A MINIMUM PRINCIPLE FOR THE QUASI-STATIC PROBLEM IN LINEAR VISCOELASTICITY

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Abstract. *A minimum principle is set up for the quasi-static boundary-value problem* (QSP) *in linear viscoelasticity. A linear homogeneous and isotropic viscoelastic solid under unidimensional displacements is considered along with the complete set of thermodynamic restrictions on the relaxation function. It is assumed that boundary conditions are of Dirichlet type and initial history data are not given. The variational formulation of* QSP *is set up through a convex functional based on a "weighted" L^2 inner product as the bilinear form and is strictly related to the thermodynamic restrictions on the relaxation function. As an aside, the same technique is proved to be applicable to analogous physical problems such as the quasi-static heat flux equation.*

0. Introduction

The connection between thermodynamic restrictions and variational formulations or, possibly, extremum principles in linear viscoelasticity was investigated in a number of recent papers (see [1] for an up-to-date survey). Because of the convolution which occurs in the stress-strain relation, very often variational formulations of linear viscoelasticity are expressed by functionals based on bilinear forms of convolution type. Quite rarely an extremum property holds for the functional under consideration. If so, however, the extremum property proves to be a direct consequence of the restrictions imposed by thermodynamics on the relaxation function [2][3][4].

Things are even more difficult when the quasi stationary approximation is concerned, *i.e.* when the inertia term is disregarded and solutions are defined on the whole time axis.

A minimum principle for the quasi-static problem (QSP) was established by Christensen [5] through a convex bilinear functional that becomes also stationary if an appropriate class is considered for the displacement fields. Roughly speaking, it states that a factorized quasi-static solution $\mathbf{u}_0(x,t) = h(t)\mathbf{k}(x)$ minimizes a suitable functional with respect to perturbations of the spatial part $\mathbf{k}(x)$ only. Such a functional is built up by considering the L^2 inner product as the bilinear form. As noticed in [3], the convexity follows from thermodynamic restrictions, but the functional does not turn out to be stationary unless we assume the same time dependence for all displacement fields. Hence the whole quasi-static solution cannot be characterized as a minimum.

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To avoid this objection a "maximum-minimum" principle was built up in [6] by constructing a suitable family of convolution type functionals involving Fourier transformation. Thereby a solution to QSP can be characterized as a saddle point relative to the additive decomposition (with respect to time) of the displacement field into its even and odd parts. What is more, this property is crucially related to the thermodynamic restrictions.

Unfortunately, there is no possibility of getting a minimum principle by considering variational formulations of this kind. Nevertheless, on the basis of the thermodynamic restrictions, we are able to get a minimum principle for QSP in a *general* class of displacement fields through a *single* functional.

In order to keep up the convexity property we search for extremum principle via a "weighted" L^2 inner product as the bilinear form. However, since a convolution inescapably occurs through the stress-strain relation, the stationary property should involve appropriate and shrinking assumptions. Indeed, unidimensional displacements only are allowed as well as homogeneity and isotropy of the relaxation function are required.

Since our approach rests heavily upon the special structure of the bilinear form, it cannot be generalized to two- and three-dimensional displacements, apparently. Nevertheless it is applicable without restrictions to other analogous physical problems such as the quasi-static heat flux equation.

1. Setting of the problem

Let us consider a linear viscoelastic homogeneous isotropic body which occupies a bounded regular domain Ω of $\mathbb{R}^3$. Moreover, we consider unidimensional displacements $u : \Omega \times \mathbb{R} \to \mathbb{R}$, so that the constitutive stress-strain relation takes the form

$$\mathbf{T}(u)(x,t) = G_0 \nabla u(x,t) + \int_0^\infty G'(s) \nabla u(x, t-s)\, ds, \tag{1.1}$$

where G_0 is the *istantaneous elastic modulus* and G' is the *Boltzmann memory function.*

For example, an equation of type (1.1) can be found in the study of vibrations of a spatially homogeneous viscoelastic membrane and in the case of a simple extension (or torsion) of a viscoelastic wire with one extremity fixed and the other subjected to the traction (or torque) $\mathbf{T}$.

As usual in materials with memory, the Boltzmann function is required to comply with a fading memory principle. Following Day [7], we may state it as follows:

$$G' \in L^1(\mathbb{R}^+). \tag{1.2}$$

Finally, we may account for the body being a *solid* by letting

$$G_\infty \stackrel{def}{=} G_0 + \int_0^\infty G'(s)\, ds > 0. \tag{1.3}$$

Substituting (1.1) into the equation of motion in the quasi-static approximation and assuming homogeneous Dirichlet boundary conditions we have

$$\begin{cases} G_0 \Delta u(x,t) + \int_0^\infty G'(s) \Delta u(x,t-s)\, ds + f(x,t) = 0 & (x,t) \in \Omega \times \mathbb{R} \\ u|_{\partial\Omega} = 0 & t \in \mathbb{R}. \end{cases} \tag{1.4}$$

We shall refer to (1.4), where G_0 and G' comply with (1.2)-(1.3) as *quasi-static problem in linear viscoelasticity* (QSP).

Taking into account that G' can be defined on the whole real line by assuming

$$G'(s) = 0 \qquad \forall s \in (-\infty, 0),$$

we can get

$$\int_0^\infty G'(s) \Delta u(x, t-s)\, ds = (G' * \Delta u)(x,t),$$

where $*$ denotes convolution on $\mathbb{R}$. Using the Dirac δ-function , we introduce

$$\Gamma \stackrel{def}{=} G_0 \delta + G', \tag{1.5}$$

so that (1.4) may be rewritten in the following compact form

$$\begin{cases} \Gamma * \Delta u + f = 0 \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.6}$$

In view of the variational formulation we introduce the following

Definition. *A function u is called a* strict solution *to* QSP (1.4) *with source function f belonging to $L^2(\mathbb{R}, L^2(\Omega))$ if u belongs to $L^2(\mathbb{R}, H_0^1(\Omega))$ and satisfies*

$$\int_{\mathbb{R}} \int_\Omega \left[(\Gamma * \nabla u)(x,t) \cdot \nabla v(x,t) - f(x,t) \cdot v(x,t) \right] dx\, dt = 0$$

for any $v \in L^2(\mathbb{R}, H_0^1(\Omega))$.

For later convenience we denote by $\hat{f}$ the (formal) Fourier transform of any function f defined on $\Omega \times \mathbb{R}$, namely

$$\hat{f}(x, \omega) \stackrel{def}{=} \int_{\mathbb{R}} f(x,t) \exp(-i\omega t)\, dt \quad ;$$

similarly, letting the subscript s (c) denote the half-range Fourier sine (cosine) transform, for any function g defined on $\Omega \times \mathbb{R}^+$ we have

$$\hat{g}_s(x,\omega) = \int_0^\infty g(x,t) \sin \omega t\, dt \qquad , \qquad \hat{g}_c(x,\omega) = \int_0^\infty g(x,t) \cos \omega t\, dt.$$

2. Thermodynamic restrictions

In the sequel we shall assume further conditions on G_0 and G' that are derived from Thermodynamics. Mainly, we recall here the so called *Graffi's inequality*

$$\omega \widehat{G}'_s(\omega) \leq 0 \qquad \forall \omega \in \mathbb{R} \tag{2.1}$$

which is a necessary and sufficient condition that the work in sinusoidal processes is non-negative. As proved by Fabrizio & Morro [8], (2.1) is quite equivalent to the Second Law of Thermodynamics in the form of the Clausius property for isothermal processes relative to isotropic and homogeneous viscoelastic materials. In the same work it is proved that from (2.1) it follows

$$G_0 \geq G_\infty. \tag{2.2}$$

It is worth remarking that, according to [9], a stronger version of the Second Law can be given so that the Clausius inequality reduces to an equality if and only if "reversible" cycles are considered. If such is the case, then (2.1) is replaced by

$$\omega \widehat{G}'_s(\omega) < 0 \qquad \forall \omega \in \mathbb{R} \setminus \{0\}. \tag{2.3}$$

and (2.2) holds with the strict inequality sign. Therefore we have the following

Proposition 2.1. *Under assumptions* (1.2)-(1.3) *and* (2.3), *there exists* δ *such that*

$$|\widehat{\Gamma}(\omega)| = |G_0 + \widehat{G}'(\omega)| \geq \delta > 0 \qquad \forall \omega \in \mathbb{R}. \tag{2.4}$$

As to the link between the solvability of QSP and thermodynamic restrictions, we recall the following result proved by Fabrizio in [8].

Proposition 2.2. *If the viscoelastic material is a solid, i.e.* (1.3) *holds, then QSP has one and only one strict solution provided that* (2.3) *is satisfied.*

Remark . *In the previous* Proposition 2.2*, the inequality* (2.3) *cannot be weakened into* (2.1) *(see* [10]*). Nevertheless, it is a sufficient but not necessary condition (see* [11]*).*

3. Preliminaries on the variational formulation

A systematic method for the derivation of variational formulations is that pertaining to the theory of inverse problems of the Calculus of Variations. Here we state the main results of this theory (see [13]) which are relevant to our purpose.

Let X be a Banach space over $\mathbb{R}$ and X^* denote the conjugate (dual) space of X. Letting $v \in X^*$ and $z \in X$, $\langle v, z \rangle$ represents a *non-degenerate bilinear form* whereby if $\langle v, z \rangle = 0$ for every $v \in X^*$ (resp. $z \in X$) then z (resp. v) is the null element of X (resp. X^*). Suppose that $U \subset X$ is an open set and let N be an operator $U \to X^*$. Let $u + \nu h \in U, \nu \in [O, \overline{\nu}]$, for some $\overline{\nu} > 0$. If N is Gâteaux differentiable we denote by

$$dN(u|h) = \frac{dN(u + \nu h)}{d\nu}\Big|_{\nu=0}$$

the (Gâteaux) differential of N at $u \in U$ in the direction h. If there exists $N'(u)$ such that

$$dN(u|h) = \langle N'(u), h \rangle$$

we regard $N'(u)$ as the derivative of N with respect to the bilinear form. Similarly, in the case of functionals $f : X \to \mathbb{R}$ we write $df(u|h) = \langle f'(u), h \rangle$. If there exists a functional f such that $N = f'$ then we say that N is a *potential operator*.

The existence of a potential for the given operator is crucially related to the choice of the bilinear form, as it is expressed by the following theorem whose proof is given by Vainberg [12]:

Theorem 3.1. *Let $\langle \cdot, \cdot \rangle$ be a non-degenerate bilinear form on X. Suppose that*

(i) N is an operator from X into X^*,
(ii) N has a linear Gâteaux differential $dN(u|h)$ at every point of the open convex set $D \subset X$,
(iii) the bilinear form $\langle dN(u|h), k \rangle$ on $h, k \in X$ is continuous at every point of D.

Then a necessary and sufficient condition for N to be a potential operator in D is that

$$\langle dN(u|h), k \rangle = \langle dN(u|k), h \rangle \tag{3.1}$$

for every $h, k \in X$ and every $u \in D$. Moreover, if (3.1) *holds then $N = f'$ with*

$$f(u) = f(u_0) + \int_0^1 \langle N(u_0 + \nu(u - u_0)), u - u_0 \rangle \, d\nu \tag{3.2}$$

u_0 being any point of D.

For any *linear* operator L which is symmetric with respect to the chosen bilinear form Theorem 3.1 yields a functional f, given by (3.2), which is a *potential* for L ; that is, in such a case, the potentialness condition (3.1) reads

$$\langle Lu, w \rangle = \langle Lw, u \rangle. \tag{3.3}$$

The application of this theorem to a given differential problem requires, as a preliminary step, the choice of a bilinear form. Most often variational formulations are established by letting X be the Hilbert space $L^2(\mathbb{R}, L^2(\Omega))$, and $\langle \cdot, \cdot \rangle$ be the scalar product in that space. In connection with viscoelasticity, and more generally in the case of constitutive functionals expressed by convolutions, this choice is unsuccessful.

Since the bilinear form is required to be non-degenerate, an alternative choice for $\langle \cdot, \cdot \rangle$ in the case of space-time dependent functions could be the L^2 scalar product with respect to space and the convolution with respect to time, as made in [12]. This approach leads to variational formulations with minimum properties in dynamical viscoelasticity [2][3][4] but with saddle-point property in the quasi-static approximation [6].

In order to reach a minimum principle for QSP, in the next section we shall introduce a "weighted" L^2 scalar product with respect to space and time . It is worth noting that although this choice leads to a non-degenerate bilinear form which is non symmetric, nevertheless the potentialness condition (3.3) is satisfied when L is the QSP operator.

4. Minimum principle

For any pair $(\mathbf{p},\mathbf{q})$ of vector- or scalar-valued functions on $\Omega \times \mathbb{R}$ we introduce the following bilinear form

$$\begin{aligned}\langle \mathbf{p},\mathbf{q}\rangle_\Gamma &\overset{def}{=} \int_{\mathbb{R}}\int_\Omega \mathbf{p}(x,t)\cdot\big(\Gamma * \mathbf{q}\big)(x,t)\,dx\,dt \\ &= \int_{\mathbb{R}}\int_0^\infty\int_\Omega \mathbf{p}(x,t)\cdot\big[G_0\mathbf{q}(x,t) + G'(t-s)\mathbf{q}(x,s)\big]\,dx\,ds\,dt. \end{aligned} \tag{4.1}$$

where Γ is defined by (1.5).

Thanks to (1.2), this bilinear form is well-defined on $L^2(\mathbb{R},L^2(\Omega))$ but it is easily seen to be non symmetric. Nevertheless, by virtue of the thermodynamic restriction (2.3), it is non-degenerate: *i.e.* if $\langle \mathbf{p},\mathbf{q}\rangle_\Gamma = 0$ for every $\mathbf{q}\in L^2(\mathbb{R},L^2(\Omega))$, then $\mathbf{p}=0$ a.e.. In order to prove this property, we apply Plancherel's formula to (4.1) and we obtain

$$\langle \mathbf{p},\mathbf{q}\rangle_\Gamma = \frac{1}{2\pi}\int_{\mathbb{R}}\int_\Omega \widehat{\Gamma}(\omega)\widehat{\mathbf{p}}(x,\omega)\cdot\widehat{\mathbf{q}}^*(x,\omega)\,dx\,d\omega \tag{4.2}$$

where z^* denotes the complex conjugate of z. Hence, if

$$\langle \mathbf{p},\mathbf{q}\rangle_\Gamma = 0 \qquad\qquad \forall \mathbf{q}\in L^2(\mathbb{R},L^2(\Omega)) \tag{4.3}$$

then we have

$$0 = 2\int_0^\infty\int_\Omega \Big\{\mathrm{Re}\big[\widehat{\Gamma}(\omega)\hat{\mathbf{p}}(x,\omega)\big]\cdot\hat{\mathbf{q}}^+(x,\omega) - \mathrm{Im}\big[\widehat{\Gamma}(\omega)\hat{\mathbf{p}}(x,\omega)\big]\cdot i\hat{\mathbf{q}}^-(x,\omega)\Big\}\,dx\,d\omega$$

where $\mathbf{q}^+$ and $\mathbf{q}^-$ are respectively the even and odd parts of $\mathbf{q}$, *i.e.*

$$\mathbf{q}^+(t) = \frac{1}{2}\big(\mathbf{q}(t)+\mathbf{q}(-t)\big) \qquad , \qquad \mathbf{q}^-(t) = \frac{1}{2}\big(\mathbf{q}(t)-\mathbf{q}(-t)\big),$$

so that the following relations hold

$$\mathrm{Re}\;\hat{\mathbf{q}}^*(\omega) = \hat{\mathbf{q}}^+(\omega) \qquad , \qquad \mathrm{Im}\;\hat{\mathbf{q}}^*(\omega) = i\hat{\mathbf{q}}^-(\omega).$$

Because of the bijectivity of the Fourier transform from L^2 into L^2, $\hat{\mathbf{q}}^+$ and $i\hat{\mathbf{q}}^-$ are arbitrary and independent functions of $L^2(\mathbb{R}^+, L^2(\Omega))$ when $\mathbf{q}$ runs over $L^2(\mathbb{R}, L^2(\Omega))$. Therefore (4.3) leads to

$$\begin{aligned} \mathrm{Re}\ \left[\widehat{\Gamma}(\omega)\hat{\mathbf{p}}(x,\omega)\right] &= 0 \\ \mathrm{Im}\ \left[\widehat{\Gamma}(\omega)\hat{\mathbf{p}}(x,\omega)\right] &= 0 \end{aligned}$$

almost everywhere on $\Omega \times \mathbb{R}$ in the sense of $L^2(\mathbb{R}, L^2(\Omega))$, that is

$$\int_{\mathbb{R}} \int_{\Omega} \left|\widehat{\Gamma}(\omega)\right|^2 |\hat{\mathbf{p}}(x,\omega)|^2 \, dx\, d\omega = 0. \tag{4.4}$$

Now, since $\widehat{\Gamma}(\omega) = G_0 + \widehat{G}'(\omega)$ is a continuous and bounded complex-valued scalar function that does not vanish for any $\omega \in \mathbb{R}$ by virtue of (2.4), it must be $\hat{\mathbf{p}} = 0$ and then $\mathbf{p} = 0$ in $L^2(\mathbb{R}, L^2(\Omega))$.

Remark. *The inequality* (2.4) *implies that*

$$|||\mathbf{f}|||_\Gamma = \int_{\mathbb{R}} \int_{\Omega} \left|\widehat{\Gamma}(\omega)\right|^2 |\hat{\mathbf{f}}(x,\omega)|^2 \, dx\, d\omega$$

is a norm in $L^2(\mathbb{R}, L^2(\Omega))$ *equivalent to the natural one.*

Finally, to apply Theorem 3.1 we have to prove the potentialness condition (3.3) for the problem (1.6) with respect to $\langle\cdot,\cdot\rangle_\Gamma$, which reads

$$\langle \Gamma * \Delta u, w\rangle_\Gamma = \langle \Gamma * \Delta w, u\rangle_\Gamma \qquad \text{for any } u, w \text{ in } L^2(\mathbb{R}, H^1_0(\Omega)).$$

Denoting by $(\cdot,\cdot)$ the usual inner product on $L^2(\mathbb{R}, L^2(\Omega))$, we have

$$\langle \Gamma * \Delta u, w\rangle_\Gamma = (\Gamma * \Delta u, \Gamma * w). \tag{4.5}$$

Then, using the fact that Γ is independent of x and integrating by parts with respect to the space variables, we find

$$(\Gamma * \Delta u, \Gamma * w) = (\Gamma * u, \Gamma * \Delta w) = (\Gamma * \Delta w, \Gamma * u) = \langle \Gamma * \Delta w, u\rangle_\Gamma$$

which leads to the required result.

The straightforward application of Theorem 3.1 gives

Theorem 4.1. *If the relaxation function G satisfies conditions* (1.2),(1.3) *and* (2.3) *then u is a strict solution to QSP if and only if it is a strict minimum on* $L^2(\mathbb{R}, H^1_0(\Omega))$ *of the functional*

$$\Phi(u) = \frac{1}{2}\langle \Gamma * \nabla u, \nabla u\rangle_\Gamma - \langle f, u\rangle_\Gamma. \tag{4.6}$$

The minimum property follows easily from the convexity of Φ with respect to the usual L^2 norm, in fact

$$\Phi(u) = \frac{1}{2}\big(\Gamma * \nabla u, \Gamma * \nabla u\big) - \big(u, \Gamma * f\big). \tag{4.6}$$

It is worth remarking that, in this case, thermodynamic restrictions are intimately related to the non degeneracy of the chosen bilinear form, instead of convexity of the functional Φ.

5. Other applications

It is easily seen that the potentialness condition (3.3) for the problem (1.6) is not satisfied if the material is not homogeneous or not isotropic. What is more, condition (3.3) fails for three-dimensional displacements even in the homogeneous and isotropic case.

The previous technique, however, applies successfully to other quasi-static problems in mathematical physics when constitutive equations like (1.1) are concerned. For example, we take under consideration a homogeneous and isotropic rigid heat conductor with linear memory occupying a fixed bounded domain $\Omega \subset \mathbb{R}^3$.

If we consider only small variations of the temperature $\theta(x,t)$ from a reference temperature θ_0 and small temperature gradients $\mathbf{g}(x,t)$, we may suppose that internal energy $\varepsilon(x,t)$ and heat flux $\mathbf{q}(x,t)$ are described by the following linearized constitutive equations (see [14]):

$$\varepsilon(x,t) = \varepsilon_0 + \alpha_0 u(x,t) + \int_0^\infty \alpha'(s) u(x,t-s)\, ds, \tag{5.1}$$

$$\mathbf{q}(x,t) = -k_0 \nabla u(x,t) + \int_0^\infty k'(s) \nabla u(x,t-s)\, ds \tag{5.2}$$

where $u(x,t) = \theta(x,t) - \theta_0$ and $\alpha', k' \in L^1(\mathbb{R}^+)$.

The ensuing evolution problem, with homogeneous Dirichlet boundary conditions, is given by

$$\begin{cases} \alpha_0 u_t(x,t) + \int_0^\infty \alpha'(s) u_t(x,t-s)\, ds - k_0 \Delta u(x,t) - \int_0^\infty k'(s) \Delta u(x,t-s)\, ds = r(x,t) \\ u|_{\partial\Omega} = 0 \end{cases}$$

so that in the quasi-static approximation we have

$$\begin{cases} k_0 \Delta u(x,t) + \int_0^\infty k'(s) \Delta u(x,t-s)\, ds + r(x,t) = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.3}$$

or, in a more compact form,

$$\begin{cases} \Gamma * \Delta u + r = 0 \\ \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.4}$$

where

$$\Gamma \stackrel{def}{=} K_0 \delta + k'. \tag{5.5}$$

The only considerable difference between (5.4) and (1.6) is that concerning thermodynamic restrictions. In fact, as proved by G.Gentili (see [14]), in a rigid heat conductor satisfying (5.2) the Second Law of Thermodynamics holds if and only if

$$k_0 + \hat{k}'_c(\omega) > 0 \qquad\qquad \forall \omega \in \mathbb{R}. \tag{5.6}$$

Nevertheless, it is easily seen that the bilinear form

$$\langle \mathbf{p}, \mathbf{q} \rangle_\Gamma \stackrel{def}{=} \int_{\mathbb{R}} \int_\Omega \mathbf{p}(x,t) \cdot \big(\Gamma * \mathbf{q}\big)(x,t)\, dx\, dt \tag{5.7}$$

is non-degenerate since (5.6) implies

$$\big|\widehat{\Gamma}(\omega)\big| \geq \mathrm{Re}\, \widehat{\Gamma}(\omega) = k_0 + \hat{k}'_c(\omega) > 0 \qquad \forall \omega \in \mathbb{R}.$$

Moreover, if $k_0 \neq 0$ (and then $k_0 > 0$ by virtue of (5.6)), the norm

$$|||\mathbf{f}|||^2_\Gamma = \int_{\mathbb{R}} \int_\Omega \big|\widehat{\Gamma}(\omega)\big|^2 \big|\hat{\mathbf{f}}(x,t)\big|^2\, dx\, d\omega$$

is equivalent to the natural one in $L^2(\mathbb{R}, L^2(\Omega))$.

Hence, by using the same procedure as in the previous section, a minimum principle for the quasi-static problem (5.4) in heat conduction with memory can be established through the functional

$$\Psi(u) = \frac{1}{2} \langle \Gamma * \nabla u, \nabla u \rangle_\Gamma - \langle r, u \rangle_\Gamma.$$

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