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STEADY STATES OF THE HINGED EXTENSIBLE BEAM WITH EXTERNAL LOAD

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We analyze from both the physical and the analytical viewpoints the equation

$$\omega u'''' - \left(\gamma + \int_0^1 [u'(x)]^2 \mathrm{d}x\right) u'' = g,$$

the solutions represent the equilibria of a thin extensible beam subject to external load.

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1. Introduction

We consider a boundary value problem, written in dimensionless form, describing the steady state solutions of the vertical deflection $u : [0,1] \to \mathbb{R}$, with respect to the reference configuration, of a thin extensible beam of natural length $\ell > 0$ hinged at the endpoints of the space interval:

$$\begin{cases} \omega u'''' - \left(\gamma + \int_0^1 [u'(x)]^2 \mathrm{d}x\right) u'' = g, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.1}$$

Here, $g \in L^2(0,1)$ is the lateral static load distribution, while the positive parameter ω is of the order $\frac{h^2}{\ell^2}$, where $h \ll \ell$ is the thickness of the beam. Finally, $\gamma \in \mathbb{R}$ represents the longitudinal displacement of the ends, proportional to the axial load acting in the reference configuration. Precisely, γ is positive when the beam is stretched, negative when compressed.

As customary in the structural mechanics literature, the investigation of the solutions to (1.1), in dependence on γ, is named *nonlinear buckling problem.* The notion of buckling, introduced by Euler more than two centuries ago, describes a static instability of structures due to inplane loading. In this respect, the main concern is to find the *critical buckling loads*, at which a bifurcation of solutions occurs, and their associated mode shapes, called *postbuckling configurations.* In the past, nonlinear buckling problems were mainly considered in the field of structural and engineering mechanics (cf. Ref. 13 and references therein). Nowadays, the study of the prebuckling, transition and postbuckling states under prescribed compressive stresses has become of particular relevance in the analysis of the static deformation of micromachined beams and microbridges (cf. Ref. 8). Indeed, the thin film material composing a micromechanical structure is normally under residual stresses, as a result of fabrication processes. Unlike microelectronics devices, a micromechanical structure is no longer constrained by its underlying silicon substrate with the exception of its ends. Therefore, residual stresses may cause bending and buckling of its configuration, and this behavior can be exploited to fabricate useful micromechanical structures.

A lot of papers on postbuckling analysis of beams axially loaded at the ends beyond the critical value are present in the literature. However, most of them deal with approximations and numerical simulations. For a detailed overview, we refer the reader to Nayfeh and Pai (see Ref. 14). To the best of our knowledge, exact solutions to (1.1), with $g = 0$ and hinged ends, have been first found in Ref. 5 and, more formally, in Ref. 15, whereas exact stationary solutions to the ended-loaded Timoshenko beam equation have been obtained in Ref. 11. Around the same period, several authors have also investigated the stability properties of the unbuckled (trivial) and the buckled stationary states (e.g. Refs. 2, 6, 11 and 15), but only in the homogeneous case $g = 0$.

On the contrary, our aim is to understand how the steady state solutions are affected by the presence of an external load. Therefore, we assume $g \in L^2(0,1)$, and we look for solutions to (1.1) in the following sense.

Definition 1.1. A (weak) solution to (1.1) is a function $u \in H^2(0,1) \cap H_0^1(0,1)$ such that

$$\omega \int_0^1 u''(x)w''(x)\mathrm{d}x + \left(\gamma + \int_0^1 [u'(\xi)]^2 \mathrm{d}\xi\right) \int_0^1 u'(x)w'(x)\mathrm{d}x = \int_0^1 g(x)w(x)\mathrm{d}x,$$

for every test function $w \in H^2(0,1) \cap H_0^1(0,1)$.

It is worth noting that (1.1) represents the static counterpart of quite many different evolution equations, arising both from elastic and viscoelastic theories (see Ref. 9 and references therein). An example is the following quasilinear equation describing the small transversal deflection of the Euler−Bernoulli beam, proposed by Woinowsky−Krieger (see Ref. 16) in the '50s:

$$\partial_{tt}u(x,t) + \omega\partial_{xxxx}u(x,t) - \left(\gamma + \int_0^1 [\partial_\xi u(\xi,t)]^2 \mathrm{d}\xi\right)\partial_{xx}u(x,t) = g(x). \tag{1.2}$$

This is the case of a beam with fixed ends where the geometric nonlinearity, accounting for the axial tension due to the elongation, is taken into consideration (see Ref. 13). Obviously, the steady state solutions remain the same in the presence of rotational inertia (as in the Kirchhoff theory), or of any kind of damping, due to structural and/or external mechanical dissipation. The global dynamics of (1.2) with linear damping and hinged ends has been addressed in Refs. 7 and 10, where the existence of the global attractor is obtained, for a general longitudinal displacement γ. The result has recently been improved in Ref. 9, which provides the optimal regularity of the attractor for the motion of both damped-elastic and viscoelastic nonlinear extensible beam models related to (1.1).

In any case, the set of solutions to (1.1) has a dramatic effect on the long-term dynamics of the corresponding evolution system, especially when its structure is nontrivial. Indeed, very different asymptotic behaviors occur, depending on whether the associated static problem has one, a finite number or infinitely many solutions, respectively (cf. Refs. 9 and 10). Nonetheless, in spite of its wide range of applications, a stringent variational derivation of (1.1) seems not to be available in the literature (see Ref. 1 for a survey on nonlinear corrections to classical beam models). Besides, it is not clear at all if and how this model could be extended to account for shear deformations in plates.

The goal of this paper is twofold. On one hand, we provide a detailed variational derivation of the model equation; this is done in Sec. 2. On the other hand, we solve (1.1), obtaining a closed-form solution for the postbuckling configurations. To this end, in Sec. 3, we actually consider an abstract generalization of the original Eq. (1.1). The analysis of the homogeneous case is carried out in Sec. 4, where we provide an explicit formula for the solutions, for all values of γ and ω. Finally, in Sec. 5, we tackle the more complicated nonhomogeneous case. Here, besides γ and ω, the multiplicity of solutions depends on the shape of the distributed lateral load g.

2. A Variational Derivation of the Model

In this section, we derive the physical model (1.1), following the classical variational approach of the minimum energy principle.

Let us consider a thin cylindrical beam of natural length $\ell > 0$, uniform cross section and thickness $h > 0$. Assuming the beam homogeneous, with unitary mass density and symmetric (along with all external loads) with respect to the vertical

xy-plane, we can restrict our attention on its section lying in the plane $z = 0$. In the sequel, we identify the beam with the section and we assume that its middle line at rest occupies the interval $[0, \ell]$ of the x-axis. The beam is subject to a distributed body force

$$\tilde{g}(x, y), \quad (x, y) \in [0, \ell] \times \left[-\frac{h}{2}, \frac{h}{2}\right],$$

in the transversal y-direction only, and to a uniform boundary tension

$$\tilde{\tau}(x), \quad x \in \partial[0, \ell] = \{0, \ell\},$$

in the axial x-direction only. On each cross x-section, their resultants amount to

$$g(x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{g}(x, y) \mathrm{d}y, \quad x \in [0, \ell]$$

and

$$\tau(x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{\tau}(x) \mathrm{d}y = h\tilde{\tau}(x), \quad x \in \{0, \ell\}.$$

Introducing the displacement vector at a generic point (x, y) of the beam

$$\mathfrak{U}(x, y) = (W(x, y), U(x, y)),$$

where W is the stretching component and U is the bending component, we consider the symmetric strain tensor of finite elasticity

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \frac{1}{2}[\nabla\mathfrak{U} + \nabla\mathfrak{U}^{\mathrm{T}}] + \frac{1}{2}\nabla\mathfrak{U}^{\mathrm{T}}\nabla\mathfrak{U}. \tag{2.1}$$

Assuming the beam to be isotropic, according to the Hooke law, the stress−tensor is given by

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbb{I}\right],$$

where $E > 0$ is the Young modulus and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio.

Besides the thinness of the beam, which amounts to require $h \ll \ell$, we make the following further assumptions:

(i) The x-component of the gradient of the stretching $W(x, y)$ is small compared to the other gradients.
(ii) The Kirchhoff assumption is fulfilled: any cross section remains perpendicular to the deformed longitudinal axis of the beam during the bending.

Within this approximation scheme (cf. Ref. 12), the only nonzero component of the stress tensor is

$$\sigma_{11} = \frac{E}{1+\nu}\left[\varepsilon_{11} + \frac{\nu}{1-2\nu}(\varepsilon_{11} + \varepsilon_{22})\right].$$

On the other hand, the equality

$$0 = \sigma_{22} = \frac{E}{1+\nu}\left[\varepsilon_{22} + \frac{\nu}{1-2\nu}(\varepsilon_{11}+\varepsilon_{22})\right]$$

yields the relation

$$\varepsilon_{22} = \frac{\nu}{\nu-1}\varepsilon_{11}$$

and we finally obtain

$$\sigma_{11} = \frac{E}{1-\nu^2}\varepsilon_{11}.$$

At this point, we assume for the components of the displacement vector $\mathfrak{U}$ the approximated forms

$$W(x,y) = w(x) - yu'(x), \quad U(x,y) = u(x), \tag{2.2}$$

where we put

$$w(x) = W(x,0), \quad u(x) = U(x,0).$$

In fact, (2.2) is rigorously justified in large deflection theory by means of an asymptotic expansion, as explained in Refs. 3 and 4. Therefore, from (2.1),

$$\varepsilon_{11} = w'(x) - yu''(x) + \frac{1}{2}[u'(x)]^2,$$

and, in turn,

$$\sigma_{11} = \frac{E}{1-\nu^2}\left[w'(x) - yu''(x) + \frac{1}{2}[u'(x)]^2\right].$$

The strain energy $\mathcal{P}$ within the beam is defined as

$$\mathcal{P} = \frac{1}{2}\int_0^\ell \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, \mathrm{d}y \mathrm{d}x,$$

having set

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sum_{i,j} \sigma_{ij}\varepsilon_{ij} = \sigma_{11}\varepsilon_{11}.$$

Hence, after an integration in y, we get

$$\mathcal{P} = \frac{Eh}{2(1-\nu^2)}\int_0^\ell \left(\frac{1}{2}[u'(x)]^2 + w'(x)\right)^2 \mathrm{d}x + \frac{Eh^3}{24(1-\nu^2)}\int_0^\ell [u''(x)]^2 \mathrm{d}x.$$

To compute the total energy, we need to consider the work done by the forces applied to the beam, given by

$$\mathcal{W} = \int_0^\ell \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{g}(x,y)\, U(x,y)\mathrm{d}y\mathrm{d}x + \int_{-\frac{h}{2}}^{\frac{h}{2}} [\tilde{\tau}(\ell)\, W(\ell,y) - \tilde{\tau}(0)\, W(0,y)]\mathrm{d}y,$$

which, exploiting (2.2) and integrating in y, reduces to

$$\mathcal{W} = \int_0^\ell g(x)u(x)\mathrm{d}x + \tau(\ell)w(\ell) - \tau(0)w(0).$$

In order to force the last two terms on the right-hand side to vanish, we might assume either homogeneous boundary conditions for w or the vanishing of the axial tension τ. Unfortunately, none of the two possibilities can be considered here. Indeed, an accurate analysis of the buckling problem requires the assumption that one end of the beam is fixed and the other moves as a thrust bearing on the x-axis under the action of an axial load, the so-called *Euler critical load*. In essence, either the external tension in the x-direction or the axial displacements of both ends of the beam must be given and nonzero. Since we are interested in equilibrium rather than dynamics, we are allowed to consider the latter occurrence. In particular, one end is assumed to be nailed in its reference configuration at $x = 0$, the other to displace in a position $x = \ell + C$. Of course, it is understood that C represents the displacement ideally produced by (and proportional to) the external tension in the x-direction. In other words, we take into account the possible elongation of the beam by setting

$$w(0) = 0, \quad w(\ell) = C. \tag{2.3}$$

From the physical viewpoint, when $C > 0$ the beam behaves as it were compressed, whereas for $C < 0$ as it were subject to traction. This choice leads to

$$\mathcal{W} = \int_0^\ell g(x)u(x)\mathrm{d}x + C\tau(\ell),$$

and we are left to assign boundary conditions only for the bending component u, for which we take the so-called hinged boundary conditions

$$u(0) = u(\ell) = u''(0) = u''(\ell) = 0. \tag{2.4}$$

We are now in a position to derive the equation describing the motions of u and w. To this end, we have to minimize the Lagrangian functional

$$\mathcal{L} = \mathcal{P} - \mathcal{W}$$

over the class of functions satisfying (2.3) and (2.4). Hence, for any $\hat{u}$ and $\hat{w}$ satisfying the boundary conditions

$$\hat{u}(0) = \hat{u}(\ell) = \hat{u}''(0) = \hat{u}''(\ell) = \hat{w}(0) = \hat{w}(\ell) = 0,$$

we consider the first variation

$$\mathcal{L}'(u, w; \hat{u}, \hat{w}) = \lim_{s\to 0} \frac{\mathcal{L}(u + s\hat{u}, w + s\hat{w}) - \mathcal{L}(u, w)}{s}.$$

A straightforward calculation yields

$$\mathcal{L}'(u, w; \hat{u}, \hat{w}) = a(u, w; \hat{u}) + b(u, w; \hat{w}) - \int_0^\ell g(x)\hat{u}(x)\mathrm{d}x,$$

where we set

$$a(u,w;\hat{u}) = \int_0^\ell \left\{ \frac{Eh^3}{12(1-\nu^2)} u''''(x) - \frac{Eh}{1-\nu^2} \left[\left(w'(x) + \frac{1}{2}[u'(x)]^2 \right) u'(x) \right]' \right\} \hat{u}(x)\mathrm{d}x$$

and

$$b(u,w;\hat{w}) = -\frac{Eh}{1-\nu^2} \int_0^\ell \left(w'(x) + \frac{1}{2}[u'(x)]^2 \right)' \hat{w}(x)\mathrm{d}x.$$

The resulting Euler–Lagrange equations follow from the condition

$$a(u,w;\hat{u}) + b(u,w;\hat{w}) - \int_0^\ell g(x)\hat{u}(x)\mathrm{d}x = 0, \quad \forall\, \hat{u}, \hat{w}.$$

Due to the arbitrariness of $\hat{u}$ and $\hat{w}$, we end up with the system of two coupled equations

$$\begin{cases} \dfrac{Eh}{1-\nu^2} \left(w' + \dfrac{1}{2}[u']^2 \right)' = 0, \\ \dfrac{Eh^3}{12(1-\nu^2)} u'''' - \dfrac{Eh}{1-\nu^2} \left[\left(w' + \dfrac{1}{2}[u']^2 \right) u' \right]' = g. \end{cases}$$

The first equation tells that the quantity

$$w'(x) + \frac{1}{2}[u'(x)]^2$$

is constant. Hence, from (2.3),

$$w'(x) + \frac{1}{2}[u'(x)]^2 = \frac{1}{\ell} \int_0^\ell \left(w'(x) + \frac{1}{2}[u'(x)]^2 \right) \mathrm{d}x = \frac{C}{\ell} + \frac{1}{2\ell} \int_0^\ell [u'(x)]^2 \mathrm{d}x.$$

Substituting this expression into the second equation of the system, we are led to

$$\frac{Eh^3}{12(1-\nu^2)} u'''' - \frac{Eh}{1-\nu^2} \left(\frac{C}{\ell} + \frac{1}{2\ell} \int_0^\ell [u'(x)]^2 \mathrm{d}x \right) u'' = g.$$

Defining the dimensionless quantities

$$x^\star = \frac{x}{\ell}, \quad u^\star(x^\star) = \frac{1}{\ell} u(\ell x^\star), \quad \gamma = \frac{2C}{\ell}, \quad \omega = \frac{h^2}{6\ell^2}, \quad g^\star(x^\star) = \frac{2\ell(1-\nu^2)}{Eh} g(\ell x^\star),$$

we obtain the final form of the equation of motion for the bending component, which reads (deleting the $\star$)

$$\omega u'''' - \left(\gamma + \int_0^1 [u'(x)]^2 \mathrm{d}x \right) u'' = g.$$

Accordingly, the boundary conditions (2.4) become

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

3. An Abstract Problem

We now analyze an abstract problem, of which (1.1) is just a particular instance.

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let A be a strictly positive self-adjoint linear operator on H with domain $\mathcal{D}(A)$. For $r \in \mathbb{R}$, we define the Hilbert spaces

$$H^r = \mathcal{D}(A^{\frac{r}{2}}), \quad \|u\|_r = \|A^{\frac{r}{2}}u\|.$$

Let $M \in C([0,\infty))$, with $M(0) = 0$, be a strictly increasing (hence positive) function. Given $\beta \in \mathbb{R}$ and $f \in H^{-1}$, we consider the equation

$$Au + (\beta + M(\|u\|_1^2))u = f. \tag{3.1}$$

Definition 3.1. A vector $u \in H^1$ is a (weak) solution to (3.1) if

$$\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}w \right\rangle + (\beta + M(\|u\|_1^2))\langle u, w\rangle = \left\langle A^{-\frac{1}{2}}f, A^{\frac{1}{2}}w \right\rangle,$$

for every $w \in H^1$.

Remark 3.1. Due to the structure of the equation, if $f \in H$ and u is a solution to (3.1), then $u \in H^2$, and so it is a solution in the strong sense.

Notation 3.1. We denote by λ_n, with $n = \{1, 2, \ldots\}$, the strictly positive (possibly finite) sequence of the distinct eigenvalues of A, and by $\mathcal{E}_n$ the eigenspace corresponding to λ_n, with (possibly infinite) orthogonal dimension $\dim(\mathcal{E}_n) = d_n$. For every n, let $e_{n,i}$, with $i \in \{1, \ldots, d_n\}$, be an orthonormal basis of $\mathcal{E}_n$. In particular, the equality

$$A^p e_{n,i} = \lambda_n^p e_{n,i}$$

holds for every $p \in \mathbb{R}$. We call $\mathbb{P}_n$ the projection of H^{-1} onto $\mathcal{E}_n$. Finally, setting

$$M_\infty = \lim_{s\to\infty} M(s) \in (0, \infty],$$

we introduce the subset of the natural numbers (depending on the given value of the parameter β)

$$\mathbb{S} = \{n : -\beta - \lambda_n \in (0, M_\infty)\}.$$

Throughout this work, we will assume

$$|\mathbb{S}| < \infty, \tag{3.2}$$

namely, $\mathbb{S}$ has finite cardinality.

Remark 3.2. Condition (3.2) is certainly satisfied (in fact, for every $\beta \in \mathbb{R}$) if A is an elliptic operator. In which case, λ_n can be ordered in such a way to be strictly increasing. Moreover, $d_n < \infty$ for every n.

Our aim is to analyze the multiplicity of solutions to (3.1). In particular, we will show that there is always at least one solution, and at most a finite number of solutions, whenever the eigenvalues not exceeding $-\beta$ are simple.

Remark 3.3. The physical model (1.1) is recovered by setting $M(s) = \frac{s}{\omega}$, $\beta = \frac{\gamma}{\omega}$, $H = L^2(0,1)$,

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1),$$

and taking $f \in H^2(0,1) \cap H_0^1(0,1)$. Then,

$$g = -\frac{1}{\omega} f'' \in L^2(0,1).$$

In this case, the (strong) solutions to (3.1) are weak solutions to (1.1), and the other way around. The eigenvalues of A are all simple and equal to

$$\lambda_n = n^2\pi^2, \quad n \in \mathbb{N},$$

with corresponding eigenvectors

$$e_n(x) = \sqrt{2}\sin n\pi x.$$

Consequently,

$$\mathbb{S} = \{n : n^2\pi^2\omega < -\gamma\}.$$

4. The Homogeneous Case

For the homogeneous case, we provide an accurate description of the solutions. To this end, in light of (3.2), we introduce the (finite) number

$$n_\star = |\mathbb{S}|. \tag{4.1}$$

Then, we have

Theorem 4.1. *Let $f = 0$. If there exists an eigenvalue λ_n which is not simple and $n \in \mathbb{S}$, then (3.1) has infinitely many solutions. Otherwise, it has exactly $2n_\star + 1$ solutions: the trivial one and*

$$u_n^\pm = C_n^\pm e_{n,1},$$

for every $n \in \mathbb{S}$, where

$$C_n^\pm = \pm\sqrt{\frac{M^{-1}(-\beta - \lambda_n)}{\lambda_n}}.$$

Proof. If $n \in \mathbb{S}$ and $d_n > 1$, then any $u \in \mathcal{E}_n$ satisfying

$$\|u\|_1^2 = M^{-1}(-\beta - \lambda_n)$$

is a solution to (3.1). Clearly, there are infinitely many such u, given by

$$u = \sum_i u_i e_{i,n}, \quad u_i \in \mathbb{R},$$

with

$$\sum_i u_i^2 = \frac{1}{\lambda_n} M^{-1}(-\beta - \lambda_n).$$

Assume then that λ_n is simple whenever $n \in \mathbb{S}$. Obviously, $u = 0$ is a solution. Let us look for a nontrivial solution u. Setting

$$\mu = \beta + M(\|u\|_1^2), \tag{4.2}$$

such a solution solves the equation

$$Au + \mu u = 0.$$

Hence,

$$\mu = -\lambda_n$$

and

$$u = Ce_{n,1},$$

for some $C \neq 0$. In particular,

$$\|u\|_1^2 = C^2 \lambda_n.$$

The value C is determined by (4.2), which yields the relation

$$M(C^2 \lambda_n) = -\beta - \lambda_n.$$

Therefore, we have nontrivial solutions if and only if $n \in \mathbb{S}$. Namely, there are exactly $2n_\star$ nontrivial solutions, explicitly computed. □

Remark 4.1. The nontrivial solutions to the homogeneous version of problem (1.1) are given by (cf. Refs. 6 and 11)

$$u_n^\pm(x) = \pm\sqrt{-\frac{2\gamma}{n^2\pi^2} - 2\omega}\,\sin n\pi x.$$

From the physical viewpoint, this means that when the beam compression exceeds the first eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, then nontrivial symmetric solutions pop up (the buckling states).

5. The Nonhomogeneous Case

In the nonhomogeneous case, the picture is much more complicated, and the shape of f plays a crucial role. Let us set

$$f_{n,i} = \langle A^{-\frac{1}{2}} f, A^{\frac{1}{2}} e_{n,i} \rangle.$$

The fact that $f \in H^{-1}$ translates into the summability of the series

$$\sum_{n,i} \frac{1}{\lambda_n} f_{n,i}^2.$$

Besides, $f_{n,i} \neq 0$ for some n and some i, otherwise $f = 0$. For every $k \in \{1, 2, \ldots\}$, we define

$$Q_k = \sum_{n \neq k,\, i} \frac{\lambda_n f_{n,i}^2}{(\lambda_n - \lambda_k)^2}.$$

Along with $n_\star$ given by (4.1), we also need to introduce the numbers

$$k_\star = |\{k : -\beta - \lambda_k \in (0, M_\infty), M(Q_k) < -\beta - \lambda_k, \mathbb{P}_k f = 0\}|,$$

$$k_\star^0 = |\{k : -\beta - \lambda_k \in (0, M_\infty), M(Q_k) = -\beta - \lambda_k, \mathbb{P}_k f = 0\}|.$$

Observe that

$$k_\star + k_\star^0 \leq n_\star.$$

Denoting by

$$\underline{M}''(s) = \liminf_{\sigma \to 0} \frac{M'(s + \sigma) - M'(s)}{\sigma}$$

the lower second derivative of M, the main result reads as follows.

Theorem 5.1. *Let $f \neq 0$. In addition to the general assumptions on M, suppose that*

- *either M is a convex function; or*
- *$M \in C^1(\mathbb{R}^+)$ fulfills the relation*

$$2s\underline{M}''(s) + 3M'(s) \in (0, \infty], \quad \forall\, s > 0. \tag{5.1}$$

Then, Eq. (3.1) has infinitely many solutions if and only if the conditions

(i) $d_k > 1$
(ii) $\mathbb{P}_k f = 0$
(iii) $M(Q_k) < -\beta - \lambda_k$

simultaneously hold for some k. Otherwise, there are $m_\star$ solutions, with

$$1 \leq m_\star \leq 2n_\star + 2k_\star + k_\star^0 + 1.$$

Therefore, if there is an eigenvalue exceeding $-\beta$, whose multiplicity is greater than one, then infinite solutions may appear, unless the projection of the external load f on the relative eigenspace is not zero. In which case, the degrees of freedom are somehow frozen.

Remark 5.1. The most interesting case $M(s) = s$ is covered by the theorem. Moreover, (5.1) is satisfied by a large class of strictly increasing concave functions, such as $M(s) = s^\vartheta$ $(0 < \vartheta < 1)$ and $M(s) = \log(1+s)$. Another example is

$$M(s) = \mathrm{e}^{-\frac{1}{\sqrt{s+c}}} - \mathrm{e}^{-\frac{1}{\sqrt{c}}},$$

which fulfills (5.1) for all $c \geq 0$, is concave for $c \geq \frac{1}{9}$ and has finite limit as $s \to \infty$.

Before proceeding, we state a straightforward corollary, which also subsumes the analogous result for the homogeneous case.

Corollary 5.1. *If $\beta \geq -\inf_n \lambda_n$, then (3.1) has only one solution.*

The proof of Theorem 5.1 requires a couple of preliminary results. The first one is more like a simple remark.

Lemma 5.1. *Let $I \subset \mathbb{R}$ be an open interval, and let $\Lambda \in C^1(I)$. If*

$$\underline{\Lambda}''(s) \in (0, \infty], \quad \forall\, s \in I \backslash J,$$

where $J \subset I$ is a discrete set, then Λ is strictly convex on I.

Lemma 5.2. *Assume (5.1). Let $a_n \geq 0$ and $b_n \in \mathbb{R}$ be two sequences such that*

$$\sum_n a_n b_n^\imath = \varrho_\imath \in \mathbb{R}, \quad \imath = 2, 3, 4,$$

with $\varrho_3 \neq 0$ (which implies $\varrho_2 > 0$ and $\varrho_4 > 0$). Then,

$$2\varrho_3^2 \underline{M}''(\varrho_2) + 3\varrho_4 M'(\varrho_2) > 0.$$

Proof. By (5.1),

$$2\varrho_3^2 \underline{M}''(\varrho_2) > -\frac{3\varrho_3^2}{\varrho_2} M'(\varrho_2).$$

Hence,

$$2\varrho_3^2 \underline{M}''(\varrho_2) + 3\varrho_4 M'(\varrho_2) > \frac{3}{\varrho_2} M'(\varrho_2)(\varrho_2 \varrho_4 - \varrho_3^2).$$

It is a standard matter to check that

$$\varrho_2 \varrho_4 = \left(\sum_n a_n b_n^2\right)\left(\sum_n a_n b_n^4\right) \geq \left(\sum_n a_n b_n^3\right)^2 = \varrho_3^2.$$

Since $M'(\varrho_2) \geq 0$, the conclusion follows. □

Proof. (of Theorem 5.1) As in the previous case, we set

$$\mu = \beta + M(\|u\|_1^2), \tag{5.2}$$

which, since $u = 0$ is not a solution anymore, yields the constraint

$$-\beta + \mu \in (0, M_\infty). \tag{5.3}$$

Writing

$$u = \sum_{n,i} u_{n,i} e_{n,i},$$

with $u_{n,i} = \langle u, e_{n,i} \rangle$, we have

$$\|u\|_1^2 = \sum_{n,i} \lambda_n u_{n,i}^2.$$

Thus, (5.2) turns into

$$\mu = \beta + M\left(\sum_{n,i} \lambda_n u_{n,i}^2\right). \tag{5.4}$$

Projecting (3.1) on the orthonormal basis, we obtain, for every n, i,

$$\lambda_n u_{n,i} + \mu u_{n,i} = f_{n,i}. \tag{5.5}$$

The solution u is known once we determined all the coefficients $u_{n,i}$ appearing in (5.5).

We begin to look for solutions u for which

$$\mu \neq -\lambda_n, \quad \forall\, n.$$

In that case, once μ is fixed, the coefficients $u_{n,i}$ are uniquely determined by (5.5) as

$$u_{n,i} = \frac{f_{n,i}}{\lambda_n + \mu}. \tag{5.6}$$

Setting

$$\Phi(\mu) = \sum_{n,i} \frac{\lambda_n f_{n,i}^2}{(\lambda_n + \mu)^2} > 0$$

and

$$\Lambda(\mu) = \beta - \mu + M(\Phi(\mu)),$$

substituting (5.6) into (5.4), and recalling (5.3), we realize at once that the admissible values of μ are the solutions to the equation

$$\Lambda(\mu) = 0 \quad \text{with } \mu \in D = (\beta, \beta + M_\infty) \backslash \{-\lambda_n\}.$$

The set D is the union (empty if $n_\star = 0$) of $n_\star$ bounded open interval and of the open interval

$$I_0 = (\alpha, \beta + M_\infty),$$

where

$$\alpha = \begin{cases} \inf\limits_{n\in\mathbb{S}} -\lambda_n & \text{if } n_\star > 0, \\ \beta & \text{if } n_\star = 0. \end{cases}$$

For every $\mu \in D$, we have

$$\Phi'(\mu) = -2\sum_{n,i} \frac{\lambda_n f_{n,i}^2}{(\lambda_n+\mu)^3}$$

and

$$\Phi''(\mu) = 6\sum_{n,i} \frac{\lambda_n f_{n,i}^2}{(\lambda_n+\mu)^4} > 0.$$

Thus, Φ is strictly convex on each bounded connected component of D. We claim that the same is true for Λ as well. Indeed, if M is convex, this is immediate: $M \circ \Phi$ is the composition of a strictly increasing convex function with a strictly convex function. Conversely, if M fulfills (5.1), for every $\mu \in D$ such that $\Phi'(\mu) \neq 0$ we have

$$\underline{\Lambda}''(\mu) = \underline{M(\Phi)}''(\mu) \geq \underline{M}''(\Phi(\mu))(\Phi'(\mu))^2 + M'(\Phi(\mu))\Phi''(\mu),$$

where the right-hand side can be possibly infinite. By applying Lemma 5.2 with

$$a_n = \lambda_n \sum_i f_{n,i}^2 \quad \text{and} \quad b_n = \frac{1}{(\lambda_n+\mu)},$$

we learn that

$$\underline{\Lambda}''(\mu) \in (0,\infty].$$

Since the equation $\Phi'(\mu) = 0$ has at most one solution on each bounded connected component of D, Lemma 5.1 yields the claim. Accordingly, $\Lambda(\mu) = 0$ can have at most two solutions on each bounded connected component of D. In the unbounded interval I_0, the function Λ is strictly decreasing. Moreover, setting $M(\Phi(M_\infty)) = 0$ if $M_\infty = \infty$,

$$\lim_{\mu\to(\beta+M_\infty)^-} \Lambda(\mu) = -M_\infty + M(\Phi(M_\infty)) < 0,$$

and

$$\lim_{\mu\to\alpha^+} \Lambda(\mu) = \begin{cases} M_\infty & \text{if } n_\star > 0, \\ M(\Phi(\beta)) & \text{if } n_\star = 0. \end{cases}$$

Noting that, if $n_\star = 0$,

$$M(\Phi(\beta)) = \lim_{\mu\to\beta^+} M(\Phi(\mu)) > 0,$$

we conclude that there is exactly one solution in I_0. In summary, equation $\Lambda(\mu) = 0$ has at least one solution and at most $2n_\star + 1$ solutions in D. In turn, (3.1) possesses the same number of solutions with the property that $\mu \neq -\lambda_n$. Indeed, for every $\mu \in D$ such that $\Lambda(\mu) = 0$, the vector u with Fourier coefficients given by (5.6) belongs to H^1. This is guaranteed by the convergence of the series

$$\sum_{n,i} \frac{\lambda_n f_{n,i}^2}{(\lambda_n + \mu)^2},$$

since μ cannot be a cluster point for $-\lambda_n$.

Next, we look for solutions u such that

$$\mu = -\lambda_k,$$

for some given k. We preliminarily observe that, due to (5.3), if $-\beta - \lambda_k \notin (0, M_\infty)$, no such solutions exist. In the other case, for $n \neq k$, the values $u_{n,i}$ are fixed by (5.6) with $\mu = -\lambda_k$. We are left to determine the values $u_{k,i}$. But (5.4) now reads

$$M\left(\lambda_k \sum_i u_{k,i}^2 + Q_k\right) = -\beta - \lambda_k.$$

Therefore, we have no solutions whenever

$$M(Q_k) > -\beta - \lambda_k.$$

Assume then that $M(Q_k) \leq -\beta - \lambda_k$. From (5.5), we have no solutions unless $f_{k,i} = 0$ for all i, that is, unless $\mathbb{P}_k f = 0$. In which case, we have one solution if $M(Q_k) = -\beta - \lambda_k$ (namely, $u_{k,i} = 0$ for all i). If $M(Q_k) < -\beta - \lambda_k$, we have two solutions provided that $\mathrm{d}_k = 1$, corresponding to

$$u_{k,1} = \pm\sqrt{\frac{M^{-1}(-\beta - \lambda_k) - Q_k}{\lambda_k}},$$

and infinitely many solutions if $\mathrm{d}_k > 1$. □

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