NEW VARIATIONAL PRINCIPLES IN QUASI-STATIC VISCOELASTICITY

C. Giorgi and A.Marzocchi (*)

Summary. *A "saddle point" (or maximum-minimum) principle is set up for the qu static boundary-value problem in linear viscoelasticity. The appropriate class of convolut type functionals for it is taken in terms of bilinear forms with a weight function invol Fourier transform. The "minimax" property is shown to hold as a direct consequence of thermodynamic restrictions on the relaxation function. This approach can be extende further linear evolution problems where initial data are not prescribed.*

0. Introduction

By *quasi-static problem* we mean the problem to find a function $\mathbf{u}(x,t)$ satisfying

$$\nabla \cdot \mathbf{T}(\mathbf{u})(x,t) + \mathbf{f}(x,t) = 0 \qquad ($$

in a cylinder $\Omega \times \mathrm{I\!R}$, together with boundary conditions that, for the sake of simplicity, assumed to be homogeneous

$$\mathbf{u}(x,t) = 0 \qquad \text{on } \partial\Omega \times \mathrm{I\!R}. \qquad ($$

Here Ω is an open bounded connected subset of the Euclidean space $\mathrm{I\!R}^3$ with Lipsc boundary.

In particular, for linear viscoelastic materials the stress tensor $\mathbf{T}$ depends on the placement field $\mathbf{u}$ as follows

$$\mathbf{T}(\mathbf{u})(x,t) = \mathbf{G}_0(x)\nabla\mathbf{u}(x,t) + \int_0^\infty \mathbf{G}'(x,s)\nabla\mathbf{u}(x,t-s)\,ds, \qquad ($$

where $\mathbf{G}_0$ and $\mathbf{G}'$ are fourth-order tensors such that

$$(\mathbf{G}_0)_{ijkl} = (\mathbf{G}_0)_{jikl} = (\mathbf{G}_0)_{ijlk} \qquad (\mathbf{G}')_{ijkl} = (\mathbf{G}')_{jikl} = (\mathbf{G}')_{ijlk}. \qquad ($$

Furthermore we assume

$$\mathbf{G}_0 \in C^0(\overline{\Omega}) \cap C^1(\Omega), \qquad \mathbf{G}' \in L^1(\mathrm{I\!R}^+, C^0(\overline{\Omega}) \cap C^1(\Omega)) \qquad ($$

in order to comply with fading memory and spatial regularity requirements. Finally, s we need dealing with variational formulations, $\mathbf{G}_0$ and $\mathbf{G}'$ must be symmetric, namely

(*) Authors'address: Università Cattolica del S.Cuore, Dipartimento di Matematica, Via Tri 17, 25121 Brescia (Italy)

$$(\mathbf{G}_0)_{ijkl} = (\mathbf{G}_0)_{klij}, \qquad (\mathbf{G}')_{ijkl} = (\mathbf{G}')_{klij}.$$

Summarizing, we shall refer to

$$\begin{cases} \nabla \cdot \big[\mathbf{G}_0(x)\nabla \mathbf{u}(x,t) + \int\limits_0^\infty \mathbf{G}'(x,s)\nabla \mathbf{u}(x,t-s)\,ds\big] + \mathbf{f}(x,t) = 0 & (x,t) \in \Omega \times \mathbb{R} \\ \\ \mathbf{u}(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}. \end{cases}$$

where $\mathbf{G}_0$ and $\mathbf{G}'$ comply with (0.4)-(0.6), as *quasi-static problem in linear viscoelast* (QSP).

Taking into account that $\mathbf{G}'$ can be defined on the whole real line by assuming

$$\mathbf{G}'(x,s) = \mathbf{0} \qquad \forall (x,s) \in \overline{\Omega} \times (-\infty, 0),$$

we can get

$$\int_0^\infty \mathbf{G}'(x,s)\nabla \mathbf{u}(x,t-s)\,ds = (\mathbf{G}' * \nabla \mathbf{u})(x,t),$$

where $*$ denotes convolution on $\mathbb{R}$, and (0.7) may be rewritten in the following compact f

$$\begin{cases} \nabla \cdot \big[(\mathbf{G}_0 + \mathbf{G}'*)\nabla \mathbf{u}\big] + \mathbf{f} = 0 \\ \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases}$$

Definition. *A function* $\mathbf{u}$ *is called a* strict solution *to QSP with source function* $\mathbf{f}$ *in* $L^1(\mathbb{R},$ *(or* $L^2(\mathbb{R}, L^2(\Omega))$*) if* $\mathbf{u}$ *belongs to* $L^1(\mathbb{R}, H_0^1(\Omega))$ *(or* $L^2(\mathbb{R}, H_0^1(\Omega))$*) and satisfies* (0.7) *alr everywhere in* $\Omega \times \mathbb{R}$.

In the sequel we shall assume further conditions on $\mathbf{G}'$ that are derived from Ther dynamics. Mainly, we recall here *Graffi's inequality* ([1])

$$\omega \widehat{\mathbf{G}}'_s(x,\omega) \le \mathbf{0} \qquad \qquad \forall \omega \in \mathbb{R}$$

where $\widehat{\mathbf{G}}'_s(x,\omega) = \int_0^\infty \mathbf{G}'(x,s)\sin\omega s\,ds$. It is a necessary and sufficient condition that work in sinusoidal Λprocesses is non-negative. As proved by Fabrizio & Morro [5], (0.9 quite equivalent to the Second Law of Thermodynamics in the form of the Clausius prop for isothermal processes.

It is worth remarking that, according to [3], a stronger version of the Second Law ca given so that the Clausius inequality reduces to an equality if and only if "reversible" cy are considered. If such is the case, then (0.9) is replaced by

([1]) For any fourth-order tensor $\mathbf{A}$ the notation $\mathbf{A} > \mathbf{0}$ ($\mathbf{A} \ge \mathbf{0}$) means that $\mathbf{A}$ is positive-defi (semi-definite) in the space of second-order symmetric tensors. For simplicity, hencefort is understood as the zero element in any vector space.

$$\omega \widehat{\mathbf{G}}'_s(x,\omega) < \mathbf{0}. \qquad \forall \omega \in \mathrm{I\!R} \setminus \{0\} \qquad (0$$

Finally, we may account for the body being a *solid* by letting

$$\mathbf{G}_\infty(x) \stackrel{def}{=} \mathbf{G}_0(x) + \int_0^\infty \mathbf{G}'(x,s)\, ds > \mathbf{0} \qquad \forall x \in \overline{\Omega}. \qquad (0$$

As to the solvability of QSP, we recall the following existence and uniqueness re
proved by FABRIZIO in [3].

Proposition 1. *If the viscoelastic material is a solid, i.e.* (0.11) *holds, then QSP has*
and only one strict solution provided that (0.10) *is satisfied.*

Remark. *In the previous* Proposition (0.10) *cannot be weakened into* (0.9) *(see* [4]*). Ne*
theless, it is a sufficient but not necessary condition (see [7]*).*

Concerning variational formulations, a minimum principle for QSP was establishec
CHRISTENSEN in [1]. Roughly speaking, it states that a strict factorized solution $\mathbf{u}_0(x,t$
$h(t)\mathbf{k}(x)$ to QSP minimizes a suitable functional with respect to perturbations of the spa
part $\mathbf{k}(x)$ only. Indeed, as noticed in [6], the convexity of the convolution-type functi
introduced by CHRISTENSEN follows from thermodynamic restrictions, but yet it does
turn out to be stationary at the solution unless we assume the same time dependence
all displacement fields. As a consequence, the converse statement of the principle canno
proved, that is to say, the whole solution $\mathbf{u}_0$ to QSP cannot be characterized as a minim

A new technique to obtain stationary and minimum principles for linear evolu
equations was early introduced by REISS [9] and further developed by many authors (
[10],[6],[2]). It rests upon the introduction of suitable bilinear forms of convolution t
involving Laplace transformation with respect to time and then applies to initial bound
value problems only. Thus QSP does not fit into this approach. Nevertheless, the techni
of REISS can be modified for this purpose.

In this paper we introduce a class of bilinear forms close to REISS' one but invol
Fourier transform and use them to construct a suitable family of convolution-type functior
Thereby new variational principles for QSP are set up. The first (Th.1) states that a solu
$\mathbf{u}_0(x,t)$ to QSP can be characterized as a stationary point of every functional in this far
with respect to general perturbations. It is worth noting that few mild assumptions
required to prove it, namely (0.4)-(0.6).

Contrary to what happens in the corresponding dynamic initial boundary-value prob
[6], the addition of thermodynamic conditions does not lead to a minimum, but yiel
"maximum-minimum" principle. In fact, our main result (Th.2) states that a solutior
QSP can be characterized as a saddle point, with respect to an appropriate decomposi
of $\mathbf{u}$, if (0.9) holds. This feature seems to be a typical property of our approach as sketc
by another example. So we hope that the ideas exposed here work as well for other lir
evolution problems in materials with memory where initial history data are not prescrib

1. The bilinear forms $< \cdot , \cdot >_y$

Let Y be a real function belonging to $L^1(\mathbb{R}^+)$ and Y_e be its odd extension, i.e.

$$Y_e(s) = \begin{cases} Y(s), & s \in \mathbb{R}^+ \\ -Y(-s), & s \in \mathbb{R}^-. \end{cases}$$

We define the function y on $\mathbb{R}$ as

$$y(t) \stackrel{def}{=} -i\widehat{Y}_e(t) = -i\int_{-\infty}^{\infty} Y_e(s)e^{-ist}\,ds \qquad (\ldots$$

where the hat denotes Fourier transform in $\mathbb{R}$. Now, since $Y_e(s)\cos st$ is an even functio
s and $Y_e(s)\sin st$ is an odd one, we have

$$\begin{aligned} y(t) &= -i\int_{-\infty}^{\infty} Y_e(s)(\cos st - i\sin st)\,ds \\ &= -2\int_0^{\infty} Y(s)\sin st\,ds. \end{aligned} \qquad (\ldots$$

The function y is easily seen to be a real, odd, absolutely continuous function which vanis
as $|t|$ tends to infinity.

For any pair $(\mathbf{p}, \mathbf{q})$ of vector- or tensor-valued functions on $\Omega \times \mathbb{R}$ we introduce
following bilinear form

$$< \mathbf{p}, \mathbf{q} >_y \stackrel{def}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t+\tau)\int_{\Omega} \mathbf{p}(x,t)\cdot\mathbf{q}(x,\tau)\,dx\,dt\,d\tau. \qquad (\ldots$$

Substituting (1.1) into (1. 3), it follows

$$\begin{aligned} < \mathbf{p}, \mathbf{q} >_y &= -i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y_e(\omega)\int_{\Omega} \mathbf{p}(x,t)e^{-i\omega t}\cdot\mathbf{q}(x,\tau)e^{-i\omega\tau}\,dx\,dt\,d\tau\,d\omega \\ &= -i\int_{-\infty}^{\infty} Y_e(\omega)\int_{\Omega} \hat{\mathbf{p}}(x,\omega)\cdot\hat{\mathbf{q}}(x,\omega)\,dx\,d\omega. \end{aligned}$$

Now, by definition, Y_e is an odd function of ω and $Re\big(\hat{\mathbf{p}}, \hat{\mathbf{q}}\big)$ an even one, hence it must b

$$< \mathbf{p}, \mathbf{q} >_y = \int_{-\infty}^{\infty} Y_e(\omega)Im\big(\hat{\mathbf{p}}(\omega), \hat{\mathbf{q}}(\omega)\big)\,d\omega \qquad (\ldots$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$ and $Im\ z$ ($Re\ z$) denotes the imaginary (real) p
of the complex number z.

Of course, this bilinear form is well-defined on $L^1(\mathbb{R}, L^2(\Omega))$ whenever $Y \in L^1(\mathbb{I}$
as well as on $L^2(\mathbb{R}, L^2(\Omega))$ whenever $Y \in C_0^\infty(\mathbb{R}^+)$. Moreover, from (1.4) it is immed
to notice that $< \cdot,\cdot >_y$ is symmetric. Taking into account that by definition (2) $\hat{f}(\omega$
$\hat{f}_c(\omega) - i\hat{f}_s(\omega)$ one gets the alternative form

(2) In the sequel $\hat{\mathbf{f}}_{\mathbf{s}}$ (resp.$\hat{\mathbf{f}}_{\mathbf{c}}$) will denote full-range Fourier sine (cosine) transform of a func
$\mathbf{f}$ defined on $\mathbb{R}$.

$$< \mathbf{p},\mathbf{q} >_y = -2\int_0^\infty Y(\omega)\big[\big(\hat{\mathbf{p}}_c(\omega),\hat{\mathbf{q}}_s(\omega)\big)+\big(\hat{\mathbf{p}}_s(\omega),\hat{\mathbf{q}}_c(\omega)\big)\big]\,d\omega. \qquad ($$

2. Variational and Saddle-point Principles

The aim of this section is to recover QSP from a variational principle. We do this in ways. First, under quite mild assumptions on $\mathbf{G}_0$ and $\mathbf{G}'$, we characterize a solution to (as a stationary point of every functional in a suitable family and we give sufficient condit for it to exist and be unique. Second, we introduce another family of functionals depend on $\mathbf{u}$ through its even and odd parts and we prove, under thermodynamic restrictions, a solution to (0.7) turns out to be a saddle-point for them.

Remembering the compact form (0.8) of QSP and the bilinear form (1.3), we introd the following family of functionals

$$\Phi_y(\mathbf{u}) \stackrel{def}{=} \frac{1}{2} < \nabla\mathbf{u},(\mathbf{G}_0+\mathbf{G}'*)\nabla\mathbf{u} >_y - < \mathbf{u},\mathbf{f} >_y, \qquad ($$

where $\mathbf{G}_0$ and $\mathbf{G}'$ comply with (0.4)-(0.6), $\mathbf{f}\in L^1(\mathbb{R},L^2(\Omega))$ and y is given by (1.2) for ev $Y\in L^1(\mathbb{R}^+)$. From these assumptions it follows that each Φ_y is well-defined on

$$V = L^1(\mathbb{R},H^1_0(\Omega)).$$

Owing to the convolution $\mathbf{G}'*\nabla\mathbf{u}$ it is worth observing that, by definition,

$$< \mathbf{p}*\mathbf{q},\mathbf{r} >_y = \int_{-\infty}^{\infty} y(t+\tau)\Big(\int_{-\infty}^{\infty}\mathbf{p}(\tau-\sigma)\mathbf{q}(\sigma)\,d\sigma,\mathbf{r}(t)\Big)\,d\tau\,dt.$$

The change of variables $\lambda=\tau-\sigma,\quad \eta=\sigma$ gives

$$< \mathbf{p}*\mathbf{q},\mathbf{r} >_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t+\lambda+\eta)\big(\mathbf{p}(\lambda)\mathbf{q}(\eta),\mathbf{r}(t)\big)\,d\lambda\,d\eta\,dt$$

and from (1.4) it follows

$$< \mathbf{p}*\mathbf{q},\mathbf{r} >_y = \int_{-\infty}^{\infty} Y_e(\omega)Im\big(\hat{\mathbf{p}}(\omega)\hat{\mathbf{q}}(\omega),\hat{\mathbf{r}}(\omega)\big)\,d\omega. \qquad ($$

Now we state the first variational principle.

Λ**Theorem 1.** *Let* $\mathbf{u}_0\in V$ *be a strict solution to QSP with* $\mathbf{f}\in L^1(\mathbb{R},L^2(\Omega))$. *Then every* $Y\in L^1(\mathbb{R}^+)$, $\mathbf{u}_0$ *is a stationary point for the functional* $\Lambda\Phi_y$. *Conversely, if* $\mathbf{u}_0$ *stationary point on* V *of every* Φ_y, $Y\in L^1(\mathbb{R}^+)$, *then* $\mathbf{u}_0$ *is a strict solution to QSP. Moreover, if conditions* $(0.10)-(0.11)$ *are satisfied then the stationary point* $\mathbf{u}_0$ *exists is unique in* V.

Proof. Suppose that $\mathbf{u}_0 \in V$ is a strict solution to QSP with $\mathbf{f} \in L^1(\mathbb{R}, L^2(\Omega))$. Then, any $\omega \in \mathbb{R}$, its Fourier transform $\hat{\mathbf{u}}_0(\omega)$ belongs to $H_0^1(\Omega)$ and satisfies

$$\nabla \cdot \left(\left[\mathbf{G}_0 + \widehat{\mathbf{G}}'(\omega) \right] \nabla \hat{\mathbf{u}}_0(\omega) \right) + \hat{\mathbf{f}}(\omega) = \mathbf{0} \qquad \text{a.e. on } \Omega. \qquad ($$

Letting

$$L_\omega \mathbf{u} \stackrel{def}{=} \nabla \cdot \left(\left[\mathbf{G}_0 + \widehat{\mathbf{G}}'(\omega) \right] \nabla \mathbf{u} \right) ,$$

using (1.4), (2.2), (0.6) and the divergence theorem we have

$$\frac{d}{d\alpha} \Phi_y(\mathbf{u}_0 + \alpha \mathbf{v})|_{\alpha=0} = - \int_{-\infty}^{\infty} Y_e(\omega) Im \left(L_\omega \hat{\mathbf{u}}_0(\omega) + \hat{\mathbf{f}}(\omega), \hat{\mathbf{v}}(\omega) \right) d\omega$$

for every $\mathbf{v} \in V$ and $Y \in L^1(\mathbb{R}^+)$. Hence from (2.3) it follows that $\mathbf{u}_0$ gives a station point of every Φ_y in the space V.

Conversely, we suppose that for every $Y \in L^1(\mathbb{R}^+)$

$$\frac{d}{d\alpha} \Phi_y(\mathbf{u}_0 + \alpha \mathbf{v})|_{\alpha=0} = 0 \qquad \forall \mathbf{v} \in V.$$

In particular, letting $\mathbf{v}$ take the form $\mathbf{v}(x,t) = h(t)\mathbf{w}(x)$, where $h \in L^1(\mathbb{R})$, $\mathbf{w} \in H_0^1(\Omega)$ have

$$0 = \frac{d}{d\alpha} \Phi_y(\mathbf{u}_0 + \alpha \mathbf{v})|_{\alpha=0} =$$

$$= -2 \int_0^{\infty} Y(\omega) \left[\hat{h}_c(\omega) Im \left(L_\omega \hat{\mathbf{u}}_0(\omega) + \hat{\mathbf{f}}(\omega), \mathbf{w} \right) - \hat{h}_s(\omega) Re \left(L_\omega \hat{\mathbf{u}}_0(\omega) + \hat{\mathbf{f}}(\omega), \mathbf{w} \right) \right] d\omega.$$

By the arbitrariness of $\mathbf{w}$, $Y\hat{h}_c$ and $Y\hat{h}_s$ we obtain the real and the imaginary part of (so that $\hat{\mathbf{u}}_0(\omega)$ must be a weak solution to (2.3) for almost every ω. Finally, by the unique of the Fourier transform, $\mathbf{u}_0$ solves the original problem (0.8).

The last part of the theorem follows trivially from Proposition 1.

A different variational principle can be set up by charaterizing the solution as a sad point. To do this, however, stronger assumptions than in Th.1 are required on $\mathbf{G}'$.

At first, we observe that a function $\mathbf{u}$ belonging to

$$W = L^2(\mathbb{R}, H_0^1(\Omega))$$

can be split into its even and odd parts with respect to time, i.e. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where

$$\mathbf{u}_1(t) \stackrel{def}{=} \frac{1}{2}(\mathbf{u}(t) + \mathbf{u}(-t)), \qquad \mathbf{u}_2(t) \stackrel{def}{=} \frac{1}{2}(\mathbf{u}(t) - \mathbf{u}(-t)) \qquad \text{a.e. in } \mathbb{R}. \qquad ($$

Moreover, the Fourier transforms $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ characterize respectively the real and imagi part of $\hat{\mathbf{u}}$, so that

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 \quad \text{and} \quad \hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_c, \;\; \hat{\mathbf{u}}_2 = -i\hat{\mathbf{u}}_s. \qquad ($$

Let W_1 be the subspace of W consisting of even functions and W_2 that of odd o Since W_1 and W_2 are orthogonal with respect to the inner product of $L^2(\mathbb{R}; L^2(\Omega))$ we h $W_1 \oplus W_2 = W$.

In view of (2.4) we can rewrite the functional (2.1) in the following form

$$\Psi_y(\mathbf{u}_1, \mathbf{u}_2) \stackrel{def}{=} \frac{1}{2} < \nabla(\mathbf{u}_1 + \mathbf{u}_2), (\mathbf{G}_0 + \mathbf{G}' *)\nabla(\mathbf{u}_1 + \mathbf{u}_2) >_y - < (\mathbf{u}_1 + \mathbf{u}_2), \mathbf{f} >_y$$

where $\mathbf{f} \in L^2(\mathbb{R}, (L^2(\Omega))$ and y is given by (1.2). Assuming that $Y \in C_0^\infty(\mathbb{R}^+)$, each Ψ well-defined on $W_1 \times W_2$ thanks to (1.4) and the bijectivity of the Fourier transform from into itself.

Theorem 2. *Let* $\mathbf{G}'$ *satisfy* (0.9) *and* $\mathbf{u} \in W$ *be a strict solution to QSP with* $\mathbf{f} \in L^2(\mathbb{R}, L^2$ *Then, for every positive* $\Lambda\Lambda Y \in C_0^\infty(\mathbb{R}^+)$, *the pair* $(\mathbf{u}_1, \mathbf{u}_2)$ *given by* (2.4) *is a saddle-p for* Ψ_y. *Conversely, if* $(\mathbf{u}_1, \mathbf{u}_2)$ *is a saddle-point on* $W_1 \times W_2$ *of every* Ψ_y, $Y \in C_0^\infty($ *positive, then* $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ *is a strict solution to QSP.*
If (0.10), (0.11) *hold instead of* (0.9) *then the saddle-point exists and is unique in* W.

Proof. If $\mathbf{u} \in W$ is a strict solution to QSP with $\mathbf{f} \in L^2(\mathbb{R}, L^2(\Omega))$ then its Fourier transf $\hat{\mathbf{u}}$ belongs to W and satisfies

$$\nabla \cdot \big(\big[\mathbf{G}_0(x) + \widehat{\mathbf{G}}'(x,\omega) \big] \nabla \hat{\mathbf{u}}(x,\omega) \big) + \hat{\mathbf{f}}(x,\omega) = \mathbf{0} \qquad \text{a.e. on } \Omega \times \mathbb{R}. \qquad ($$

Remembering that $\widehat{\mathbf{G}}' = \widehat{\mathbf{G}}'_c - i\widehat{\mathbf{G}}'_s$, the pair $(\hat{\mathbf{u}}_c, \hat{\mathbf{u}}_s) \in W_1 \times W_2$ given by (2.4), (2.5) m satisfy a.e. the system

$$\begin{cases} \nabla \cdot \big(\big[\mathbf{G}_0 + \widehat{\mathbf{G}}'_c \big] \nabla \hat{\mathbf{u}}_c - \widehat{\mathbf{G}}'_s \nabla \hat{\mathbf{u}}_s \big) + \hat{\mathbf{f}}_c = \mathbf{0} \\ \\ \nabla \cdot \big(\big[\mathbf{G}_0 + \widehat{\mathbf{G}}'_c \big] \nabla \hat{\mathbf{u}}_s + \widehat{\mathbf{G}}'_s \nabla \hat{\mathbf{u}}_c \big) + \hat{\mathbf{f}}_s = \mathbf{0}. \end{cases} \qquad ($$

On the other hand, using (1.5), (2.2), (2.5) and the symmetry assumptions (0.6 straightforward calculation leads to

$$\Psi_y(\mathbf{u}_1, \mathbf{u}_2) = \int_0^\infty Y \big[\big(\nabla \hat{\mathbf{u}}_s, \widehat{\mathbf{G}}'_s \nabla \hat{\mathbf{u}}_s\big) - \big(\nabla \hat{\mathbf{u}}_c, \widehat{\mathbf{G}}'_s \nabla \hat{\mathbf{u}}_c\big) - 2\big(\nabla \hat{\mathbf{u}}_s, (\mathbf{G}_0 + \widehat{\mathbf{G}}'_c)\nabla \hat{\mathbf{u}}_c\big) +$$

$$+ 2\big(\hat{\mathbf{u}}_c, \hat{\mathbf{f}}_s\big) + 2\big(\hat{\mathbf{u}}_s, \hat{\mathbf{f}}_c\big) \big] \, d\omega$$

where the dependence on ω is understood. Thereby we have

$$\frac{d}{d\alpha} \Psi_y(\mathbf{u}_1 + \alpha \mathbf{v}_1, \mathbf{u}_2)|_{\alpha=0} = 2 \int_0^\infty Y \big(\nabla \cdot \{ \big[\mathbf{G}_0 + \widehat{\mathbf{G}}'_c \big] \nabla \hat{\mathbf{u}}_s + \widehat{\mathbf{G}}'_s \nabla \hat{\mathbf{u}}_c \} + \hat{\mathbf{f}}_s, \hat{\mathbf{v}}_1 \big) \, d\omega$$

for every $\mathbf{v}_1 \in W_1$, since $\hat{\mathbf{v}}_1 = \hat{\mathbf{v}}_{1c}$ belongs to W_1, and

$$\frac{d}{d\alpha}\Psi_y(\mathbf{u}_1, \mathbf{u}_2 + \alpha\mathbf{v}_2)|_{\alpha=0} = 2\int_0^\infty Y\big(\nabla \cdot \{[\mathbf{G}_0 + \widehat{\mathbf{G}}'_c]\nabla\hat{\mathbf{u}}_c - \widehat{\mathbf{G}}'_s\nabla\hat{\mathbf{u}}_s\} + \hat{\mathbf{f}}_c, i\hat{\mathbf{v}}_2\big)\, d\omega$$

for every $\mathbf{v}_2 \in W_2$, since $i\hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_{2s}$ belongs to W_2.

Finally, from (2.7) it follows

$$\frac{d}{d\alpha}\Psi_y(\mathbf{u}_1 + \alpha\mathbf{v}_1, \mathbf{u}_2)|_{\alpha=0} = 0 \qquad \forall \mathbf{v}_1 \in W_1$$

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$$\frac{d}{d\alpha}\Psi_y(\mathbf{u}_1, \mathbf{u}_2 + \alpha\mathbf{v}_2)|_{\alpha=0} = 0 \qquad \forall \mathbf{v}_2 \in W_2$$

for every $Y \in C_0^\infty(\mathrm{I\!R}^+)$, and from (0.9) we have

$$\frac{d^2}{d\alpha^2}\Psi_y(\mathbf{u}_1 + \alpha\mathbf{v}_1, \mathbf{u}_2)|_{\alpha=0} = -2\int_0^\infty Y(\omega)\big(\nabla\hat{\mathbf{v}}_1(\omega), \widehat{\mathbf{G}}'_s(\omega)\nabla\hat{\mathbf{v}}_1(\omega)\big)\, d\omega > 0 \qquad \forall \mathbf{v}_1 \in W$$

$$\frac{d^2}{d\alpha^2}\Psi_y(\mathbf{u}_1, \mathbf{u}_2 + \alpha\mathbf{v}_2)|_{\alpha=0} = -2\int_0^\infty Y(\omega)\big(\nabla\hat{\mathbf{v}}_2(\omega), \widehat{\mathbf{G}}'_s(\omega)\nabla\hat{\mathbf{v}}_2(\omega)\big)\, d\omega =$$

$$= 2\int_0^\infty Y(\omega)\big(\nabla\hat{\mathbf{v}}_{2s}(\omega), \widehat{\mathbf{G}}'_s(\omega)\nabla\hat{\mathbf{v}}_{2s}(\omega)\big)\, d\omega < 0 \qquad \forall \mathbf{v}_2 \in W$$

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for every positive $Y \in C_0^\infty(\mathrm{I\!R}^+)$. This proves that $(\mathbf{u}_1, \mathbf{u}_2)$ is a saddle point on $W_1 \times W_2$ every Ψ_y.

Conversely, let assume that (2.8) and (2.9) hold for every Ψ_y, $Y \in C_0^\infty(\mathrm{I\!R}^+)$ posit From the arbitrariness of $Y_e\hat{\mathbf{v}}_1$ in W_2 and $iY_e\hat{\mathbf{v}}_2$ in W_1, it follows that the pair $(\hat{\mathbf{u}}_c, \hat{\mathbf{u}}_s$ $(\hat{\mathbf{u}}_1, i\hat{\mathbf{u}}_2) \in W_1 \times W_2$ solves (2.7), and so $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2$ satisfies (2.6). Finally, the bijecti of the Fourier transform from L^2 into itself, $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ must be a strict solution to QS W.

The last part of the theorem follows trivially from Proposition 1.

3. Concluding remarks

Some other evolution problem in linear viscoelasticity can be fitted into the pre approach, for instance the dynamical boundary-value problem on the whole time axis consists in finding a function $\mathbf{u}$ defined on $\Omega \times \mathrm{I\!R}$ which satisfies a.e.

$$\begin{cases} \mathbf{u}_{tt} - \nabla \cdot \big[(\mathbf{G}_0 + \mathbf{G}'*)\nabla\mathbf{u}\big] = \mathbf{f} \\ \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases}$$

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where $\mathbf{G}_0$ and $\mathbf{G}'$ comply with (0.4)-(0.6) and $\mathbf{f} \in L^2(\mathrm{I\!R}, L^2(\Omega))$ (see [8]).

Following along the lines of previous sections, we introduce the functional

$$\Lambda_y(\mathbf{u}) \stackrel{def}{=} \frac{1}{2} < \mathbf{u}_t, \mathbf{u}_t >_y + \frac{1}{2} < \nabla \mathbf{u}, (\mathbf{G}_0 + \mathbf{G}' *) \nabla \mathbf{u} >_y - < \mathbf{u}, \mathbf{f} >_y \quad .$$

In view of (1.4), (1.5) we have

$$< \mathbf{u}_t, \mathbf{u}_t >_y = - \int_{-\infty}^{\infty} \omega^2 Y_e(\omega) Im\big(\hat{\mathbf{u}}(\omega), \hat{\mathbf{u}}(\omega)\big)\, d\omega =$$
$$= 4 \int_0^{\infty} \omega^2 Y(\omega) \big(\hat{\mathbf{u}}_c(\omega), \hat{\mathbf{u}}_s(\omega)\big)\, d\omega.$$

For every $Y \in C_0^\infty(\mathrm{I\!R}^+)$ the functional Λ_y is well-defined on

$$H = L^2(\mathrm{I\!R}, H_0^1(\Omega)) \cap H^1(\mathrm{I\!R}, L^2(\Omega)).$$

and is stationary at $\mathbf{u}^0 \in H$ if and only if $\mathbf{u}^0$ is a weak solution to (3.1). This is easily s
by taking the Fourier transform of (3.1), namely

$$\omega^2 \hat{\mathbf{u}}(x,\omega) + \nabla \cdot \Big(\big[\mathbf{G}_0(x) + \widehat{\mathbf{G}}'(x,\omega)\big] \nabla \hat{\mathbf{u}}(x,\omega)\Big) + \hat{\mathbf{f}}(x,\omega) = \mathbf{0},$$

and paralleling the proof of Th.1.

Furthermore, if (0.9) holds, the pair $(\mathbf{u}_1^0, \mathbf{u}_2^0)$ obtained by (2.4) from the solution $\mathbf{u}$
a saddle point of every functional

$$\Gamma_y(\mathbf{u}_1, \mathbf{u}_2) \stackrel{def}{=} \Lambda_y(\mathbf{u}_1 + \mathbf{u}_2)$$

defined on $H_1 \times H_2$ (the analog of $W_1 \times W_2$). Really, we have

$$\frac{d^2}{d\alpha^2} \Gamma_y(\mathbf{u}_1^0 + \alpha \mathbf{v}_1, \mathbf{u}_2^0)|_{\alpha=0} \equiv \frac{d^2}{d\alpha^2} \Psi_y(\mathbf{u}_1^0 + \alpha \mathbf{v}_1, \mathbf{u}_2^0)|_{\alpha=0} > 0 \qquad \forall \mathbf{v}_1 \in H_1$$
$$\frac{d^2}{d\alpha^2} \Gamma_y(\mathbf{u}_1^0, \mathbf{u}_2^0 + \alpha \mathbf{v}_2)|_{\alpha=0} \equiv \frac{d^2}{d\alpha^2} \Psi_y(\mathbf{u}_1^0, \mathbf{u}_2^0 + \alpha \mathbf{v}_2)|_{\alpha=0} < 0 \qquad \forall \mathbf{v}_2 \in H_2,$$

for every positive $Y \in C_0^\infty(\mathrm{I\!R}^+)$.

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