Proceeding of the 1987 Melecon Conference, Rome, March 25-27, 1987.

ADAPTIVE LEAST SQUARE APPROXIMATION OF SIGNALS

Riccardo LEONARDI
Signal Processing Laboratory
Swiss Federal Institute of Technology
16, ch. de Bellerive - CH-1007 LAUSANNE
(Switzerland)

Abstract:

The purpose of this paper is to present an adaptive algorithm to find the best approximation in the least square sense of a given signal.

The proposed method takes advantage of the fact that the least square approximation of a given signal over a chosen domain D can be directly obtained from the corresponding optimal least square approximations of this signal over any set of domains that constitutes a partition of D. The approximation characteristics and a parameter that takes into account of the relative position and geometry of these domains are sufficient to provide the overall best approximation over D. This property is shown to be independent of the basis functions used in the approximation. It is also shown how the total least square error can be obtained from the least square errors that define a partition of D.

The paper presents as well the computational efficiency of the algorithm. As an example, an application in the context of image segmentation is presented.

I) Introduction:

Least square approximation (LSA) has been widely applied in Science for a great variety of problems. Even if its use has been sometimes criticized in the context of some specific application such as Image Processing, its computational efficiency, the good adequacy that exists between most of the original data and the approximated ones still makes it popular. Besides, no other approximation criterion has been shown to be more adapted to such a great number of applications.

Various properties of the LSA can be demonstrated. In this paper, we shall present how to obtain the LSA of a signal over a domain D from any set of regions that defines a partition of D. This property will be used to obtain an adaptive approximation of the original data. As an example, the strategy is then applied to segment images.

In section II, the basis of LSA is summarized. Section III demonstrates the above mentionned property. Section IV discusses how to use it to make the approximation adaptive. Finally, section V presents an adaptive split-and-merge algorithm coupled with a least square approximation by means of 2-D polynomial functions to segment images.

II) Least Square Approximation:

Let us consider a digital signal $x(k)$ $(k=1,\ldots,N)$ to be approximated by a set of r complex valued functions $\Psi_i(t)$. $k=1,\ldots,N$ defines a discrete interval I. The approximated signal $x_a(k)$ can be expressed by:

$$x_a(k) = \sum_{i=1}^{r} u_i \Psi_i(k) \tag{1}$$

or using a vectorial notation with $\mathbf{u}^T=[u_1\, u_2 \ldots u_r]$, $\Psi(k)^T=[\Psi_1(k)\, \Psi_2(k) \ldots \Psi_r(k)]$,

$$x_a(k) = \mathbf{u}^T \Psi(k) \tag{2}$$

u_i define the r parameters of the approximation. The optimal LSA is obtained by minimizing the square error:

$$E^2 = \sum_{k=1}^{N} [x(k) - x_a(k)]^2 \tag{3}$$

or using a vectorial notation with $\mathbf{x}^T=[x(1)\ x(2) \ldots x(N)]$ and $\mathbf{x_a}^T=[x_a(1)\ x_a(2) \ldots x_a(N)]$:

$$\begin{aligned} E^2 &= [\mathbf{x}-\mathbf{x_a}]^T[\mathbf{x}-\mathbf{x_a}] \\ &= [\mathbf{x}-\mathbf{Zu}]^T[\mathbf{x}-\mathbf{Zu}] \end{aligned} \tag{4}$$

where the rxN transpose matrix of $\mathbf{Z}$ is defined by $\mathbf{Z}^T = [\Psi(1)\ \Psi(2) \ldots \Psi(N)]$.

The optimal solution is given by setting to 0 the derivative of E^2 with respect to **u**, which is equivalent to solve the linear equation system [1]:

$$(Z^TZ).u = Z^T \cdot x \tag{5}$$

(Z^TZ) defines an rxr symmetric square matrix that will be called for simplicity **S**. Similarly, vector $Z^T \cdot x$ will be called **h**.

In the most general case, the computation of (5) requires :

1) the evaluation of vectors $\Psi(k)$ (k=1,2,...,N).

2) Nr(r-1)/2 multiplications and (N-1)r(r-1)/2 additions to compute the rxr matrix **S**. O(Nr^2) operations will be considered as sufficient.

3) the matrix vector multiplication to estimate vector **h**, which requires O(Nr) operations.

4) Solving the equation **S.u = h** as long as S is regular, which corresponds to O(r^3) operations [2]..

Therefore, it can be said that step 1 to 3 represent the most important cost as in general N»r.

The property that will be proved in the next section will reduce the computation of the LSA to step 4 if the LSA is known for a set of intervals that define a partition of I.

III) Combination of LSA:

The LSA criterion has the following interesting property: *Given a domain D, the optimal LSA over this domain can be obtained from the LSA over the domains that define a partition of D.*

To prove this, let us consider a one-dimensional discrete interval I that is composed from two disjoint intervals I_1 and I_2. I_1 and I_2 can be separated by a certain number of points. Using a similar notation to that of the previous section, **u**, $\mathbf{u_1}$ and $\mathbf{u_2}$ define the optimal LSA over I, I_1 and I_2, respectively if the same set of approximating functions $\Psi_i(t)$ (i=1,...,r) is considered. In the same manner, we define as N, N_1 and N_2, the number of data within each interval. Naturally, the relation $N=N_1+N_2$ holds. Vector **x** of size N is simply obtained by concatenating vectors $\mathbf{x_1}$ of size N_1 and $\mathbf{x_2}$ of size N_2, relative to each interval I_1 and I_2. The corresponding approximated vectors will be represented by $\mathbf{x_a}$, $\mathbf{x_{1a}}$ and $\mathbf{x_{2a}}$ when the LSA is estimated on I, I_1 and I_2. Similarly to section II, we define with respect to each interval matrices **Z** of size Nxr, $\mathbf{Z_1}$ of size N_1xr, and $\mathbf{Z_2}$ of size N_2xr; rxr square matrices **S**, $\mathbf{S_1}$, and $\mathbf{S_2}$ as well as vectors **h**, $\mathbf{h_1}$ and $\mathbf{h_2}$.

The following equations hold:

$$(Z^TZ).u = Z^T \cdot x = h \tag{6}$$
$$(Z_1{}^TZ_1).u_1 = Z_1{}^T \cdot x_1 = h_1 \tag{7}$$
$$(Z_2{}^TZ_2).u_2 = Z_2{}^T \cdot x_2 = h_2 \tag{8}$$

Solving (6) corresponds to minimize the square error E^2 expressed by (3) within the interval I:

$$E^2 = [x - x_a]^T[x - x_a] = [x - Zu]^T[x - Zu] \tag{9}$$

This error can also be estimated separately for each interval:

$$E^2 = [x_1 - Z_1u]^T[x_1 - Z_1u] + [x_2 - Z_2u]^T[x_2 - Z_2u] \tag{10}$$

Note that in this case the optimal solution for the contribution of the approximation on each interval is represented by **u**. If we differentiate (10) with respect to **u**, the optimal solution is obtained by setting this derivative to 0 which gives the linear equation:

$$-2\, Z_1{}^T[x_1 - Z_1u] - 2\, Z_2{}^T[x_2 - Z_2u] = 0$$

$$(Z_1{}^TZ_1 + Z_2{}^TZ_2).u = Z_1{}^Tx_1 + Z_2{}^Tx_2 \tag{11}$$

Every term of this equation is already known as it can be seen by refering to equations (7) and (8). The optimal LSA over I can be simply written as:

$$u = (S_1 + S_2)^{-1}.\,(S_1u_1 + S_2u_2) \tag{12}$$

This equation holds for any number of intervals that define a partition of I, as all the previous equations remain valid if more disjoint intervals are considered to make a partition of I. The property can be applied independently of the functions $\Psi_i(t)$ used as long as these do not vary with respect to u_i.

It can also be generalized to m-dimensional signals. These are then approximated using a set of r m-dimensional functions. Every sample is put into a vector and intervals become domains of an m-dimensional space.

It can be seen from equation (12), that storing matrices $\mathbf{S_1}$ and $\mathbf{S_2}$ that are relevant of the respective position and geometry of both intervals, as well as the parameters of the approximation $\mathbf{u_1}$ and $\mathbf{u_2}$ allows to get the desired LSA. The computation is then reduced to solve a linear system of r equations involving just O(r^3) operations.

Using the same formalism as above, it can be hown that the least square error over I can be xpressed using the least square errors over I_1 and I_2 that will be denoted by E_1^2 and E_2^2, respectively. We have:

$$E^2 = E_1^2 + E_2^2 + \mathbf{u_1}^T\mathbf{S_1}\mathbf{u_1} + \mathbf{u_2}^T\mathbf{S_2}\mathbf{u_2} - \mathbf{u}^T\mathbf{S}\mathbf{u} \tag{13}$$

The property that has just been pointed out nay find useful applications. It can serve to estimate how the LSA evolve when one is considering an increasing number of data. Another application will be presented in the next section.

IV) Adaptive LSA of Signals

It is often difficult to approximate a large number of data with a small set of approximating functions, especially if these data correspond to non-stationary signals. By changing the parameter values, the approximation may fit harmoniously part of the data.

The idea is to use an adaptive strategy to find parts of the signal that can be well represented by a least square approximation using a set of functions Ψ_i. The approximated signal corresponds to a conjunction of intervals over which the original data are approximated in the least square sense.

One starts to decompose the original N samples of the signal into a set of sequences of samples. Each sequence is defined over a certain domain D. The concatenation of all sequences gives the original signal. On each domain D, the LSA is then estimated. If the parameters of the LSA are similar for two different domains, these are merged into one and the optimal LSA over this new domain is evaluated using (12). The process will go on as long as there exist similar domains or when a certain number of domains is reached.

The way of decomposing initially the original ignal sets the name of the adaptive algorithm. If he signal is subdivided initially into equal length equences, the process is called *region growing*; if it s cut into pieces with respect to some homogeneity riterion before starting the adaptive merge process, he overall approximation strategy will be called *plit-and-merge*. In the next section, this approach vill be used to adaptively segment images.

V) Example: Image Segmentation:

The proposed method is similar to the split-and-merge algorithm suggested by Horowitz and Pavlidis [3]. It is made more powerfull by combining it with a LSA approximation by means of 2-D polynomial functions.

In the split process, the original image is divided iteratively into a set of squares of different sizes. A square is divided into four identical subsquares whenever some suitable error measure between the approximated image in the least square sense within the square and the original image exceeds a certain threshold.

In the merge process, the various squares are associated on the basis of some similarity measure. During the whole segmentation, distorsions are measured with respect to the original image. Before merging any two neighbouring regions, the analysis is performed on all possible couple of two contiguous regions. Those being the most similar will be merged. The process will stop whenever a certain number of regions is reached or as soon as a further merging will cause an unacceptable distorsion to the original image.

Figure 1 shows an image representing a couple. Figure 2 gives the relative approximated image with 70 regions represented by 3rd order 2-D polynomial functions (10 coefficients). The corresponding segmentation is illustrated by figure 3.

Figure 1: Original couple picture.

Figure 2: Approximated picture by 3rd order polynomial functions with 70 regions.

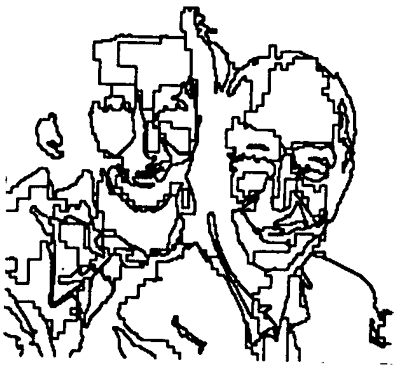

Figure 3: Segmentation shape.

V) Conclusion:

The present paper has shown that LSA over certain domain can be obtained from the LSA ove any set of domains that define a partition of th previous one.

Using this property, it was possible to derive a adaptive algorithm for approximating signals. A application of it to image segmentation wa presented in section V.

It is our concern to seek new areas in which th above mentioned property would find valuabl applications

References:

[1] P.J. Davis, Interpolation and Approximation Blaisdell Publishing Company (1963).

[2] T.K. Sarkar, K.R. Siarkiewicz and R.F. Stratton. *Survey of Numerical Methods for Solution of Large Systems of Linear Equations for Electromagnetic Field Problems*, IEEE Trans. on Antennas and Propagation, Vol. AP-29, No.6, pp. 847-856 (Nov. 1981).

[3] S.L. Horowitz and T. Pavlidis, *Picture Segmentation Using a Tree Traversal Algorithm*, Journal of the Association for Computing Machinery, Vol. 23, pp. 368-388 (1976).