0$.

Then, by multiplying (6.2a) by $k_0\xi + k_1\theta$ and integrating over Ω, we obtain

$$\int_\Omega (c_1\xi + c_2\theta)(k_0\xi + k_1\theta)\,dx + \frac{1}{2}\frac{d}{dt}\left\| \int_0^t (k_0\nabla\xi + k_1\nabla\theta)\,d\tau \right\|^2 = I_2, \tag{6.5}$$

where

$$I_2 = \int_\Omega \left[\frac{1}{2}\mathbf{M}\cdot(\mathbf{M}_1 + \mathbf{M}_2) + \int_0^t \hat{r}\,d\tau + c_1\xi_0 + c_2\theta_0 - \frac{1}{2}\big(|\mathbf{M}_{01}|^2 - |\mathbf{M}_{02}|^2\big)\right](k_0\xi + k_1\theta)\,dx.$$

The integral I_2 can be estimated as

$$I_2 \leqslant C\left[\big(\|\mathbf{M}_1\|_{H^2} + \|\mathbf{M}_2\|_{H^2}\big)\|\mathbf{M}\| + \left\|\int_0^t \hat{r}\,d\tau\right\| + \|\xi_0\| + \|\theta_0\| + \|\mathbf{M}_0\|\right]\|k_0\xi + k_1\theta\|.$$

Therefore, by applying the Young inequality, we prove

$$I_2 \leqslant \frac{1}{2}\big(k_0c_1\|\xi\|^2 + k_1c_2\|\theta\|^2\big) + C\left[\big(\|\mathbf{M}_1\|^2_{H^2} + \|\mathbf{M}_2\|^2_{H^2}\big)\|\mathbf{M}\|^2 + \int_0^t \|\hat{r}\|^2\,d\tau + \|\xi_0\|^2 + \|\theta_0\|^2 + \|\mathbf{M}_0\|^2\right]. \tag{6.6}$$

By substituting (6.4), (6.6) into (6.3), (6.5), adding the resulting equations and choosing properly δ, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\gamma\|\mathbf{M}\|^2 + \frac{1}{\sigma}\left\|\int_0^t \nabla\times\mathbf{H}\,d\tau\right\|^2 + \left\|\int_0^t (k_0\nabla\xi + k_1\nabla\theta)\,d\tau\right\|^2\right) \\ &\quad + \frac{1}{2}\big(\nu\|\nabla\mathbf{M}\|^2 + \mu\|\mathbf{H}\|^2 + c_1k_0\|\xi\|^2 + c_2k_1\|\theta\|^2\big) + \int_\Omega (c_1k_1 + c_2k_0)\xi\theta\,dx \\ &\quad \leqslant \zeta(t)\|\mathbf{M}\|^2 + C\big(\|\mathbf{M}_0\|^2 + \|\mathbf{H}_0\|^2 + \|\xi_0\|^2 + \|\theta_0\|^2\big) + C\int_0^t \|\hat{r}\|^2\,d\tau, \end{aligned} \tag{6.7}$$

where $\zeta \in L^1(0,t)$.

Accordingly, Gronwall's inequality yields

$$\|\mathbf{M}\|^2 \leqslant C(T)\left[\|\mathbf{M}_0\|^2 + \|\mathbf{H}_0\|^2 + \|\xi_0\|^2 + \|\theta_0\|^2 + \int_0^T \|\hat{r}\|^2\,d\tau\right]$$

and

$$\int_0^T \big(\|\mathbf{H}\|^2 + \|\mathbf{M}\|^2_{H^1} + \|\xi\|^2 + \|\theta\|^2\big)\,d\tau \leqslant C(T)\left[\|\mathbf{M}_0\|^2 + \|\mathbf{H}_0\|^2 + \|\xi_0\|^2 + \|\theta_0\|^2 + \int_0^T \|\hat{r}\|^2\,d\tau\right].$$

This completes the proof of Theorem 3.2.

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