Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 326 (2007) 754–771
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The longtime behavior of a nonlinear Reissner–Mindlin plate with exponentially decreasing memory kernels

Claudio Giorgi, Federico M. Vegni *

Dipartimento di Matematica, Facoltà di Ingegneria, Università di Brescia, Via Valotti 9, 25133 Brescia, Italy

Received 15 November 2004
Available online 18 April 2006
Submitted by K.A. Lurie

Abstract

In this paper we investigate the longtime behavior of the mathematical model of a homogeneous viscoelastic plate based on Reissner–Mindlin deformation shear assumptions. According to the approximation procedure due to Lagnese for the Kirchhoff viscoelastic plate, the resulting motion equations for the vertical displacement and the angle deflection of vertical fibers are derived in the framework of the theory of linear viscoelasticity. Assuming that in general both Lame's functions, λ and μ, depend on time, the coupling terms between the equations of displacement and deflection depend on hereditary contributions. We associate to the model a nonlinear semigroup and show the behavior of the energy when time goes on. In particular, assuming that the kernels λ and μ decay exponentially, and not too weakly with respect to the physical properties considered in the model, then the energy decays uniformly with respect to the initial conditions; i.e., we prove the existence of an absorbing set for the semigroup associated to the model.

Keywords: Linear viscoelasticity; Reissner–Mindlin plates; Dynamical systems

1. Introduction

In order to describe the mechanical behavior of the plate, we first formulate proper constitutive equations in the framework of the well-established linear theory of viscoelasticity. Then, by means of the Mindlin strain-displacement relations (first assumed in [12]) and the Reissner

* Corresponding author.
E-mail addresses: giorgi@ing.unibs.it (C. Giorgi), federico@ing.unibs.it (F.M. Vegni).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.03.024

assumption on deformation shear [15], we deduce the set of model equations for the plate, by paralleling the procedure performed in [9] and [10].

The resulting system models the evolution of vertical displacement and deflections of a homogeneous, isotropic Mindlin plate with memory.

2. The Reissner–Mindlin plate model with memory

The first step of this paper is to formulate a mathematical model of the motion of a thin plate obeying a stress–strain hereditary constitutive equation.

We consider a thin plate of uniform thickness d. When the plate is in equilibrium, we assume it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^3$ placed in a reference frame $\mathbf{x} = (x_1, x_2, x_3)$. Let $U_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, denote the components of the displacement vector of the point $\mathbf{x}$. The plate has a middle surface midway between its faces in a region $\Omega \subset \mathbb{R}^2$ of the plane $x_3 = 0$. We suppose that its smooth boundary $\Gamma = \partial\Omega$ is composed for two parts, $\Gamma = \Gamma_0 \cup \Gamma_1$, such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$ and we assume that the plate is rigidly clamped along Γ_0 and simply supported along Γ_1. Henceforth, we denote by $u_i(x_1, x_2)$, $i = 1, 2, 3$, the components of the displacement vector of the points of the middle surface Ω of the plate which have coordinates $(x_1, x_2, 0)$ at equilibrium.[1] Moreover, let $\Omega^- = \Omega \times \{-d/2\}$ and $\Omega^+ = \Omega \times \{d/2\}$ denote its faces, and $\Lambda = \partial\Omega \times (-d/2, d/2)$ its edge.

As is well known, in standard linear elasticity the stress–strain relation is given by

$$\mathbf{S}(\mathbf{x}, t) = \mathbb{L}_0\mathbf{E}(\mathbf{x}, t),$$

where the *elastic strain* $\mathbf{E}$ and the *stress* $\mathbf{S}$ are second-order tensors. In small displacement theory, $\mathbf{E}$ is given by

$$\mathbf{E}(\mathbf{x}, t) = \frac{1}{2}\big(\nabla\mathbf{U}(\mathbf{x}, t) + (\nabla\mathbf{U})^T(\mathbf{x}, t)\big), \tag{2.1}$$

$\mathbf{U}$ being the *displacement vector*. In the isotropic case, the fourth-order tensor $\mathbb{L}_0$ involves two independent *Lame's constants* λ and μ, namely

$$\mathbb{L}_0 = \lambda\mathbf{I} \otimes \mathbf{I} + 2\mu\mathbb{I}, \quad \text{for } \tau \geqslant 0.$$

According to this constitutive equation, a mathematical model for the isotropic elastic plate can be derived by assuming the Mindlin–Timoshenko hypothesis (see, for instance, [10]). Namely, the linear filaments of the plate initially perpendicular to the middle surface Ω are required to remain straight and undergo neither contraction nor extension, but the Kirchhoff assumption that they remain perpendicular to the deformed middle surface is removed.

In small displacement theory this assumption implies that transverse shear effects may be no longer neglected and the small displacement $\mathbf{U}(x_1, x_2, x_3)$ is related to the displacement of the middle surface $\mathbf{u}(x_1, x_2)$ by the approximate relations

$$U_1 = u_1 - x_3\psi, \qquad U_2 = u_2 - x_3\phi, \qquad U_3 = u_3, \tag{2.2}$$

where $\psi(x_1, x_2)$ and $\phi(x_1, x_2)$ are the angles of deflection of the filament with respect to the normal direction along the x_1- and x_2-axis, respectively. In virtue of (2.1), this assumption leads to the strain-displacement relations of the Mindlin–Timoshenko model, namely

[1] Observe that we denote with the pedex $1, 2, \ldots$ the component of a vector, and with the pedex x, t, s the derivative with respect to the variable.

$$\begin{cases} E_{11} = \dfrac{\partial u_1}{\partial x_1} - x_3 \dfrac{\partial \psi}{\partial x_1}, \\ E_{22} = \dfrac{\partial u_2}{\partial x_2} - x_3 \dfrac{\partial \phi}{\partial x_2}, \\ E_{12} = \dfrac{1}{2}\left[\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} - x_3\left(\dfrac{\partial \psi}{\partial x_2} + \dfrac{\partial \phi}{\partial x_1}\right)\right], \\ E_{13} = \dfrac{1}{2}\left[\dfrac{\partial w}{\partial x_1} - \psi\right], \\ E_{23} = \dfrac{1}{2}\left[\dfrac{\partial w}{\partial x_2} - \phi\right], \\ E_{33} = 0, \end{cases} \tag{2.3}$$

where $w = u_3$ is the bending component of the displacement $\mathbf{u}$. Here, in order to generalize the model, we assume that the plate is composed of an *isotropic linear viscoelastic* material. As a consequence, the *stress–strain* law is given by

$$\mathbf{S}(\mathbf{x}, t) = D\mathbb{L} * \mathbf{E}(\mathbf{x}, t), \tag{2.4}$$

where $*$ denotes a convolution

$$D\mathbb{L} * \mathbf{E}(\mathbf{x}, t) = \int_{-\infty}^{\infty} D_\tau \mathbb{L}(\tau)\mathbf{E}(\mathbf{x}, t - \tau)\, d\tau$$

and D_τ is the distributional derivative with respect to τ. Moreover, we observe that $\mathbb{L}(\tau)$ is an isotropic fourth-order tensor which vanishes for $\tau < 0$ and involves two independent *relaxation functions* λ and μ, namely

$$\mathbb{L}(\tau) = \lambda(\tau)\mathbf{I} \otimes \mathbf{I} + 2\mu(\tau)\mathbb{I}, \quad \text{for } \tau \geqslant 0.$$

Because of the relation $D_\tau \mathbb{L} = \mathbb{L}(0)\delta + \mathbb{L}'$, where $' = d/dt$ and δ represents the Dirac delta distribution, we have

$$D\mathbb{L} * \mathbf{E}(\mathbf{x}, t) = \mathbb{L}(0)\mathbf{E}(\mathbf{x}, t) + \int_0^{\infty} \mathbb{L}'(\tau)\mathbf{E}(\mathbf{x}, t - \tau)\, d\tau. \tag{2.5}$$

In order to apply a variational principle and attain the motion equation, we shall transform the original stress–strain constitutive equation by means of the Laplace transform. Let $\varphi(t)$ be a function taking values in a Hilbert space, and let denote by $\hat{\varphi}$ its Laplace transform, namely

$$\hat{\varphi}(s) = \int_0^{\infty} e^{-st}\varphi(t)\, dt.$$

In particular, we have $(D\mathbb{L})\hat{}\,(s) = s\hat{\mathbb{L}}(s)$ where

$$\hat{\mathbb{L}}(s) = \hat{\lambda}(s)\mathbf{I} \otimes \mathbf{I} + 2\hat{\mu}(s)\mathbb{I}.$$

Formally applying the Laplace transform in (2.4)–(2.5), we get

$$\hat{S}_{ij} = s\hat{\lambda}\delta_{ij}\hat{E}_{kk} + 2s\hat{\mu}\hat{E}_{ij}. \tag{2.6}$$

Now, we introduce the *viscoelastic Poisson's ratio* ν and *viscoelastic Young's modulus* E so that their Laplace transforms are respectively defined by

$$\hat{\nu} = \frac{\hat{\lambda}}{2s(\hat{\lambda} + \hat{\mu})}, \qquad \hat{E} = \frac{\hat{\mu}(3\hat{\lambda} + 2\hat{\mu})}{\hat{\lambda} + \hat{\mu}}. \tag{2.7}$$

Then, we have

$$2\hat{\mu} = \frac{\hat{E}}{1 + s\hat{\nu}}, \qquad \hat{\lambda} = \frac{\hat{E}}{1 + s\hat{\nu}} \cdot \frac{s\hat{\nu}}{1 - 2s\hat{\nu}} \tag{2.8}$$

and (2.6) becomes

$$\hat{S}_{ij} = \frac{s\hat{E}}{1 + s\hat{\nu}}\left(\hat{E}_{ij} + \frac{s\hat{\nu}}{1 - 2s\hat{\nu}}\hat{E}_{kk}\delta_{ij}\right). \tag{2.9}$$

As is customary in thin plate theory, we assume that the exterior normal force acting on the faces Ω^+ and Ω^- is zero, so that the transverse normal stress is negligible compared to other stress:

$$\hat{S}_{33} = S_{33} \approx 0 \quad \text{in } \Omega^+ \cup \Omega^-.$$

This allows $\hat{E}_{33}$ to be expressed as a function of $\hat{E}_{11}$ and $\hat{E}_{22}$, namely

$$\hat{E}_{33} = -\frac{s\hat{\nu}}{1 - s\hat{\nu}}(\hat{E}_{11} + \hat{E}_{22}). \tag{2.10}$$

Substituting (2.10) into (2.9), we get

$$\begin{cases} \hat{S}_{11} = \dfrac{s\hat{E}}{1 - s^2\hat{\nu}^2}[\hat{E}_{11} + s\hat{\nu}\hat{E}_{22}], \\ \hat{S}_{22} = \dfrac{s\hat{E}}{1 - s^2\hat{\nu}^2}[s\hat{\nu}\hat{E}_{11} + \hat{E}_{22}], \\ \hat{S}_{33} = 0, \\ \hat{S}_{ij} = K\dfrac{s\hat{E}}{1 + s\hat{\nu}}\hat{E}_{ij}, \quad i \neq j, \end{cases} \tag{2.11}$$

where, as is customary in the theory of Mindlin–Timoshenko beams and plates, consistency with the presence of transverse shear justifies the introduction of a suitable scalar factor K to correct its expression.

Discrepancy between the last relation in (2.3) and (2.10) is due to the fact that terms of order d^2 and higher have been ignored here. In addition, consistency with the absence of transverse shear requires that the plate is subject to an external distribution of loads per unit mass $\mathbf{f} = (f_1, f_2, f_3)$ with f_1 and f_2 independent of x_3.

The motion equation, via a variational formulation, can be obtained by introducing the *viscoelastic energy* $\mathcal{P}(t)$ whose transform is given by

$$\hat{\mathcal{P}}(s) = \frac{1}{2}\int_{-d/2}^{d/2}\int_{\Omega}\left(\hat{S}_{ij}\hat{E}_{ij} + \rho_0 s^2\hat{U}_i\hat{U}_i\right)dx_1\,dx_2\,dx_3. \tag{2.12}$$

If integration with respect to x_3 is carried out, the stretching components $\hat{u}_1$, $\hat{u}_2$ uncouple from the bending component $\hat{w}$ of the plate displacement, so that both the *strain energy* and the *kinetic energy* split into two parts (see [10]).

Substituting (2.11) and (2.2)–(2.3) into (2.12) and considering only the part $\hat{\mathcal{P}}_b(s)$ of $\hat{\mathcal{P}}(s)$ containing the bending terms, we obtain

$$\begin{aligned}\hat{\mathcal{P}}_b(s) = \frac{s\hat{N}}{2}\int_\Omega \bigg[\left(\frac{\partial\hat{\psi}}{\partial x_1}\right)^2 + \left(\frac{\partial\hat{\phi}}{\partial x_2}\right)^2 + 2s\hat{\nu}\frac{\partial\hat{\psi}}{\partial x_1}\frac{\partial\hat{\phi}}{\partial x_2} \\ + K\frac{(1-s\hat{\nu})}{2}\left(\frac{\partial\hat{\psi}}{\partial x_2} + \frac{\partial\hat{\phi}}{\partial x_1}\right)^2\bigg]\,dx_1\,dx_2 \\ + \frac{s\hat{H}}{2}\int_\Omega \bigg[\left(\frac{\partial\hat{w}}{\partial x_1} - \hat{\psi}\right)^2 + \left(\frac{\partial\hat{w}}{\partial x_2} - \hat{\phi}\right)^2\bigg]\,dx_1\,dx_2 \\ + \frac{s^2\rho_0 d}{2}\int_\Omega \left\{\frac{d^2}{12}\big(\hat{\psi}^2 + \hat{\phi}^2\big) + \hat{w}^2\right\}dx_1\,dx_2.\end{aligned} \tag{2.13}$$

Here N and H are assumed to be regular causal functions such that their Laplace transforms satisfy the relations

$$\hat{N}(s) = \frac{\hat{E}(s)d^3}{12(1 - s^2\hat{\nu}^2(s))}, \tag{2.14}$$

$$\hat{H}(s) = \frac{\hat{E}(s)d^3 K}{48(1 + s^2\hat{\nu}(s))}. \tag{2.15}$$

3. The variational setting

We transform (2.13) in the time domain and study the associated variational problem.

Let $v = \begin{bmatrix}\psi \\ \phi\end{bmatrix}$ and introduce the spatial operator

$$A = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x_2^2} & \frac{1+\nu}{2}\frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{1+\nu}{2}\frac{\partial^2}{\partial x_1 \partial x_2} & \frac{1-\nu}{2}\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \end{bmatrix},$$

where ν is a constant such that $0 < \nu < 1/2$.

We follow Dafermos [1] (see also [7]), in order to formulate the problem in a history space setting and we introduce the additional variables

$$\eta^t(s) = v(t) - v(t-s), \tag{3.1}$$

$$\rho^t(s) = w(t) - w(t-s) \tag{3.2}$$

for $t \in [0, +\infty)$ and $s \in (0, +\infty)$.

The dynamics of η and ρ are governed by the equations

$$\eta_t + \eta_s = v_t \quad \text{in } \Omega \times (0, +\infty)^2,$$
$$\rho_t + \rho_s = w_t \quad \text{in } \Omega \times (0, +\infty)^2$$

along with the initial and boundary conditions

$$\eta^0 = \eta_0, \quad \rho^0 = \rho_0, \quad \text{in } \Omega \times (0, +\infty),$$
$$\eta^t(0) = 0, \quad \rho^t(0) = 0, \quad \text{in } \Omega \times (0, +\infty),$$

where $\eta_0 = v(0) - v(-s)$ and $\rho_0 = w(0) - w(-s)$ describe the past history; the homogeneous boundary conditions are a consequence of definitions (3.1), (3.2).

Then, we associate the variables η and ρ to the memory kernels μ and κ, such that

$$\mu(\sigma) = -d^2 N'(\sigma),$$
$$\kappa(\sigma) = -d^2 H'(\sigma).$$

We observe that the introduction of the past history as a new state variable seems unavoidable if one is interested in analyzing the structural stability of the system with respect to initial data.

The analytic purpose of this work is to prove some continuous dependence and uniqueness results which allow to interpret the model as a dynamical system. Then, we show the dissipativity of the semigroup associated to the problem by constructing an *absorbing set* (see, e.g., [8,16]). This analysis extends the studies proposed in [7] where the beam case is examined; we note that the results concerning the plate model are not trivial and that we remove the balance hypothesis on the structural constants of the model, only asking that the memory kernels should dissipate sufficient energy. We also remark that the longtime behavior of the plate model can be investigated by a direct approach, which does not imply the use of semigroup techniques, and that a structural–polynomial-stability of the solutions can be proved if the memory kernels decay polynomially (see [14]).

Introducing the nonlinear functions f and g we are able to state our problem.

Problem P. Our purpose is to study the rate of decay of the solution (v, w, η, ρ) associated to the semilinear system of equations

$$v_{tt} - Av - \int_0^\infty \mu(s) A\eta(s)\, ds + \kappa_0(v + \nabla w) + \int_0^\infty \kappa(s)\big(\eta(s) + \nabla\rho(s)\big)\, ds + f(v) = 0, \tag{3.3}$$

$$w_{tt} - \kappa_0 \nabla\cdot(v + \nabla w) - \int_0^\infty \kappa(s)\nabla\cdot\big(\eta(s) + \nabla\rho(s)\big)\, ds + g(w) = 0, \tag{3.4}$$

$$\eta_t + \eta_s = v_t, \tag{3.5}$$

$$\rho_t + \rho_s = w_t \tag{3.6}$$

in Ω, for any $t > 0$, $s > 0$, which satisfy the boundary problem for the mechanical variables

$$\left(\frac{\partial\psi}{\partial x_1} + \nu\frac{\partial\phi}{\partial x_2}\right)n_1 + \frac{1-\nu}{2}\left(\frac{\partial\psi}{\partial x_2} + \frac{\partial\phi}{\partial x_1}\right)n_2 = 0, \tag{3.7}$$

$$\frac{1-\nu}{2}\left(\frac{\partial\psi}{\partial x_2} + \frac{\partial\phi}{\partial x_1}\right)n_1 + \left(\nu\frac{\partial\psi}{\partial x_1} + \frac{\partial\phi}{\partial x_2}\right)n_2 = 0, \tag{3.8}$$

$$(v + \nabla w)\cdot n = 0 \tag{3.9}$$

in $\partial\Omega \times [0, +\infty)$ along with the boundary problem for the history variables

$$\left(\frac{\partial\eta_1}{\partial x_1} + \nu\frac{\partial\eta_2}{\partial x_2}\right)n_1 + \frac{1-\nu}{2}\left(\frac{\partial\eta_1}{\partial x_2} + \frac{\partial\eta_2}{\partial x_1}\right)n_2 = 0,$$

$$\frac{1-\nu}{2}\left(\frac{\partial\eta_1}{\partial x_2} + \frac{\partial\eta_2}{\partial x_1}\right)n_1 + \left(\nu\frac{\partial\eta_1}{\partial x_1} + \frac{\partial\eta_2}{\partial x_2}\right)n_2 = 0,$$

$$(\eta + \nabla\rho)\cdot n = 0$$

in $\partial\Omega \times [0,+\infty) \times (0,+\infty)$; and

$$\eta^t(0) = 0,$$
$$\rho^t(0) = 0$$

on $\Omega \times [0,+\infty)$. We study the initial value problem

$$\begin{aligned}
&v(0) = v_0 && \text{in } \Omega,\\
&v_t(0) = v_1 && \text{in } \Omega,\\
&\eta^0 = \eta_0 && \text{in } \Omega \times (0,\infty),\\
&w(0) = w_0 && \text{in } \Omega,\\
&w_t(0) = w_1 && \text{in } \Omega,\\
&\rho^0 = \rho_0 && \text{in } \Omega \times (0,\infty).
\end{aligned}$$

4. Hypothesis and preliminary results

4.1. Hypothesis on memory kernels

As far as existence of the solution to Problem P is concerned, we let the memory kernels $\mu(s)$ and $\kappa(s)$ satisfy the following conditions:

$$\mu(s), \kappa(s) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{4.1}$$

$$\mu(s), \kappa(s) \geqslant 0, \tag{4.2}$$

$$\mu'(s), \kappa'(s) \leqslant 0. \tag{4.3}$$

In view of (4.1) and (4.2) we define

$$\mu_0 = \int_0^\infty \mu(s)\,ds \quad \text{and} \quad \kappa_{00} = \int_0^\infty \kappa(s)\,ds.$$

In order to obtain energy uniform estimates, we will suppose also

$$\begin{cases} \mu' \in L^2(0, s_\mu), \quad \kappa' \in L^2(0, s_\kappa), \\ \mu'(s) + \delta_\mu \mu(s) \leqslant 0, \quad \kappa'(s) + \delta_\kappa \kappa(s) \leqslant 0 \quad \forall s \in \mathbb{R}^+, \end{cases} \tag{4.4}$$

and that

$$\mu'(s) + M_\mu \mu(s) \geqslant 0 \quad \forall s \geqslant s_\mu, \qquad \kappa'(s) + M_\kappa \kappa(s) \geqslant 0 \quad \forall s \geqslant s_\kappa \tag{4.5}$$

for some $0 < \delta_\mu < M_\mu$, $0 < \delta_\kappa < M_\kappa$ and $s_\mu, s_\kappa > 0$, where M_μ, M_κ are constants depending on s_μ, s_κ and increasing as σ_μ and σ_κ decrease, respectively.

The conditions (4.4) imply the exponential decay of the kernels. This hypothesis seems unavoidable in order to have exponential decay of the associated linear problem and it is commonly assumed (cf., e.g., [3,11]). On the other hand, it seems quite obvious that to have exponential decay of the energy, the kernel must show the same rate of decay (see [6]).

We suppose also

$$\mu_0 \geqslant 2, \tag{4.6}$$

$$\kappa_{00} \geqslant 2. \tag{4.7}$$

Remark 4.1. We point out that the only reason of being dissipative for the model we have introduced is due to the presence of the memory terms. Hence, hypothesis (4.6) and (4.7) expresses the fact that the—only—dissipation in the model has to be "strong enough," in order to prove uniform decay of the solution with respect to initial data.[2]

Notation. Let us introduce some notation. We consider the Hilbert spaces $L^2(\Omega)$ and $H^1(\Omega)$ with the standard norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ and Hilbert products $(\cdot,\cdot)_{L^2}$, $(\cdot,\cdot)_{H^1}$. Let us use the same notation $\|\cdot\|_{L^2}$ ($\|\cdot\|_{H^1}$) both for the norm in $L^2(\Omega)$ ($H^1(\Omega)$) and for that in $L^2(\Omega,\mathbb{R}^2)$ ($H^1(\Omega,\mathbb{R}^2)$, respectively).

Let us consider two functions $v, z \in H^1(\Omega,\mathbb{R}^2)$, on account of (3.7)–(3.8), we can set

$$\begin{aligned}\langle -Av, z\rangle_{L^2} = \int_\Omega \bigg\{&\left(\frac{\partial v_1}{\partial x_1} + \nu\frac{\partial v_2}{\partial x_2}\right)\frac{\partial z_1}{\partial x_1} + \frac{1-\nu}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\frac{\partial z_1}{\partial x_2} \\ &+ \frac{1-\nu}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\frac{\partial z_2}{\partial x_1} + \left(\nu\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right)\frac{\partial z_2}{\partial x_2}\bigg\}\, d\Omega. \end{aligned} \tag{4.8}$$

Now, we observe that

$$\langle -Av, v\rangle_{L^2} = \int_\Omega \left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_2}{\partial x_2}\right)^2 + 2\nu\frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \frac{1-\nu}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)^2\right\} d\Omega.$$

Let us introduce the Hilbert space

$$V = \left\{(v_1, v_2) \in L^2(\Omega,\mathbb{R}^2)\colon\ \frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \in L^2(\Omega)\right\}$$

with the norm

$$\|v\|_V^2 = \int_\Omega \left\{v_1 + v_2 + \left|\frac{\partial v_1}{\partial x_1}\right|^2 + \left|\frac{\partial v_2}{\partial x_2}\right|^2 + \left|\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right|^2\right\} d\Omega.$$

The kernel of $-A$ is the subspace of V of the linear function of the form

$$\begin{bmatrix} c_0x_2 + c_1 \\ -c_0x_1 + c_2 \end{bmatrix},$$

where c_0, c_1, and c_2 are constants. It follows that the linear subspace of V which is orthogonal in $L^2(\Omega,\mathbb{R}^2)$ to the kernel of $-A$ is characterized by the conditions

$$\int_\Omega (x_2v_1 - x_1v_2)\, d\Omega = 0, \qquad \int_\Omega v_1\, d\Omega = 0, \qquad \int_\Omega v_2\, d\Omega = 0$$

and we denote by V_A this space.

[2] Obviously the value 2 is connected with the technique used in calculation, and it is none interest to find a sharper value. The thing which seems unavoidable is the lower bound to the $L^1(\mathbb{R}^+)$-norms of memory kernels.

Remark 4.2. The operator $-A$ is coercive on V_A. Exploiting Korn inequality (see, e.g., [2,4]), it is possible to prove that the norm $\|\cdot\|_V$ introduced by the operator $-A$ in V_A is equivalent to the usual norm in $H^1(\Omega,\mathbb{R}^2)$. Hence, in what follows, we shall often exploit the inequality

$$c_1\|v\|_V \leqslant \|v\|_{H^1} \leqslant c_2\|v\|_V$$

for every $v \in V_A$ and some $0 < c_1 < c_2$.

Thanks to (4.1), (4.2) and Remark 4.2, we introduce the Hilbert spaces

$$\mathcal{M} = L^2_\mu\big(\mathbb{R}^+; V_A\big) \cap L^2_\kappa\big(\mathbb{R}^+; L^2\big(\Omega,\mathbb{R}^2\big)\big),$$
$$\mathcal{N} = L^2_\kappa\big(\mathbb{R}^+; H^1(\Omega)\big),$$

where the natural scalar product is

$$(\xi,\zeta)_{\mathcal{M}} = \int_0^\infty \mu(s)\big(\xi(s),\zeta(s)\big)_V\,ds + \int_0^\infty \kappa(s)\big(\xi(s),\zeta(s)\big)_{L^2}\,ds,$$

$$(\xi,\zeta)_{\mathcal{N}} = \int_0^\infty \kappa(s)\big(\xi(s),\zeta(s)\big)_{H^1}\,ds.$$

4.2. Hypothesis on the nonlinearities

We assume that $f \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ and $g \in C^1(\mathbb{R},\mathbb{R})$ are such that there exist F and G as

$$f = \nabla F, \qquad g = G'$$

and $F(0,0) = G(0) = 0$. Let $s \in \mathbb{R}$ and $x \in \mathbb{R}^2$ and we make the following assumptions (see [13]):

$$\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \geqslant 0, \tag{4.9}$$

$$\liminf_{|s|\to\infty} \frac{sg(s) - \varepsilon_0 G(s)}{s^2} \geqslant 0, \tag{4.10}$$

$$\liminf_{|x|\to\infty} \frac{F_{x_i}(x)}{|x|^2} \geqslant 0, \quad i = 1,2, \tag{4.11}$$

$$\liminf_{|x|\to\infty} \frac{f(x)\cdot x - \varepsilon_0 F(x)}{|x|^2} \geqslant 0 \tag{4.12}$$

for some $\varepsilon_0 > 0$. We will use the following lemmas (cf. [5] for proofs).

Lemma 4.1. *Assume* (4.9)–(4.12) *hold. Then, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\big(g(w),w\big)_{L^2} - \varepsilon_0 \int_\Omega G(w)\,d\Omega \geqslant -\epsilon\|w\|^2_{L^2} - C_\epsilon, \tag{4.13}$$

$$\int_\Omega G(w)\,d\Omega \geqslant -\epsilon\|w\|^2_{L^2} - C_\epsilon, \tag{4.14}$$

$$\big(f(v),v\big)_{L^2}-\varepsilon_0\int_\Omega F(v)\,d\Omega\geqslant -\epsilon\|v\|_{L^2}^2-C_\epsilon, \tag{4.15}$$

$$\int_\Omega F(v)\,d\Omega\geqslant -\epsilon\|v\|_{L^2}^2-C_\epsilon \tag{4.16}$$

for every $w\in H_0^1(\Omega)$ *and* $v\in H_0^1(\Omega,\mathbb{R}^2)$.

To prove uniform energy estimates, we have to require

$$\big|g'(s)\big|\leqslant \Lambda_g, \tag{4.17}$$

$$\left|\frac{\partial f_i(x_1,x_2)}{\partial x_i}\right|\leqslant \Lambda_f\quad \forall i=1,2. \tag{4.18}$$

Lemma 4.2. *We assume that* (4.17) *and* (4.18) *hold. Then, inequalities* (4.14), (4.16) *are implied. Furthermore, for every* $\epsilon>0$ *there exists* $C_\epsilon>0$ *such that*

$$\big(g(w),w\big)_{L^2}-\int_\Omega G(w)\,d\Omega\geqslant -\epsilon\|w\|_{L^2}^2-C_\epsilon, \tag{4.19}$$

$$\big(f(v),v\big)_{L^2}-\int_\Omega F(v)\,d\Omega\geqslant -\epsilon\|v\|_{L^2}^2-C_\epsilon \tag{4.20}$$

for every $w\in H^1(\Omega,\mathbb{R})$ *and* $v\in H^1(\Omega,\mathbb{R}^2)$.

Remark 4.3. On account of hypothesis on the nonlinearities, multiple solutions to the stationary problem are admitted, in particular when f and g are nonmonotonic functions.

5. Well posedness

Theorem 5.1. *Suppose conditions* (4.1)–(4.3) *and* (4.9)–(4.12) *hold true. We assume also*

$$v_0\in V_A,\quad w_0\in H^1(\Omega),\quad \eta_0\in\mathcal{M},$$
$$v_1\in L^2\big(\Omega,\mathbb{R}^2\big),\quad w_1\in L^2(\Omega),\quad \rho_0\in\mathcal{N}.$$

Then, for every $T>0$*, Problem* P *admits a unique continuous solution, i.e.,* $(v,w,\eta,\ \rho)$ *such that*

$$v\in C(0,T;V_A),\quad w\in C\big(0,T;H^1(\Omega)\big),\quad \eta\in C(0,T;\mathcal{M}),$$
$$v\in C\big(0,T;L^2(\Omega,\mathbb{R}^2)\big),\quad w\in C\big(0,T;L^2(\Omega)\big),\quad \rho\in C(0,T;\mathcal{N}).$$

Furthermore, we suppose (v^i,w^i,η^i,ρ^i), $i=1,2$*, are solutions corresponding to initial data* $(v_0^i,v_1^i,w_0^i,w_1^i,\eta^i,\rho^i)$*, then there exists an increasing function* $C=C(T)$ *such that the following estimate holds*:

$$\begin{aligned}&\big\|v^1-v^2\big\|_V^2+\big\|v_t^1-v_t^2\big\|_{L^2}^2+\big\|w^1-w^2\big\|_{H^1}^2+\big\|w_t^1-w_t^2\big\|_{L^2}^2\\&\qquad+\big\|\eta^1-\eta^2\big\|_{\mathcal{M}}^2+\big\|\rho^1-\rho^2\big\|_{\mathcal{N}}^2\\&\quad\leqslant C(T)\big\{\big\|v_0^1-v_0^2\big\|_V^2+\big\|v_1^1-v_1^2\big\|_{L^2}^2+\big\|w_0^1-w_0^2\big\|_{H^1}^2+\big\|w_1^1-w_1^2\big\|_{L^2}^2\\&\qquad+\big\|\eta_0^1-\eta_0^2\big\|_{\mathcal{M}}^2+\big\|\rho_0^1-\rho_0^2\big\|_{\mathcal{N}}^2\big\}.\end{aligned}$$

The proofs of the stated results are omitted. They can be carried out via Faedo–Galerkin method and using Gronwall type inequalities (with due technical modifications, see [7], where the beam case is analyzed in detail; see also [17]).

6. Absorbing set

We agree to denote the solution

$$z(t) = \begin{bmatrix} v(t) \\ v_t(t) \\ w(t) \\ w_t(t) \\ \eta^t \\ \rho^t \end{bmatrix}$$

to Problem P, with initial data

$$z(0) = z_0 \in \mathcal{H} = V_A \times L^2\big(\Omega, \mathbb{R}^2\big) \times H^1(\Omega) \times L^2(\Omega) \times \mathcal{M} \times \mathcal{N}$$

by $S(t)z_0$. Hence, on account of Theorem 5.1, the solution is described by the continuous semigroup $S(t)$ acting on the space $\mathcal{H}$, i.e., $S(t)$ enjoys the following properties:

(i) $S(t): \mathcal{H} \to \mathcal{H}$, continuous for every $t \geqslant 0$,
(ii) $S(0) = \mathbb{I}$ (identity on $\mathcal{H}$),
(iii) $\lim_{t \to 0^+} S(t)z = z$ for every $z \in \mathcal{H}$,
(iv) $S(t)S(\tau) = S(t + \tau)$.

We prove here the dissipative nature of the system. This is equivalent to proving the existence of an absorbing set, i.e., of a bounded set into which all the orbits corresponding to different initial data and evolving according to the action of the semigroup $S(t)$ eventually enter. We recall its definition (cf., e.g., [8,16,17]).

Definition 6.1. Let (X, d) be a metric space. A *set* $\mathcal{B}_0 \subset X$ is said to be *absorbing* for the semigroup $\{S(t)\colon t \geqslant 0\}$ acting on X if for any bounded set $\mathcal{B} \subset X$ there exists $t_{\mathcal{B}} \geqslant 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \geqslant t_{\mathcal{B}}$.

We introduce the energy associated to the semigroup $S(t)$ as

$$\begin{aligned} \mathcal{E}(t) = {} & \big\|v_t(t)\big\|^2_{L^2} + \big\|v(t)\big\|^2_V + \big\|w_t(t)\big\|^2_{L^2} + \big\|\eta(t)\big\|^2_{L^2_\mu(V)} \\ & + \big\|\eta + \nabla\rho(t)\big\|^2_{L^2_\kappa(L^2)} + \kappa_0\big\|v(t) + \nabla w(t)\big\|^2_{L^2}. \end{aligned} \tag{6.1}$$

Concerning the dissipation in energy of the semigroup $S(t)$, hypothesis (4.4)–(4.7) implies the exponential decay of (6.1), and we are able to state our main result.

Theorem 6.1. *Assume* (4.1)–(4.7), (4.17) *and* (4.18). *Then, there exist two positive constants* C_1 *and* C_0*—depending only on the structures of the nonlinearities and of the memory kernels, hence* $C_0 = C_0(\mu_0, \kappa_0, \kappa_{00}, \varepsilon_0)$ *and* $C_1 = C_1(\mu_0, \kappa_0, \kappa_{00}, \varepsilon_0)$*—and* $\epsilon > 0$*, such that the following estimate holds*

$$\mathcal{E}(t) \leqslant C_1 e^{-\epsilon t}\mathcal{E}(0) + C_0$$

for every $t \geqslant 0$.

Corollary 6.1. *The semigroup $\{S(t)\colon t \geqslant 0\}$, acting on the space $\mathcal{H}$, has a bounded absorbing set $\mathcal{B}_0$; i.e., any ball of $\mathcal{H}$ with radius strictly greater than C_0, can be chosen as an absorbing set $\mathcal{B}_0$ for the semigroup $\{S(t)\colon t \geqslant 0\}$.*

Remark 6.1. When the nonlinearities vanish, we have that $C_0 = 0$, and Theorem 6.1 implies exponential vanishing of the energy associated to the semigroup $S(t)$.

Proof of Theorem 6.1. We consider (3.3) multiplied by v_t in $L^2(\Omega, \mathbb{R}^2)$, to get

$$\frac{1}{2}\frac{d}{dt}\big(\|v_t\|_{L^2}^2 + \|v\|_V^2\big) + \int_0^\infty \mu(s)\big(\eta(s), v_t\big)_V\,ds + \kappa_0(v + \nabla w, v_t)_{L^2} + \int_0^\infty \kappa(s)\big(\eta(s) + \nabla\rho(s), v_t\big)_{L^2}\,ds + \big(f(v), v_t\big)_{L^2} = 0. \tag{6.2}$$

Equation (3.4) multiplied by w_t in $L^2(\Omega)$, and integration by parts give

$$\frac{1}{2}\frac{d}{dt}\|w_t\|_{L^2}^2 + \kappa_0(v + \nabla w, \nabla w_t)_{L^2} + \int_0^\infty \kappa(s)\big(\eta(s) + \nabla\rho(s), \nabla w_t\big)_{L^2}\,ds + \big(g(w), w_t\big)_{L^2} = 0. \tag{6.3}$$

Consider Eq. (3.5) multiplied in $L^2_\mu(\mathbb{R}^+, V_A)$ by η; we get, with obvious notation:

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{L^2_\mu(V_A)}^2 + \int_0^\infty \mu(s)\big(\eta_s(s), \eta(s)\big)_V\,ds = \int_0^\infty \mu(s)\big(v_t, \eta(s)\big)_V\,ds. \tag{6.4}$$

Consider Eq. (3.5), add the gradient of (3.6), and multiply the result by $\eta + \nabla\rho$ in $L^2_\kappa(\mathbb{R}^+, L^2(\Omega, \mathbb{R}^2))$; we have

$$\frac{1}{2}\frac{d}{dt}\|\eta + \nabla\rho\|_{L^2_\kappa(L^2)}^2 + \int_0^\infty \kappa(s)\big((\eta + \nabla\rho)_s(s), (\eta + \nabla\rho)(s)\big)_{L^2}\,ds = \int_0^\infty \kappa(s)\big((v + \nabla w)_t, (\eta + \nabla\rho)(s)\big)_{L^2}\,ds. \tag{6.5}$$

Adding up (6.4) and (6.5) and using hypothesis (4.4), we get

$$\frac{1}{2}\frac{d}{dt}\big(\|\eta\|_{L^2_\mu(V_A)}^2 + \|\eta + \nabla\rho\|_{L^2_\kappa(L^2)}^2\big) + (\eta_s, \eta)_{L^2_\mu(V_A)} + \big((\eta + \nabla\rho)_s, \eta + \nabla\rho\big)_{L^2_\kappa(L^2)} \leqslant \int_0^\infty \mu(s)\big(v_t, \eta(s)\big)_V\,ds + \int_0^\infty \kappa(s)\big((v + \nabla w)_t, (\eta + \nabla\rho)(s)\big)_{L^2}\,ds. \tag{6.6}$$

Add Eqs. (6.2), (6.3) to (6.6); we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|v_t\|^2_{L^2} + \|v\|^2_V + \|w_t\|^2_{L^2} + \kappa_0\|v+\nabla w\|^2_{L^2} + \|\eta\|^2_{L^2_\mu(V_A)} + \|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)}\big)\\
&\quad + (\eta_s,\eta)_{L^2_\mu(V_A)} + \big((\eta+\nabla\rho)_s, \eta+\nabla\rho\big)_{L^2_\kappa(L^2)} + \big(f(v), v_t\big)_{L^2} + \big(g(w), w_t\big)_{L^2} = 0.
\end{aligned} \tag{6.7}
$$

We stress that the operator A does not act on the s variable. Then, by integration by parts and exploiting (4.4), we have that

$$
\begin{aligned}
(\eta_s,\eta)_{L^2_\mu(V_A)} &= \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\big\|\eta(s)\big\|^2_V\,ds = -\frac{1}{2}\int_0^\infty \mu'(s)\big\|\eta(s)\big\|^2_V\,ds\\
&\geqslant \frac{\delta_\mu}{2}\int_0^\infty \mu(s)\big\|\eta(s)\big\|^2_V\,ds.
\end{aligned} \tag{6.8}
$$

By the same technique, we are yield to

$$
\big((\eta+\nabla\rho)_s, \eta+\nabla\rho\big)_{L^2_\kappa(L^2)} \geqslant \frac{\delta_\kappa}{2}\int_0^\infty \kappa(s)\big\|(\eta+\nabla\rho)(s)\big\|^2_{L^2}\,ds. \tag{6.9}
$$

Using (6.8), (6.9) in (6.7), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg(\|v_t\|^2_{L^2} + \|v\|^2_V + \|w_t\|^2_{L^2} + \|\eta\|^2_{L^2_\mu(V_A)} + \|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)} + \kappa_0\|v+\nabla w\|^2_{L^2}\\
&\quad + 2\int_\Omega F(v)\,d\Omega + 2\int_\Omega G(w)\,d\Omega\bigg) + \frac{\delta}{2}\big(\|\eta\|^2_{L^2_\mu(V)} + \|\rho\|^2_{L^2_\kappa(L^2)}\big) \leqslant 0,
\end{aligned} \tag{6.10}
$$

where $\delta = \min(\delta_\mu, \delta_\kappa)$.

We now consider the product in $L^2(\Omega)$ of (3.4) and w

$$
(w_{tt}, w)_{L^2} + \kappa_0(v+\nabla w, \nabla w)_{L^2} + \int_0^\infty \kappa(s)\big(\eta(s)+\nabla\rho(s), \nabla w\big)_{L^2} + \big(g(w), w\big)_{L^2} = 0 \tag{6.11}
$$

and the product in $L^2(\Omega, \mathbb{R}^2)$ of (3.3) and v

$$
\begin{aligned}
&(v_{tt}, v)_{L^2} + \kappa_0(v+\nabla w, v)_{L^2} + \|v\|^2_V + \int_0^\infty \mu(s)\big(\eta(s), v\big)_V\\
&\quad + \int_0^\infty \kappa(s)\big(\eta(s)+\nabla\rho(s), v\big)_{L^2} + \big(f(v), v\big)_{L^2} = 0.
\end{aligned} \tag{6.12}
$$

Using Young inequality, we have that

$$
\left|\int_0^\infty \mu(s)\big(\eta(s), v\big)_V\,ds\right| \leqslant \int_0^\infty \mu(s)\big\|\eta(s)\big\|_V\|v\|_V\,ds
$$

$$\leqslant \int_0^\infty \mu(s)\left(\frac{\epsilon}{\mu_0}\|v\|_V^2 + \frac{\mu_0}{4\epsilon}\big\|\eta(s)\big\|_V^2\right) ds$$

$$\leqslant \epsilon\|v\|_V^2 + \frac{\mu_0}{4\epsilon}\|\eta\|_{\mathcal{M}}^2 \tag{6.13}$$

and, using the same estimate, we also get

$$\left|\int_0^\infty \kappa(s)\big(\eta(s)+\nabla\rho(s), \nabla w + v\big)_{L^2} ds\right| \leqslant \epsilon\|\nabla w + v\|_{L^2}^2 + \frac{\kappa_{00}}{4\epsilon}\|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2. \tag{6.14}$$

Adding (6.11) to (6.12) and using (6.13), (6.14), we are yield to

$$(w_{tt}, w)_{L^2} + (v_{tt}, v)_{L^2} + (1-\epsilon)\|v\|_V^2 + (\kappa_0-\epsilon)\|v+\nabla w\|_{L^2}^2$$
$$+ \big(f(v), v\big)_{L^2} + \big(g(w), w\big)_{L^2} \leqslant \frac{\mu_0}{4\epsilon}\|\eta\|_{\mathcal{M}}^2 + \frac{\kappa_{00}}{4\epsilon}\|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2. \tag{6.15}$$

Then, adding (6.10) to (6.15), multiplied by a coefficient α such small that

$$\frac{\mu_0\alpha}{\epsilon} < \delta \quad \text{and} \quad \frac{\kappa_{00}\alpha}{\epsilon} < \delta,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\bigg(\|v_t\|_{L^2}^2 + \|v\|_V^2 + \|w_t\|_{L^2}^2 + \|\eta\|_{L^2_\mu(V)}^2 + \|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2$$
$$+ \kappa_0\|v+\nabla w\|_{L^2}^2 + 2\int_\Omega F(v)\,d\Omega + 2\int_\Omega G(w)\,d\Omega\bigg)$$
$$+ \alpha(w_{tt}, w)_{L^2} + \alpha(v_{tt}, v)_{L^2} + \alpha(\kappa_0-\epsilon)\|v+\nabla w\|_{L^2}^2 + \alpha(1-\epsilon)\|v\|_V^2$$
$$+ \frac{\delta}{4}\big(\|\eta\|_{L^2_\mu(V)}^2 + \|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2\big) + \alpha\big(f(v), v\big)_{L^2} + \alpha\big(g(w), w\big)_{L^2} \leqslant 0. \tag{6.16}$$

We set

$$\mathcal{L} = \mathcal{E} + 2\int_\Omega F(v)\,d\Omega + 2\int_\Omega G(w)\,d\Omega$$

and observe that exploiting hypothesis (4.17) and (4.18) the energy is bounded when $\mathcal{L}$ is bounded and $\mathcal{L}$ is positively defined.

We introduce the auxiliary variables $\bar{v} = v_t + \alpha v$ and $\bar{w} = w_t + \alpha w$, and use the trivial identity

$$\alpha(v, v_{tt})_{L^2} = \frac{1}{2}\frac{d}{dt}\big(\|\bar{v}\|_{L^2}^2 + \alpha^2\|v\|_{L^2}^2\big) + \alpha^3\|v\|_{L^2}^2 - \alpha\|\bar{v}\|_{L^2}^2 - (v_{tt}, v_t)_{L^2}$$

(and the same for w and $\bar{w}$). Then, (6.16) becomes

$$\frac{1}{2}\frac{d}{dt}\bigg(\|v\|_V^2 + \|\eta\|_{L^2_\mu(V)}^2 + \|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2 + \kappa_0\|v+\nabla w\|_{L^2}^2$$
$$+ 2\int_\Omega F(v)\,d\Omega + 2\int_\Omega G(w)\,d\Omega + \|\bar{v}\|_{L^2}^2 + \alpha^2\|v\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2 + \alpha^2\|w\|_{L^2}^2\bigg)$$
$$+ \alpha(\kappa_0-\epsilon)\|v+\nabla w\|_{L^2}^2 + \alpha(1-\epsilon)\|v\|_V^2 + \alpha^3\|v\|_{L^2}^2$$

$$-\alpha\|\bar{v}\|_{L^2}^2+\alpha^3\|w\|_{L^2}^2-\alpha\|\bar{w}\|_{L^2}^2+\frac{\delta}{4}\big(\|\eta\|_{L^2_\mu(V)}^2+\|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2\big)$$
$$+\alpha\big(f(v),v\big)_{L^2}+\alpha\big(g(w),w\big)_{L^2}\leqslant 0. \tag{6.17}$$

Now, we look to control the terms in $\bar{v}$ and $\bar{w}$. Let us introduce the functionals:

$$\mathcal{L}_1=-\int_\Omega v_t\cdot\int_0^\infty\mu(s)\eta(s)\,ds\,dx, \tag{6.18}$$

$$\mathcal{L}_2=-\int_\Omega w_t\int_0^\infty\kappa(s)\rho(s)\,ds\,dx. \tag{6.19}$$

We note that

$$\left(\int_0^\infty\mu(s)\theta(s)\,ds\right)^2=\left(\int_0^\infty\mu^{1/2}(s)\mu^{1/2}(s)\theta(s)\,ds\right)^2\leqslant\mu_0\int_0^\infty\mu(s)\theta^2(s)\,ds \tag{6.20}$$

for every $\theta\in L^2_\mu(\mathbb{R}^+)$. Using Hölder inequality and (6.20), we get

$$\left(\int_0^\infty\kappa(s)\big(\eta(s)+\nabla\rho(s)\big)\,ds,\int_0^\infty\mu(s)\eta(s)\,ds\right)_{L^2}$$
$$\leqslant\kappa_{00}^{1/2}\left(\int_0^\infty\kappa(s)\big\|\eta(s)+\nabla\rho(s)\big\|_{L^2}^2\,ds\right)^{1/2}\mu_0^{1/2}\left(\int_0^\infty\mu(s)\big\|\eta(s)\big\|_{L^2}^2\,ds\right)^{1/2}$$
$$\leqslant\frac{\kappa_{00}}{2}\|\eta+\nabla\rho\|_{L^2_\kappa(L^2)}^2+\frac{\mu_0}{2}\|\eta\|_{L^2_\mu(L^2)}^2. \tag{6.21}$$

Using (3.3)–(3.6) in (6.18) and (6.19), we get

$$\frac{d\mathcal{L}_1}{dt}=-\int_0^\infty\mu(s)\big(Av,\eta(s)\big)_{L^2}\,ds+\left\|\int_0^\infty\mu(s)A^{1/2}\eta(s)\,ds\right\|_{L^2}$$
$$+\int_0^\infty\mu(s)\big(\eta(s),f(v)\big)_{L^2}\,ds+\left(\int_0^\infty\kappa(s)\big(\eta(s)+\nabla\rho(s)\big)\,ds,\int_0^\infty\mu(s)\eta(s)\,ds\right)_{L^2}$$
$$+\kappa_0\int_0^\infty\mu(s)\big(\eta(s),v+\nabla w\big)_{L^2}\,ds-\mu_0\|v_t\|_{L^2}^2-\int_0^\infty\mu'(s)\big(\eta(s),v_t\big)_{L^2}\,ds \tag{6.22}$$

and

$$\frac{d\mathcal{L}_2}{dt}=\int_0^\infty\kappa(s)\big(\rho(s),g(w)\big)_{L^2}\,ds-\kappa_{00}\|w_t\|_{L^2}^2+\kappa_0\int_0^\infty\kappa(s)\big(\nabla\rho(s),v+\nabla w\big)_{L^2}\,ds$$
$$+\left(\int_0^\infty\kappa(s)\nabla\rho(s)\,ds,\int_0^\infty\kappa(s)\big(\eta(s)+\nabla\rho(s)\big)\,ds\right)_{L^2}$$

$$-\int_0^\infty \kappa'(s)\big(\rho(s), w_t\big)_{L^2}\, ds. \tag{6.23}$$

We study each term of the right-hand side of (6.22) and (6.23). In view of (4.5), we obtain (cf. [6,17])

$$-\mu_0\|v_t\|^2_{L^2} - \int_0^\infty \mu'(s)\big(\eta(s), v_t\big)_{L^2}\, ds \leqslant -\frac{\mu_0}{2}\|v_t\|^2_{L^2} + C\int_0^\infty \mu(s)\big\|\nabla\eta(s)\big\|^2_{L^2}\, ds \tag{6.24}$$

and

$$-\kappa_{00}\|w_t\|^2_{L^2} - \int_0^\infty \kappa'(s)\big(\rho(s), w_t\big)_{L^2}\, ds \leqslant -\frac{\kappa_{00}}{2}\|w_t\|^2_{L^2} + C\int_0^\infty \kappa(s)\big\|\nabla\rho(s)\big\|^2_{L^2}\, ds, \tag{6.25}$$

where C is a constant depending on s_μ, s_κ, μ_0 and κ_{00}.

Using Hölder and Young inequalities and (6.20), we have

$$-\int_0^\infty \mu(s)\big(Av, \eta(s)\big)_{L^2}\, ds + \left\|\int_0^\infty \mu(s)A^{1/2}\eta(s)\, ds\right\|_{L^2} \leqslant \epsilon\big\|A^{1/2}v\big\|^2_{L^2} + C\|\eta\|^2_{\mathcal{M}} \tag{6.26}$$

and

$$\begin{aligned}&\kappa_0\left(v + \nabla w, \int_0^\infty \kappa(s)\nabla\rho(s)\, ds\right)_{L^2} + \left(\int_0^\infty \kappa(s)\big(\eta(s) + \nabla\rho(s)\big)\, ds, \int_0^\infty \kappa(s)\nabla\rho(s)\, ds\right)_{L^2}\\ &\quad\leqslant \epsilon\|v + \nabla w\|^2_{L^2} + C\big(\|\eta\|^2_{L^2_\mu(V)} + \|\eta + \nabla\rho\|^2_{L^2_\kappa(L^2)}\big),\end{aligned} \tag{6.27}$$

where the constants Cs depend on κ_0, κ_{00}, μ_0 and ϵ. Analogously,

$$\kappa_0\int_0^\infty \mu(s)\big(\eta(s), v + \nabla w\big)_{L^2} \leqslant \epsilon\|v + \nabla w\|^2_{L^2} + C\|\eta\|^2_{L^2_\mu(V)}. \tag{6.28}$$

Exploiting (4.17), we get

$$\begin{aligned}\int_0^\infty \kappa(s)\big(\rho(s), g(w)\big)_{L^2}\, ds &\leqslant \kappa_{00}^{1/2}\int_\Omega \big(\Lambda_g|w| + g(0)\big)\left(\int_0^\infty \kappa(s)\big|\rho(s)\big|^2\, ds\right)^{1/2} dx\\ &\leqslant \epsilon\|w\|^2_{L^2} + C\|\eta + \nabla\rho\|^2_{L^2_\kappa(L^2)} + C\|\eta\|^2_{L^2_\mu(V)} + C_g,\end{aligned} \tag{6.29}$$

where C_g is a constant depending on $g(0)$, κ_{00}, $|\Omega|$ and on ϵ. Likely, using (4.18), we get

$$\int_0^\infty \mu(s)\big(\eta(s), f(v)\big)_{L^2}\, ds \leqslant \epsilon\|v\|^2_{L^2} + C\|\eta\|^2_{L^2_\mu(V)} + C_f, \tag{6.30}$$

where C_f is a constant depending on $f(0,0)$, μ_0, $|\Omega|$ and on ϵ.

Adding (6.22) to (6.23), and using (6.24)–(6.30), we are yield to

$$\begin{aligned}
&\frac{d(\mathcal{L}_1+\mathcal{L}_2)}{dt}+\frac{\mu_0}{2}\|v_t\|_{L^2}^2+\frac{\kappa_{00}}{2}\|w_t\|_{L^2}^2\\
&\quad\leqslant C\big(\|\eta\|^2_{L^2_\mu(V)}+\|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)}\big)+\epsilon\big(\big\|A^{1/2}v\big\|^2_{L^2}+2\|v+\nabla w\|^2_{L^2}+\|v\|^2_{L^2}+\|w\|^2_{L^2}\big)\\
&\qquad+C_g+C_f.
\end{aligned}\tag{6.31}$$

Recalling that

$$\|v_t\|^2_{L^2}=\|\bar{v}\|^2_{L^2}-\alpha\frac{d}{dt}\|v\|^2_{L^2}-\alpha^2\|v\|^2_{L^2}$$

(the same estimate holds for w_t), hence we can rewrite (6.31) as

$$\begin{aligned}
&\frac{d}{dt}\big(\mathcal{L}_1+\mathcal{L}_2-\alpha\|v\|^2_{L^2}-\alpha\|w\|^2_{L^2}\big)+\frac{\mu_0}{2}\|\bar{v}_t\|^2_{L^2}+\frac{\kappa_{00}}{2}\|\bar{w}_t\|^2_{L^2}\\
&\quad\leqslant C\big(\|\eta\|^2_{L^2_\mu(V)}+\|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)}\big)+\epsilon\|v\|^2_V+2\epsilon\|v+\nabla w\|^2_{L^2}+\big(\epsilon+\alpha^2\big)\|v\|^2_{L^2}\\
&\qquad+\big(\epsilon+\alpha^2\big)\|w\|^2_{L^2}+C_g+C_f.
\end{aligned}\tag{6.32}$$

Let us introduce the functional

$$\begin{aligned}
\mathcal{L}&=\|\bar{v}\|^2_{L^2}+\alpha^2\|v\|^2_{L^2}+\kappa_0\|v+\nabla w\|^2_{L^2}+\|v\|^2_V+\|\bar{w}\|^2_{L^2}+\alpha^2\|w\|^2_{L^2}\\
&\quad+\|\eta\|^2_{L^2_\mu(V)}+\|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)}+2\int_\Omega F(v)\,d\Omega+2\int_\Omega G(w)\,d\Omega
\end{aligned}\tag{6.33}$$

and remark that, in view of (4.14), (4.16) and Poincaré inequality, there exist two positive constants c_1 and c_2 such that

$$c_1+\mathcal{L}(t)\geqslant c_2\mathcal{E}(t)\geqslant 0$$

for every $t\geqslant 0$.

Let us add (6.17), multiplied by a constant M, which will be chosen suitably large, to (6.32), to get

$$\begin{aligned}
&\frac{d}{dt}\big(M\mathcal{L}+\mathcal{L}_1+\mathcal{L}_2-\alpha\|v\|^2_{L^2}-\alpha\|w\|^2_{L^2}\big)+\left(\frac{\mu_0}{2}-M\alpha\right)\|\bar{v}\|^2_{L^2}+\left(\frac{\kappa_{00}}{2}-M\alpha\right)\|\bar{w}\|^2_{L^2}\\
&\quad+\big(M\alpha(\kappa_0-\epsilon)-2\epsilon\big)\|v+\nabla w\|^2_{L^2}+\big(M\alpha(1-\epsilon)-\epsilon\big)\|v\|^2_V\\
&\quad+\big(M\alpha^3-\epsilon-\alpha^2\big)\big(\|v\|^2_{L^2}+\|w\|^2_{L^2}\big)+\left(\frac{M\delta}{4}-C\right)\big(\|\eta\|^2_{\mathcal{M}}+\|\eta+\nabla\rho\|^2_{L^2_\kappa(L^2)}\big)\\
&\quad+M\alpha\big(f(v),v\big)_{L^2}+M\alpha\big(g(w),w\big)_{L^2}\leqslant C_g+C_f.
\end{aligned}\tag{6.34}$$

The positivity of the multiplying coefficients appearing in (6.34) is ensured by hypothesis (4.6) and (4.7). Indeed,

$$\begin{aligned}
&\frac{M\delta}{4}>C,\qquad \frac{\mu_0}{2}-M\alpha>0,\qquad \frac{\kappa_{00}}{2}-M\alpha>0,\\
&M\alpha^3-\epsilon\alpha^2>0,\qquad M\alpha(1-\epsilon)>0,\qquad M\alpha(\kappa_0-\epsilon)-2\epsilon>0.
\end{aligned}$$

A possible choice of ϵ,α,M follows $0<\epsilon=\alpha^2<1$ and $\epsilon<\kappa_0$ and M such that $M\delta>4C$.

Thanks to equivalence between $\mathcal{L}$ and $M\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2$, we have that there exists $\delta > 0$ such that

$$\frac{d}{dt}\big(M\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2\big) + \delta\big(M\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2\big) \leqslant C_f + C_g.$$

The application of a generalized Gronwall lemma (see, e.g., [17]) gives exponential decay of $\mathcal{L}$ and hence of $\mathcal{E}$. □

References

[1] Constantine M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37 (1970) 297–308.

[2] Georges Duvaut, Jaques-L. Lions, Inequalities in Mechanics and Physics, Grundlehren Math. Wiss., vol. 219, Springer-Verlag, Berlin, 1976. Translated from French by C.W. John.

[3] Mauro Fabrizio, Barbara Lazzari, On the existence and the asymptotic stability of solutions for linear viscoelastic solids, Arch. Ration. Mech. Anal. 116 (2) (2001) 139–152.

[4] Gaetano Fichera, Existence theorems in linear and semi-linear elasticity, in: Vorträge der Wissenschaftlichen Jahrestagung der Gesellschaft für Angewandte Mathematik und Mechanik, Munich, 1973, Z. Angew. Math. Mech. 54 (1974) T24–T36.

[5] Jean-M. Ghidaglia, Roger Temam, Attractors for damped nonlinear hyperbolic equations, J. Math. Pures Appl. (9) 66 (3) (1987) 273–319.

[6] Claudio Giorgi, Jaime Muñoz Rivera, Vittorino Pata, Global attractors for a semilinear hyperbolic equation in viscoelasticity, J. Math. Anal. Appl. 260 (1) (2001) 83–99.

[7] Claudio Giorgi, Federico M.G. Vegni, Uniform energy estimates for a semilinear evolution equation of the Mindlin–Timoshenko beam with memory, Math. Comput. Modelling 39 (9–10) (2004) 1005–1021.

[8] Alain Haraux, Systèmes dynamiques dissipatifs et applications, Recherches en Mathématiques Appliquées (Research in Applied Mathematics), vol. 17, Masson, Paris, 1991.

[9] John E. Lagnese, Boundary stabilization of thin plates, SIAM Stud. Appl. Math., vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[10] John E. Lagnese, Jacques-L. Lions, Modelling Analysis and Control of Thin Plates, Recherches en Mathématiques Appliquées (Research in Applied Mathematics), vol. 6, Masson, Paris, 1988.

[11] Zhuangyi Liu, Chang Peng, Exponential stability of a viscoelastic Timoshenko beam, Adv. Math. Sci. Appl. 8 (1) (1998) 343–351.

[12] Raymond D. Mindlin, Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates, J. Appl. Mech. 18 (1951) 31–38.

[13] Vittorino Pata, Attractors for a damped wave equation on R^3 with linear memory, Math. Methods Appl. Sci. 23 (7) (2000) 633–653.

[14] Higidio Portillo Oquendo, Jaime E. Muñoz Rivera, Asymptotic behavior of a Mindlin–Timoshenko plate with viscoelastic dissipation on the boundary, Funkcial. Ekvac. 46 (3) (2003) 363–382.

[15] Eric Reissner, On the theory of bending of elastic plates, J. Math. Phys. 23 (1944) 184–191.

[16] Roger Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, second ed., Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1997.

[17] Federico M.G. Vegni, Dissipativity of a conserved phase-field system with memory, Discrete Contin. Dyn. Syst. 9 (4) (2003) 949–968.