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Uniform Energy Estimates for a Semilinear Evolution Equation of the Mindlin-Timoshenko Beam with Memory

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Abstract—In this paper, we investigate a mathematical model for viscoelastic beams, based on the Mindlin-Timoshenko assumptions. The resulting constitutive equations are derived in the framework of well-established theory of linear viscoelasticity, and according to the approximation procedure due to Lagnese for the Kirchhoff viscoelastic beams and plates. Assuming a nonlinear body force acting on the beam, we show that this model generates a strongly continuous semigroup which acts on the appropriate phase space. Uniform energy estimates are then given. The existence of an absorbing set for the solution of the problem is also studied. Furthermore, we remark the necessary conditions to extend this approach to the study of the longtime behavior of the Mindlin-Timoshenko plates.

Keywords—Semigroup, Mindlin-Timoshenko beam, Absorbing set, Dissipative system.

1. THE MODEL

We consider the motion of a beam of uniform thickness d in the vertical plane. When the beam is in equilibrium, we assume it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^2$ placed in a reference frame $\mathbf{x} = (x_1, x_3)$. The middle line of the beam (called *elastic line*) lies midway between its upper and lower faces in a region $(0, L) \subset \mathbb{R}$ of the axis $x_3 = 0$. We assume that it is rigidly clamped in 0 and in L. Henceforth, we denote by $u_i(x_1)$, $i = 1, 3$, the components of the displacement vector of the points of the elastic line which have coordinates $(x_1, 0)$ at equilibrium. Since d is the uniform thickness of the beam, let $\Omega^- = (0, L) \times \{-d/2\}$ and $\Omega^+ = (0, L) \times \{d/2\}$ denote its faces.

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As is well known, in standard three-dimensional linear elasticity the stress-strain relation is given by

$$\mathbf{S}(\mathbf{x},t) = \mathbb{L}_0(\mathbf{E}(\mathbf{x},t)),$$

where the *elastic strain* $\mathbf{E}$ and the *stress* $\mathbf{S}$ are second-order tensors, and $\mathbb{L}_0$ is a fourth-order tensor. In small displacement theory, $\mathbf{E}$ is given by

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{2}\left(\nabla\mathbf{U}(\mathbf{x},t) + (\nabla\mathbf{U})^{\top}(\mathbf{x},t)\right) \tag{1.1}$$

$\mathbf{U}$ being the *displacement vector*.

In the isotropic case, $\mathbb{L}_0$ involves two independent *Lame's constants* λ and μ, namely

$$\mathbb{L}_0 = \lambda\,\mathbf{I}\otimes\mathbf{I} + 2\,\mu\,\mathbb{I}, \qquad \text{for } \tau \geq 0.$$

According to this constitutive equation, a mathematical model for isotropic elastic plates and beams can be derived by assuming the Mindlin-Timoshenko hypothesis (see, for instance [1]). Namely, the linear filaments of the plate (or beam) initially perpendicular to the middle surface (or line) are required to remain straight and undergo no strain in deformation, but the Kirchhoff assumption that they remain perpendicular to the deformed middle surface (or line) is removed.

In a linearized theory for the beam, this assumption implies that transverse shear effects may be no longer neglected, and the small displacement $\mathbf{U}$ is related to the displacement of the elastic line $\mathbf{u}(x,t) = (u(x,t),\, w(x,t))$, $u = u_1$, $w = u_3$, and $x = x_1$, by the approximate relations

$$U_1 = u - x_3\,\psi, \qquad U_3 = w, \qquad -\frac{d}{2} \leq x_3 \leq \frac{d}{2}, \tag{1.2}$$

where ψ is the angle of deflection of the filament with respect to the normal direction.

In virtue of (1.1), this assumption leads to the strain-displacement relations of the Mindlin-Timoshenko model, namely

$$\begin{aligned} E_{11} &= u_x - x_3\,\psi_x, \\ E_{13} &= \frac{1}{2}\left(w_x - \psi\right), \\ E_{12} &= E_{22} = E_{23} = E_{33} = 0, \end{aligned} \tag{1.3}$$

where we denote with the pedex x the partial derivative with respect to x_1.

Here, in order to generalize the model, we assume that the plate is composed of an *isotropic linear viscoelastic* material. As a consequence, the *stress-strain* law is given by

$$\mathbf{S}(\mathbf{x},t) = D\mathbb{L} * (\mathbf{E})(\mathbf{x},t), \tag{1.4}$$

where $*$ denotes convolution,

$$D\mathbb{L} * \mathbf{E}(\mathbf{x},t) = \int_{-\infty}^{\infty} D_\tau\,\mathbb{L}(\tau)\,\mathbf{E}(\mathbf{x},t-\tau)\,d\tau$$

and D_τ is the distributional derivative with respect to τ. Moreover, we observe that $\mathbb{L}(\tau)$ is an isotropic fourth-order tensor-valued function which vanishes for $\tau < 0$ and involves two independent *relaxation functions* λ and μ, namely

$$\mathbb{L}(\tau) = \lambda(\tau)\,\mathbf{I}\otimes\mathbf{I} + 2\,\mu(\tau)\,\mathbb{I}, \qquad \forall\,\tau \geq 0.$$

Because of the relation $D_\tau\mathbb{L} = \mathbb{L}(0)\,\delta + \mathbb{L}_t$, where $\mathbb{L}_t = \frac{d\mathbb{L}}{dt}$ and δ represents the Dirac delta distribution, we have

$$D\mathbb{L} * \mathbf{E}(\mathbf{x},t) = \mathbb{L}(0)\,\mathbf{E}(\mathbf{x},t) + \int_0^{\infty} \mathbb{L}_t(\tau)\,\mathbf{E}(\mathbf{x},t-\tau)\,d\tau.$$

In order to apply a variational principle and attain the motion equation, we shall transform the original stress-strain constitutive equation by means of the Laplace transform. Let $\varphi(t)$ be a function taking values in a Hilbert space, and let denote by $\hat{\varphi}$ its Laplace transform, namely

$$\hat{\varphi}(s) = \int_0^\infty e^{-st}\varphi(t)\,dt.$$

In particular, we have $(D\mathbb{L})\hat{}\,(s) = s\,\hat{\mathbb{L}}(s)$ where

$$\hat{\mathbb{L}}(s) = \hat{\lambda}(s)\,\mathbf{I}\otimes\mathbf{I} + 2\,\hat{\mu}(s)\,\mathbb{I}.$$

Formally applying the Laplace transform in (1.4), we get

$$\hat{S}_{ij} = s\,\hat{\lambda}\,\delta_{ij}\,\hat{E}_{kk} + 2s\,\hat{\mu}\,\hat{E}_{ij}. \tag{1.5}$$

Now, we introduce the *viscoelastic Poisson's ratio* ν and *viscoelastic Young's modulus* E so that their Laplace transforms are, respectively, defined by

$$\hat{\nu} = \frac{\hat{\lambda}}{2s\left(\hat{\lambda}+\hat{\mu}\right)}, \qquad \hat{E} = \frac{\hat{\mu}\left(3\hat{\lambda}+2\hat{\mu}\right)}{\hat{\lambda}+\hat{\mu}}. \tag{1.6}$$

Then, we have

$$2\hat{\mu} = \frac{\hat{E}}{1+s\hat{\nu}}, \qquad \hat{\lambda} = \frac{\hat{E}}{1+s\hat{\nu}}\cdot\frac{s\hat{\nu}}{1-2s\hat{\nu}}, \tag{1.7}$$

and (1.5) becomes

$$\hat{S}_{ij} = \frac{s\hat{E}}{1+s\hat{\nu}}\left(\hat{E}_{ij} + \frac{s\hat{\nu}}{1-2s\hat{\nu}}\,\hat{E}_{kk}\,\delta_{ij}\right). \tag{1.8}$$

Taking into account (1.3), we get

$$\begin{aligned} \hat{S}_{11} &= \frac{s\hat{E}\,(1-s\hat{\nu})}{(1+s\hat{\nu})\,(1-2s\hat{\nu})}\,\hat{E}_{11}, \\ \hat{S}_{12} &= S_{22} = S_{23} = S_{33} = 0, \\ \hat{S}_{13} &= \frac{s\hat{E}}{1+s\hat{\nu}}\,\hat{E}_{13}. \end{aligned} \tag{1.9}$$

As is customary in the theory of Mindlin-Timoshenko beams and plates, consistency with the presence of transverse shear requires the introduction of a suitable scalar factor K to correct its expression. Namely, the last relation in (1.9) is replaced by

$$\hat{S}_{13} = K\,\frac{s\hat{E}}{1+s\hat{\nu}}\,\hat{E}_{13}. \tag{1.10}$$

This procedure is contrived even if routinely done; in applications, it is customary to fix K to a precise value strictly less than 1.

Let ρ_0 the constant mass density. The motion equation, via a variational formulation, can be obtained by introducing the viscoelastic (kinetic and internal) total energy $\mathcal{P}(t)$. This procedure (see [1,2]) is carried out in the frequency domain, regarding the viscoelastic material as an elastic one with complex valued elastic constants. Accordingly, the transformed (visco) elastic total energy is given by

$$\hat{\mathcal{P}}(s) = \frac{1}{2}\int_{-d/2}^{d/2}\int_0^L \left(\rho_0 s^2\hat{U}_i\hat{U}_i + \hat{S}_{ij}\,\hat{E}_{ij}\right)\,dx_1\,dx_3. \tag{1.11}$$

If an integration with respect to x_3 is carried out, the stretching component $\hat{u}_1 = \hat{u}$ uncouples from the bending component $\hat{u}_3 = \hat{w}$ of the displacement and the deflection angle $\hat{\psi}$, so that the *strain energy* and the *kinetic energy* split into two parts (see [1], for instance).

Substituting (1.3) and (1.9),(1.10) into (1.11) and considering only the bending part $\hat{\mathcal{P}}_b(s)$ of $\hat{\mathcal{P}}(s)$, which contains the terms w and ψ, we obtain

$$\begin{aligned}\hat{\mathcal{P}}_b(s) = \frac{d}{8}\int_0^L \left[\frac{d^2 s}{3}\hat{N}(s)\hat{\psi}_x^2(s) + \hat{H}(s)\left(\hat{w}_x(s) - \hat{\psi}(s)\right)^2\right] dx \\ + \frac{s^2\rho_0 d}{2}\int_0^L \left[\frac{d^2}{12}\hat{\psi}^2(s) + \hat{w}^2(s)\right] dx,\end{aligned} \tag{1.12}$$

where N and H are assumed to be regular causal functions such that their Laplace transforms satisfy the relations

$$\hat{N}(s) = \frac{\hat{E}(s)}{1 - s^2\hat{\nu}^2(s)}, \qquad \hat{H}(s) = \frac{2Ks\hat{E}(s)}{1 + s\hat{\nu}(s)}. \tag{1.13}$$

In order to account for a nonlinear body force $\mathbf{f}(\mathbf{U})$, depending on the whole displacement and distributed on the whole bar, the external work W has to be considered. In the frequency domain, it is given by

$$\hat{W} = \int_{-d/2}^{d/2}\int_0^L \hat{f}_i(U_1, U_3)\hat{U}_i\, dx_1\, dx_3.$$

For simplicity, we assume that the external force is independent of U_1, and normal to the beam ($f_1 = f_2 = 0$). Then we write

$$\hat{W} = d\int_0^L \left[\hat{f}_3(w)\hat{w}\right] dx_1.$$

Thus, the total bending functional is given by

$$\hat{\mathcal{P}}_b(s) + \hat{w}.$$

Studying the associated variational problem and using assumption (1.15), it easily follows that $\hat{w}$ and $\hat{v}$ solve the Dirichlet boundary value problem

$$\begin{aligned}\rho_0\, d^2 s^2\hat{\psi} - d^2 s\hat{N}\,\hat{\psi}_{xx} + 3\hat{H}\left(\hat{\psi} - \hat{w}_x\right) &= 0, && \text{in } (0, L),\\ 12\rho_0\, s^2\hat{w} + 3\hat{H}\left(\hat{\psi} - \hat{w}_x\right)_x &= \hat{f}_3, && \text{in } (0, L),\\ \hat{w} = \hat{v} &= 0, && \text{at } x = 0, L,\end{aligned} \tag{1.14}$$

where the Dirichlet boundary conditions for the deflection angle ψ and the vertical displacement w model a beam which is clamped at the ends. Problem (1.14) has been studied in [3].

In order to achieve a nonhereditary coupling term, we suppose that $\hat{H}(s)$ is a constant function with positive value H. As a consequence, we obtain

$$\hat{\mu}(s) = \frac{H}{2Ks}, \tag{1.15}$$

which is equivalent to assume that μ is the constant $H/2K$. Finally, we set $\rho_1 = \rho_0 d^2$, $\rho_2 = 12\rho_0$, $d_0 = d^2 N(0)$, and $k = 3H$, to get

$$\begin{aligned}\rho_1 s^2\hat{\psi} - d_0 N(0)^{-1} s\hat{N}\,\hat{\psi}_{xx} + k\left(\hat{\psi} - \hat{w}_x\right) &= 0, && \text{in } (0, L),\\ \rho_2 s^2\hat{w} + k\left(\hat{\psi} - \hat{w}_x\right)_x &= \hat{f}_3, && \text{in } (0, L),\\ \hat{w} = \hat{v} &= 0, && \text{at } x = 0, L.\end{aligned} \tag{1.16}$$

It is worth mentioning that assumption (1.15) makes the model more difficult to study. In particular, a balance on the constants is needed and condition (4.1) is required here; even if artificial (the material does not show stress in shear), this request is consistent with the fact that we study a limit case, when the dissipation due to coupling term vanishes. This assumption is unnecessary when the coupling term is of hereditary type (in this case, of course, some hypothesis on the corresponding memory kernel is also needed, see [3]).

2. THE ANALYTIC SETTING

We transform (1.16) into the time domain and setting $v = -\psi$, we get the initial-boundary value problem, which we study in the semilinear case of forcing terms f and g

$$\rho_1 v_{tt} - d_0 v_{xx} - \int_0^\infty b(\sigma) v_{xx}(t-\sigma)\, d\sigma + k(v + w_x) = f(v), \tag{2.1}$$

$$\rho_2 w_{tt} - k(v + w_x)_x = g(w), \tag{2.2}$$

in $(0, L) \times (0, \infty)$, where $b(\sigma) = d^2 N'(\sigma)$ is a memory kernel. We have the following boundary conditions:

$$v(0,t) = v(L,t) = 0, \qquad t \geq 0, \tag{2.3}$$

$$w(0,t) = w(L,t) = 0, \qquad t \geq 0, \tag{2.4}$$

and initial conditions

$$\begin{aligned} v(\cdot,0) &= v_0, \quad \text{in } (0,L), \qquad & v_t(\cdot,0) &= v_1, \quad \text{in } (0,L), \\ w(\cdot,0) &= w_0, \quad \text{in } (0,L), \qquad & w_t(\cdot,0) &= w_1, \quad \text{in } (0,L). \end{aligned}$$

The past history of the rotation angle is to be given, and we set

$$v(-s) = v_2(s), \qquad \text{in } (0,L),$$

for every $s \geq 0$. We suppose that the beam has been clamped in its whole past, then $v_2(s,0) = v_2(s,L) = 0$.

REMARK 2.1. This model is an extension of the well-known viscoelastic Kirchhoff beam. Indeed, if we eliminate the term $k(v + w_x)$ between the two equations and we introduce the Kirchhoff assumption

$$v = -w_x,$$

we get

$$\rho_2 w_{tt} - \rho_1 w_{xxtt} + d_0 w_{xxxx} + \int_0^\infty b(\sigma) w_{xxxx}(t-\sigma)\, d\sigma = g(w).$$

The corresponding homogeneous problem has been studied in [4], where the exponential and the polynomial decay of the solution is obtained by energy estimates, when the memory kernel decay exponentially or polynomially, respectively, if a condition on the coefficients is satisfied. It is worth mentioning the analysis of the homogeneous problem carried out in [5], where the exponential stability of the solution is showed using a contraction argument.

Here, in order to show that this model represents a dynamical system, we introduce the new variable

$$\eta^t(s) = v(t) - v(t-s), \tag{2.5}$$

$s \geq 0$ (cf. [6–9], and references therein). Then, from (2.3), (2.5), and setting $\eta_0(s) = v_0 - v_2(s)$, η undergoes the following boundary and initial conditions:

$$\eta^t(0) = 0, \qquad \forall\, t \geq 0, \tag{2.6}$$
$$\eta^t(0,s) = \eta^t(L,s) = 0, \qquad \forall\, s,t \geq 0, \tag{2.7}$$
$$\eta^0(s) = \eta_0(s), \qquad \forall\, s \geq 0. \tag{2.8}$$

Differentiation in t of (2.5) gives

$$\eta^t_t(s) = v_t(t) - \eta^t_s(s). \tag{2.9}$$

When the kernel b is summable, we add and subtract to equation (2.1) the term $v_{xx}\int_0^\infty b(\sigma)\,d\sigma$, to get

$$\rho_1 v_{tt} - d_1 v_{xx} - \int_0^\infty \mu(\sigma)\eta^t_{xx}(\sigma)\,d\sigma + k(v + w_x) = 0, \tag{2.10}$$

where we have set

$$d_1 = \left(d_0 + \int_0^\infty b(\sigma)\,d\sigma\right) \qquad \text{and} \qquad \mu(\sigma) = -b(\sigma).$$

PROBLEM P. Find (v, w, η) solution to the system

$$\rho_1 v_{tt} - d_1 v_{xx} - \int_0^\infty \mu(\sigma)\eta^t_{xx}(\sigma)\,d\sigma + k(v + w_x) = f(v), \tag{2.11}$$
$$\rho_2 w_{tt} - k(v + w_x)_x = g(w), \tag{2.12}$$
$$\eta_t + \eta_s = v_t, \tag{2.13}$$

in $(0, L)$, for any $t \geq 0$ and any $s \geq 0$, with the initial conditions

$$\begin{aligned} v(0) = v_0 \text{ and } v_t(0) = v_1, &\qquad \text{in } (0,L),\\ w(0) = w_0 \text{ and } w_t(0) = w_1, &\qquad \text{in } (0,L),\\ \eta^0 = \eta_0, &\qquad \text{in } (0,L)\times(0,\infty), \end{aligned} \tag{2.14}$$

and the boundary conditions

$$v(0,t) = v(L,t) = 0, \qquad \forall\, t \geq 0, \tag{2.15}$$
$$w(0,t) = w(L,t) = 0, \qquad \forall\, t \geq 0, \tag{2.16}$$
$$\eta^t(0) = 0, \qquad \forall\, t \geq 0, \tag{2.17}$$
$$\eta^t(0,s) = \eta^t(L,s) = 0, \qquad \forall\, t,s \geq 0. \tag{2.18}$$

The plan of this note is as follows. We first sketch some analytic tools necessary to the functional setting of the problem, and we formulate the hypothesis under which a variational formulation is given. We prove existence and uniqueness of the solution to the problem, as well as continuous dependence results, which allow to interpret the solution as a strongly continuous semigroup acting on the appropriate phase space. Its dissipativity is, then, achieved via uniform energy estimates. The last section remarks on Mindlin-Timoshenko *plates*, which can be dealt with using the same technique showed herein, if an appropriate additional condition is satisfied.

2.1. Hypothesis and Technical Tools

With standard notation, we introduce the Hilbert spaces

$$H = L^2(0, L)$$

with its usual norm $\|\cdot\|$ and product $(\cdot,\cdot)$, and

$$V = H_0^1(0, L)$$

with the product $(u,v)_V = (u_x, v_x)$, which induces the norm $\|\cdot\|_V$ equivalent to the standard norm in V, thanks to Poincaré inequality

$$\|u\| \leq C_P \|u_x\|,$$

where we choose $C_P = 1$. Let us denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V.

We write $C(I; X)$ to denote the Banach space of the functions which are continuous on the-possibly unbounded-interval I with values in the Hilbert space $(X, (\cdot,\cdot)_X)$. In general, we use also the notation $L^2(I; X)$ and $H^1(I; X)$, with obvious meaning. Furthermore, if μ is a positive function on I, we can introduce the Hilbert space

$$\mathcal{M} = L^2_\mu(I; X)$$

endowed with the inner product

$$(u,v)_{\mathcal{M}} = \int_I \mu(\sigma)(u(\sigma), v(\sigma))_X \, d\sigma.$$

We assume

$$\begin{gathered} v_0 \in V, \qquad w_0 \in V, \qquad \eta_0 \in \mathcal{M}, \\ v_1 \in H, \qquad w_1 \in H. \end{gathered} \tag{2.19}$$

We consider the memory kernel μ which satisfies the following assumptions:

(k1) $$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),$$

(k2) $$\mu(\sigma) \geq 0, \qquad \forall\, \sigma \in \mathbb{R}^+,$$

(k3) $$\mu'(\sigma) \leq 0, \qquad \forall\, \sigma \in \mathbb{R}^+.$$

In order to obtain energy uniform estimates, we suppose also

(k4) $$\begin{gathered} \mu' \in L^2(0, \sigma_0), \\ \mu'(\sigma) + \delta\mu(\sigma) \leq 0, \qquad \forall\, \sigma \in \mathbb{R}^+, \\ \mu'(\sigma) + M\mu(\sigma) \geq 0, \qquad \forall\, \sigma \geq \sigma_0, \end{gathered}$$

for some $0 < \delta < M$ and $\sigma_0 > 0$, where M is a constant depending on σ_0 and it increases as σ_0 decreases. This exponentially decaying condition on kernel is standardly considered (see, e.g., [6,10]). On the other hand, it seems quite obvious that to have exponential decay of the energy, the kernel must show the same rate of decay (see also [11,12]).

In view of (k1) and (k2), we set

$$\mu_0 = \int_0^\infty \mu(\sigma)\, d\sigma.$$

We assume that both forcing terms f and g are Lipschitz functions of their variable, i.e.,

$$|f(r_1) - f(r_2)| \leq \Lambda_f |r_1 - r_2|, \qquad |g(r_1) - g(r_2)| \leq \Lambda_g |r_1 - r_2|, \tag{2.20}$$

for every $r_1, r_2 \in \mathbb{R}$, and where Λ_f and Λ_g are positive constants. Some other assumptions on the nonlinearity are in order. Introducing the functions

$$F(r) = -\int_0^r f(s)\, ds, \qquad G(r) = -\int_0^r g(s)\, ds,$$

we let

$$\liminf_{|r|\to\infty} \frac{F(r)}{r^2} \geq 0, \tag{2.21}$$

$$\liminf_{|r|\to\infty} \frac{G(r)}{r^2} \geq 0, \tag{2.22}$$

$$\liminf_{|r|\to\infty} \frac{-rf(r) - F(r)}{r^2} \geq 0, \tag{2.23}$$

(see [13,14]).

3. WELL-POSEDNESS RESULTS

We discuss now existence of the solution to Problem **P**. First, we observe that (2.21)–(2.23) and Poincaré inequality imply that there exist two positive constants c_0 and c_1, depending on $\varepsilon > 0$, such that for every v, $w \in V$

$$(F(v), 1) \geq -c_0 - \varepsilon\|v_x\|^2, \qquad (G(w), 1) \geq -c_0 - \varepsilon\|w_x\|^2, \tag{3.1}$$

$$-(f(v), v) + \varepsilon\|v_x\|^2 \geq -c_1 + (F(v), 1), \tag{3.2}$$

(see [14]).

THEOREM 3.1. *Assume (k1)–(k3) and (2.19)–(2.23), then Problem* **P** *has a solution, i.e., there exists a triplet* (v, w, η), *such that*

$$\begin{aligned} v &\in C\left(\mathbb{R}^+; V\right), \\ v_t &\in C\left(\mathbb{R}^+; H\right), \\ w &\in C\left(\mathbb{R}^+; V\right), \\ w_t &\in C\left(\mathbb{R}^+; H\right), \\ \eta &\in C\left(\mathbb{R}^+; \mathcal{M}\right), \end{aligned}$$

which solves the variational problem

$$\begin{gathered} \rho_1\langle v_{tt}, \phi\rangle + d_1(v_x, \phi_x) + \int_0^\infty \mu(\sigma)(\eta_x(\sigma), \phi_x)\, d\sigma \\ +k(v + w_x, \phi) = (f(v), \phi), \qquad \forall\, \phi \in V, \end{gathered} \tag{3.3}$$

$$\rho_2\langle w_{tt}, \phi\rangle + k(v + w_x, \phi_x) = (g(w), \phi), \qquad \forall\, \phi \in V, \tag{3.4}$$

$$(\eta_t + \eta_s, \zeta)_{\mathcal{M}} = (v_t, \zeta)_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M}, \tag{3.5}$$

almost everywhere in $\mathbb{R}^+$.

Furthermore, let $\{v_{0i},\, v_{1i},\, w_{0i},\, w_{1i},\, \eta_{0i}\}$, $i = 1, 2$ *be two sets of data, which satisfy (2.19), and let* $\{v_i,\, w_i,\, \eta_i\}$ *be the corresponding solutions to Problem* **P**. *Then, the following estimate holds:*

$$\begin{gathered} \|v_1(t) - v_2(t)\|_V^2 + \|\partial_t v_1(t) - \partial_t v_2(t)\|_H^2 + \|w_1(t) - w_2(t)\|_V^2 \\ +\|\partial_t w_1(t) - \partial_t w_2(t)\|_H^2 + \|\eta_1(t) - \eta_2(t)\|_{\mathcal{M}}^2 \\ \leq c\left(\|v_{01} - v_{02}\|_V^2 + \|v_{11} - v_{12}\|_H^2\right. \\ \left. + \|w_{01} - w_{02}\|_V^2 + \|w_{11} - w_{12}\|_H^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2\right). \end{gathered} \tag{3.6}$$

PROOF OF THEOREM 3.1. We make use of a Faedo-Galerkin approximating scheme. Let $\{\phi_j\}$, $j \in \mathbb{N}$ be an orthonormal basis in H, orthogonal in V, too, and let $\{\zeta_j\}$ be an orthonormal base of $\mathcal{M}$. We fix $n \in \mathbb{N}$ and consider the projections

$$\begin{aligned} P^n &: V \longrightarrow V^n = \operatorname{Span}\{\phi_1, \dots, \phi_n\}, \\ Q^n &: \mathcal{M} \longrightarrow \mathcal{M}^n = \operatorname{Span}\{\zeta_1, \dots, \zeta_n\}. \end{aligned}$$

We let $a_{ij}(t)$, $i = 1, 2, 3$ be some regular function such that

$$\begin{aligned} v^n(t) &= \sum_{j=1,\dots,n} a_{1j}^n(t)\phi_j, \\ w^n(t) &= \sum_{j=1,\dots,n} a_{2j}^n(t)\phi_j, \\ \eta^n(t) &= \sum_{j=1,\dots,n} a_{3j}^n(t)\zeta_j, \end{aligned}$$

satisfy

$$
\begin{aligned}
&\rho_1\langle v_{tt}^n,\phi\rangle + d_1\left(v_x^n,\phi_x\right) + \int_0^\infty \mu(\sigma)\left(\eta_x^n(\sigma),\phi_x\right)\,d\sigma \\
&\quad + k\left(v^n + w_x^n,\phi\right) = \left(P^n f\left(v^n\right),\phi\right), \qquad \forall\,\phi\in V,
\end{aligned}
\tag{3.7}
$$

$$
\rho_2\langle w_{tt}^n,\phi\rangle + k\left(v^n + w_x^n,\phi_x\right) = \left(P^n g\left(w^n\right),\phi\right), \qquad \forall\,\phi\in V, \tag{3.8}
$$

$$
\left(\eta_t^n + \eta_s^n,\zeta\right)_{\mathcal{M}} = \left(v_t^n,\zeta\right)_{\mathcal{M}}, \qquad \forall\,\zeta\in\mathcal{M}, \tag{3.9}
$$

almost everywhere in I, with initial data

$$
\begin{aligned}
v^n(0) &= P^n v_0,\\
v_t^n(0) &= P^n v_1,\\
w^n(0) &= P^n w_0,\\
w_t^n(0) &= P^n w_1,\\
\eta^n(0) &= Q^n \eta_0.
\end{aligned}
$$

We choose $\phi = \phi_\kappa$, and $\zeta = \zeta_\kappa$, $\kappa = 1,\ldots,n$ to have a Cauchy problem for a nonlinear system with of ordinary differential equations in $a_{ij}(t)$, whose local C^1 solution exists on an interval $(0,t_n)$, by classic theorems.

The following estimates ensure that, for every n we can take $t_n = T$. We introduce the functional

$$
\mathcal{E}^n = \frac{1}{2}\left(\rho_1\left\|v_t^n\right\|^2 + \rho_2\left\|w_t^n\right\|^2 + k\left\|v^n + w_x^n\right\|^2 + \left\|\eta^n\right\|_{\mathcal{M}}^2 + d_1\left\|v_x^n\right\|^2\right). \tag{3.10}
$$

Take equations (3.7) and (3.8), and test them with v_t^n and by w_t^n, respectively, then add. After integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\left\|v_t^n\right\|^2 + \rho_2\left\|w_t^n\right\|^2 + k\left\|v^n + w_x^n\right\|^2 + d_1\left\|v_x^n\right\|^2 + \left(F^n\left(v^n\right),1\right) \\
&\qquad + \left(G^n\left(w^n\right),1\right)\Big) - \int_0^\infty \mu(\sigma)\left(\eta_{xx}^n(\sigma),v_t^n\right)\,d\sigma = 0.
\end{aligned}
\tag{3.11}
$$

We test (3.9) with η^n to get

$$
\frac{1}{2}\frac{d}{dt}\left\|\eta^n\right\|_{\mathcal{M}}^2 + \int_0^\infty \mu(\sigma)\left(\eta_{sx}^n(\sigma),\eta_x^n(\sigma)\right)\,d\sigma = \int_0^\infty \mu(\sigma)\left(v_{tx}^n,\eta_x^n(\sigma)\right)\,d\sigma
$$

after integration by parts, on account of (k3), we get

$$
\frac{1}{2}\frac{d}{dt}\left\|\eta^n\right\|_{\mathcal{M}}^2 \le -\int_0^\infty \mu(\sigma)\left(v_t^n,\eta_{xx}^n(\sigma)\right)\,d\sigma. \tag{3.12}
$$

Now, adding (3.11) to (3.12), we are led to

$$
\frac{d}{dt}\mathcal{E}_1^n \le 0, \tag{3.13}
$$

where

$$
\mathcal{E}_1^n = \mathcal{E}^n + \left(F^n\left(v^n\right),1\right) + \left(G^n\left(w^n\right),1\right).
$$

Then, $\mathcal{E}_1^n$ decreases and it is bounded, uniformly in time, by a constant C which depends on $\mathcal{E}_1^n(0)$, and then on the physical settings of Problem **P**, i.e., on ρ_1, ρ_2, k, d_1, on initial data, and on the forcing terms f and g, and on L. In view of (3.1), we obtain the uniformly in n boundedness of $\mathcal{E}^n$. Then, for every $I \subset \mathbb{R}^+$, and

$$\begin{aligned}
\|v^n\|_{L^\infty(I;V)} &\le C,\\
\|v^n_t\|_{L^\infty(I;H)} &\le C,\\
\|w^n\|_{L^\infty(I;V)} &\le C,\\
\|w^n_t\|_{L^\infty(I;H)} &\le C,\\
\|\eta^n\|_{L^\infty(I;\mathcal{M})} &\le C.
\end{aligned}$$

Exploiting a well-known compactness property (cf. Corollary 4, Section 8 in [15]), we have that, up to some subsequence, there exist a triplet (v, w, η) such that

$$\begin{aligned}
v^n &\longrightarrow v \text{ strongly in } C(I;H),\\
w^n &\longrightarrow w \text{ strongly in } C(I;H),\\
\eta^n &\longrightarrow \eta \text{ weakly* in } L^\infty(I;\mathcal{M}).
\end{aligned}$$

It is easy to see that (v, w, η) satisfy (3.3)–(3.5), since continuity of projections F^n and G^n.

Consider, now, two solutions (v^n, w^n, η^n) and (v^m, w^m, η^m), respectively, corresponding to initial data $P^n v_{01}$, $P^n v_{11}$, $P^n w_{01}$, $P^n w_{11}$, $Q^n \eta_{01}$, and $P^m v_{01}$, $P^m v_{11}$, $P^m w_{01}$, $P^m w_{11}$, $Q^m \eta_{01}$. We observe that

$$\begin{aligned}
v^n, w^n, v^m, w^m &\in C(I;V),\\
v^n_t, w^n_t, v^m_t, w^m_t &\in C(I;H),\\
\eta^n, \eta^m &\in C(I;\mathcal{M}).
\end{aligned}$$

The differences

$$\begin{aligned}
v^{nm} &= v^n - v^m,\\
w^{nm} &= w^n - w^m,\\
\eta^{nm} &= \eta^n - \eta^m,
\end{aligned}$$

solve

$$\begin{aligned}
\rho_1 \langle v^{mn}_{tt}, \phi\rangle + d_1 (v^{mn}_x, \phi_x) + \int_0^\infty \mu(\sigma) \left(\eta^{mn}_x(\sigma), \phi_x\right) d\sigma &\\
+ k \left(v^{mn} + w^{mn}_x, \phi\right) = \left(f(v^m) - f(v^m), \phi\right), &\qquad \forall\, \phi \in V,\\
\rho_2 \langle w^{mn}_{tt}, \phi\rangle + k \left(v^{mn} + w^{mn}_x, \phi_x\right) = \left(g(w^n) - g(w^m), \phi\right), &\qquad \forall\, \phi \in V,\\
\left(\eta^{mn}_t + \eta^{mn}_s, \zeta\right)_{\mathcal{M}} = \left(v^{mn}_t, \zeta\right)_{\mathcal{M}}, &\qquad \forall\, \zeta \in \mathcal{M}.
\end{aligned}$$

Repeating the steps bringing to (3.13), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\rho_1 \|v^{nm}_t\|^2 + \rho_2 \|w^{nm}_t\|^2 + k \|v^{nm} + w^{nm}_x\|^2 + d_1 \|v^{nm}_x\|^2 + \|\eta^{nm}\|_{\mathcal{M}}\right)&\\
\le \left(f(v^m) - f(v^n), v^{nm}_t\right) + \left(g(w^m) - g(w^n), w^{nm}_t\right)&
\end{aligned}$$

and by (2.20) and Hölder inequality, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\rho_1 \|v^{nm}_t\|^2 + \rho_2 \|w^{nm}_t\|^2 + k \|v^{nm} + w^{nm}_x\|^2 + d_1 \|v^{nm}_x\|^2 + \|\eta^{nm}\|^2_{\mathcal{M}}\right)&\\
\le \Lambda_f \|v^{mn}_t\| \, \|v^{nm}\| + \Lambda_g \|w^{mn}\| \, \|w^{nm}_t\|.&
\end{aligned}$$

Set

$$\mathcal{E}^{mn} = \rho_1 \left\|v_t^{nm}\right\|^2 + \rho_2 \left\|w_t^{nm}\right\|^2 + k \left\|v^{nm} + w_x^{nm}\right\|^2 + d_1 \left\|v_x^{nm}\right\|^2 + \left\|\eta^{nm}\right\|_{\mathcal{M}}^2$$

then, Young inequality yields

$$\frac{d}{dt}\mathcal{E}^{mn} \leq c\mathcal{E}^{mn}.$$

Thus, Gronwall lemma ensures that

$$\mathcal{E}^{mn}(t) \leq 2e^{cT}\mathcal{E}^{mn}(0) \tag{3.14}$$

for every $t \in [0,T]$. Now, let $m, n \longrightarrow \infty$, we get

$$\begin{gathered}
\rho_1\|\partial_t v_1(t) - \partial_t v_2(t)\|_H^2 + \rho_2\|\partial_t w_1(t) - \partial_t w_2(t)\|_H^2 \\
+k\|v_1(t) + \partial_x w_1(t) - v_2(t) - \partial_x w_2(t)\|_H^2 \\
+d_1\|\partial_x v_1(t) - \partial_x v_2(t)\|_H^2 + \|\eta_1(t) - \eta_2(t)\|_{\mathcal{M}}^2 \\
\leq c\left(\|v_{01} - v_{02}\|_V^2 + \|v_{11} - v_{12}\|_H^2 + \|w_{01} - w_{02}\|_V^2\right. \\
\left. + \|w_{11} - w_{12}\|_H^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2\right),
\end{gathered}$$

which entails (3.6).

Consider again (3.14), and take

$$\begin{gathered}
v_{01} = v_{01}, \qquad w_{01} = w_{02}, \qquad \eta_{01} = \eta_{02}, \\
v_{11} = v_{12}, \qquad w_{11} = w_{12},
\end{gathered}$$

then, $\max_{t\in[0,T]} \mathcal{E}^{mn}(t) \longrightarrow 0$ as $m, n \to \infty$ and $\{(v^m, w^m, \eta^m)\}$ is a Cauchy sequence in a Banach space, hence converging to (v, w, η) such that

$$\begin{aligned}
w, v &\in C(0,T;V), \\
v_t, w_t &\in C(0,T;H), \\
\eta &\in C(0,T;\mathcal{M}).
\end{aligned}$$

It is also clear that no explosions arise, than the solution can be prolonged to $\mathbb{R}^+$, and the theorem is proved. ∎

4. THE DISSIPATIVE SEMIGROUP ASSOCIATED TO THE SOLUTION

We set now

$$\begin{aligned}
z(t) &= \left(v(t), v_t(t), w(t), w_t(t), \eta^t\right), \\
\mathcal{Z} &= V \times H \times V \times H \times \mathcal{M}, \\
z_0 &= (v_0, v_1, w_0, w_1, \eta_0),
\end{aligned}$$

and we agree to denote the solution $z(t)$ of Problem **P** with initial data $z_0 \in \mathcal{Z}$ by $S(t)z_0$. Then, Theorem 3.1 ensures that the family of one parameter operators $S(t)$ possesses the following properties:

(i)	$S(0)$ is the identity operator on $\mathcal{Z}$,	
(ii)	$S(t)z \in C(0,\infty;\mathcal{Z})$,	$\forall\, z \in \mathcal{Z}$,
(iii)	$S(t)$ is continuous from $\mathcal{Z}$ to $\mathcal{Z}$,	$\forall\, t \geq 0$,
(iv)	$S(t)S(s) = S(t+s)$,	$\forall\, s, \quad t \geq 0$.

Following [16,17], we say that a family of one parameter operators satisfying (i)–(iv) is a strongly continuous nonlinear semigroup. The set $\mathcal{B}_0 \subset \mathcal{Z}$ is an absorbing set for $S(t)$ if for any bounded set $\mathcal{B} \subset \mathcal{Z}$ there exists a time $T = T(\mathcal{B})$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall\, t \geq T.$$

We can state the following main result.

THEOREM 4.1. *Assume that (k1)–(k4), (2.14)–(2.18), and (2.20) hold. Furthermore, we set*

$$\frac{\rho_1}{\rho_2} = \frac{d_1 + \mu_0}{k}. \tag{4.1}$$

Then, the semigroup $S(t)$, which denotes the solution to Problem **P**, *acting on $\mathcal{Z}$, possesses an absorbing set $\mathcal{B}_0$.*

COROLLARY 4.2. *If, in addition to assumptions of Theorem 4.1, we assume $f = g = 0$, the semigroup $S(t)$ decays exponentially and its absorbing set $\mathcal{B}_0$ is $\{0\}$.*

REMARK 4.3. Condition (4.1) may be rewritten as

$$\frac{\rho_1}{\rho_2} = \frac{d}{k} \tag{4.2}$$

and it expresses that are equal the propagation speeds of the two deformation waves, associated to v and w (this condition is also necessary to the exponential stability of the solution, cf. [4]). Note that (4.2) suggests the choice of the scalar factor K in (1.10). Indeed, using (1.15), from (4.2) we get

$$K = \frac{2N(0)}{\mu},$$

where μ is constant, by assumption, and $N(0)$ depends on the material function ν through (1.13).

PROOF OF THEOREM 4.1. We remark that constants, appearing in the whole proof, do not depend on initial data, but on the physical setting of the problem, only. We observe that, using an integration by parts and Hölder inequality, (k4) implies

$$\begin{aligned}
&-\int_0^\infty \mu'(\sigma)(u,\eta(\sigma))\,d\sigma \\
&\leq \frac{1}{\mu^{1/2}(\sigma_0)}\int_0^{\sigma_0} \mu^{1/2}(\sigma)\,|\mu'(\sigma)|\,|(u,\eta(\sigma))|\,d\sigma + M\int_{\sigma_0}^\infty \mu(\sigma)(u,\eta(\sigma))\,d\sigma \\
&\leq \frac{\|u\|}{\mu^{1/2}(\sigma_0)}\int_0^{\sigma_0} \mu^{1/2}(\sigma)\,|\mu'(\sigma)|\,\|\eta(\sigma)\|_V\,d\sigma \\
&\quad + M\|u\|\int_{\sigma_0}^\infty \mu^{1/2}(\sigma)\mu^{1/2}(\sigma)\|\eta(\sigma)\|_V\,d\sigma \\
&\leq \frac{\|u\|}{\mu^{1/2}(\sigma_0)}\left(\int_0^{\sigma_0} |\mu'(\sigma)|^2\,d\sigma\right)^{1/2}\|\eta\|_{\mathcal{M}} + M\mu_0\|u\|\|\eta\|_{\mathcal{M}} \\
&\leq \varepsilon\|u\|^2 + c(\varepsilon)\|\eta\|^2_{\mathcal{M}},
\end{aligned} \tag{4.3}$$

for every $\eta \in \mathcal{M}$ and $u \in H$, where $\varepsilon > 0$ (cf. also [12]).

We introduce the quantities

$$\mathcal{L}_1 = -\rho_1\int_0^\infty \mu(\sigma)(\eta(\sigma), v_t)\,d\sigma, \tag{4.4}$$

$$\mathcal{L}_2 = \rho_1(v_t, v + w_x) + \frac{\rho_2 d_1}{k}(w_t, v_x) + \frac{\rho_2}{k}\int_0^\infty \mu(\sigma)(\eta_x(\sigma), w_t)\,d\sigma, \tag{4.5}$$

$$\mathcal{L}_3 = -\rho_2(w_t, w), \tag{4.6}$$

$$\mathcal{L}_4 = \rho_1(v_t, v). \tag{4.7}$$

Using (2.11)–(2.13), we are led to

$$\begin{aligned}
\frac{d\mathcal{L}_1}{dt} &= -d_1\int_0^\infty \mu(\sigma)(\eta(\sigma), v_{xx})\,d\sigma - \int_0^L\left|\int_0^\infty \mu(\sigma)\eta(\sigma)\,d\sigma\right|^2 dx \\
&\quad + k\int_0^\infty \mu(\sigma)(\eta(\sigma), v + w_x)\,d\sigma - \int_0^\infty \mu(\sigma)(\eta(\sigma), f(v))\,d\sigma \\
&\quad + \rho_1\int_0^\infty \mu(\sigma)(\eta_s(\sigma), v_t)\,d\sigma - \rho_1\mu_0\|v_t\|^2
\end{aligned}$$

and integration by parts gives

$$\begin{aligned}\frac{d\mathcal{L}_1}{dt} &\le -\rho_1\mu_0\|v_t\|^2 + d_1\int_0^\infty \mu(\sigma)(\eta_x(\sigma), v_x)\,d\sigma\\ &+k\int_0^\infty \mu(\sigma)(\eta(\sigma), v+w_x)\,d\sigma + \mu_0\|\eta\|^2_{\mathcal{M}}\\ &-\int_0^\infty \mu(\sigma)(\eta(\sigma), f(v))\,d\sigma - \rho_1\int_0^\infty \mu'(\sigma)(\eta(\sigma), v_t)\,d\sigma.\end{aligned}$$

By means of Hölder inequality, (2.20) and (4.3), we have

$$\frac{d\mathcal{L}_1}{dt} \le -\rho_1\mu_0\|v_t\|^2 + \varepsilon_1\left(\|v_x\|^2 + \|v_t\|^2 + \|v+w_x\|^2\right) + c(\varepsilon_1)\|\eta\|^2_{\mathcal{M}} + \varepsilon_1\|v\|^2 + c.$$

Consider now,

$$\begin{aligned}\frac{d\mathcal{L}_2}{dt} &= \rho_1(v_{tt}, v+w_x) + \rho_1(v_t, (v+w_x)_t)\\ &+\frac{\rho_2 d_1}{k}(w_{tt}, v_x) + \frac{\rho_2 d_1}{k}(w_t, v_{xt})\\ &+\frac{\rho_2}{k}\int_0^\infty \mu(\sigma)(\eta_{xt}(\sigma), w_t)\,d\sigma + \frac{\rho_2}{k}\int_0^\infty \mu(\sigma)(\eta_x(\sigma), w_{tt})\,d\sigma.\end{aligned}$$

Using (2.11)–(2.13), we get

$$\begin{aligned}\frac{d\mathcal{L}_2}{dt} &= \left(d_1 v_{xx} + \int_0^\infty \mu(\sigma)\eta_{xx}(\sigma)\,d\sigma - k(v+w_x) + f(v), v+w_x\right)\\ &+\rho_1\|v_t\|^2 + \rho_1(v_t, w_{xt})\\ &+d_1((v+w_x)_x, v_x) + \frac{d_1}{k}(g(w), v_x) + \frac{\rho_2 d_1}{k}(w_t, v_{xt})\\ &+\frac{\rho_2}{k}\int_0^\infty \mu(\sigma)((v_t-\eta_s)_x(\sigma), w_t)\,d\sigma\\ &+\frac{1}{k}\int_0^\infty \mu(\sigma)(\eta_x(\sigma), k(v+w_x)_x + g(w))\,d\sigma.\end{aligned}$$

Then, integration by parts leads to

$$\begin{aligned}\frac{d\mathcal{L}_2}{dt} &= -k\|v+w_x\|^2 + \rho_1\|v_t\|^2\\ &+\left(\rho_1 - \frac{\rho_2 d_1}{k} - \frac{\rho_2\mu_0}{k}\right)(w_{xt}, v_t) + \frac{\rho_2}{k}\int_0^\infty \mu'(\sigma)(\eta(\sigma), w_t)\,d\sigma\\ &+(f(v), v+w_x) + \frac{d_1}{k}(g(w), v_x) + \frac{1}{k}\int_0^\infty \mu(\sigma)(\eta_x(\sigma), g(w))\,d\sigma.\end{aligned}$$

Here, we have exploited the cancellation of three terms: $d_1(v_{xx}, w_x)$ and $d_1\|v_x\|^2$, which comes from multiplying (2.11) by $v+w_x$ and (2.12) by d_1/kv_x, in H; and

$$\int_0^\infty \mu(\sigma)(\eta_{xx}(\sigma), v+w_x)\,d\sigma,$$

which comes from multiplying (2.11) by $v+w_x$, and (2.12) by $(1/k)\int_0^\infty \mu(\sigma)\eta_x(\sigma)\,d\sigma$ in H. Indeed, since (4.1) holds, we have

$$\begin{aligned}\frac{d\mathcal{L}_2}{dt} &= -k\|v+w_x\|^2 + \rho_1\|v_t\|^2 + \frac{\rho_2}{k}\int_0^\infty \mu'(\sigma)(\eta(\sigma), w_t)\,d\sigma\\ &\quad + (f(v), v+w_x) + \frac{d_1}{k}(g(w), v_x) + \frac{1}{k}\int_0^\infty \mu(\sigma)(\eta_x(\sigma), g(w))\,d\sigma.\end{aligned}$$

Then, by Hölder inequality, and exploiting (2.20),(4.3), we deduce

$$\frac{d\mathcal{L}_2}{dt} \le -k\|v+w_x\|^2 + \rho_1\|v_t\|^2 + \varepsilon_2\|w_t\|^2 + c(\varepsilon_2)\|\eta\|^2_{\mathcal{M}} \\ + \varepsilon_2\|w\|^2 + c(\varepsilon_2)\|v_x\|^2 + \varepsilon_2\|v+w_x\|^2 + c(\varepsilon_2)\left(1+\|v\|^2\right).$$

Now, Poincaré inequality gives

$$\frac{d\mathcal{L}_2}{dt} \le -(k-2\varepsilon_2)\|v+w_x\|^2 + \rho_1\|v_t\|^2 + \varepsilon_2\|w_t\|^2 \\ + c(\varepsilon_2)\left(\|\eta\|^2_{\mathcal{M}} + \|v_x\|^2\right) + c. \tag{4.8}$$

Using (2.12), and integrating by parts, exploiting Young and Poincaré inequalities and the lipschitz growth of g, we have

$$\begin{aligned}\frac{d\mathcal{L}_3}{dt} &= -\rho_2\|w_t\|^2 + k(v+w_x, w_x) - (g(w), w) \\ &\le -\rho_2\|w_t\|^2 + 2k\|v+w_x\|^2 + \frac{k}{4}\|v\|^2 - (g(w), w) \\ &\le -\rho_2\|w_t\|^2 + \frac{k}{4}\|v_x\|^2 + c\left(1 + \|w_x+v\|^2 + \|v_x\|^2\right).\end{aligned}$$

Finally, exploiting Hölder and Young inequalities and (3.2), we get

$$\begin{aligned}\frac{d\mathcal{L}_4}{dt} &= -d_1\|v_x\|^2 - \int_0^\infty \mu(\sigma)(\eta_x(\sigma), v_x)\,d\sigma - k(v+w_x, v) + (f(v), v) + \rho_1\|v_t\|^2 \\ &\le -\frac{d_1}{2}\|v_x\|^2 + c\|\eta\|^2_{\mathcal{M}} + c\|v+w_x\|^2 + \rho_1\|v_t\|^2 - (F(v), 1) + c.\end{aligned}$$

Now, and we have

$$\frac{d\mathcal{L}_1}{dt} + (\rho_1\mu_0 - \varepsilon_1)\|v_t\|^2 \le 2\varepsilon_1\left(\|v_x\|^2 + \|v+w_x\|^2\right) + c\|\eta\|^2_{\mathcal{M}} + c, \tag{4.9}$$

$$\frac{d\mathcal{L}_2}{dt} + (k-2\varepsilon_2)\|v+w_x\|^2 \le \rho_1\|v_t\|^2 + \varepsilon_2\|w_t\|^2 + c(\varepsilon_2)\left(\|\eta\|^2_{\mathcal{M}} + \|v_x\|^2\right) + c, \tag{4.10}$$

$$\frac{d\mathcal{L}_3}{dt} + \rho_2\|w_t\|^2 \le 2k\|v+w_x\|^2 + \frac{k}{4}\|v_x\|^2 + c\left(\|w_x+v\|^2 + \|v_x\|^2\right) + c, \tag{4.11}$$

$$\frac{d\mathcal{L}_4}{dt} + \frac{d_1}{2}\|v_x\|^2 + (F(v), 1) \le c\left(\|\eta\|^2_{\mathcal{M}} + \|v+w_x\|^2\right) + \rho_1\|v_t\|^2 + c. \tag{4.12}$$

Recall (3.1), take $\hat{\varepsilon}$, $\tilde{\varepsilon} > 0$, and define $\mathcal{L}_5 = \mathcal{L}_2 + \hat{\varepsilon}\mathcal{L}_3 + \tilde{\varepsilon}\mathcal{L}_4$, we can choose $\hat{\varepsilon}$, $\tilde{\varepsilon}$, and ε_2 small enough that constants

$$(k-2\varepsilon_2) - (2k+c)\hat{\varepsilon} - c\tilde{\varepsilon}, \\ \hat{\varepsilon}\rho_2 - \varepsilon_2, \\ \frac{\tilde{\varepsilon}d_1}{2} - \frac{\hat{\varepsilon}k}{4},$$

are positive. Again, take $\bar{\varepsilon} > 0$ and define $\mathcal{L}_6 = \mathcal{L}_1 + \bar{\varepsilon}\mathcal{L}_5$; choose ε_1 and $\bar{\varepsilon}$ small enough to have

$$\frac{d\mathcal{L}_6}{dt} + \varepsilon\left(\|v_x\|^2 + \|w_t\|^2 + \|v_t\|^2 + \|v+w_x\|^2 \right.\\ \left. + (F(v), 1) + (G(w), 1)\right) \le c\|\eta\|^2_{\mathcal{M}} + c, \tag{4.13}$$

where we set $\varepsilon > 0$, and we exploit the inequality $\varepsilon G(w) \le \varepsilon\|w_x + v\|^2 + \varepsilon\|v_x\|^2 + c$, thanks to the Lipschitz condition on g. Finally, we choose $N > 0$ big enough and set

$$\mathcal{L} = N\mathcal{E}_1 + \mathcal{L}_6.$$

Then, if N is big enough, we multiply (3.13) by N and add to (4.13), to get

$$\frac{d\mathcal{L}}{dt} + \varepsilon\mathcal{L} \le c. \tag{4.14}$$

We recall a uniform Gronwall-type lemma (cf. Lemma 2.5 in [8]).

LEMMA 4.4. *Let Φ, be a nonnegative locally summable and absolutely continuous function on the interval $[\tau,\infty)$, satisfying for some $\varepsilon > 0$ and $m_0 \geq 0$ the differential inequality*

$$\frac{d}{dt}\,\Phi^2(t) + \varepsilon\Phi^2(t) \leq m_0$$

almost everywhere in $t\in[\tau,\infty)$. *Then, for all* $t \in [\tau,\infty)$,

$$\Phi^2(t) \leq 2\Phi^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{2m_0}{\varepsilon}.$$

An application of the Lemma 4.4 to equation (4.14) gives

$$\mathcal{L}(t) \leq 2\mathcal{L}(0)e^{-\varepsilon t} + \frac{2c}{\varepsilon}, \tag{4.15}$$

for every $t \geq 0$.

We observe that an estimate on $\mathcal{L}$ is an estimate on $\mathcal{E}$ and vice versa. Indeed, by means of Young and Poincaré inequality, we have

$$\int_0^\infty \mu(\sigma)(\eta_x(\sigma),w_t)\,d\sigma \leq c\left(\|\eta\|^2_{\mathcal{M}} + \|w_t\|^2\right),$$

$$\begin{aligned}|(w_t,w)| &\leq c\left(\|w\|^2 + \|w\|^2\right) \leq c\left(\|w_t\|^2 + \|w_x\|^2\right)\\ &\leq c\left(\|w_t\|^2 + \|w_x+v\| + \|v\|^2\right)\\ &\leq c\left(\|w_t\|^2 + \|w_x+v\|^2 + \|v_x\|^2\right),\end{aligned}$$

then, it is easy to get $\mathcal{L} \leq |\mathcal{L}| \leq c\mathcal{E}$. Vice versa, the trivial inequality $\|a\|^2 + \|b\|^2 \leq 3/2(\|a\|^2 + \|b\|^2) + (a,b)$ for every $a,b \in H$ leads to $\mathcal{E} \leq c\mathcal{L}$.

Then, from (4.15), we can choose, e.g., $R_0 = 4C/\varepsilon$, and there exists a finite T, such that

$$\|S(t)(v_0,v_1,w_0,w_1,\eta_0)\|_{\mathcal{Z}} \leq R_0,$$

for every $t \geq T$. In particular, any ball in $\mathcal{Z}$ of radius strictly greater than $2c/\varepsilon$ may be taken as absorbing set for the semigroup $S(t)$ acting on $\mathcal{Z}$. ∎

5. REMARK ON THE MINDLIN-TIMOSHENKO PLATES

We consider now the Mindlin-Timoshenko plate defined on the regular domain $\Omega \subset \mathbb{R}^2$ (see, e.g., [1,2]). In particular, we set

$$v = \begin{bmatrix}\psi\\ \phi\end{bmatrix}, \qquad \eta = \begin{bmatrix}\eta^1\\ \eta^2\end{bmatrix}, \qquad f = \begin{bmatrix}f^1\\ f^2\end{bmatrix},$$

and introduce the operator

$$A = \begin{bmatrix}\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1-\nu}{2}\dfrac{\partial^2}{\partial x_2^2} & \dfrac{1+\nu}{2}\dfrac{\partial^2}{\partial x_1\partial x_2}\\ \dfrac{1+\nu}{2}\dfrac{\partial^2}{\partial x_1\partial x_2} & \dfrac{1-\nu}{2}\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\end{bmatrix},$$

where ν is a constant such that $0 < \nu < 1/2$, and μ is the memory kernel. The Mindlin-Timoshenko model (with memory), describing the behavior of the rotation angles ψ and of ϕ and the transverse displacement w of the plate with memory, is the following:

$$\frac{\rho h^3}{12}\,v_{tt} - d_1Av - \int_0^\infty \mu(\sigma)A\eta(\sigma)\,d\sigma + k(v+\nabla w) = f(v), \tag{5.1}$$

$$\rho h w_{tt} - k\nabla\cdot(v+\nabla w) = g(w), \tag{5.2}$$

$$\eta_t + \eta_s = v_t, \tag{5.3}$$

(we abuse notation, concerning the name of the constants). We have introduced the additional vector variable η, to deal this situation with the same technique we have used in the beam case.

We consider the Hilbert spaces

$$
\begin{aligned}
H &= L^2(\Omega), & H^2 &= H \times H,\\
V &= H_0^1(\Omega), & V^2 &= V \times V,
\end{aligned}
$$

and we use the equivalent norm, induced by the scalar product

$$
(u, v)_{V^2} = \int_\Omega \left(\nabla u^1 \cdot \nabla v^1 + \nabla u^2 \cdot \nabla v^2\right)\, d\Omega,
$$

for every $v, u \in V^2$, since we suppose the plate is clamped at its boundary with common notation. We denote by the pedex $\dot{v} = 1, 2$ the derivative along the direction i.

The operator $-A$ can be viewed as an operator from V^2 onto V^{2*}. Indeed, for every v, $z \in V^2$, we can set

$$
\begin{aligned}
\langle -Av, z\rangle_{V^2} \equiv \int_\Omega &\Big\{ v_1^1 z_1^1 + v_2^2 z_2^2 + \nu\left(v_2^2 z_1^1 + v_1^1 z_2^2\right)\\
&+ \frac{1-\nu}{2}\left(v_2^1 z_2^1 + v_1^2 z_2^1 + v_2^1 z_1^2 + v_1^2 z_1^2\right)\Big\}\, d\Omega.
\end{aligned}
\tag{5.4}
$$

We observe that the operator $-A$ is coercive, by means of the Korn inequality, in the appropriate phase space. We finally point out that condition (4.1) becomes

$$
\frac{h^2}{12} = \frac{d_1 + \mu_0}{k}. \tag{5.5}
$$

THE ADDITIONAL CONDITION. It is possible to extend every step of the study carried out in the previous section to the two-dimensional case. In addition, we suppose that there exist a scalar function $u = u(x_1, x_2)$ such that

$$
Av = \nabla u, \tag{5.6}
$$

namely,

$$
\begin{aligned}
u_1 &= \psi_{11} + \frac{1-\nu}{2}\psi_{22} + \frac{1+\nu}{2}\phi_{12},\\
u_2 &= \phi_{22} + \frac{1-\nu}{2}\psi_{11} + \frac{1+\nu}{2}\phi_{12}.
\end{aligned}
$$

This assumption enables us to simplify the terms

$$
\langle Av, v + \nabla w\rangle
$$

and

$$
\langle \nabla \cdot (v + \nabla w), u\rangle,
$$

where u is an admissible test function in V^2, which we use to deduce an inequality equivalent to (4.8). Then, Schwartz rule requires

$$
(\psi_2 - \phi_1)_{11} = (\phi_1 - \psi_2)_{22}
$$

with some boundary conditions. We remark that this condition is satisfied by the Kirchhoff assumption

$$
\begin{aligned}
\psi &= -\frac{\partial w}{\partial x_1},\\
\phi &= -\frac{\partial w}{\partial x_2},
\end{aligned}
$$

namely, $v + \nabla w = 0$. In this case, the variables v and w uncouple and (5.1) becomes

$$\frac{\rho h^3}{12} v_{tt} - d_1 A v - \int_0^\infty \mu(\sigma) A\eta(\sigma)\, d\sigma = f(v),$$
$$\eta_t + \eta_s = v_t.$$

Following the same strategy for the variable η, we need to introduce the scalar function $\bar{u} = \bar{u}(x_1, x_2, s)$ such that:

$$A\eta = \nabla \bar{u}, \tag{5.7}$$

where the symbol ∇ is intended to act on the two space variables only.

REMARK 5.1. Assumption (5.6) seems quite strict to describe viscoelastic plates. In a forthcoming paper, we shall structure the problem by removing hypothesis (1.15).

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