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Global Attractors for a Semilinear Hyperbolic Equation in Viscoelasticity

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A semilinear partial differential equation of hyperbolic type with a convolution term describing simple viscoelastic materials with fading memory is considered. Regarding the past history (memory) of the displacement as a new variable, the equation is transformed into a dynamical system in a suitable Hilbert space. The dissipation is extremely weak, and it is all contained in the memory term. Longtime behavior of solutions is analyzed. In particular, in the autonomous case, the existence of a global attractor for solutions is achieved.

Key Words: viscoelasticity; memory kernel; absorbing set; global attractor.

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1. INTRODUCTION

We consider the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^3$, arising in the theory of isothermal viscoelasticity (cf. [4, 25]),

$$\begin{aligned} &u_{tt} - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(t-s)\,ds + g(u) = f \qquad \text{in } \Omega \times \mathbb{R}^+ \\ &u(x,t) = 0, \qquad x \in \partial\Omega, t \in \mathbb{R} \\ &u(x,t) = u_0(x,t), \qquad x \in \Omega, t \leq 0 \end{aligned} \tag{1.1}$$

with $k(0)$, $k(\infty) > 0$, and $k'(s) \leq 0$ for every $s \in \mathbb{R}^+$.

As is well known (see [9]), this equation describes a homogeneous and isotropic viscoelastic solid. If the solid is not homogeneous and isotropic, the function k has to be replaced by a fourth order tensor depending on $x \in \Omega$, satisfying certain additional hypothesis (see, e.g., [8, p. 140]), and the above Eq. (1.1) has to be written in divergence form. However, all the arguments presented here can be easily generalized to the non-homogeneous non-isotropic case.

Notice that if $k' \equiv 0$, (1.1) reduces to the (semilinear) wave equation, where g represents some displacement-dependent body force density. Thus, neglecting the contributions of the nonlinearity, that is, taking $g \equiv 0$, all the dissipation is contained in the convolution integral. In particular, the existence of a genuine memory induces a damping mechanism, and asymptotic stability is to be expected (see [5, 8, 17]).

A problem similar to (1.1) has been studied in [23]. In that paper, however, the situation was in some sense more favorable, because of the presence of an instantaneous damping term. Clearly, the trade off in considering a weaker dissipation is a much stronger requirement on the structure of the nonlinearity. Here, as in [23], we introduce the new variable (see [5])

$$\eta^t(x,s) = u(x,t) - u(x,t-s). \tag{1.2}$$

We set for simplicity $\mu(s) = -k'(s)$ and $k(\infty) = 1$. In view of (1.2), adding and subtracting the term Δu, Eq. (1.1) transforms into the system

$$\begin{cases} u_{tt} = \Delta u + \int_0^\infty \mu(s)\Delta\eta(s)\,ds - g(u) + f \\ \eta_t = -\eta_s + u_t, \end{cases} \tag{1.3}$$

where the second equation is obtained differentiating (1.2). Initial-boundary conditions are then given by

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega, t \geq 0 \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+, t \geq 0 \\ u(x,0) = u_0(x), & x \in \Omega \\ u_t(x,0) = v_0(x), & x \in \Omega \\ \eta^0(x,s) = \eta_0(x,s), & (x,s) \in \Omega \times \mathbb{R}^+ \end{cases} \tag{1.4}$$

having set

$$\begin{cases} u_0(x) = u_0(x,0) \\ v_0(x) = \partial_t u_0(x,t)_{|t=0} \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s). \end{cases}$$

The memory kernel μ is required to satisfy the following hypotheses:

(h1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)\ \forall s \in \mathbb{R}^+$;

(h2) $\mu(s) \geq 0$ and $\mu'(s) \leq 0\ \forall s \in \mathbb{R}^+$;

(h3) $\int_0^\infty \mu(s)\, ds = k_0 > 0$;

(h4) $\mu'(s) + \delta\mu(s) \leq 0\ \forall s \in \mathbb{R}^+$ and some $\delta > 0$;

(h5) There exists $s_0 \geq 0$ such that $\mu' \in L^2((0, s_0))$ and $\mu'(s) + M\mu(s) \geq 0\ \forall s \geq s_0$ and some $M > 0$.

We can clearly weaken (h5) asking that μ' is square summable in a neighborhood of zero. This automatically implies, thanks to (h1), that $\mu' \in L^2((0, s_0))$, for every $s_0 > 0$. Conditions (h4)–(h5), which are not needed in the existence and uniqueness result, imply the exponential decay of $\mu(s)$. In particular, notice that $\mu(0)$ has to be finite. For instance, sums of exponentials fulfill (h1)–(h5). This rather strong decay rate of the kernel seems to be unavoidable in order to have the exponential decay of the associated linear problem (see, e.g., [5, 8, 12, 17]). On the other hand, since all the dissipation of the system is contained only in the memory term, we also have to require that $\mu \not\equiv 0$, and this explains (h3).

Concerning the nonlinear term, we assume that g is differentiable with bounded derivative. Clearly, this is a rather strong condition. Indeed, if we restrict our analysis to existence and uniqueness results, we may ask much weaker condition on g such as those in [23] (see also [10]). Nonetheless, this condition is the best one to obtain, for instance, the absorbing set. We point out that in this work we are able to obtain the exponential decay of the associated linear homogeneous system via energy estimates. The reader should compare this result with the analogous ones of [8, 17], where the exponential decay is obtained employing semigroup techniques. Our approach allows us to improve the asymptotic analysis to the nonlinear case. However, due to the extremely weak dissipation of the linear semigroup, we cannot arrive at further Lipschitz nonlinearities.

Let us mention some others papers related to the problem we address. For the nonlinear one-dimensional equation, Dafermos [6], exploring the dissipative properties of the equation, showed that the system is well posed provided the initial data are small enough, whereas for the n-dimensional linear system the author proved the asymptotic stability of the solutions, but without providing an explicit rate of decay (see [5]). For 3-dimensional isotropic and homogeneous materials, Dassios and Zafiropoulus [7], using an asymptotic analysis, proved that the solution of the viscoelastic system of memory type has a uniform decay to zero provided that the relaxation kernel is the exponential function. This result was improved for more general relaxations functions in the articles [16, 18, 19, 21]. On the other hand, when the relaxation has a polynomial decay, it was proved that the solution of the corresponding model has a polynomial decay either, with the same rate of decay as the relaxation (see [20]). Finally, for boundary stabilization of viscoelastic plates see [15]. Unfortunately, the methods used to achieve uniform rate of decay in those works are based on second order estimates, which are time dependent in our problem. Thus these techniques fail in the case of semilinear problems, and a new asymptotic analysis has to be devised.

The main result of this paper is to show the existence of a global attractor for the solution of Eq. (1.1). The method we use introduces a new multiplier and applies the concept of *strongly continuous semigroup of operators*, and the techniques developed in [23].

The plan of the paper is as follows. In Section 2 we introduce the notation. In Section 3 we give the definition of weak solution, along with existence and uniqueness results. Section 4 is devoted to uniform energy estimates and to the existence of absorbing sets for the solutions. Finally, in Section 5, we prove that the semigroup associated to our problem admits a global attractor in the phase-space.

2. NOTATION

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. With usual notation, we introduce the spaces H^{-1}, L^2, and H_0^1 acting on Ω. The symbol $\langle \cdot , \cdot \rangle$ will be used to denote the duality map between H^{-1} and

H_0^1. Exploiting the Poincaré inequality

$$\lambda_0 \int_\Omega |v|^2 \, dx \le \int_\Omega |\nabla v|^2 \, dx, \qquad \forall v \in H_0^1, \tag{2.1}$$

for some $\lambda_0 > 0$, the norm in H_0^1 is given by

$$\|v\|_{H_0^1}^2 = \int_\Omega |\nabla v|^2 \, dx.$$

In view of (h1), let $L^2_\mu(\mathbb{R}^+, H_0^1)$ be the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_\mu(\mathbb{R}^+, H_0^1)} = \int_\Omega \left(\int_0^\infty \mu(s) \nabla\varphi(s) \nabla\psi(s) \, ds \right) dx.$$

Finally we introduce the Hilbert space

$$\mathscr{H} = H_0^1 \times L^2 \times L^2_\mu\big(\mathbb{R}^+, H_0^1\big).$$

To describe the asymptotic behavior of the solutions of our system we need also to introduce the space $\mathscr{T}$ of L^1_{loc}-*translation bounded* L^2-valued functions on $\mathbb{R}^+$, namely

$$\mathscr{T} = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^+, L^2) : \|f\|_{\mathscr{T}} = \sup_{\xi \ge 0} \int_\xi^{\xi+1} \left(\int_\Omega |f(t)|^2 \, dx \right)^{1/2} dy < \infty \right\}.$$

3. EXISTENCE AND UNIQUENESS

We first formulate precisely the conditions on the nonlinearity. Let $g \in C^1(\mathbb{R})$ and denote

$$G(s) = \int_0^s g(y) \, dy$$

and

$$\mathscr{G}(u) = \int_\Omega G(u(x)) \, dx, \quad \text{for } u \in H_0^1.$$

The assumptions are as follows: there exist $C_0 > 0$ and $\Gamma > 0$ such that

(g1) $$\liminf_{|y|\to\infty} \frac{G(y)}{y^2} \geq 0;$$

(g2) $$\liminf_{|y|\to\infty} \frac{yg(y) - C_0 G(y)}{y^2} \geq 0;$$

(g3) $$|g'(y)| \leq \Gamma.$$

The following inequalities are direct consequences of (g1)–(g2) (cf. [10]),

$$\mathscr{G}(u) + \frac{1}{4}\int_\Omega |\nabla u|^2\, dx \geq -C_1, \qquad \forall u \in H_0^1, \tag{3.1}$$

$$\int_\Omega ug(u)\,dx - C_0\mathscr{G}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx \geq -C_2, \qquad \forall u \in H_0^1, \tag{3.2}$$

for some $C_1, C_2 > 0$.

The solution to the initial-boundary value problem (1.3)–(1.4) is defined in the following manner.

DEFINITION 3.1. Set $I = [0, T]$, for $T > 0$, and let $f \in L^1(I, L^2)$. We say a function $z = (u, u_t, \eta) \in C(I, \mathscr{H})$ is a solution to problem (1.3)–(1.4) in the time interval I, with initial data $z(0) = z_0 = (u_0, v_0, \eta_0) \in \mathscr{H}$, provided

$$\langle u_{tt}, \tilde{v} \rangle = -\int_\Omega \nabla u \nabla \tilde{v}\, dx - \int_\Omega \left(\int_0^\infty \mu(s) \nabla \eta(s)\, ds \right) \nabla \tilde{v}\, dx$$
$$- \int_\Omega g(u)\tilde{v}\, dx + \int_\Omega f\tilde{v}\, dx$$

$$\int_\Omega \left(\int_0^\infty \mu(s)(\eta_t(s) + \eta_s(s)) \Delta\tilde{\eta}(s)\, ds \right) dx = \int_\Omega u_t \left(\int_0^\infty \mu(s) \Delta \tilde{\eta}(s)\, ds \right) dx$$

for all $\tilde{v} \in H_0^1$ and $\tilde{\eta} \in L^2_\mu(\mathbb{R}^+, H^2 \cap H_0^1)$, and a.e. $t \in I$.

The proof of the next two theorems is omitted, since existence and uniqueness are proved exactly like the analogous results of [23], where the only difference is the presence of a damping term, which however plays a significant role only in time-independent estimates.

THEOREM 3.2 (Existence). *Let* (h1)–(h2) *and* (g1)–(g3) *hold. Then, given any* $T > 0$, *problem* (1.3)–(1.4) *has a solution* z *in the time interval* $I = [0, T]$, *with initial data* z_0.

THEOREM 3.3 (Continuous Dependence). *Let* (h1)–(h2) *and* (g1)–(g3) *hold. For* $i = 1, 2$, *let* $\{z_{0i}, f_i\}$ ($z_{0i} \in \mathscr{H}$ *and* $f_i \in L^1(I, L^2)$) *be two sets of data, and denote by* z_i *two corresponding solutions to problem* (1.3)–(1.4) *in the time interval* $I = [0, T]$. *Assume also that* $\|z_{0i}\|_{\mathscr{H}} < R$ *and* $\|f_i\|_{L^1(I, L^2)} < R$, *for some* $R > 0$. *Then the following estimate holds*,

$$\|z_1 - z_2\|_{\mathscr{H}}^2 \le C_R(T)\left(\|z_{01} - z_{02}\|_{\mathscr{H}}^2 + \|f_1 - f_2\|_{L^1(I, L^2)}^2\right) \tag{3.3}$$

for some constant $C_R(T) > 0$. *In particular, problem* (1.3)–(1.4) *has a unique solution*.

4. UNIFORM ENERGY ESTIMATES

In the sequel of the paper, we agree to denote the solution $z(t)$ of (1.3)–(1.4) with initial data z_0 by $S(t)z_0$. When the system is autonomous, namely, when f is independent of time, $S(t)$ is a strongly continuous semigroup of continuous (nonlinear) operators on $\mathscr{H}$ (see [24] for a detailed presentation of the theory). This follows directly from Theorem 3.2 and estimate (3.3) of Theorem 3.3.

The energy associated to (1.3) at time t is given by

$$\mathscr{E}(t) = \frac{1}{2}\left(\int_\Omega |\nabla u(t)|^2\,dx + \int_\Omega |u_t(t)|^2\,dx + \int_\Omega \left(\int_0^\infty \mu(s)|\nabla \eta^t(s)|^2\,ds\right)dx\right). \tag{4.1}$$

The main result of the section is

THEOREM 4.1. *Assume* (h1)–(h5) *and* (g1)–(g3), *and let* $F \subset \mathscr{T}$ *be a bounded set. Then there exists positive constants* C, Λ, ε (*depending on* F) *such that the relation*

$$\mathscr{E}(t) \le Ce^{-\varepsilon t}\mathscr{E}(0) + \Lambda \tag{4.2}$$

holds for every $t \ge 0$ *and every* $f \in F$. *In particular, if* $g \equiv 0$ *and* F *reduces to the null function* (*that is, the linear homogeneous case*), *then* $\Lambda = 0$.

Proof. Set

$$\Phi = \sup_{h \in F} \|h\|_{\mathscr{T}}.$$

Let $f \in F$ and denote

$$m_f(t) = \left(\int_\Omega |f(t)|^2 \, dx \right)^{1/2}.$$

Clearly,

$$\sup_{\xi \geq 0} \int_\xi^{\xi+1} m_f(\xi) \, d\xi \leq \Phi. \tag{4.3}$$

Taking the derivative with respect to t of (4.1) we have

$$\frac{d}{dt}\mathscr{E}(t) = \int_\Omega \nabla u(t) \nabla u_t(t) \, dx + \int_\Omega u_t(t) u_{tt}(t) \, dx$$
$$+ \int_\Omega \left(\int_0^\infty \mu(s) \nabla \eta_t^t(s) \nabla \eta^t(s) \, ds \right) dx.$$

Substituting (1.3) in the above inequality, we obtain

$$\frac{d}{dt}\mathscr{E}(t) = -\frac{1}{2} \int_\Omega \left(\int_0^\infty \mu(s) \frac{d}{ds} |\nabla \eta^t(s)|^2 \, ds \right) dx$$
$$- \frac{d}{dt}\mathscr{G}(u(t)) + \int_\Omega f(t) u_t(t) \, dx. \tag{4.4}$$

Integration by parts in s and (h4) yield (cf. [11] for the details)

$$-\frac{1}{2} \int_\Omega \left(\int_0^\infty \mu(s) \frac{d}{ds} |\nabla \eta^t(s)|^2 \, ds \right) dx \leq -\frac{\delta}{2} \int_\Omega \left(\int_0^\infty \mu(s) |\nabla \eta^t(s)|^2 \, ds \right) dx,$$

whereas the Hölder inequality gives

$$\int_\Omega f(t) u_t(t) \, dx \leq m_f(t) \left(\int_\Omega |u_t(t)|^2 \, dx \right)^{1/2}.$$

Hence from (4.4) we get the estimate

$$\frac{d}{dt}(\mathscr{E}(t) + \mathscr{G}(u(t)))$$
$$\leq -\frac{\delta}{2} \int_\Omega \left(\int_0^\infty \mu(s) |\nabla \eta^t(s)|^2 \, ds \right) dx + m_f(t) \left(\int_\Omega |u_t(t)|^2 \, dx \right)^{1/2}$$
$$\leq -\frac{\delta}{2} \int_\Omega \left(\int_0^\infty \mu(s) |\nabla \eta^t(s)|^2 \, ds \right) dx + m_f(t) \mathscr{E}^{1/2}(t). \tag{4.5}$$

Introduce now the functional

$$\mathscr{F}(t) = -\int_\Omega u_t(t)\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx. \tag{4.6}$$

Throughout the proof, we will make extensive use of the Hölder and Young inequalities, (h3), and (2.1).

The derivative with respect to t of $\mathscr{F}(t)$ entails

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t) = &-\int_\Omega u_{tt}(t)\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx\\ &-\int_\Omega u_t(t)\left(\int_0^\infty \mu(s)\eta_t^t(s)\,ds\right)dx.\end{aligned} \tag{4.7}$$

Exploiting (h5), and recalling (h2), we have

$$\begin{aligned}
&-\int_\Omega u_t(t)\left(\int_0^\infty \mu(s)\eta_t^t(s)\,ds\right)dx\\
&\quad= -\int_\Omega u_t(t)\left(\int_0^\infty \mu(s)\big(u_t(t) - \eta_s^t(s)\big)ds\right)dx\\
&\quad= -k_0\int_\Omega |u_t(t)|^2\,dx - \int_\Omega u_t(t)\left(\int_0^\infty \mu'(s)\eta^t(s)\,ds\right)dx\\
&\quad\le -k_0\int_\Omega |u_t(t)|^2\,dx + \int_\Omega |u_t(t)|\left(\int_0^{s_0} \frac{|\mu'(s)|}{\mu^{1/2}(s_0)}\mu^{1/2}(s)|\eta^t(s)|\,ds\right.\\
&\qquad\left. + M\int_{s_0}^\infty \mu(s)|\eta^t(s)|\,ds\right)dx\\
&\quad\le -k_0\int_\Omega |u_t(t)|^2\,dx\\
&\qquad+\frac{\sqrt{2}}{\lambda_0}\max\left\{\frac{1}{\mu^{1/2}(s_0)}\left(\int_0^{s_0}|\mu'(s)|^2\,ds\right)^{1/2}, Mk_0^{1/2}\right\}\\
&\qquad\cdot\int_\Omega |u_t(t)|\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)^{1/2}dx\\
&\quad\le -\frac{k_0}{2}\int_\Omega |u_t(t)|^2\,dx + C_3\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx,
\end{aligned} \tag{4.8}$$

where $C_3 = C_3(\lambda_0, k_0, \mu(s_0), M)$; whereas, due to (1.3)

$$\begin{aligned}
&-\int_\Omega u_{tt}(t)\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx \\
&\qquad = \int_\Omega \nabla u(t)\left(\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\right)dx + \int_\Omega\left(\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\right)^2 dx \\
&\qquad\quad + \int_\Omega g(u(t))\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx - \int_\Omega f(t)\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx.
\end{aligned} \tag{4.9}$$

Let us examine in detail the four terms appearing in the right-hand side of (4.9). Concerning the first two, choosing $\varrho > 0$ to be specified later, we get

$$\begin{aligned}
&\int_\Omega \nabla u(t)\left(\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\right)dx + \int_\Omega\left(\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\right)^2 dx \\
&\qquad \le \frac{\varrho}{2}\int_\Omega |\nabla u(t)|^2\,dx + k_0\left(1 + \frac{|\Omega|}{2\varrho}\right)\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx.
\end{aligned} \tag{4.10}$$

By force of (g3),

$$\begin{aligned}
&\int_\Omega g(u(t))\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx \\
&\qquad \le k_0^{1/2}\int_\Omega (\Gamma|u(t)| + |g(0)|)\left(\int_0^\infty \mu(s)|\eta^t(s)|^2\,ds\right)^{1/2} dx \\
&\qquad \le \frac{\varrho}{2}\int_\Omega |\nabla u|^2\,dx + \frac{k_0^{1/2}}{2\lambda_0}\left(1 + \frac{\Gamma^2 k_0^{1/2}}{\varrho\lambda_0}\right)\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx \\
&\qquad\quad + \frac{|g(0)|^2|\Omega| k_0^{1/2}}{2}.
\end{aligned} \tag{4.11}$$

The last term of (4.9) is controlled as

$$\begin{aligned}
&-\int_\Omega f(t)\left(\int_0^\infty \mu(s)\eta^t(s)\,ds\right)dx \\
&\qquad \le \frac{k_0^{1/2}}{\lambda_0^{1/2}} m_f(t)\left(\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx\right)^{1/2}.
\end{aligned} \tag{4.12}$$

Collecting (4.8)–(4.12), we obtain from (4.7) the estimate

$$\frac{d}{dt}\mathcal{F}(t) \le -\frac{k_0}{2}\int_\Omega |u_t(t)|^2\,dx + \varrho\int_\Omega |\nabla u(t)|^2\,dx + C_4\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx + C_5 m_f(t)\mathcal{E}^{1/2}(t) + C_6 \tag{4.13}$$

for some positive constants $C_4 = C_4(\varrho)$, C_5, C_6. Notice that if $g \equiv 0$, then $C_6 = 0$.

Finally we consider the equality

$$\frac{d}{dt}\int_\Omega u(t)u_t(t)\,dx = \int_\Omega |u_t(t)|^2\,dx + \int_\Omega u(t)u_{tt}(t)\,dx.$$

Thus, exploiting again (1.3), and appealing to (3.2), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(t)u_t(t)\,dx &= \int_\Omega |u_t(t)|^2\,dx - \int_\Omega |\nabla u(t)|^2\,dx \\ &\quad - \int_\Omega \nabla u(t)\left(\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\right)dx \\ &\quad - \int_\Omega u(t)g(t)\,dx + \int_\Omega u(t)f(t)\,dx. \\ &\le \int_\Omega |u_t(t)|^2\,dx - \frac{1}{4}\int_\Omega |\nabla u(t)|^2\,dx - C_0\mathcal{G}(u(t)) + C_2 \\ &\quad + k_0\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\,ds\right)dx + m_f(t)\mathcal{E}^{1/2}(t).\end{aligned} \tag{4.14}$$

To conclude the proof, for $N > 0$ and $\nu > 0$, introduce the functional

$$\mathcal{L}(t) = N\mathcal{E}(t) + N\mathcal{G}(u(t)) + NC_1 + \mathcal{F}(t) + \nu\int_\Omega u(t)u_t(t)\,dx.$$

Recalling (3.1), it is easy to check that, provided N is big enough and ν small enough, there exists two constants $C_7 > 1$ and $C_8 > 0$ depending on N and ν (with $C_8 = 0$ when $g \equiv 0$) such that

$$\frac{1}{C_7}\mathcal{E}(t) \le \mathcal{L}(t) \le C_7\mathcal{E}(t) + C_8. \tag{4.15}$$

Collecting (4.5), (4.13), and (4.14), we end up with

$$\begin{aligned}
&\frac{d}{dt}\mathscr{L}(t) + \left(\frac{\nu}{4} - \varrho\right)\int_\Omega |\nabla u(t)|^2\, dx \\
&\quad + \left(\frac{k_0}{2} - \nu\right)\int_\Omega |u_t(t)|^2\, dx + \frac{\nu}{8}\mathscr{G}(u(t)) + \frac{\nu}{8N}\mathscr{F}(t) \\
&\quad + \left(\frac{N\delta}{2} - C_4 - \nu k_0\right)\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\, ds\right)dx \\
&\quad + \frac{\nu^2}{8N}\int_\Omega u(t)u_t(t)\, dx \\
&\quad \le \left(\frac{\nu}{8} - \nu C_0\right)\mathscr{G}(u(t)) + \frac{\nu}{8N}\mathscr{F}(t) + \frac{\nu^2}{8N}\int_\Omega u(t)u_t(t)\, dx \\
&\qquad + (N + C_5 + \nu)m_f(t)\mathscr{E}^{1/2}(t) + (\nu C_2 + C_6). \qquad (4.16)
\end{aligned}$$

At this point, it is easy to see that, due to (2.1) and (3.3), and assuming without loss of generality $N > 1$ and $\nu < 1$.

$$\begin{aligned}
&\left(\frac{\nu}{8} - \nu C_0\right)\mathscr{G}(u(t)) + \frac{\nu}{8N}\mathscr{F}(t) + \frac{\nu^2}{8N}\int_\Omega u(t)u_t(t)\, dx \\
&\quad \le \frac{\nu}{16}\int_\Omega |\nabla u(t)|^2\, dx + \nu C_9\int_\Omega |u_t(t)|^2\, dx \\
&\qquad + \nu C_9\int_\Omega\left(\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2\, ds\right)dx + C_{10}, \qquad (4.17)
\end{aligned}$$

for some C_9, $C_{10} > 0$. Again, $C_{10} = 0$ when $g \equiv 0$. Choose now ν small enough such that

$$\frac{k_0}{2} - \nu - C_9\nu \ge \frac{\nu}{8},$$

set $\varrho = \nu/16$ (which automatically fixes the value of C_4), and choose N big enough such that

$$\frac{N\delta}{2} - C_4 - \nu k_0 - \nu C_9 \ge \frac{\nu}{8}.$$

Then let $\varepsilon = \nu/8N$. Denoting

$$C_{11} = C_7^{1/2}(N + C_5 + \nu)$$

and

$$C_{12} = \nu\varepsilon N C_1 + C_2 + C_6 + C_{10}$$

(notice that $C_{12} = 0$ when $g \equiv 0$), from (4.15)–(4.17) we get the differential inequality

$$\frac{d}{dt}\mathscr{L}(t) + \varepsilon\mathscr{L}(t) \leq C_{11} m_f(t) \mathscr{L}^{1/2}(t) + C_{12}.$$

By virtue of a generalization of the Gronwall Lemma (see, e.g., [22, 23]), keeping in mind (4.3), we obtain

$$\mathscr{L}(t) \leq 2\mathscr{L}(0)e^{-\varepsilon t} + \frac{2C_{12}}{\varepsilon} + \frac{e^{\varepsilon}}{(1 - e^{-\varepsilon/2})}\Phi^2.$$

The proof is carried out applying once more (4.15). The constants C and Λ of the statement turn out to be

$$C = 2C_7^2 \qquad \text{and} \qquad \Lambda = 2C_7C_8 + \frac{2C_7C_{12}}{\varepsilon} + \frac{C_7 e^{\varepsilon}}{(1 - e^{-\varepsilon/2})}\Phi^2.$$

Notice that when $\Phi = 0$ (that is, $F = \{0\}$) and $g \equiv 0$, then $\Lambda = 0$. ∎

Remark 4.2. The uniform energy estimate (4.2) implies the existence of a bounded *absorbing set* $\mathscr{B}^* \subset \mathscr{H}$ for $S(t)$, which is uniform as f is allowed to run in a bounded set $F \subset \mathscr{T}$. Indeed, if $\mathscr{B}^*$ is any ball of $\mathscr{H}$ of radius less than $\sqrt{2\Lambda}$, for any bounded set $\mathscr{B} \subset \mathscr{H}$ it is immediate to see that there exists $t(\mathscr{B}) \geq 0$ such that

$$S(t)\mathscr{B} \subset \mathscr{B}^*$$

for every $t \geq t(\mathscr{B})$ and every $f \in F$.

Moreover, if we define

$$\mathscr{B}_0 = \bigcup_{t \geq 0} S(t)\mathscr{B}^*$$

it is clear that $\mathscr{B}_0$ is still a bounded absorbing set which is also invariant for $S(t)$, that is, $S(t)\mathscr{B}_0 \subset \mathscr{B}_0$ for every $t \geq 0$. In particular, if F is connected, then $\mathscr{B}_0$ is connected as well.

5. EXISTENCE OF A GLOBAL ATTRACTOR

In this section we assume

$$f \in H \text{ constant in time}.$$

In this case, as mentioned before, $S(t)$ is a strongly continuous semigroup on $\mathscr{H}$. Our aim is to show $S(t)$ admits a global attractor. Recall that the (unique) global attractor of $S(t)$ acting on $\mathscr{H}$ is the compact set $\mathscr{A} \subset \mathscr{H}$ enjoying the following properties:

(1) $\mathscr{A}$ is fully invariant for $S(t)$, that is, $S(t)\mathscr{A} = A$ for every $t \geq 0$;

(2) $\mathscr{A}$ is an attracting set, namely, for any bounded set $\mathscr{B} \subset \mathscr{H}$,

$$\lim_{t \to \infty} \delta_{\mathscr{H}}(S(t)\mathscr{B}, \mathscr{A}) = 0,$$

where $\delta_{\mathscr{H}}$ denotes the semidistance on $\mathscr{H}$.

More details on the subject can be found in the classical books [1, 13, 26].

Actually, we could extend with minor modifications our analysis to some particular nonautonomous situations, introducing the notion of *strongly continuous process of operators* (see [14]), and prove the existence of an attractor which is uniform as f belongs to the hull of a translation-compact function in a suitable space (see, e.g., [2, 3]).

In the sequel, denote $A = -\Delta$, the Laplacian with Dirichlet boundary conditions. It is well known that A is a positive operator on L^2 with domain $\mathscr{D}(A) = H^2 \cap H_0^1$. Moreover, one can define the powers A^s of A for $s \in \mathbb{R}$. The space $V_{2s} = \mathscr{D}(A^s)$ turns out to be a Hilbert space with the inner product

$$\langle u, v \rangle_{V_{2s}} = \langle A^s u, A^s v \rangle.$$

In particular, $V_{-1} = H^{-1}$, $V_0 = L^2$, $V_1 = H_0^1$. The injection $V_{s_1} \hookrightarrow V_{s_2}$ is compact whenever $s_1 > s_2$. For further convenience, for $s \in \mathbb{R}$, introduce the Hilbert space

$$\mathscr{H}_s = V_{1+s} \times V_s \times L^2_\mu(\mathbb{R}^+, V_{1+s}).$$

Clearly, $\mathscr{H}_0 = \mathscr{H}$.

Let now $z_0 = (u_0, v_0, \eta_0) \in \mathscr{B}_0$, where $\mathscr{B}_0$ is the invariant, connected, bounded absorbing set of $S(t)$ given by Remark 4.2.

Following a standard procedure (cf. [23]) we write the solution $z = (u, v, \eta)$ to (1.3)–(1.4) as $z = z_L + z_N$, with $z_L = (u_L, v_L, \eta_L)$ and $z_N = (u_N, v_N, \eta_N)$, where z_L and z_N are the solutions (in the sense of Definition 3.1) to the problems

$$\begin{aligned}
\partial_{tt} u_L &= \Delta u_L + \int_0^\infty \mu(s) \Delta \eta_L(s)\, ds - g(u) + f \\
\partial_t \eta_L &= -\partial_s \eta_L + \partial_t u_L \\
z_L(0) &= z_0
\end{aligned}$$

and

$$\begin{aligned}\partial_{tt} u_N &= \Delta u_N + \int_0^\infty \mu(s) \Delta \eta_N(s)\, ds\\ \partial_t \eta_N &= -\partial_s \eta_N + \partial_t u_N\\ z_N(0) &= 0.\end{aligned}$$

In the remaining of the section, let K denote a generic constant, which is independent of $z_0 \in \mathscr{B}_0$.

It is apparent from Theorem 4.1 that z_L fulfills estimate (4.2) with $\Lambda = 0$; therefore

$$\|z_L(t)\|_{\mathscr{H}} \leq K e^{-\varepsilon t} \qquad \forall t \in \mathbb{R}^+. \tag{5.1}$$

Concerning z_N, for every $t \geq 0$ there exists $K(t) > 0$ (independent of $z_0 \in \mathscr{B}_0$), such that

$$\|z_N(t)\|_{\mathscr{H}_{1/2}} \leq K(t) \qquad \forall t \in \mathbb{R}^+. \tag{5.2}$$

Proof of (5.2) is obtained repeating the argument of Lemma 5.4 in [23]. The only difference is that here we get a time-dependent estimate, whereas in [23] (due to the presence of a weak damping in the equation for u_N) the estimate is uniform as $t \geq 0$.

Finally (see Lemma 5.5 in [23]), we have the compact embedding

$$\mathscr{C}(t) = \bigcup_{z_0 \in B_0} \eta_N^t \hookrightarrow L^2_\mu(\mathbb{R}^+, H^1_0). \tag{5.3}$$

Denote the closure of $\mathscr{C}$ in $L^2_\mu(\mathbb{R}^+, H^1_0)$ by $\overline{\mathscr{C}}$. With reference to (5.2)–(5.3), for $t \geq 0$ let $\mathscr{R}(t)$ be the ball of $V_{3/2} \times V_{1/2}$ of radius $K(t)$ centered at zero and introduce the set

$$\mathscr{K}(t) = \mathscr{R}(t) \times \overline{\mathscr{C}} \subset \mathscr{H}.$$

From the compact embedding $V_{3/2} \times V_{1/2} \hookrightarrow H^1_0 \times L^2$ and (5.3), $\mathscr{K}(t)$ is compact in $\mathscr{H}$. By construction, $z_N(t) \in \mathscr{K}(t)$ for every $t \geq 0$.

Due to the compactness of $\mathscr{K}(t)$, for every fixed $t \geq 0$ and every $d > K e^{-\varepsilon t}$, there exist finitely many balls of $\mathscr{H}$ of radius d such that $z(t)$ belongs to the union of such balls, for every $z_0 \in \mathscr{B}_0$. This implies that

$$\alpha_{\mathscr{H}}(S(t)\mathscr{B}_0) \leq K e^{-\varepsilon t}, \qquad \forall t \geq 0, \tag{5.4}$$

where $\alpha_{\mathscr{H}}$ is the *Kuratowski measure of noncompactness*, defined by (cf.

[13])

$$\alpha_{\mathscr{H}}(\mathscr{B}) = \inf\{d : \mathscr{B} \text{ has a finite cover of balls of } \mathscr{H} \text{ of diameter less that } d\}.$$

Since the invariant, connected, bounded absorbing set $\mathscr{B}_0$ fulfills (5.4), exploiting a classical result of the theory of attractors of semigroups (see [13]), we conclude that the ω-limit set of $\mathscr{B}_0$, that is,

$$\omega(\mathscr{B}_0) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)\mathscr{B}_0}^{\mathscr{H}},$$

is the connected and compact global attractor of $S(t)$.

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REFERENCES

1. A. V. Babin and M. I. Vishik, "Attractors of Evolution Equations," North-Holland, Amsterdam, 1992.
2. V. V. Chepyzhov and M. I. Vishik, Nonautonomous evolution equations and their attractors, *Russian J. Math. Phys.* **1** (1993), 165–190.
3. V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, *J. Math. Pures Appl.* **73** (1994), 279–333.
4. B. D. Coleman and W. Noll, Foundations of linear viscoelasticity, *Rev. Mod. Phys.* **33** (1961), 239–249.
5. C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.* **37** (1970), 297–308.
6. C. M. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear one dimensional viscoelasticity, *J. Differential Equations* **6** (1969), 71–86.
7. G. Dassios and F. Zafiropoulos, Equipartition of energy in linearized 3-D viscoelasticity, *Quart. Appl. Math.* **48** (1990), 715–730.
8. M. Fabrizio and B. Lazzari, On the existence and asymptotic stability of solutions for linear viscoelastic solids, *Arch. Rational Mech. Anal.* **116** (1991), 139–152.
9. M. Fabrizio and A. Morro, "Mathematical Problems in Linear Viscoelasticity," SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1992.
10. J. M. Ghidaglia and R. Temam, Attractors for damped nonlinear hyperbolic equations, *J. Math. Pures Appl.* **66** (1987), 273–319.
11. C. Giorgi, A. Marzocchi, and V. Pata, Asymptotic behavior of a semilinear problem in heat conduction with memory, *NoDEA Nonlinear Differential Equations Appl.* **5** (1998), 333–354.

12. C. Giorgi, M. G. Naso, and V. Pata, Exponential stability in linear heat conduction with memory: A semigroup approach, *Comm. Appl. Anal.* **5** (2001), 121–134.
13. J. Hale, Asymptotic behavior and dynamics in infinite dimensions, *in* "Nonlinear Differential Equations" (J. Hale and P. Martínez-Amores, Eds.), Pitman, Boston, 1985.
14. A. Haraux, "Systèmes dynamiques dissipatifs et applications," Coll. RMA, Vol. 17, Masson, Paris, 1990.
15. J. E. Lagnese, Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping, *in* "Internat. Ser. Numer. Math.," Vol. 91, Birkhäuser, Basel/Boston, 1989.
16. Z. Liu and C. Peng, Exponential stability of a viscoelastic Timoshenko beam, *Adv. Math. Sci. Appl.* **8** (1998), 343–351.
17. Z. Liu and S. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, *Quart. Appl. Math.* **54** (1996), 21–31.
18. Z. Liu and S. Zheng, Exponential stability of the Kirchhoff plate with thermal or viscoelastic damping, *Quart. Appl. Math.* **55** (1997), 551–564.
19. J. E. Muñoz Rivera, Asymptotic behaviour in linear viscoelasticity, *Quart. Appl. Math.* **3** (1994), 629–648.
20. J. E. Muñoz Rivera and R. K. Barreto, Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels, *Appl. Anal.* **60** (1996), 341–357.
21. J. E. Muñoz Rivera and S. Jiang, A global existence theorem for the Dirichlet problem in nonlinear n-dimensional viscoelasticity, *Differential Integral Equations* **9** (1996), 791–810.
22. V. Pata, G. Prouse, and M. I. Vishik, Traveling waves of dissipative non-autonomous hyperbolic equations in a strip, *Adv. Differential Equations* **3** (1998), 249–270.
23. V. Pata and A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, submitted for publication.
24. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
25. M. Renardy, W. J. Hrusa, and J. A. Nohel, Mathematical problems in viscoelasticity, *in* "Pitman Monographs and Surveys in Pure and Applied Mathematics," Vol. 35, Wiley, New York, 1987.
26. R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.