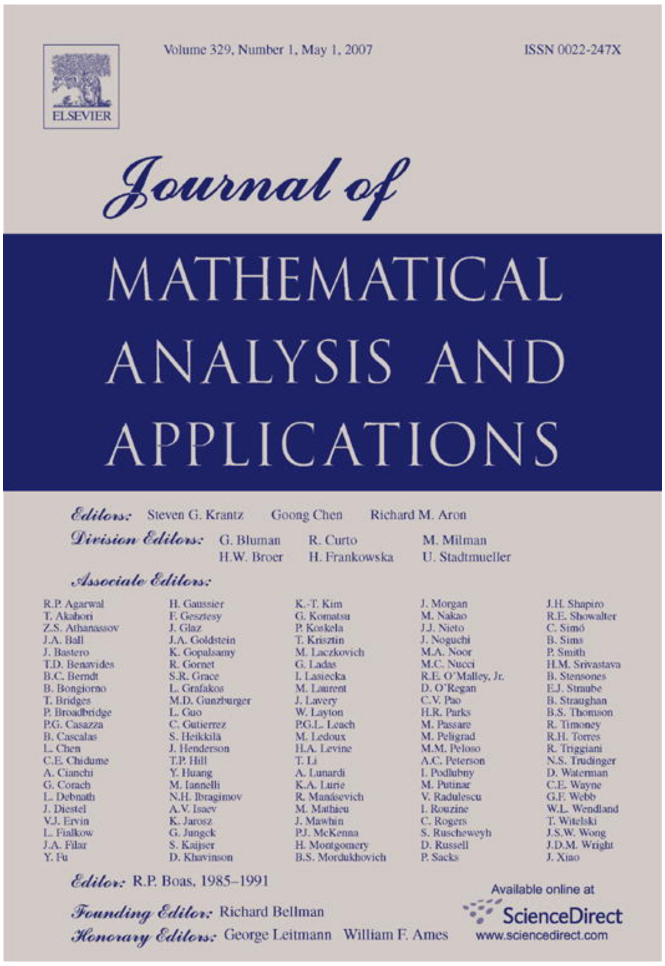
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Gauge invariance and asymptotic behavior for the Ginzburg–Landau equations of superconductivity ☆

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Abstract

In this paper we study the gauge invariance of the time-dependent Ginzburg–Landau equations through the introduction of a model which uses observable variables. We observe that the various choices of gauge lead to a different representation of such variables and therefore to a different definition of the weak solution of the problem. With a suitable decomposition of the unknown fields, related to the choice of London gauge, we examine the Ginzburg–Landau equations and deduce some energy estimates which prove the existence of a maximal attractor for the system.

Keywords: Superconductivity; Gauge invariance; Global attractor

1. Introduction

This paper has two different aims. In the first part we examine the gauge invariance of the time-dependent Ginzburg–Landau equations (also called Gor'kov–Èliashberg equations [6,11]), which describe the behavior of a superconductor during the phase transition between the normal and the superconducting state. As already pointed out by several authors [2,7], such equations are invariant up to a gauge transformation and the invariance of the model means that the physical

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problem cannot be affected by the particular choice of the gauge. This property certainly holds in the class of regular solutions, however when we consider the weak solutions, we observe that different choices of the gauge lead to different weak formulations, since the functional spaces in which the problem is set, depend on the choice of the gauge.

Therefore, in this paper, we prove the existence of the global attractor for the time-dependent Ginzburg–Landau equations in the London gauge, although the long-time behavior has been studied also in [10] with the Lorentz gauge.

The plan of the paper is the following. In Section 2 we present a gauge-invariant model by writing the time-dependent Ginzburg–Landau equations by means of observable variables. Moreover we introduce a decomposition of the velocity of superconducting electrons and observe that the choice of the gauge in the classical formulations is equivalent to the choice of a particular decomposition. In Section 3, we recall the theorem of existence and uniqueness of the solution proved in [12] with the choice of London gauge. Finally in Section 4, we deduce some energy estimates which allow to prove the existence of the global attractor. The estimates are established for the system obtained by means of the decomposition of the observable variables, which is equivalent to the classical Gor'kov–Èliashberg system. Our method differs from the technique used in [12], where the authors study the asymptotic behavior of the solutions without making use of energy estimates.

2. Superconductivity and gauge invariance of the Ginzburg–Landau equations

The main property of a superconductor is the complete disappearance of the electrical resistivity at some low critical temperature T_c, which is characteristic of the material. However, there exists a second effect which is equally meaningful. This phenomenon, called Meissner effect, is the perfect diamagnetism. In other words, the magnetic field is expelled from the superconductor, independently of whether the field is applied in the superconductive state (zero-field-cooled) or already in the normal state (field-cooled).

In the London theory [8,9] and in the paper [4] it is assumed that the supercurrent $\mathbf{J}_s$ inside the superconductor is related to the magnetic field $\mathbf{H}$ by the constitutive equation

$$\nabla \times \Lambda \mathbf{J}_s = -\mu \mathbf{H}, \tag{1}$$

where $\Lambda(x)$ is a scalar coefficient characteristic of the material and μ is the magnetic permeability. Equation (1) is able to describe both the effects of superconductivity, namely the complete disappearance of the electrical resistivity and the Meissner effect.

An important step in the phenomenological description of superconductivity was the Ginzburg–Landau theory [5], which describes the phase transition between the normal and the superconducting state.

Landau argued that this transition induces a sudden change in the symmetry of the material and suggested that the symmetry can be measured by a complex-valued parameter ψ, called order parameter. The physical meaning of ψ is specified by saying that $f^2 = |\psi|^2$ is the number density, n_s, of superconducting electrons. Hence $\psi = 0$ means that the material is in the normal state, i.e., $T > T_c$, while $|\psi| = 1$ corresponds to the state of a perfect superconductor ($T = 0$).

There must exist a relation between ψ and the absolute temperature T and this occurs through the free energy e. If the magnetic field is zero, at constant pressure and around the critical temperature T_c the free energy e_0 is written as

$$e_0 = -a(T)|\psi|^2 + b(T)|\psi|^4,$$

where higher-order terms in $|\psi|^4$ are neglected, so that the model is valid around the critical temperature T_c for small values of $|\psi|$.

Suppose that the superconductor occupies a bounded domain Ω, with regular boundary $\partial\Omega$ and denote by $\mathbf{n}$ the unit outward normal to $\partial\Omega$. If a magnetic field occurs, then the free energy of the material is given by

$$\int_\Omega e(\psi, T, \mathbf{H})\,dx = \int_\Omega \left[e_0(\psi, T) + \mu|\mathbf{H}|^2 + \frac{1}{2m_*}|-i\hbar\nabla\psi - e_*\mathbf{A}\psi|^2\right]dx - \int_{\partial\Omega} \mathbf{A}\times\mathbf{H}_{ex}\cdot\mathbf{n}\,da, \tag{2}$$

where m_* is the mass of the superelectron and e_* is its effective charge, $\mathbf{A}$ is the vector potential related to $\mathbf{H}$ and $\hbar$ is Planck's constant. The vector $\mathbf{H}_{ex}$ represents the external magnetic field on the boundary $\partial\Omega$ and we suppose $\nabla\times\mathbf{H}_{ex} = \mathbf{0}$.

The generalization of the Ginzburg–Landau theory to the evolution problem was analyzed by Schmid [11], Gor'kov and Èliashberg [6] in the context of the BCS theory of superconductivity. Now the total current density $\mathbf{J}$ is given by $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_n$, where $\mathbf{J}_n$ obeys the Ohm's law

$$\mathbf{J}_n = \sigma\mathbf{E},$$

while the supercurrent $\mathbf{J}_s$ satisfies the London equation (1). In order to describe the physical state of the evolution system, Gor'kov and Èliashberg consider three variables, the wave function ψ, the vector and scalar potentials $\mathbf{A}$ and ϕ, which are related to the electrical and magnetic fields $\mathbf{E}$, $\mathbf{H}$ by means of the equations

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi, \qquad \mu\mathbf{H} = \nabla\times\mathbf{A}. \tag{3}$$

The evolution model of superconductivity is governed by the differential system [6,11]

$$\gamma\left(\frac{\partial\psi}{\partial t} - i\frac{e_*}{\hbar}\phi\psi\right) = -\frac{1}{2m_*}(i\hbar\nabla + e_*\mathbf{A})^2\psi + \alpha\psi - \beta|\psi|^2\psi, \tag{4}$$

$$\sigma\left(\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi\right) = -\frac{1}{\mu}\nabla\times\nabla\times\mathbf{A} + \mathbf{J}_s \tag{5}$$

with

$$\mathbf{J}_s = -\frac{i\hbar e_*}{2m_*}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e_*^2}{m_*}|\psi|^2\mathbf{A} \tag{6}$$

and γ a suitable coefficient representing a relaxation time. The associated boundary conditions are given by

$$(i\hbar\nabla + e_*\mathbf{A})\psi\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{A})\times\mathbf{n}|_{\partial\Omega} = \mu\mathbf{H}_{ex}\times\mathbf{n}. \tag{7}$$

The system (4)–(6) must be invariant under a gauge transformation

$$(\psi, \mathbf{A}, \phi) \longleftrightarrow \left(\psi e^{i\frac{e_*}{\hbar}\chi}, \mathbf{A} + \nabla\chi, \phi + \frac{\partial\chi}{\partial t}\right), \tag{8}$$

where the gauge χ can be any smooth scalar function of (x, t).

Various gauges have been considered [2,7,12,15]. In the London gauge, χ is chosen so that $\nabla\cdot\mathbf{A} = 0$, $\mathbf{A}\cdot\mathbf{n}|_{\partial\Omega} = 0$. In the Lorentz gauge we have $\phi = -\frac{1}{\mu\sigma}\nabla\cdot\mathbf{A}$ and the boundary condition

$\mathbf{A}\cdot\mathbf{n}|_{\partial\Omega}=0$. Finally, in the zero electrical potential gauge we have $\phi=0$. It is not possible to have both $\phi=0$, and the London gauge simultaneously.

The gauge invariance of the system (4)–(6) has been stated in many papers, where it is emphasized that the choice of the gauge is technical and does not affect the physical meaning of the solutions. As shown by Eq. (6), the choice of the gauge is related to the decomposition of the supercurrent in the form

$$\mathbf{J}_s=\Lambda(f)^{-1}(-\mathbf{A}+\nabla\theta),$$

where $\psi=fe^{i\frac{e_*}{\hbar}\theta}$ and $\Lambda(f)=\frac{m_*}{e_*^2}f^{-2}$. Accordingly, the different conditions on the magnetic potential $\mathbf{A}$ depending on the choice of the gauge, lead to different differential systems. In order to explain this assertion, we observe that the system (4)–(6) can be written by means of the observable variables $f,\mathbf{J}_s,\mathbf{H},\mathbf{E}$, which are necessarily independent by the choice of the gauge. From (4) we deduce the equation [3]

$$\gamma\frac{\partial f}{\partial t}=\frac{\hbar^2}{2m_*}\Delta f-\frac{m_*}{2e_*^2}\mathbf{J}_s^2f^{-3}+\alpha f-\beta f^3 \tag{9}$$

and in view of (6) we obtain London's equation

$$\nabla\times\left[\Lambda(f)\mathbf{J}_s\right]=-\mu\mathbf{H}. \tag{10}$$

Equation (5) is essentially Ampere's law

$$\nabla\times\mathbf{H}=\mathbf{J}_s+\mathbf{J}_n+\varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

when $\frac{\partial\mathbf{E}}{\partial t}$ is supposed negligible, namely when we consider the quasi-steady approximation.

Finally, by substituting the relation (10) in Maxwell equation

$$\nabla\times\mathbf{E}=-\mu\frac{\partial\mathbf{H}}{\partial t}, \tag{11}$$

we have

$$\frac{\partial}{\partial t}\left[\Lambda(f)\mathbf{J}_s\right]=\mathbf{E}-\nabla\phi_s, \tag{12}$$

where $\phi_s(x,t)$ is a smooth scalar function. Equation (12) corresponds to the Euler equation for a non-viscous electronic liquid (see [8, p. 59]) "where ϕ_s is the thermodynamic potential per electron; a function, in particular, of the concentrations of the superelectrons."

In order to obtain the complete equivalence with the problem (4)–(6), "the pressure" ϕ_s has to be related to the $\nabla\cdot\mathbf{E}$ by means of the identity [1]

$$\phi_s=\frac{\hbar^2\sigma}{2m_*\gamma}\Lambda(f)\nabla\cdot\mathbf{E}. \tag{13}$$

Hence, in the quasi-steady approximation, Eqs. (9)–(12) can be written also in the new form

$$\gamma\frac{\partial f}{\partial t}=\frac{\hbar^2}{2m_*}\Delta f-\frac{e_*^2}{2m_*}\mathbf{p}_s^2f+\alpha f-\beta f^3, \tag{14}$$

$$\frac{1}{\mu}\nabla\times\nabla\times\mathbf{p}_s+\Lambda^{-1}(f)\mathbf{p}_s+\sigma\mathbf{E}=0, \tag{15}$$

$$\mathbf{E}=\frac{\partial\mathbf{p}_s}{\partial t}+\nabla\phi_s, \tag{16}$$

where $\mathbf{p}_s = \Lambda(f)\mathbf{J}_s$ denotes the velocity of superelectrons.

Moreover by means of (13) and (15), we get

$$\nabla \cdot \left(\Lambda^{-1}(f)\mathbf{p}_s\right) = -\sigma \nabla \cdot \mathbf{E} = -\frac{2m_*\gamma}{\hbar^2}\Lambda^{-1}(f)\phi_s. \tag{17}$$

Concerning the boundary conditions, we assume

$$\mathbf{E} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{18}$$

Together with the conditions (7), the previous relation yields

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \mathbf{p}_s) \times \mathbf{n}|_{\partial\Omega} = -\mu \mathbf{H}_{ex} \times \mathbf{n},$$
$$f\mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad f\nabla\phi_s \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{19}$$

In order to simplify our notations, hereafter we consider the non-dimensional form of Eqs. (14)–(17), namely

$$\dot{f} - \frac{1}{k^2}\Delta f + \left(f^2 - 1\right)f + f|\mathbf{p}_s|^2 = 0, \tag{20}$$
$$\eta(\dot{\mathbf{p}}_s + \nabla\phi_s) + \nabla \times \nabla \times \mathbf{p}_s + f^2\mathbf{p}_s = 0, \tag{21}$$
$$k^2 f\phi_s + f\nabla \cdot \mathbf{p}_s + 2\nabla f \cdot \mathbf{p}_s = 0, \tag{22}$$

where we have denoted by a superimposed dot the partial derivative with respect to the variable t.

The problem (20)–(22) with boundary conditions (19) cannot be considered equivalent to the system (4)–(7) for any choice of the gauge, since a theorem of uniqueness for the solution of the first problem has not been proved. As already observed, each choice of the gauge is related to a particular decomposition of $\mathbf{p}_s$ of the form

$$\mathbf{p}_s = -\mathbf{A} + \nabla\theta. \tag{23}$$

Consequently, this choice leads to the differential system

$$\dot{f} - \frac{1}{k^2}\Delta f + \left(f^2 - 1\right)f + f|\mathbf{A} - \nabla\theta|^2 = 0, \tag{24}$$
$$\eta(\dot{\mathbf{A}} - \nabla\phi) + \nabla \times \nabla \times \mathbf{A} + f^2(\mathbf{A} - \nabla\theta) = 0, \tag{25}$$
$$k^2 f(\dot{\theta} - \phi) + f\nabla \cdot (\mathbf{A} - \nabla\theta) + 2\nabla f \cdot (\mathbf{A} - \nabla\theta) = 0, \tag{26}$$

where

$$\phi = \phi_s + \dot{\theta}. \tag{27}$$

The previous system is equivalent to the original Gor'kov–Èliashberg system

$$\dot{\psi} - ik\phi\psi + \left(\frac{i}{k}\nabla + \mathbf{A}\right)^2\psi - \left(1 - |\psi|^2\right)\psi = 0, \tag{28}$$

$$\eta(\dot{\mathbf{A}} - \nabla\phi) + \nabla\times\nabla\times\mathbf{A} = -\frac{i}{2}\left[\psi^*(\nabla\psi - i\mathbf{A}\psi) - \psi(\nabla\psi^* + i\mathbf{A}\psi^*)\right], \tag{29}$$

which coincides with the non-dimensional form of (4)–(5).

From a physical point of view, the representation (23) means that $\mathbf{p}_s$ is decomposed as the sum of an irrotational field and a vector $\mathbf{A}$. Of course, this decomposition is not unique, but depends on the particular choice of the gauge. For instance, if we consider London gauge, $\mathbf{A}$ will be a solenoidal field. Hence the properties of the vector $\mathbf{p}_s$ could change when we choose a different gauge.

In the following we will perform a choice of the decomposition (23), namely we will suppose

$$\nabla \cdot \mathbf{A} = 0, \qquad \mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \int_{\Omega} \phi \, dx = 0. \tag{30}$$

Accordingly, we restrict our attention to the system (24)–(26) and associate the corresponding boundary conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \mathbf{A}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{H}_{ex} \times \mathbf{n}, \tag{31}$$

$$f \nabla\theta \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{32}$$

Moreover, by taking the divergence of (25) and using $(30)_1$, we obtain the following equation

$$\eta\Delta\phi - \nabla \cdot \left[f^2(\mathbf{A} - \nabla\theta) \right] = 0. \tag{33}$$

Hence, Eq. (26) yields

$$\eta\Delta\phi + k^2 f^2(\dot{\theta} - \phi) = 0. \tag{34}$$

3. Existence, uniqueness and properties of the solutions

With different choices of gauge, existence and uniqueness results have been proved for the system (28)–(29) with the initial and boundary conditions

$$\psi(x,0) = \psi_0(x), \qquad \mathbf{A}(x,0) = \mathbf{A}_0(x), \tag{35}$$

$$\nabla\psi \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \mathbf{A}) \times \mathbf{n}|_{\partial\Omega} = -\mathbf{H}_{ex} \times \mathbf{n}|_{\partial\Omega}, \tag{36}$$

$$\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{37}$$

We recall here some results proved in [12,14] which make use of London gauge. In order to obtain a precise formulation of the problem we introduce the following functional space

$$\mathbf{V}_0 = \left\{ \mathbf{A} \in H^1(\Omega) \colon \nabla \cdot \mathbf{A} = 0, \ \mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\}.$$

Moreover we denote by $\|\cdot\|_p$ and $\|\cdot\|_{H^s}$ the norms in $L^p(\Omega)$ and $H^s(\Omega)$, respectively. For each $\mathbf{A} \in \mathbf{V}_0$, the inequalities

$$\|\mathbf{A}\|_{H^1} \leqslant K_1 \|\nabla \times \mathbf{A}\|_2, \tag{38}$$

$$\|\mathbf{A}\|_{H^{1/2}(\partial\Omega)} \leqslant K_2 \|\nabla \times \mathbf{A}\|_2 \tag{39}$$

hold with K_1, K_2 positive constants depending on the domain Ω.

The following theorem, proved in [12], ensures the well posedness of the problem.

Theorem 1. *If* $(\psi_0, \mathbf{A}_0) \in H^1(\Omega) \times \mathbf{V}_0$, *there exists a unique solution* $(\psi, \mathbf{A})$ *of the problem* (28)–(29) *with boundary and initial conditions* (35)–(37) *such that* $\psi \in L^2(0,T; H^2(\Omega)) \cap C(0,T; H^1(\Omega))$, $\mathbf{A} \in L^2(0,T; \mathbf{V}_0 \cap H^2(\Omega)) \cap C(0,T; \mathbf{V}_0)$.

In view of the equivalence between the systems (24)–(26) and (28)–(29), we can obtain an existence and uniqueness theorem for the first problem, by writing the functional spaces of Theorem 1 in terms of the variables f, $\nabla\theta$, $\mathbf{A}$, ϕ.

It is worth noting that the choice of the gauge plays an important role in the definition of the functional spaces in which the problem is set. In particular another choice of the gauge leads to a different formulation of the problem.

We conclude this section by showing a property of the solutions of the Ginzburg–Landau equations [2], which will be useful for the proof of the estimates in the following section.

Proposition 2. *If* $(f, \mathbf{p}_s, \phi_s)$ *is a solution such that* $f_0(x)^2 \leqslant 1$ *almost everywhere in* Ω*, then* $f(x,t)^2 \leqslant 1$ *a.e. in* $\Omega \times [0,T]$.

Proof. By multiplying Eq. (20) by f we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}f^2 + \frac{1}{k^2}|\nabla f|^2 - \frac{1}{2k^2}\Delta f^2 + \left(f^2-1\right)^2 + \left(f^2-1\right) + f^2\mathbf{p}_s^2 = 0,$$

so that

$$\frac{\partial}{\partial t}\left(f^2-1\right) - \frac{1}{k^2}\Delta\left(f^2-1\right) + 2\left(f^2-1\right) \leqslant 0.$$

Now let us multiply the previous inequality by $h = (f^2-1)_+ = \max\{f^2-1, 0\}$. In this way we deduce

$$\frac{\partial}{\partial t}\frac{1}{2}h^2 - \frac{1}{k^2}h\Delta h + 2h^2 \leqslant 0.$$

Hence, by integrating on Ω, we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\|h\|_2^2 + \frac{1}{k^2}\|\nabla h\|_2^2 + 2\|h\|_2^2 \leqslant 0.$$

The assumption $f_0(x)^2 \leqslant 1$ allows to conclude that

$$\frac{1}{2}\|h\|_2^2 + \int_0^t \left[\frac{1}{k^2}\|\nabla h\|_2^2 + 2\|h\|_2^2\right] d\tau \leqslant 0$$

for each $t \in [0,T]$, so that $f^2 \leqslant 1$ almost everywhere in $\Omega \times [0,T]$. □

4. Energy estimates

In this section we examine the asymptotic behavior of the solution of the Ginzburg–Landau system. To this end, we will define an energy functional $\mathcal{E}_0$ and prove the inequality which guarantees the existence of an absorbing set for the system. Let

$$\mathcal{E}_0(f, \mathbf{p}_s) = \frac{1}{2}\int_\Omega \left[\frac{1}{k^2}|\nabla f|^2 + \frac{1}{2}\left(f^2-1\right)^2 + |\nabla \times \mathbf{p}_s|^2 + f^2|\mathbf{p}_s|^2\right] dx \tag{40}$$

the energy associated to the system (20)–(22). By means of the decomposition (23) we can express the energy functional (40) in terms of the variables $(f, \nabla\theta, \mathbf{A})$, namely

$$\mathcal{E}_0(f, \nabla\theta, \mathbf{A}) = \frac{1}{2}\int_\Omega \left[\frac{1}{k^2}|\nabla f|^2 + \frac{1}{2}\left(f^2-1\right)^2 + |\nabla \times \mathbf{A}|^2 + f^2|\mathbf{A} - \nabla\theta|^2\right] dx.$$

Moreover, we observe that $\mathcal{E}_0$ can be written as a function of the variables $(\psi, \mathbf{A})$ in the form

$$\mathcal{E}_0(\psi, \mathbf{A}) = \frac{1}{2} \int_\Omega \left[\left| \left(\frac{i}{k} \nabla + \mathbf{A} \right) \psi \right|^2 + \frac{1}{2} \left(|\psi|^2 - 1 \right)^2 + |\nabla \times \mathbf{A}|^2 \right] dx.$$

Note that $\mathcal{E}_0(\psi, \mathbf{A})$ has to be invariant up to gauge transformations of the form (8), since the energy depends on the observable variables $(f, \mathbf{p}_s)$ through the relation (40).

Theorem 3. *If the initial data satisfy $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \leqslant M$, then there exists a constant Γ, depending on Ω, $\mathbf{H}_{ex}$, k and η, such that for each $\Gamma' > \Gamma$, $\mathcal{E}_0(f, \nabla\theta, \mathbf{A}) \leqslant \Gamma'$ holds for $t > t_0$, where t_0 depends on M and $\Gamma' - \Gamma$.*

Proof. Henceforth, we denote by c_j, $j \in \mathbb{N}$, an arbitrary positive constant. By multiplying Eq. (24) by $\dot{f} + f$, integrating on Ω and keeping (31) into account, we obtain the equation

$$\int_\Omega \left[\dot{f}^2 + \frac{1}{k^2} \nabla f \cdot \nabla \dot{f} + f \dot{f} |\mathbf{A} - \nabla\theta|^2 + \left(f^3 - f \right) \dot{f} \right] dx$$
$$+ \int_\Omega \left[f \dot{f} + \frac{1}{k^2} |\nabla f|^2 + f^2 |\mathbf{A} - \nabla\theta|^2 + \left(f^2 - 1 \right) f^2 \right] dx = 0.$$

Hence

$$\frac{d}{dt} \int_\Omega \left[\frac{1}{2k^2} |\nabla f|^2 + \left(\frac{f^4}{4} - \frac{f^2}{2} \right) + \frac{1}{2} f^2 \right] dx$$
$$+ \int_\Omega \left[\dot{f}^2 + f \dot{f} |\mathbf{A} - \nabla\theta|^2 + \frac{1}{k^2} |\nabla f|^2 + \left(f^4 - f^2 \right) + f^2 |\mathbf{A} - \nabla\theta|^2 \right] dx = 0. \tag{41}$$

Similarly, by multiplying Eq. (25) by $\dot{\mathbf{A}} + c_1 \mathbf{A}$, integrating by parts and using (31), we have

$$\frac{d}{dt} \int_\Omega \left[\frac{1}{2} |\nabla \times \mathbf{A}|^2 + \frac{\eta c_1}{2} |\mathbf{A}|^2 \right] dx + \frac{d}{dt} \int_{\partial\Omega} \mathbf{A} \cdot \mathbf{H}_{ex} \times \mathbf{n} \, da$$
$$+ \int_\Omega \left[\eta |\dot{\mathbf{A}}|^2 + f^2 (\mathbf{A} - \nabla\theta) \cdot \dot{\mathbf{A}} + c_1 |\nabla \times \mathbf{A}|^2 + c_1 f^2 (\mathbf{A} - \nabla\theta) \cdot \mathbf{A} \right] dx$$
$$+ \int_{\partial\Omega} c_1 \mathbf{A} \cdot \mathbf{H}_{ex} \times \mathbf{n} \, da = 0.$$

Note that, in the previous equation the term involving $\nabla\phi$ vanishes as a consequence of $(30)_1$.

Finally, if we multiply Eqs. (26) and (34) by $f\dot{\theta}$ and $-\phi$ respectively and integrate on Ω, we obtain the relations

$$\int_\Omega \left[k^2 f^2 \dot{\theta}^2 - k^2 f^2 \phi \dot{\theta} - f^2 (\mathbf{A} - \nabla\theta) \cdot \nabla\dot{\theta} \right] dx = 0, \tag{42}$$

$$\int_\Omega \left[\eta |\nabla\phi|^2 - k^2 f^2 \phi \dot{\theta} + k^2 f^2 \phi^2 \right] dx = 0, \tag{43}$$

where the boundary integrals vanish in view of (32).

Equations (41)–(43) yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\bigg[\frac{1}{k^2}|\nabla f|^2+\frac{1}{2}\big(f^2-1\big)^2+f^2+|\nabla\times\mathbf{A}|^2+\eta c_1|\mathbf{A}|^2\\
&\quad+f^2|\mathbf{A}-\nabla\theta|^2\bigg]\,dx+\frac{d}{dt}\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da\\
&\quad+\int_\Omega\bigg[\frac{1}{k^2}|\nabla f|^2+\big(f^4-f^2\big)+c_1|\nabla\times\mathbf{A}|^2+f^2|\mathbf{A}-\nabla\theta|^2\bigg]\,dx\\
&\quad+c_1\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da+\int_\Omega\big[\dot f^2+k^2f^2(\dot\theta-\phi)^2+\eta\dot{\mathbf{A}}^2+\eta|\nabla\phi|^2\big]\,dx\\
&=-\int_\Omega c_1f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\,dx. \qquad (44)
\end{aligned}$$

Let us introduce the functional

$$\begin{aligned}
\mathcal{F}=&\frac{1}{2}\int_\Omega\bigg[\frac{1}{k^2}|\nabla f|^2+\frac{1}{2}\big(f^2-1\big)^2+f^2+|\nabla\times\mathbf{A}|^2+\eta c_1|\mathbf{A}|^2\\
&+f^2|\mathbf{A}-\nabla\theta|^2\bigg]\,dx+\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da+2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)},
\end{aligned}$$

where the constant K_2 is defined in (39). Note that $\mathcal{F}$ is positive definite since the relation (39) implies

$$\begin{aligned}
\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da&\geqslant-\|\mathbf{A}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)}\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)}\\
&\geqslant-K_2\|\nabla\times\mathbf{A}\|_2\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)}\\
&\geqslant-\frac{1}{4}\|\nabla\times\mathbf{A}\|_2^2-2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)}.
\end{aligned}$$

Therefore $\mathcal{F}\geqslant\frac{1}{2}\mathcal{E}_0\geqslant0$.

On the other hand, the functional $\mathcal{F}$ can be written as

$$\mathcal{F}=\mathcal{E}_0+\int_\Omega\big[f^2+\eta c_1|\mathbf{A}|^2\big]\,dx+\int_{\partial\Omega}\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da+2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)},$$

so that

$$\begin{aligned}
\mathcal{F}\leqslant&\,\mathcal{E}_0+\int_\Omega\big(f^2-1\big)\,dx+\eta c_1K_1\|\nabla\times\mathbf{A}\|_2^2\\
&+K_2\|\nabla\times\mathbf{A}\|_2\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)}+2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)}+\mathrm{vol}(\Omega).
\end{aligned}$$

Therefore, we can prove the existence of two positive constants C_1, C_2, depending on $\mathbf{H}_{ex}$, k, η and Ω, such that

$$\frac{1}{2}\mathcal{E}_0\leqslant\mathcal{F}\leqslant C_1\mathcal{E}_0+C_2. \qquad (45)$$

The relation (44) yields

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F} + \int_\Omega \biggl[\frac{1}{k^2}|\nabla f|^2 + \bigl(f^2-1\bigr)^2 + f^2 + c_1|\nabla\times\mathbf{A}|^2 + c_2|\mathbf{A}|^2 \\
&\quad + f^2|\mathbf{A}-\nabla\theta|^2\biggr]dx + \int_{\partial\Omega} c_1\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da + \frac{K_2^2}{2}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} \\
&\quad + \int_\Omega \bigl[\dot f^2 + k^2 f^2(\dot\theta-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\bigr]dx \\
&= \int_\Omega \bigl[c_2|\mathbf{A}|^2 - c_1 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\bigr]dx + 2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} \\
&\quad + k^2\operatorname{vol}(\Omega).
\end{aligned} \tag{46}$$

Concerning the right-hand side, observe that

$$\begin{aligned}
I_\Omega &:= \int_\Omega \bigl[c_2|\mathbf{A}|^2 - c_1 f^2(\mathbf{A}-\nabla\theta)\cdot\mathbf{A}\bigr]dx \\
&\leqslant c_2\|\mathbf{A}\|_2^2 + c_1\bigl\|f(\mathbf{A}-\nabla\theta)\bigr\|_2\|f\mathbf{A}\|_2 \\
&\leqslant K_1c_2\|\nabla\times\mathbf{A}\|_2^2 + c_1\biggl(\frac{1}{2c_3}\bigl\|f(\mathbf{A}-\nabla\theta)\bigr\|_2^2 + \frac{c_3}{2}\|f\mathbf{A}\|_2^2\biggr).
\end{aligned}$$

Moreover, in view of Proposition 2, we have

$$\|f\mathbf{A}\|_2^2 \leqslant \|\mathbf{A}\|_2^2 \leqslant K_1\|\nabla\times\mathbf{A}\|_2^2,$$

so that with the choices of $c_1 = c_3 = \frac{1}{2K_1}$, $c_2 = \frac{c_1}{4K_1}$, we obtain

$$I_\Omega \leqslant \frac{c_1}{2}\|\nabla\times\mathbf{A}\|_2^2 + \frac{1}{2}\bigl\|f(\mathbf{A}-\nabla\theta)\bigr\|_2^2.$$

Substitution in (46) leads to the inequality

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F} + \int_\Omega \biggl[\frac{1}{k^2}|\nabla f|^2 + \bigl(f^2-1\bigr)^2 + f^2 + \frac{c_1}{2}|\nabla\times\mathbf{A}|^2 + c_2|\mathbf{A}|^2 \\
&\quad + \frac{1}{2}f^2|\mathbf{A}-\nabla\theta|^2\biggr]dx + \int_{\partial\Omega} c_1\mathbf{A}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da + 2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} \\
&\quad + \int_\Omega \bigl[\dot f^2 + k^2 f^2(\dot\theta-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\bigr]dx \leqslant C,
\end{aligned}$$

where

$$C = 2K_2^2\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)} + \operatorname{vol}(\Omega).$$

By putting $\lambda = 2\min\{\frac{c_1}{2}, \frac{c_2}{\eta c_1}, 1\}$, we have proved the inequality

$$\frac{d}{dt}\mathcal{F} + \lambda\mathcal{F} + \int_\Omega \bigl[\dot f^2 + k^2 f^2(\dot\theta-\phi)^2 + \eta|\dot{\mathbf{A}}|^2 + \eta|\nabla\phi|^2\bigr]dx \leqslant C. \tag{47}$$

Hence

$$\frac{d}{dt}\mathcal{F}+\lambda\mathcal{F}\leqslant C.$$

The application of Gronwall lemma yields

$$\mathcal{F}(t)\leqslant\mathcal{F}(0)e^{-\lambda t}+\frac{C}{\lambda}\left(1-e^{-\lambda t}\right)\leqslant\mathcal{F}(0)e^{-\lambda t}+\frac{C}{\lambda}.$$

Therefore, in view of the relation (45) we obtain the inequality

$$\mathcal{E}_0(t)\leqslant 2\mathcal{F}(t)\leqslant 2C_1\mathcal{E}_0(0)e^{-\lambda t}+\Gamma,$$

where $\Gamma=2C_1+\frac{C}{\lambda}$. The assumption on the initial data allows to prove the inequality

$$\mathcal{E}_0(t)\leqslant 2C_1Me^{-\lambda t}+\Gamma.$$

Hence, for each $\Gamma'>\Gamma$, the inequality

$$\mathcal{E}_0(t)\leqslant\Gamma'$$

holds if $t>t_0=\max\{0,\frac{1}{\lambda}\log\frac{2C_1M}{\Gamma'-\Gamma}\}$. □

5. Higher-order energy estimates

We introduce now the higher-order energy functional defined as

$$\mathcal{E}_1(\psi,\mathbf{A})=\frac{1}{2}\int_\Omega\left[\left|\left(\frac{i}{k}\nabla+\mathbf{A}\right)^2\psi\right|^2+|\nabla\times\nabla\times\mathbf{A}|^2\right]dx.$$

Like the functional $\mathcal{E}_0$, the energy $\mathcal{E}_1$ can be written by means of the variables $(f,\nabla\theta,\mathbf{A})$ as

$$\begin{aligned}\mathcal{E}_1(f,\nabla\theta,\mathbf{A})=\frac{1}{2}\int_\Omega\biggl[&\left(-\frac{1}{k^2}\Delta f+f|\mathbf{A}-\nabla\theta|^2\right)^2\\&+\left(-\frac{1}{k}f\Delta\theta+\frac{2}{k}\nabla f\cdot(\mathbf{A}-\nabla\theta)\right)^2+|\nabla\times\nabla\times\mathbf{A}|^2\biggr]dx,\end{aligned}\tag{48}$$

or by means of $(f,\mathbf{p}_s)$ as

$$\mathcal{E}_1(f,\mathbf{p}_s)=\frac{1}{2}\int_\Omega\left[\left(-\frac{1}{k^2}\Delta f+f\mathbf{p}_s^2\right)^2+\left(\frac{1}{k}f\nabla\cdot\mathbf{p}_s+\frac{2}{k}\nabla f\cdot\mathbf{p}_s\right)^2+|\nabla\times\nabla\times\mathbf{p}_s|^2\right]dx.$$

We prove now some energy estimates for the functional (48). In order to simplify our notations we define

$$P=-\frac{1}{k^2}\Delta f+f|\mathbf{A}-\nabla\theta|^2,\qquad Q=-\frac{1}{k}f\Delta\theta+2\nabla f\cdot(\mathbf{A}-\nabla\theta).\tag{49}$$

By multiplying Eq. (24) by $\dot{P}+P-k\dot{\theta}Q$ and integrating in Ω, we obtain

$$\int_\Omega\left[\frac{d}{dt}\frac{P^2}{2}+P^2+\dot{f}\dot{P}+f\left(f^2-1\right)(\dot{P}+P-k\dot{\theta}Q)+\dot{f}P-k\dot{f}\dot{\theta}Q-k\dot{\theta}PQ\right]dx=0.\tag{50}$$

Similarly, by multiplying (26) by $\dot{Q} + Q + k\dot{\theta}P$, we have

$$\int_\Omega \left[\frac{d}{dt}\frac{Q^2}{2} + Q^2 + f(\dot{\theta} - \phi)\dot{Q} + f(\dot{\theta} - \phi)Q + f(\dot{\theta} - \phi)\dot{\theta}P + \dot{\theta}PQ\right]dx = 0.$$

Now we consider Eq. (34) and multiply it by $\Delta\phi$, obtaining

$$\int_\Omega \left[\frac{\eta}{k^2}(\Delta\phi)^2 + f^2(\dot{\theta} - \phi)\Delta\phi\right]dx = 0. \tag{51}$$

The relations (50)–(51) yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(P^2 + Q^2\right)dx + \int_\Omega \left[P^2 + Q^2 + \frac{\eta}{k^2}(\Delta\phi)^2\right]dx + I_1 + I_2 = 0, \tag{52}$$

where I_1 and I_2 are defined as

$$\begin{aligned} I_1 = \int_\Omega \big[&\dot{f}\dot{P} + \dot{f}P - k\dot{f}\dot{\theta}Q + kf(\dot{\theta} - \phi)\dot{Q} + kf(\dot{\theta} - \phi)Q + k^2 f(\dot{\theta} - \phi)\dot{\theta}P \\ &+ f^2(\dot{\theta} - \phi)\Delta\phi\big]dx, \\ I_2 = \int_\Omega &f\left(f^2 - 1\right)[\dot{P} + P - k\dot{\theta}Q]\,dx. \end{aligned}$$

By integrating by parts and keeping the boundary conditions (31), (32) into account, we get

$$\begin{aligned} I_1 = \int_\Omega \bigg[&\frac{1}{k^2}|\nabla\dot{f}|^2 + \dot{f}^2|\mathbf{A} - \nabla\theta|^2 + 2f\dot{f}(\mathbf{A} - \nabla\theta)\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta}) \\ &+ \frac{1}{k^2}\nabla f\cdot\nabla\dot{f} + f\dot{f}|\mathbf{A} - \nabla\theta|^2 - 2\dot{f}\dot{\theta}\nabla f\cdot(\mathbf{A} - \nabla\theta) + f\dot{f}\phi\Delta\theta \\ &- f^2(\dot{\theta} - \phi)\Delta\dot{\theta} + 2f(\dot{\theta} - \phi)\nabla\dot{f}\cdot(\mathbf{A} - \nabla\theta) \\ &+ 2f(\dot{\theta} - \phi)\nabla f\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta}) - f^2(\dot{\theta} - \phi)\Delta\theta \\ &+ 2f(\dot{\theta} - \phi)\nabla f\cdot(\mathbf{A} - \nabla\theta) - f\dot{\theta}(\dot{\theta} - \phi)\Delta f \\ &+ k^2 f^2(\dot{\theta} - \phi)\dot{\theta}|\mathbf{A} - \nabla\theta|^2 - 2f(\dot{\theta} - \phi)\nabla f\cdot\nabla\phi \\ &- f^2(\nabla\dot{\theta} - \nabla\phi)\cdot\nabla\phi\bigg]dx. \end{aligned}$$

Since $\nabla\cdot\mathbf{A} = 0$, in the previous expression we can replace $\Delta\theta$ and $\Delta\dot{\theta}$ by $-\nabla\cdot(\mathbf{A} - \nabla\theta)$ and $-\nabla\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta})$ respectively and integrate by parts. A straightforward computation proves that I_1 can be written as

$$\begin{aligned} I_1 = \int_\Omega \bigg[&|\mathbf{R}|^2 + |\mathbf{S}|^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{1}{k^2}|\nabla f|^2 + f^2|\mathbf{A} - \nabla\theta|^2\right) - \frac{k^2}{4}\phi^2|\nabla f|^2 \\ &- \frac{1}{4}f^2\phi^2|\mathbf{A} - \nabla\theta|^2 + f(\phi - \dot{\theta})\nabla f\cdot\nabla\phi - f^2(\dot{\mathbf{A}} - \nabla\phi)\cdot(\mathbf{A} - \nabla\theta) \\ &- f^2|\dot{\mathbf{A}}|^2 - f\dot{f}(\mathbf{A} - \nabla\theta)\cdot\nabla\phi + f\phi\nabla f\cdot\dot{\mathbf{A}}\bigg]dx, \end{aligned}$$

where

$$\mathbf{R} = \dot{f}(\mathbf{A} - \nabla\theta) + f(\dot{\mathbf{A}} - \nabla\dot{\theta}) + f\nabla\phi - \dot{\theta}\nabla f + \frac{1}{2}\phi\nabla f,$$
$$\mathbf{S} = \frac{1}{k}\nabla\dot{f} + kf\dot{\theta}(\mathbf{A} - \nabla\theta) - \frac{k}{2}f\phi(\mathbf{A} - \nabla\theta).$$

Concerning I_2, we observe that

$$\begin{aligned} I_2 = \int_\Omega \Big[&-\frac{1}{k^2}\big(f^3 - f\big)\Delta\dot{f} + \dot{f}\big(f^3 - f\big)|\mathbf{A} - \nabla\theta|^2 \\ &+ 2\big(f^4 - f^2\big)(\mathbf{A} - \nabla\theta)\cdot(\dot{\mathbf{A}} - \nabla\dot{\theta}) - \frac{1}{k^2}\big(f^3 - f\big)\Delta f \\ &+ f^2\big(f^2 - 1\big)|\mathbf{A} - \nabla\theta|^2 + \big(f^4 - f^2\big)\dot{\theta}\Delta\theta \\ &- 2\dot{\theta}\big(f^3 - f\big)\nabla f\cdot(\mathbf{A} - \nabla\theta)\Big]\,dx \end{aligned}$$

and, by integrating by parts, we obtain

$$\begin{aligned} I_2 = \int_\Omega \Big[&\frac{1}{k^2}\big(3f^2 - 1\big)\nabla f\cdot\nabla\dot{f} + \big(f^3 - f\big)(\mathbf{A} - \nabla\theta)\cdot\big[\dot{f}(\mathbf{A} - \nabla\theta) \\ &+ f(\dot{\mathbf{A}} - \nabla\dot{\theta})\big] + \big(f^4 - f^2\big)\dot{\mathbf{A}}\cdot(\mathbf{A} - \nabla\theta) + 2\dot{\theta}\big(2f^3 - f\big)\mathbf{A}\cdot\nabla f \\ &+ \frac{1}{k^2}\big(3f^2 - 1\big)|\nabla f|^2 + f^2\big(f^2 - 1\big)|\mathbf{A} - \nabla\theta|^2 \\ &- 2f\dot{\theta}\big(2f^2 - 1\big)\nabla\theta\cdot\nabla f - 2\dot{\theta}\big(f^3 - f\big)\nabla f\cdot(\mathbf{A} - \nabla\theta)\Big]\,dx. \end{aligned}$$

The definition of $\mathbf{R}$ and $\mathbf{S}$ yields

$$\begin{aligned} I_2 = \int_\Omega \Big\{ &\big(f^3 - f\big)(\mathbf{A} - \nabla\theta)\cdot\mathbf{R} + \frac{1}{k}\big(3f^2 - 1\big)\nabla f\cdot\mathbf{S} \\ &+ f^3\phi(\mathbf{A} - \nabla\theta)\cdot\nabla f - \big(f^4 - f^2\big)(\mathbf{A} - \nabla\theta)\cdot(\nabla\phi - \dot{\mathbf{A}}) \\ &+ \frac{1}{k^2}\big(3f^2 - 1\big)|\nabla f|^2 + f^2\big(f^2 - 1\big)|\mathbf{A} - \nabla\theta|^2\Big\}\,dx. \end{aligned}$$

Let us consider Eq. (25), multiply it by $\nabla\times\nabla\times\dot{\mathbf{A}} + \nabla\times\nabla\times\mathbf{A}$ and integrate in Ω. Keeping the boundary conditions (31) into account, we get the relation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \big[|\nabla\times\nabla\times\mathbf{A}|^2 + \eta|\nabla\times\mathbf{A}|^2\big]\,dx + \int_\Omega \big[\eta|\nabla\times\dot{\mathbf{A}}|^2 \\ &\quad + |\nabla\times\nabla\times\mathbf{A}|^2 + \nabla\times\dot{\mathbf{A}}\cdot\big[2f\nabla f\times(\mathbf{A} - \nabla\theta) + f^2\nabla\times\mathbf{A}\big] \\ &\quad + f^2(\mathbf{A} - \nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}\big]\,dx + \int_{\partial\Omega}\dot{\mathbf{A}}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da = 0. \end{aligned} \tag{53}$$

From the relations (52)–(53), we obtain

$$\frac{d}{dt}\mathcal{E}_1 + \frac{d}{dt}\int_\Omega \left[\frac{1}{2k^2}|\nabla f|^2 + \frac{1}{2}\left|f(\mathbf{A}-\nabla\theta)\right|^2 + \frac{\eta}{2}|\nabla\times\mathbf{A}|^2 + \frac{1}{4}\left(f^2-1\right)^2\right]dx$$
$$+\int_\Omega \left[P^2 + Q^2 + \frac{\eta}{k^2}(\Delta\phi)^2 + \eta|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2\right.$$
$$\left. + |\mathbf{R}|^2 + |\mathbf{S}|^2\right]dx \leqslant I_3,$$

where $\mathcal{E}_1$ is defined by (48) and

$$I_3 = \int_\Omega \left[\frac{1}{4}\phi^2|\nabla f|^2 + \frac{k^2}{4}f^2\phi^2|\mathbf{A}-\nabla\theta|^2 - f(\phi-\dot\theta)\nabla f\cdot\nabla\phi\right.$$
$$\left. + f^2(\dot{\mathbf{A}}-\nabla\phi)\cdot(\mathbf{A}-\nabla\theta) + f^2|\dot{\mathbf{A}}|^2 + f\dot f(\mathbf{A}-\nabla\theta)\cdot\nabla\phi - f\phi\nabla f\cdot\dot{\mathbf{A}}\right]dx$$
$$-\int_\Omega \left\{\left(f^3-f\right)(\mathbf{A}-\nabla\theta)\cdot\mathbf{R} + \frac{1}{k}\left(3f^2-1\right)\nabla f\cdot\mathbf{S} + f^3\phi(\mathbf{A}-\nabla\theta)\cdot\nabla f\right.$$
$$-\left(f^4-f^2\right)(\mathbf{A}-\nabla\theta)\cdot(\nabla\phi-\dot{\mathbf{A}}) + \frac{1}{k^2}\left(3f^2-1\right)|\nabla f|^2$$
$$\left. + f^2\left(f^2-1\right)|\mathbf{A}-\nabla\theta|^2 + \left(f^2-1\right)f\dot f\right\}dx - \int_{\partial\Omega}\dot{\mathbf{A}}\cdot\mathbf{H}_{ex}\times\mathbf{n}\,da$$
$$-\int_\Omega \left\{\nabla\times\dot{\mathbf{A}}\cdot\left[2f\nabla f\times(\mathbf{A}-\nabla\theta) + f^2\nabla\times\mathbf{A}\right] + f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}\right\}dx. \tag{54}$$

Hence

$$\frac{d}{dt}(\mathcal{E}_1+\gamma\mathcal{E}_0) + \int_\Omega\left[P^2 + Q^2 + \frac{\eta}{k^2}(\Delta\phi)^2 + \eta|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2\right.$$
$$\left. + |\mathbf{R}|^2 + |\mathbf{S}|^2\right]dx \leqslant I_3, \tag{55}$$

where $\gamma = \min\{\eta, 1\}$.

In order to estimate the right-hand side of (55), we need some lemmas. We use repeatedly Theorem 3 with $\Gamma' = 2\Gamma$. Moreover we denote by $C(\Gamma)$ a generic constant depending on Γ (i.e., depending on Ω, $\mathbf{H}_{ex}$, k, η), which may vary even in the same formula.

Lemma 4. *If the initial data satisfy the inequality $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \leqslant M$, then*

$$\|\nabla\phi\|_2 \leqslant C(\Gamma) \tag{56}$$

for $t > t_0$.

Proof. Consider Eq. (33) and multiply it by ϕ. An integration by parts and use of the boundary conditions (31), (32) yield

$$\int_\Omega \eta|\nabla\phi|^2\,dx \leqslant -\int_\Omega f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\phi\,dx.$$

In view of Proposition 2, we have

$$\|\nabla\phi\|_2^2 \leqslant \frac{1}{\eta}\int_\Omega \left|f^2(\mathbf{A}-\nabla\theta)\cdot\nabla\phi\right|dx \leqslant \frac{1}{\eta}\left\|f(\mathbf{A}-\nabla\theta)\right\|_2\|\nabla\phi\|_2,$$

so that

$$\|\nabla\phi\|_2^2 \leqslant \frac{1}{\eta^2}\left\|f(\mathbf{A}-\nabla\theta)\right\|_2^2 \leqslant \frac{2}{\eta^2}\mathcal{E}_0(f,\nabla\theta,\mathbf{A}).$$

The application of Theorem 3 proves (56). □

Lemma 5. *If $\Omega\subset\mathbb{R}^2$ and the initial data satisfy $\mathcal{E}_0(f_0,\nabla\theta_0,\mathbf{A}_0)\leqslant M$, there exist positive constants $C_1(\Gamma)$, $C_2(\Gamma)$ and $C_3(\Gamma)$ such that*

$$\begin{aligned}I_3 &\leqslant C_1(\Gamma)+C_2(\Gamma)\left(\|P\|_2^2+\|Q\|_2^2+\|\nabla\times\nabla\times\mathbf{A}\|_2^2\right)\\ &\quad + C_3(\Gamma)\left[\left\|kf(\dot\theta-\phi)\right\|_2^2+\|\dot{\mathbf{A}}\|_2^2+\|\dot f\|_2^2\right]\\ &\quad + \frac{1}{2}\left(\|\mathbf{R}\|_2^2+\|\mathbf{S}\|_2^2+\eta\|\nabla\times\dot{\mathbf{A}}\|_2^2+\frac{\eta}{k^2}\|\Delta\phi\|_2^2\right).\end{aligned}$$

Proof. In view of the definitions (49), we have

$$P^2+Q^2=\left|\left(\frac{i}{k}\nabla+\mathbf{A}\right)^2\psi\right|^2=\left|-\frac{1}{k^2}\Delta\psi+\mathbf{A}^2\psi+\frac{2i}{k}\mathbf{A}\cdot\nabla\psi\right|^2.$$

Therefore, by means of the inequality $|x+y|^2\leqslant 2|x|^2+2|y|^2$, $\forall x,y\in\mathbb{C}$, we obtain

$$\begin{aligned}|\Delta\psi|^2 &\leqslant 2k^4\left(P^2+Q^2\right)+2k^4\left|\frac{2i}{k}\mathbf{A}\cdot\nabla\psi+\mathbf{A}^2\psi\right|^2\\ &\leqslant 2k^4\left(P^2+Q^2\right)+\frac{16}{k^2}|\mathbf{A}|^2|\nabla\psi|^2+4k^4|\mathbf{A}|^4|\psi|^2.\end{aligned}$$

The previous inequality and the condition $|\psi|\leqslant 1$, yield

$$\|\Delta\psi\|_2^2\leqslant 2k^4\left(\|P\|_2^2+\|Q\|_2^2\right)+\left(4k^4+\frac{8}{\nu k^2}\right)\|\mathbf{A}\|_4^4+\frac{8\nu}{k^2}\|\nabla\psi\|_4^4$$

for each $\nu>0$. Moreover, when $\Omega\subset\mathbb{R}^2$, the classical interpolation inequality

$$\|h\|_4^2\leqslant K_3\|h\|_2\|h\|_{H^1},\quad h\in H^1(\Omega), \tag{57}$$

implies

$$\|\nabla\psi\|_4^4\leqslant K_3^2\|\nabla\psi\|_2^2\|\nabla\psi\|_{H^1}^2\leqslant K_3^2\left(\|\nabla\psi\|_2^4+\|\nabla\psi\|_2^2\|\Delta\psi\|_2^2\right),$$

so that, in view of Theorem 3, we obtain

$$\|\Delta\psi\|_2^2\leqslant 2k^4\left(\|P\|_2^2+\|Q\|_2^2\right)+C(\Gamma)+8\nu C(\Gamma)\|\Delta\psi\|_2^2.$$

By choosing ν such that $8\nu C(\Gamma)<\frac{1}{2}$, we have

$$\|\Delta\psi\|_2^2\leqslant 4k^4\left(\|P\|_2^2+\|Q\|_2^2\right)+C(\Gamma). \tag{58}$$

In order to estimate the terms of (54), we will use Holder's and Young's inequalities, Sobolev embeddings, the interpolation inequality (57), the relations (56), (58) and the condition $f^2 \leqslant 1$. Accordingly, we have

$$
\begin{aligned}
J_1 &:= \int_\Omega \phi^2\big(|\nabla f|^2 + k^2 f^2 |\mathbf{A} - \nabla\theta|^2\big)\,dx \\
&\leqslant \int_\Omega \big[\phi^2|\nabla\psi|^2 + 2k^2\phi^2 f^2(\mathbf{A} - \nabla\theta)\cdot\mathbf{A} - k^2\phi^2 f^2|\mathbf{A}|^2\big]\,dx \\
&\leqslant \|\phi\|_4^2\|\nabla\psi\|_4^2 + 2k^2\big\|f(\mathbf{A} - \nabla\theta)\big\|_2\|\phi\|_6^2\|\mathbf{A}\|_6 + k^2\|\mathbf{A}\|_4^2\|\phi\|_4^2 \\
&\leqslant K_3\|\phi\|_4^2\|\nabla\psi\|_2\|\nabla\psi\|_{H^1} + C(\Gamma) \\
&\leqslant C(\Gamma) + \|\Delta\psi\|_2^2 \leqslant C(\Gamma) + 4k^2\big(\|P\|_2^2 + \|Q\|_2^2\big), \\
J_2 &:= \int_\Omega f(\phi - \dot\theta)\nabla f\cdot\nabla\phi \leqslant \big\|f(\phi - \dot\theta)\big\|_2\|\nabla\psi\|_4\|\nabla\phi\|_4 \\
&\leqslant \big\|f(\phi - \dot\theta)\big\|_2^2 + K_3^2\|\nabla\psi\|_2\|\nabla\phi\|_2\|\nabla\psi\|_{H^1}\|\nabla\phi\|_{H^1} \\
&\leqslant \big\|f(\phi - \dot\theta)\big\|_2^2 + C(\Gamma)\|\Delta\psi\|_2^2 + \frac{\eta}{4k^2}\|\Delta\phi\|_2^2 + C(\Gamma), \\
J_3 &:= \int_\Omega \big[f^2(\dot{\mathbf{A}} - \nabla\phi)\cdot(\mathbf{A} - \nabla\theta) + f^2|\dot{\mathbf{A}}|^2\big]\,dx \\
&\leqslant \big\|f(\mathbf{A} - \nabla\theta)\big\|_2\big(\|\dot{\mathbf{A}}\|_2 + \|\nabla\phi\|_2\big) + \|\dot{\mathbf{A}}\|_2^2 \leqslant C(\Gamma) + 2\|\dot{\mathbf{A}}\|_2^2, \\
J_4 &:= \int_\Omega \big[f\dot f(\mathbf{A} - \nabla\theta)\cdot\nabla\phi - f\phi\nabla f\cdot\dot{\mathbf{A}}\big]\,dx \\
&\leqslant \|\dot f\|_2\|\nabla\phi\|_4\big(\|f\mathbf{A}\|_4 + \|f\nabla\theta\|_4\big) + \|\dot{\mathbf{A}}\|_2\|\nabla f\|_4\|\phi\|_4 \\
&\leqslant \|\dot f\|_2\|\nabla\phi\|_4\left(\|\mathbf{A}\|_4 + \frac{1}{k}\|\nabla\psi\|_4\right) + \|\dot{\mathbf{A}}\|_2\|\nabla\psi\|_4\|\phi\|_4 \\
&\leqslant C(\Gamma)\big(\|\dot f\|_2^2 + \|\dot{\mathbf{A}}\|_2^2\big) + K_3\|\Delta\phi\|_2\|\nabla\phi\|_2 \\
&\quad + K_3^2\|\nabla\phi\|_2\|\nabla\psi\|_2\|\Delta\phi\|_2\|\Delta\psi\|_2 + K_3\|\Delta\psi\|_2\|\nabla\psi\|_2 \\
&\leqslant C(\Gamma)\big(\|\dot f\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|P\|_2^2 + \|Q\|_2^2\big) + \frac{\eta}{4k^2}\|\Delta\phi\|_2^2 + C(\Gamma), \\
J_5 &:= \int_\Omega \left[-f\big(f^2 - 1\big)(\mathbf{A} - \nabla\theta)\cdot\mathbf{R} - \frac{1}{k}\big(3f^2 - 1\big)\nabla f\cdot\mathbf{S}\right]dx \\
&\leqslant 2\big\|f(\mathbf{A} - \nabla\theta)\big\|_2\|\mathbf{R}\|_2 + \frac{4}{k}\|\nabla f\|_2\|\mathbf{S}\|_2 \leqslant \frac{1}{2}\|\mathbf{R}\|_2^2 + \frac{1}{2}\|\mathbf{S}\|_2^2 + C(\Gamma), \\
J_6 &:= \int_\Omega f^3\phi(\mathbf{A} - \nabla\theta)\cdot\nabla f\,dx \leqslant \big\|f(\mathbf{A} - \nabla\theta)\big\|_2\|\phi\|_4\|\nabla f\|_4 \\
&\leqslant C(\Gamma) + K_3\|\nabla\psi\|_2\|\nabla\psi\|_{H^1} \\
&\leqslant C(\Gamma) + \|\Delta\psi\|_2^2 \leqslant C(\Gamma) + 4k^2\big(\|P\|_2^2 + \|Q\|_2^2\big),
\end{aligned}
$$

$$
\begin{aligned}
J_7 &:= \int_\Omega \Big[-\big(f^4 - f^2\big)(\mathbf{A} - \nabla\theta)\cdot(\nabla\phi - \dot{\mathbf{A}}) + \frac{1}{k^2}\big(3f^2 - 1\big)|\nabla f|^2 \\
&\qquad + f^2\big(f^2 - 1\big)|\mathbf{A} - \nabla\theta|^2 + \big(f^2 - 1\big) f \dot f \Big]\, dx \\
&\leqslant C(\Gamma) + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2, \\
J_8 &:= -\int_{\partial\Omega} \dot{\mathbf{A}}\cdot \mathbf{H}_{ex} \times \mathbf{n}\, da \leqslant K_2 \|\nabla\times\dot{\mathbf{A}}\|_2 \|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \\
&\leqslant \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{K_2^2}{\eta}\|\mathbf{H}_{ex}\times\mathbf{n}\|^2_{H^{-1/2}(\partial\Omega)}, \\
J_9 &:= -\int_\Omega \nabla\times\dot{\mathbf{A}}\cdot\big[2f\nabla f\times(\mathbf{A} - \nabla\theta) + f^2\nabla\times\mathbf{A}\big]\, dx \\
&\leqslant 2\|\nabla\times\dot{\mathbf{A}}\|_2\|\nabla f\|_4\big\|f(\mathbf{A} - \nabla\theta)\big\|_4 + \|\nabla\times\dot{\mathbf{A}}\|_2\|\nabla\times\mathbf{A}\|_2 \\
&\leqslant \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{6}{\eta}\|\nabla f\|_4^2\big(\|f\mathbf{A}\|_4 + \|f\nabla\theta\|_4\big)^2 + \frac{3}{\eta}\|\nabla\times\mathbf{A}\|_2^2 \\
&\leqslant \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{12}{\eta}\|\nabla f\|_4^2\big(\|\mathbf{A}\|_4^2 + \|f\nabla\theta\|_4^2\big) + C(\Gamma) \\
&\leqslant \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2 + C(\Gamma)\|\nabla\psi\|_4^2 + \frac{12K_3^2}{\eta k^2}\|\nabla\psi\|_2^2\|\Delta\psi\|_2^2 + C(\Gamma) \\
&\leqslant \frac{\eta}{4}\|\nabla\times\dot{\mathbf{A}}\|_2^2 + C(\Gamma)\|\Delta\psi\|_2^2 + C(\Gamma), \\
J_{10} &:= -\int_\Omega f^2(\mathbf{A} - \nabla\theta)\cdot\nabla\times\nabla\times\mathbf{A}\, dx \leqslant C(\Gamma) + \|\nabla\times\nabla\times\mathbf{A}\|_2^2.
\end{aligned}
$$

From the previous estimates, we deduce the inequality

$$
\begin{aligned}
I_3 &\leqslant C_1(\Gamma) + C_2(\Gamma)\big(\|P\|_2^2 + \|Q\|_2^2 + \|\nabla\times\nabla\times\mathbf{A}\|_2^2\big) \\
&\quad + C_3(\Gamma)\big[\big\|kf(\dot\theta - \phi)\big\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2\big] \\
&\quad + \frac{1}{2}\left(\|\mathbf{R}\|_2^2 + \|\mathbf{S}\|_2^2 + \eta\|\nabla\times\dot{\mathbf{A}}\|_2^2 + \frac{\eta}{k^2}\|\Delta\phi\|_2^2\right). \qquad \square
\end{aligned}
$$

Lemma 6. *If $\mathcal{E}_0(f_0, \nabla\theta_0, \mathbf{A}_0) \leqslant M$, the following inequalities hold*

$$
\int_t^{t+1} \big[\|\dot f\|_2^2 + \big\|kf(\dot\theta - \phi)\big\|_2^2 + \eta\|\mathbf{A}\|_2^2\big]\, d\tau \leqslant C(\Gamma), \tag{59}
$$

$$
\int_t^{t+1} \mathcal{E}_1(\tau)\, d\tau \leqslant C(\Gamma) \tag{60}
$$

for $t > t_0$.

Proof. Let us consider the relation (47) and integrate in the time interval $[t, t+1]$. We have

$$\mathcal{F}(t+1) - \mathcal{F}(t) + \lambda \int_t^{t+1} \mathcal{F}(\tau)\,d\tau + \int_t^{t+1} \big[\|\dot f\|_2^2 + \|kf(\dot\theta - \phi)\|_2^2 + \eta\|\mathbf{A}\|_2^2\big]\,d\tau \leqslant C(\Gamma).$$

Since the functional $\mathcal{F}$ is positive definite, we obtain

$$\int_t^{t+1} \big[\|\dot f\|_2^2 + \|kf(\dot\theta - \phi)\|_2^2 + \eta\|\mathbf{A}\|_2^2\big]\,d\tau \leqslant C(\Gamma) + \mathcal{F}(t).$$

Thus, keeping (45) into account, by Theorem 3, we prove (59).

In order to prove (60) we observe that, by definition (48), we obtain

$$\int_t^{t+1} \mathcal{E}_1(\tau)\,d\tau = \frac{1}{2}\int_t^{t+1}\int_\Omega \big[P^2 + Q^2 + |\nabla\times\nabla\times\mathbf{A}|^2\big]\,dx\,d\tau.$$

Moreover, by using Eqs. (24)–(26), we have

$$\begin{aligned}\int_t^{t+1} \mathcal{E}_1(\tau)\,d\tau &= \frac{1}{2}\int_t^{t+1}\int_\Omega \big\{\big[\dot f + f\big(f^2-1\big)\big]^2 + k^2 f^2(\dot\theta-\phi)^2 \\ &\quad + \big[\eta(\dot{\mathbf{A}} - \nabla\phi) + f^2(\mathbf{A}-\nabla\theta)\big]^2\big\}\,dx\,d\tau \\ &\leqslant \int_t^{t+1} \big[\|\dot f\|_2^2 + \big\|f\big(f^2-1\big)\big\|_2^2 \\ &\quad + \frac{k^2}{2}\big\|f(\dot\theta-\phi)\big\|_2^2 + 2\eta^2\big(\|\dot{\mathbf{A}}\|_2^2 + \|\nabla\phi\|_2^2\big) + \big\|f^2(\mathbf{A}-\nabla\theta)\big\|_2^2\big]\,d\tau,\end{aligned}$$

so that in view of (59), of Lemma 4 and of Theorem 3, we obtain (60). □

Theorem 7. *If $\Omega \subset \mathbb{R}^2$, the system* (24)–(26) *possesses a maximal attractor.*

Proof. In view of Lemma 5, from (55) we have

$$\begin{aligned}&\frac{d}{dt}(\mathcal{E}_1 + \gamma\mathcal{E}_0) + \int_\Omega \bigg[P^2 + Q^2 + \frac{\eta}{2k^2}(\Delta\phi)^2 + \frac{\eta}{2}|\nabla\times\dot{\mathbf{A}}|^2 + |\nabla\times\nabla\times\mathbf{A}|^2 \\ &\qquad + \frac{1}{2}|\mathbf{R}|^2 + \frac{1}{2}|\mathbf{S}|^2\bigg]\,dx \\ &\quad \leqslant C_1(\Gamma) + 2C_2(\Gamma)\mathcal{E}_1 + C_3(\Gamma)\big[\|kf(\dot\theta-\phi)\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2\big].\end{aligned}$$

Hence

$$\frac{d}{dt}(\mathcal{E}_1 + \gamma\mathcal{E}_0) \leqslant C_1(\Gamma) + 2C_2(\Gamma)(\mathcal{E}_1 + \gamma\mathcal{E}_0) + C_3(\Gamma)\big[\|kf(\dot\theta-\phi)\|_2^2 + \|\dot{\mathbf{A}}\|_2^2 + \|\dot f\|_2^2\big].$$

The inequalities (59) and (60) allow to apply the uniform Gronwall lemma (see [13]) which proves that $\mathcal{E}_1(t)$ is bounded for $t > t_0$. This guarantees the existence of the maximal attractor for the system. □

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