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On the equivalence of certain quasi-Hermitian varieties

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Abstract

By Aguglia et al., new quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ in $\mathrm{PG}(r, q^2)$ depending on a pair of parameters α, β from the underlying field $\mathrm{GF}(q^2)$ have been constructed. In the present paper we study the structure of the lines contained in $\mathcal{M}_{\alpha,\beta}$ and consequently determine the projective equivalence classes of such varieties for q odd and $r = 3$. As a byproduct, we also prove that the collinearity graph of $\mathcal{M}_{\alpha,\beta}$ is connected with diameter 3 for $q \equiv 1 \pmod 4$.

KEYWORDS

collineation, Hermitian variety, quasi-Hermitian variety

1 | INTRODUCTION

It is a well-known problem in finite geometry to characterize the absolute points of a polarity in terms of their combinatorial properties. In this line of investigation, one of the most celebrated results is Segre's Theorem stating that in a Desarguesian projective plane $\mathrm{PG}(2, q)$ of odd order q a set Ω which has the same number of points, namely, $q + 1$, and the same intersections with lines as a conic (i.e., 0, 1, or 2) is indeed a conic; see [15].

As the dimension grows, the combinatorics of the intersection with subspaces turns out not to be enough as to characterize the absolute points of a polarity neither in the orthogonal case nor in the unitary one.

The set of the absolute points of a Hermitian polarity of $\mathrm{PG}(r, q^2)$ is a *nonsingular Hermitian variety*.

Quasi-Hermitian varieties of $\mathrm{PG}(r, q^2)$ are a generalization of nonsingular Hermitian varieties defined as follows. Let q be any prime power and assume $r \geq 2$; a *quasi-Hermitian variety* of $\mathrm{PG}(r, q^2)$ is a set of points having the same size and the same intersection numbers with hyperplanes as a nonsingular Hermitian variety $\mathcal{H}(r, q^2)$. In particular, the intersection numbers with hyperplanes of $\mathcal{H}(r, q^2)$ only take two values thus, quasi-Hermitian varieties are two-character sets; see [8, 9] for an overview of their applications. The Hermitian variety

$\mathcal{H}(r, q^2)$ can be viewed trivially as a quasi-Hermitian variety; as such it is called the *classical quasi-Hermitian variety of* PG(r, q^2).

For $r = 2$, a quasi-Hermitian variety of PG$(2, q^2)$ is called a *unital* or *Hermitian arc*. Nonclassical unitals have been extensively studied and characterized [6] and many constructions are known; see, for instance, [4]. As far as we know, the only known nonclassical quasi-Hermitian varieties of PG(r, q^2), $r \geq 3$ were constructed in [2, 3, 10, 14] and they are not isomorphic among themselves; see [14].

In [3], quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ of PG(r, q^2) with $r \geq 2$, depending on a pair of parameters α, β from the underlying field GF(q^2), were constructed. For $r = 2$ these varieties are Buekenhout–Metz (BM) unitals, see [5, 6, 11, 12]. As such, for $r \geq 3$ we shall call $\mathcal{M}_{\alpha,\beta}$ the *BM quasi-Hermitian varieties* of parameters α and β of PG(r, q^2).

The number of projectively inequivalent BM unitals in PG$(2, q^2)$ has been computed in [5] for q odd and in [11] for q even. In the present paper we shall enumerate the BM quasi-Hermitian varieties in PG$(3, q^2)$ with q odd and show that they behave under this respect in a similar way as BM unitals in PG$(2, q^2)$. Our long-term aim is to try to find a characterization of the BM quasi-Hermitian varieties among all possible quasi-Hermitian varieties in spaces of the same dimension and order.

Apart from the Introduction, this paper is organized into four sections. In Section 2 we describe the construction of the BM quasi-Hermitian varieties in PG$(3, q^2)$ whereas in Section 3 we determine the number of lines of PG$(3, q^2)$, q odd, through a point of $\mathcal{M}_{\alpha,\beta}$ which are entirely contained in $\mathcal{M}_{\alpha,\beta}$. By using this result in Section 4, we prove that the collinearity graph of $\mathcal{M}_{\alpha,\beta}$ is connected for $q \equiv 1 \pmod 4$ (which is the only interesting case, as for $q \equiv 3 \pmod 4$ the only lines contained in $\mathcal{M}_{\alpha,\beta}$ are those of a pencil of $(q+1)$-lines, all contained in a plane). Finally, in Section 5, we prove our main result:

Theorem 1.1. *Let $q = p^n$ with p an odd prime. Then the number N of projectively inequivalent quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ of* PG$(3, q^2)$ *is*

$$N = \frac{1}{n}\left(\sum_{k|n} \Phi\left(\frac{n}{k}\right) p^k\right) - 2,$$

where Φ is the Euler Φ-function.

As a byproduct of our arguments, we also obtain a simple way to determine when two quasi-Hermitian varieties are equivalent, see Lemmas 5.5 and 5.6 for the details.

2 | PRELIMINARIES

In this section we recall the construction of the BM quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ of PG$(3, q^2)$ described in [3].

Fix a projective frame in PG$(3, q^2)$ with homogeneous coordinates (J, X, Y, Z), and consider the affine space AG$(3, q^2)$ with infinite hyperplane Σ_∞ of equation $J = 0$. Then, the affine coordinates for points of AG$(3, q^2)$ are denoted by (x, y, z), where $x = X/J$, $y = Y/J$, and $z = Z/J$. Set

$$\mathcal{F} = \{(0, X, Y, Z) : X^{q+1} + Y^{q+1} = 0\};$$

this can be viewed as a Hermitian cone of $\Sigma_\infty \cong \mathrm{PG}(2, q^2)$ projecting a Hermitian variety of $\mathrm{PG}(1, q^2)$. Now take $\alpha \in \mathrm{GF}(q^2)^*$ and $\beta \in \mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ and consider the algebraic variety $\mathcal{B}_{\alpha,\beta}$ of projective equation

$$\begin{aligned}\mathcal{B}_{\alpha,\beta}:\ & Z^qJ^q - ZJ^{2q-1} + \alpha^q(X^{2q} + Y^{2q}) - \alpha(X^2 + Y^2)J^{2q-2} \\ & = (\beta^q - \beta)(X^{q+1} + Y^{q+1})J^{q-1}.\end{aligned} \tag{1}$$

We observe that

- $\mathcal{B}_\infty := \mathcal{B}_{\alpha,\beta} \cap \Sigma_\infty$ is the union of two lines $\ell_1 : X - \nu Y = 0 = J$ and $\ell_2 : X + \nu Y = 0 = J$, with $\nu \in \mathrm{GF}(q^2)$ such that $\nu^2 + 1 = 0$ if q is odd.
- Let $P_\infty := (0, 0, 0, 1)$. Then, $\ell_1 \cap \ell_2 = P_\infty$.
- $\mathcal{B}_\infty \subseteq \mathcal{F}$ if $q \equiv 1 \pmod 4$ or q is even.

It is shown in [3] that the point set

$$\mathcal{M}_{\alpha,\beta} := (\mathcal{B}_{\alpha,\beta}\backslash\Sigma_\infty) \cup \mathcal{F}, \tag{2}$$

that is, the union of the affine points of $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{F}$, is a quasi-Hermitian variety of $\mathrm{PG}(3, q^2)$ for any $q > 2$ even or for q odd and $4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0$. This is the variety we shall consider in the present paper limited to the case in which q is odd.

We stress that (1) is not the equation of $\mathcal{M}_{\alpha,\beta}$. However, any set of points in a finite projective space can be endowed of the structure of an algebraic variety, so we shall speak of the *variety* $\mathcal{M}_{\alpha,\beta}$ even if we do not provide an equation for it.

3 | COMBINATORIAL PROPERTIES OF $\mathcal{M}_{\alpha,\beta}$

We first determine the number of lines passing through each point of $\mathcal{M}_{\alpha,\beta}$ of $\mathrm{PG}(3, q^2)$, for q odd. We recall the following (see [13, Corollary 1.24]).

Lemma 3.1. *Let q be an odd prime power. The equation*

$$X^q + aX + b = 0$$

admits exactly one solution in $\mathrm{GF}(q^2)$ if and only if $a^{q+1} \neq 1$. When $a^{q+1} = 1$, the aforementioned equation has either q solutions when $b^q = a^q b$ or no solution when $b^q \neq a^q b$.

Lemma 3.2. *Let $\mathcal{B}_{\alpha,\beta}$ be the projective variety of Equation (1) and $\mathcal{B}_\infty$ be the intersection of the variety $\mathcal{B}_{\alpha,\beta}$ with the hyperplane at infinity $\Sigma_\infty : J = 0$ of $\mathrm{PG}(3, q^2)$.*

- *If $q \equiv 1 \pmod 4$, then, for any affine point L of $\mathcal{B}_{\alpha,\beta}$ there are exactly two lines contained in $\mathcal{B}_{\alpha,\beta}$ through L; for any point $L_\infty \in \mathcal{B}_\infty$ with $L_\infty \neq P_\infty$, there are $q + 1$ lines of a pencil through L_∞ contained in $\mathcal{B}_{\alpha,\beta}$. If $q \equiv 3 \pmod 4$ then no line of $\mathcal{B}_{\alpha,\beta}$ passes through any affine point of $\mathcal{B}_{\alpha,\beta}$ whereas through a point at infinity of $\mathcal{B}_\infty$ different from P_∞ there pass only one line contained in $\mathcal{B}_{\alpha,\beta}$.*

- There are exactly two lines of $\mathcal{B}_{\alpha,\beta}$ through P_∞ for all odd q.

Proof. Let ℓ be a line of $\mathrm{PG}(3, q^2)$ passing through an affine point of $\mathcal{B}_{\alpha,\beta}$. The affine points $P(x, y, z)$ of $\mathcal{B}_{\alpha,\beta}$ satisfy the equation:

$$\mathcal{B}_{\alpha,\beta}:\ z^q - z + \alpha^q(x^{2q} + y^{2q}) - \alpha(x^2 + y^2) = (\beta^q - \beta)(x^{q+1} + y^{q+1}).$$

From [3, Section 4], it can be directly seen that the collineation group of $\mathcal{B}_{\alpha,\beta}$ acts transitively on its affine points. Thus, we can assume that ℓ passes through the origin $O = (1, 0, 0, 0)$ of the fixed frame and hence it has affine parametric equations:

$$\begin{cases} x = m_1 t, \\ y = m_2 t, \\ z = m_3 t \end{cases}$$

with t ranging over $\mathrm{GF}(q^2)$. We study the following system:

$$\begin{cases} z^q - z + \alpha^q(x^{2q} + y^{2q}) - \alpha(x^2 + y^2) = (\beta^q - \beta)(x^{q+1} + y^{q+1}), \\ x = m_1 t, \\ y = m_2 t, \\ z = m_3 t. \end{cases} \tag{3}$$

As proved in [1, Theorem 4.3] ℓ can be contained in $\mathcal{B}_{\alpha,\beta}$ only if $m_3 = 0$. Thus assume $m_3 = 0$ and replace the parametric values of (x, y, z) in the first equation of (3). We obtain that

$$\left(t^2\alpha\left(m_1^2 + m_2^2\right)\right)^q - t^2\alpha\left(m_1^2 + m_2^2\right) = t^{q+1}(\beta^q - \beta)\left(m_1^{q+1} + m_2^{q+1}\right) \tag{4}$$

must hold for all $t \in \mathrm{GF}(q^2)$. Considering separately the cases $t \in \mathrm{GF}(q)$ and $t = \lambda$ with $\lambda \in \mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$ we obtain the following system, $\forall\ \lambda \in \mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$:

$$\begin{cases} \left(\alpha^q\left(m_1^2 + m_2^2\right)^q - \alpha\left(m_1^2 + m_2^2\right)\right) = (\beta^q - \beta)\left(m_1^{q+1} + m_2^{q+1}\right), \\ \lambda^{2q}\alpha^q\left(m_1^2 + m_2^2\right)^q - \lambda^2\alpha\left(m_1^2 + m_2^2\right) = \lambda^{q+1}(\beta^q - \beta)\left(m_1^{q+1} + m_2^{q+1}\right). \end{cases}$$

Replacing the first equation in the second, we get

$$\forall\ \lambda \in \mathrm{GF}(q^2)\setminus\mathrm{GF}(q): \lambda^{2q}\alpha^q\left(m_1^2 + m_2^2\right)^q(1 - \lambda^{1-q}) = \lambda^2\alpha\left(m_1^2 + m_2^2\right)(1 - \lambda^{q-1}).$$

Observe that $(1 - \lambda^{1-q}) = \frac{\lambda^{q-1} - 1}{\lambda^{q-1}}$. Suppose $m_1^2 + m_2^2 \neq 0$. Then,

$$\lambda^{2q-2}\alpha^{q-1}\left(m_1^2+m_2^2\right)^{q-1}=-\lambda^{q-1},$$

whence $\left(\lambda\alpha\left(m_1^2+m_2^2\right)\right)^{q-1}=-1$ for all $\lambda\in GF(q^2)\setminus GF(q)$. This is clearly not possible, as the equation $X^{q-1}=-1$ cannot have more than $q-1$ solutions. So $m_1^2+m_2^2=0$, which yields $m_2=\pm\nu m_1$ where $\nu^2=-1$. On the other hand, if $m_2=\pm\nu m_1$ and $q\equiv 1 \pmod 4$, then $m_1^{q+1}+m_2^{q+1}=m_1^{q+1}(1+\nu^{q+1})=0$, so (4) is satisfied and the lines $\ell: y-\nu x=z=0$ and $\ell: y+\nu x=z=0$ are contained in $\mathcal{B}_{\alpha,\beta}$. On the other hand, if $q\equiv 3 \pmod 4$, then $m_1^{q+1}+m_2^{q+1}=2m_1^{q+1}\neq 0$; so (4) is not satisfied and there is no line contained in $\mathcal{B}_{\alpha,\beta}$.

Now, take $L_\infty=(0,a,b,c)\in\mathcal{B}_\infty\setminus\{P_\infty\}$; hence $a^2+b^2=0$ and $a,b\neq 0$. Let r be a line through L_∞. We may assume that r has (affine) parametric equations

$$\begin{cases} x=l+at,\\ y=m+bt,\\ z=n+ct,\end{cases}$$

where t ranges over $GF(q^2)$. Assume also that (l,m,n) are the affine coordinates of a point in $\mathcal{B}_{\alpha,\beta}$, that is,

$$n^q-n+\alpha^q(m^{2q}+l^{2q})-\alpha(m^2+l^2)=(\beta^q-\beta)(l^{q+1}+m^{q+1}). \tag{5}$$

Now, r is contained in $\mathcal{B}_{\alpha,\beta}$ if and only if $q\equiv 1 \pmod 4$ and the following condition holds:

$$c+2\alpha(al+bm)+(\beta-\beta^q)(al^q+bm^q)=0. \tag{6}$$

Since $b=\nu a$ where $\nu^2=-1$ and $\nu\in GF(q)$, setting $k=l+\nu m$ Equation (6) becomes

$$c+2a\alpha k+a(\beta-\beta^q)k^q=0. \tag{7}$$

From Lemma 3.1, the above equation has exactly one solution if and only if

$$(2\alpha)^{q+1}\neq(\beta-\beta^q)^{q+1}. \tag{8}$$

Considering that $2\in GF(q)$ and $(\beta-\beta^q)^q=(\beta^q-\beta)$, we obtain that (8) is equivalent to

$$4\alpha^{q+1}+(\beta^q-\beta)^2\neq 0,$$

which holds true.

Let $\bar{k}$ be the unique solution of (7). Since $\bar{k}=l+\nu m$, we find q^2 pairs (l,m) satisfying (6). For any fixed pair (l,m), because of (5), there are q possible values of n. Thus we obtain that the number of affine lines through the point P and contained in $\mathcal{B}_{\alpha,\beta}$ is $q^2q/q^2=q$.

Furthermore, the $q+1$ lines through L_∞ lie on the plane of (affine) equation: $x+\nu y=\bar{k}$. The theorem follows. □

Lemma 3.3. *If $q \equiv 1 \pmod 4$ then, for any point $P \in \mathcal{B}_{\alpha,\beta} \setminus \mathcal{B}_\infty$ there are two lines $r_1(P)$ and $r_2(P)$ through P contained in $\mathcal{B}_{\alpha,\beta}$ such that $r_i(P) \cap (\ell_i \setminus \{P_\infty\}) \neq \emptyset$.*

Proof. As we already know, $\mathcal{B}_\infty$ is the union of two lines ℓ_1 and ℓ_2, through the point P_∞. By considering (4), we see that the point at infinity of the two lines through the origin $O \in \mathcal{B}_{\alpha,\beta}$ are one on ℓ_1 and one on ℓ_2. The semilinear automorphism group G of $\mathcal{B}_{\alpha,\beta}$ is transitive on its affine points, see [3] and maps lines into lines. Also, by Lemma 3.2 follows directly that G must fix the hyperplane at infinity. If $q \equiv 3 \pmod 4$, the point P_∞ is the only point of $\mathcal{B}_{\alpha,\beta}$ incident with two lines therein contained. So P_∞ must be stabilized by G. If $q \equiv 1 \pmod 4$, we see that P_∞ is the only point at infinity incident with just two lines of the variety, while the remaining points at infinity are incident with $q+1$ lines. So, again P_∞, which is the only point of $\mathcal{B}_\infty$ which is on no affine line of $\mathcal{B}_{\alpha,\beta}$, is fixed by G. It follows that for each affine point P we have that one of the lines intersects with ℓ_1 and the other with ℓ_2. □

Theorem 3.4. *Let $\mathcal{M}_{\alpha,\beta}$ be the BM quasi-Hermitian variety described in (2).*

- If $q \equiv 1 \pmod 4$ then through each affine point of $\mathcal{M}_{\alpha,\beta}$ there pass two lines of $\mathcal{M}_{\alpha,\beta}$ whereas through a point at infinity of $\mathcal{M}_{\alpha,\beta}$ on the union of the two lines $\ell_1 \cup \ell_2$ there pass $q+1$ lines of a pencil contained in $\mathcal{M}_{\alpha,\beta}$; finally through a point at infinity of $\mathcal{M}_{\alpha,\beta}$ which is not on $\ell_1 \cup \ell_2$ there passes only one line of $\mathcal{M}_{\alpha,\beta}$.
- If $q \equiv 3 \pmod 4$ then no line of $\mathcal{M}_{\alpha,\beta}$ passes through any affine point of $\mathcal{M}_{\alpha,\beta}$ whereas through a point at infinity of $(\mathcal{M}_{\alpha,\beta} \cap \Sigma_\infty) \setminus P_\infty$ there passes only one line contained in $\mathcal{M}_{\alpha,\beta}$.
- Through the point P_∞ there are always $q+1$ lines contained in $\mathcal{M}_{\alpha,\beta}$.

Proof. We observe that the affine points of $\mathcal{M}_{\alpha,\beta}$ are the same as those of $\mathcal{B}_{\alpha,\beta}$, whereas the set $\mathcal{F}$ of points at infinity of $\mathcal{M}_{\alpha,\beta}$ consists of the points $P = (0, x, y, z)$ such that $x^{q+1} + y^{q+1} = 0$. Furthermore, $\mathcal{B}_\infty = \ell_1 \cup \ell_2$ is contained in $\mathcal{F}$ if $q \equiv 1 \pmod 4$. Hence, from Lemma 3.2 we get the result. □

4 | CONNECTED GRAPHS FROM $\mathcal{M}_{\alpha,\beta}$ IN PG$(3, q^2)$, $q \equiv 1$ (*mod* 4)

Let $\mathcal{V}$ be an algebraic variety in PG$(n-1, q^2)$ or, more in general, just a set of points and suppose that $\mathcal{V}$ contains some projective lines. Then we can define the *collinearity graph* of $\mathcal{V}$, say $\Gamma(\mathcal{V}) = (\mathcal{P}, \mathcal{E})$ as the graph whose vertices $\mathcal{P}$ are the points of $\mathcal{V}$ and such that two points P and Q are collinear in $\Gamma(\mathcal{V})$ if and only if the line $[\langle P, Q\rangle]$ is contained in $\mathcal{V}$.

When $\mathcal{V}$ is a (nondegenerate) quadric or Hermitian variety, the graph $\Gamma(\mathcal{V})$ has a very rich structure for it is strongly regular and admits a large automorphism group; this has been widely investigated; see [7, Chapter 2, 16].

More in general, the properties of the graph $\Gamma(\mathcal{V})$ provide insight on the geometry of $\mathcal{V}$ since any automorphism of $\mathcal{V}$ is also naturally an automorphism of $\Gamma(\mathcal{V})$, even if the converse is not true in general.

Lemma 4.1. *Let $\mathcal{V}$ be an algebraic variety containing some lines and let $\mathcal{V}_\infty = \mathcal{V} \cap \Sigma_\infty$ where Σ_∞ is a hyperplane of $\mathrm{PG}(n-1, q^2)$. If the graph $\Gamma(\mathcal{V}_\infty)$ is connected and through each point of $\mathcal{V}$ there passes at least one line of $\mathcal{V}$ then the collinearity graph $\Gamma(\mathcal{V})$ is connected and its diameter $d(\Gamma(\mathcal{V}))$ is at most $d(\Gamma(\mathcal{V}_\infty)) + 2$.*

Proof. Each line of $\mathcal{V}$ has at least a point at infinity hence, given two points P and Q there exists a path from P to a point at infinity P' and from Q to another point at infinity Q' and finally a path consisting of points in $\mathcal{V}_\infty$ from P' to Q'. □

Let $\mathcal{M}_{\alpha,\beta}$ be as in (2).

Theorem 4.2. *If* $q \equiv 1 \pmod 4$, *then the graph* $\Gamma(\mathcal{M}_{\alpha,\beta})$ *is connected and its diameter is* 3.

Proof. We recall that $\mathcal{B}_{\alpha,\beta} \subseteq \mathcal{M}_{\alpha,\beta}$, $\mathcal{B}_{\alpha,\beta} \setminus \mathcal{B}_\infty = \mathcal{M}_{\alpha,\beta} \setminus \Sigma_\infty$ and that $\mathcal{B}_\infty$ splits in the union of the two distinct lines ℓ_1, ℓ_2 through P_∞. In particular, $\Gamma(\mathcal{B}_\infty)$ is a connected graph of diameter 2. Take now two points $P, Q \in \mathcal{B}_{\alpha,\beta}$. If $P, Q \in \mathcal{B}_\infty$, then we have $d(P, Q) \leq 2$ and there is nothing to prove. Suppose now $P \in \mathcal{B}_{\alpha,\beta} \setminus \mathcal{B}_\infty$ and $Q \in \mathcal{B}_\infty$. Suppose $Q \in \ell_i$. Then, from Lemma 3.3 we can consider a point $P' = r_i(P) \cap \ell_i$ where $r_i(P)$ is one of the two lines through P which is contained in $\mathcal{B}_{\alpha,\beta}$. If $P' = Q$, then $d(P, Q) = 1$; otherwise $d(P, Q) = 2$.

Take now $P, Q \in \mathcal{B}_{\alpha,\beta} \setminus \mathcal{B}_\infty$. Then, again from Lemma 3.3, the lines $r_1(P)$ and $r_1(Q)$ meet ℓ_1. Put $P' = r_1(P) \cap \ell_1$ and $Q' = r_1(Q) \cap \ell_1$. If $P' = Q'$, then $d(P, Q) \leq 2$; otherwise $d(P, Q) \leq 3$. We now show that there are pairs of points in $\mathcal{M}_{\alpha,\beta}$ which are at distance 3. Take $P \in \mathcal{B}_{\alpha,\beta} \setminus \mathcal{B}_\infty$ and $Q \in \mathcal{M}_{\alpha,\beta} \setminus \mathcal{B}_{\alpha,\beta}$. Then, Q is not collinear with any affine point by construction; also Q is not collinear with $P_i := \ell_i \cap r_i(P)$, $i = 1, 2$. So, the shortest paths from P to Q are of the form $PP_iP_\infty Q$. It follows that $d(P, Q) = 3$ and thus the diameter of the graph is 3. □

5 | MAIN RESULT

In this section we show that the arguments of [5] for classifying BM unitals in $\mathrm{PG}(2, q^2)$ can be extended to BM quasi-Hermitian varieties in $\mathrm{PG}(3, q^2)$, q odd. We keep all previous notations.

Two BM quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ of $\mathrm{PG}(3, q^2)$ are *projectively equivalent* if there exists a semilinear collineation $\psi \in \mathrm{P\Gamma}(4, q^2)$ such that $\psi(\mathcal{M}_{\alpha,\beta}) = \mathcal{M}_{\alpha',\beta'}$.

Lemma 5.1. *Let ψ be a semilinear collineation of $\mathrm{PG}(3, q^2)$, q odd, such that $\psi(\mathcal{M}_{\alpha,\beta}) = \mathcal{M}_{\alpha',\beta'}$ where $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ are two BM quasi-Hermitian varieties. Then ψ fixes P_∞ and stabilizes Σ_∞. Also, if* $q \equiv 1 \pmod 4$ *then* $\psi(\mathcal{B}_{\alpha,\beta}) = \mathcal{B}_{\alpha',\beta'}$.

Proof. First, we show that ψ fixes P_∞ for $q \equiv 3 \pmod 4$. From Theorem 3.4 we have that P_∞ is the only point of the two varieties contained in $q + 1$ lines and hence $\psi(P_\infty) = P_\infty$. Furthermore, we observe that Σ_∞ is the plane through P_∞ meeting both $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ in $q^3 + q^2 + 1$ points which are on the $q + 1$ lines through P_∞. All of the $q^3 - q^2$ points of $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ lying on exactly one line contained in the respective variety are in this plane, and these points also span Σ_∞. So also Σ_∞ is left invariant by ψ.

Now assume $q \equiv 1 \pmod 4$; from Theorem 3.4, for each point in $\ell_1 \cup \ell_2$ there pass $q+1$ lines of the quasi-Hermitian varieties however P_∞ is the only point on $\ell_1 \cup \ell_2$ such that the other $q-1$ lines through it are not incident with other lines of the two varieties, hence we again obtain $\psi(P_\infty) = P_\infty$. In this case $\mathcal{B}_{\alpha,\beta} \subseteq \mathcal{M}_{\alpha,\beta}$. Since $\psi(\Sigma_\infty) = \Sigma_\infty$, we have

$$\psi(\mathcal{B}_{\alpha,\beta} \backslash \Sigma_\infty) = \psi(\mathcal{M}_{\alpha,\beta} \backslash \Sigma_\infty) = \mathcal{M}_{\alpha',\beta'} \backslash \Sigma_\infty = \mathcal{B}_{\alpha',\beta'} \backslash \Sigma_\infty,$$

that is, ψ stabilizes the affine part of $\mathcal{B}_{\alpha,\beta}$.

Furthermore $\mathcal{B}_\infty = \mathcal{B}_{\alpha,\beta} \cap \Sigma_\infty$ consists of the union of the two lines, ℓ_1 and ℓ_2. Observe also that the lines through the affine points of $\mathcal{M}_{\alpha,\beta}$ are also lines of $\mathcal{B}_{\alpha,\beta}$ (see Theorem 3.4) and, in particular they are incident either ℓ_1 or ℓ_2. This is equivalent to say that the points of $\ell_1 \cup \ell_2$ different from P_∞ are exactly the points of Σ_∞ through which there pass some affine lines of $\mathcal{M}_{\alpha,\beta}$. This implies that $\psi(\ell_1 \cup \ell_2) = \ell_1 \cup \ell_2$ and, consequently

$$\psi(\mathcal{B}_{\alpha,\beta}) = \psi(\mathcal{B}_{\alpha,\beta} \backslash \Sigma_\infty) \cup \psi(\ell_1 \cup \ell_2) = (\mathcal{M}_{\alpha',\beta'} \backslash \Sigma_\infty) \cup (\ell_1 \cup \ell_2) = \mathcal{B}_{\alpha',\beta'}. \qquad \square$$

Theorem 5.2. *Suppose* $q \equiv 1 \pmod 4$. *Let* $\mathcal{G}$ *be the group of collineations* $\mathcal{G} = \mathrm{Aut}(\mathcal{M}_{\alpha,\beta}) \subseteq \mathrm{P\Gamma L}(4, q^2)$ *and* $\mathfrak{G}$ *the group of graph automorphisms* $\mathfrak{G} = \mathrm{Aut}(\Gamma(\mathcal{M}_{\alpha,\beta}))$. *Then the sets*

- $\Omega_0 := \{P_\infty\}$;
- Ω_1 consisting of the points at infinity of $\mathcal{B}_{\alpha,\beta}$ different from P_∞;
- $\Omega_2 := \mathcal{M}_{\alpha,\beta} \backslash \Sigma_\infty$

are all stabilized by both $\mathcal{G}$ *and* $\mathfrak{G}$. *Furthermore,* $\Omega_3 = \mathcal{M}_{\alpha,\beta} \backslash \mathcal{B}_{\alpha,\beta}$ *is an orbit for* $\mathfrak{G}$.

Proof. By [3, Section 4], we know that there is a subgroup of $\mathcal{G}$ which is transitive on the affine points of $\mathcal{M}_{\alpha,\beta}$, that is, on Ω_2. By Lemma 5.1, any collineation in $\mathcal{G}$ must stabilize the plane Σ_∞; so any element of $\mathcal{G}$ maps points of Ω_2 into points of Ω_2 and Ω_2 is an orbit of $\mathcal{G}$. Also by Lemma 5.1, $\Omega_0 := \{P_\infty\}$ is fixed by any $\gamma \in \mathcal{G}$. So we have that the points at infinity of $\mathcal{B}_{\alpha,\beta} \backslash \{P_\infty\}$, as well as the points of $\mathcal{M}_{\alpha,\beta} \backslash \mathcal{B}_{\alpha,\beta}$, are the union of orbits. Let ℓ_1, ℓ_2 be the two lines of $\mathcal{B}_{\alpha,\beta}$ at infinity. Using Lemma 3.3, we see that $\mathcal{G}$ is transitive on $\Omega_1 = (\ell_1 \cup \ell_2) \backslash \{P_\infty\}$. Indeed, for any two points $P, Q \in \ell_1 \backslash \{P_\infty\}$, by Lemma 3.2, there are points $P_0, Q_0 \in \Omega_2$ such that $r_1(P_0) \cap \Sigma_\infty = \{P\}$ and $r_1(Q_0) \cap \Sigma_\infty = \{Q\}$.

Since $\mathcal{G}$ is transitive on Ω_2, there is $\gamma \in \mathcal{G}$ such that $\gamma(P_0) = Q_0$. It follows that $\gamma((r_2(P_0) \cap \Sigma_\infty) \cup \{P\}) = (r_2(Q_0) \cap \Sigma_\infty) \cup \{Q\}$. If $\gamma(P) = Q$, then we are done. Otherwise, consider the element $\theta : (J, X, Y, Z) \to (J, X, -Y, Z)$ of $\mathcal{G}$. Observe that $\theta(r_2(Q_0)) \cap \Sigma_\infty = r_1(Q_0) \cap \Sigma_\infty$. Hence, $\theta\gamma(P) = Q$. Also, $\theta(\ell_1) = \ell_2$; so it follows that $\Omega_1 := (\ell_1 \cup \ell_2) \backslash \{P_\infty\}$ is an orbit of $\mathcal{G}$.

Since $\mathfrak{G}$ contains $\mathcal{G}$, the orbits of $\mathfrak{G}$ are possibly unions of orbits of $\mathcal{G}$. However, observe that the points of Ω_3 are the only points of $\mathcal{M}_{\alpha,\beta}$ which are on exactly one line of $\mathcal{M}_{\alpha,\beta}$ through the point P_∞. So these points must be permuted among each other also by $\mathfrak{G}$.

The same argument shows that Ω_0 is also an orbit for $\mathfrak{G}$. Now, consider the points of Ω_2. They are the points of $\mathcal{B}_{\alpha,\beta} \backslash \Omega_0$ incident with exactly two lines, while the points of Ω_1

are incident with more than two lines. So $\mathfrak{G}$ cannot map a vertex in Ω_2 into a vertex in Ω_1 and these orbits are distinct.

Put $\Gamma := \Gamma(\mathcal{M}_{\alpha,\beta})$. Observe that the graph $\Gamma \setminus \{P_\infty\}$ is the disjoint union of $\Gamma(\Omega_3)$ and $\Gamma(\Omega_1 \cup \Omega_2)$. In turn, $\Gamma(\Omega_3)$ consists of the disjoint union $K_1 \cup K_2 \cup \cdots \cup K_{q-1}$ of $q-1$ copies of the complete graph on q^2 elements. Write $\left\{v_i^j\right\}_{j=1,\dots,q^2}$ for the list of vertices of K_i with $i = 1, \dots, q-1$.

Also, each vertex of $\Gamma(\Omega_3 \cup \{P_\infty\})$ is collinear with P_∞. Let S_{q^2} be the symmetric group on q^2 elements, and consider its action on Γ given by

$$\forall\, \xi \in S_{q^2} : \check{\xi}\left(v_1^j\right) := v_1^{\xi(j)}$$

if $v_1^j \in K_1$ and fixing all remaining vertices. Obviously $\check{S}_{q^2} < \mathfrak{G}$ and $\check{S}_{q^2}$ is transitive on K_1. Let S_{q-1} be the symmetric group on $\{1, \dots, q-1\}$ and consider its action on Γ given by

$$\forall\, \sigma \in S_{q-1} : \hat{\sigma}\left(v_i^j\right) := v_{\sigma(i)}^j, j = 1, \dots, q^2$$

and all the remaining vertices of Γ are fixed. We also have $\hat{S}_{q-1} < \mathfrak{G}$ and $\hat{S}_{q-1}$ permutes the sets K_i for $i = 1, \dots, q-1$. By construction, we see that the wreath product $\check{S}_q \wr \hat{S}_{q-1}$ is a subgroup of $\mathfrak{G}$, it acts naturally on Γ, fixes all vertices not in Ω_3 and acts transitively on Ω_3. It follows that $\mathfrak{G}$ is transitive on Ω_3. □

Remark 5.3. It can be easily seen that the automorphism group of $\Gamma := \Gamma(\mathcal{M}_{\alpha,\beta})$ is in general much larger than the subgroup of collineations stabilizing $\mathcal{M}_{\alpha,\beta}$. In particular the elements of $\check{S}_q \wr \hat{S}_{q-1}$ are not, in general, collineations. For instance, in the case $q = 5$ with $\alpha = \beta = \varepsilon$ where ε is a primitive element of GF(25), root of $x^2 - x + 2$ in GF(5), the group $\mathcal{G}$ has order $2^6 5^5$, while $\mathfrak{G}$ has order $2^{99}3^{42}5^{30}7^{12}11^8 13^4 17^4 19^4 23^4$. In this case also $\mathcal{G}$ is transitive on Ω_3.

Lemma 5.4. *If $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ are two projectively equivalent BM quasi-Hermitian varieties then there is a semilinear collineation $\phi : \mathcal{M}_{\alpha,\beta} \to \mathcal{M}_{\alpha',\beta'}$ of the following type:*

$$\phi(j, x, y, z) = (j^\sigma, x^\sigma, y^\sigma, z^\sigma)M, \quad \text{where}$$
$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & -b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$, $a \in \mathrm{GF}(q) \setminus \{0\}$, $b, c \in \mathrm{GF}(q^2)$, $b^2 + c^2 \neq 0$ *and if* $b \neq 0 \neq c$ *then* $c = \lambda b$ *with* $\lambda \in \mathrm{GF}(q) \setminus \{0\}$ *such that* $\lambda^2 + 1 \neq 0$.

Proof. By Lemma 5.1, ϕ fixes the point P_∞ and stabilizes Σ_∞. As the automorphism group of $\mathcal{M}_{\alpha,\beta}$ is transitive on its affine points, we can also assume that $\phi(1,0,0,0) = (1,0,0,0)$.

More in detail, let G' be the collineation group of $\mathcal{M}_{\alpha',\beta'}$ fixing P_∞, leaving $\mathcal{F}\backslash P_\infty$ invariant and transitive on the affine points of $\mathcal{M}_{\alpha',\beta'}$. If $\phi(1,0,0,0) \neq (1,0,0,0)$ we can consider the collineation $\phi' \in G'$ mapping $\phi(1,0,0,0)$ to $(1,0,0,0)$ and then we replace ϕ by $\phi\phi'$. This implies that ϕ has the following form up to scalar multiple:

$$\phi(j,x,y,z) = (j^\sigma, x^\sigma, y^\sigma, z^\sigma)\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & d \\ 0 & e & f & g \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$, $a,b,c,d,e,f,g \in \mathrm{GF}(q^2)$ and $a \neq 0 \neq bf - ce$.

Since $(1,0,0,c)$, belongs to $\mathcal{M}_{\alpha,\beta}$ if and only if $c \in \mathrm{GF}(q)$, it follows that $\phi(1,0,0,c) = (a,0,0,c) \in \mathcal{M}_{\alpha',\beta'}$ implies $ca^{-1} \in \mathrm{GF}(q)$, and thus $a \in \mathrm{GF}(q)^*$. Now we observe that the affine plane $Y = 0$ has in common with $\mathcal{M}_{\alpha,\beta}$ the points $(1,x,0,z)$ for which $-\alpha x^2 + \beta x^{q+1} - z \in \mathrm{GF}(q)$; so, $a^{-1}(-\alpha^\sigma x^{2\sigma} + \beta^\sigma x^{\sigma(q+1)} - z^\sigma) \in \mathrm{GF}(q)$. Thus, suppose that $(1,x,0,z) \in \mathcal{M}_{\alpha,\beta}$; we have $\phi(1,x,0,z) \in \mathcal{M}_{\alpha',\beta'}$ and therefore

$$(\alpha^\sigma - \alpha'(b^2+c^2)/a)x^{2\sigma} - (\beta^\sigma - \beta'(b^{q+1}+c^{q+1})/a)x^{\sigma(q+1)} - dx^\sigma \in \mathrm{GF}(q), \tag{9}$$

as σ stabilizes $\mathrm{GF}(q)$. Let $\eta \in \mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ such that η^2 is a primitive element of $\mathrm{GF}(q)$. Considering $x^\sigma = 1, -1, \eta, -\eta, 1+\eta$ in (9), we get

$$\begin{aligned} d &= 0, \\ \alpha^\sigma - \alpha'(b^2+c^2)/a &= 0, \\ \beta^\sigma - \beta'(b^{q+1}+c^{q+1})/a &\in \mathrm{GF}(q). \end{aligned} \tag{10}$$

Similarly if we consider the affine points in common between the plane $X = 0$ and $\mathcal{M}_{\alpha,\beta}$, arguing as before, we obtain

$$\begin{aligned} g &= 0, \\ \alpha^\sigma - \alpha'(e^2+f^2)/a &= 0, \\ \beta^\sigma - \beta'(e^{q+1}+f^{q+1})/a &\in \mathrm{GF}(q). \end{aligned} \tag{11}$$

In particular,

$$b^2 + c^2 = e^2 + f^2 \neq 0. \tag{12}$$

Also, since $\beta' \notin \mathrm{GF}(q)$,

$$b^{q+1} + c^{q+1} = e^{q+1} + f^{q+1} \neq 0. \tag{13}$$

Now we recall that a generic point $(1,x,y,z) \in \mathcal{M}_{\alpha,\beta}$ if and only if $\phi(1,x,y,z) \in \mathcal{M}_{\alpha',\beta'}$. On the other hand,

$$(1,x,y,z)\in\mathcal{M}_{\alpha,\beta}\Leftrightarrow -\alpha(x^2+y^2)+\beta(x^{q+1}+y^{q+1})-z\in \mathrm{GF}(q).$$

Since $a\in \mathrm{GF}(q)\backslash\{0\}$ and σ stabilizes $\mathrm{GF}(q)$, the former equation is equivalent to

$$a^{-1}\{-\alpha^{\sigma}(x^{2\sigma}+y^{2\sigma})+\beta^{\sigma}[x^{\sigma(q+1)}+y^{\sigma(q+1)}]-z^{\sigma}\}\in \mathrm{GF}(q). \tag{14}$$

Next, we observe that $\phi(1,x,y,z)=(1,\frac{bx^{\sigma}+ey^{\sigma}}{a},\frac{cx^{\sigma}+fy^{\sigma}}{a},\frac{z^{\sigma}}{a})$ and this point belongs to $\mathcal{M}_{\alpha',\beta'}$ if and only if

$$\begin{aligned}a^{-1}\Bigg\{&-\alpha'\left[\frac{(bx^{\sigma}+ey^{\sigma})^2}{a}+\frac{(cx^{\sigma}+fy^{\sigma})^2}{a}\right]\\&+\beta'\left[\frac{(bx^{\sigma}+ey^{\sigma})^{(q+1)}}{a}+\frac{(cx^{\sigma}+fy^{\sigma})^{(q+1)}}{a}\right]-z^{\sigma}\Bigg\}\in \mathrm{GF}(q).\end{aligned} \tag{15}$$

From (14) and (15), we get that for all $(1,x,y,z)\in\mathcal{M}_{\alpha,\beta}$ the following holds:

$$\begin{aligned}&\alpha^{\sigma}(x^{2\sigma}+y^{2\sigma})-\alpha'\left[\frac{(bx^{\sigma}+ey^{\sigma})^2}{a}+\frac{(cx^{\sigma}+fy^{\sigma})^2}{a}\right]\\&+\beta'\left[\frac{(bx^{\sigma}+ey^{\sigma})^{(q+1)}}{a}+\frac{(cx^{\sigma}+fy^{\sigma})^{(q+1)}}{a}\right]-\beta^{\sigma}[x^{\sigma(q+1)}+y^{\sigma(q+1)}]\\&\in \mathrm{GF}(q),\end{aligned}$$

that is, using (12) and (13),

$$-\alpha'[2x^{\sigma}y^{\sigma}(be+cf)]+\beta'[(b^qe+c^qf)x^{\sigma q}y^{\sigma}+(be^q+cf^q)x^{\sigma}y^{\sigma q}]\in \mathrm{GF}(q). \tag{16}$$

We are going to prove that $b^qe+c^qf=0$. Thus, let $\nu\in \mathrm{GF}(q^2)$ be any solution of $X^{q+1}=-1$. The semilinear collineation ϕ has to leave invariant the Hermitian cone $\mathcal{F}$, that is, $\phi(0,x,\nu x,z)\in\mathcal{F}$, and because of the first equation in (13) this means

$$(b^qe+c^qf)\nu^{\sigma}+(be^q+cf^q)\nu^{\sigma q}=0$$

for any of the $q+1$ different solutions of $X^{q+1}=-1$. If $(b^qe+c^qf)\neq 0$ then the equation $(b^qe+c^qf)X+(b^qe+c^qf)^qX^q=0$ would have more than q solutions which are impossible. Thus,

$$b^qe+c^qf=0 \tag{17}$$

and since $\alpha'\notin \mathrm{GF}(q)$ (16) gives

$$be+cf=0. \tag{18}$$

Since $\det(M)\neq 0$, it cannot be $ce=0=bf$, so either $c\neq 0\neq e$ or $b\neq 0\neq f$. Thus, from (12) and (18) we also get $(e,f)=(c,-b)$ or $(e,f)=(-c,b)$. Thus from (17) we also obtain

$$b^q c - bc^q = 0. \tag{19}$$

Hence if $b \neq 0 \neq c$ then $c = \lambda b$ where $\lambda \in \mathrm{GF}(q)$ and $\lambda^2 + 1 \neq 0$. So the lemma follows. □

From the previous lemma, taking into account conditions from (10) to (11), we get that if $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ are projectively equivalent, then

$$(\alpha', \beta') = (a\alpha^\sigma/(b^2 + c^2), a\beta^\sigma/(b^{q+1} + c^{q+1}) + u) \tag{20}$$

for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$, $a \in \mathrm{GF}(q)^*$, $u \in \mathrm{GF}(q)$, $b, c \in \mathrm{GF}(q^2) : b^2 + c^2 \neq 0$ and if $b \neq 0 \neq c$ then $c = \lambda b$ with $\lambda \in \mathrm{GF}(q)\setminus\{0\}$. Conversely, if condition (20) holds, there is a semilinear collineation $\mathcal{M}_{\alpha,\beta} \to \mathcal{M}_{\alpha',\beta'}$; so $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}_{\alpha',\beta'}$ are projectively equivalent.

In this case we write $(\alpha, \beta) \sim (\alpha', \beta')$ where ~ is in particular an equivalence relation on the ordered pairs $(\alpha, \beta) \in \mathrm{GF}(q^2)^2$ such that $4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0$.

Lemma 5.5. *Let $\mathcal{M}_{\alpha,\beta}$ be a BM quasi-Hermitian variety of $\mathrm{PG}(3, q^2)$, q odd and ε be a primitive element of $\mathrm{GF}(q^2)$. Then, there exists $\alpha' \in \mathrm{GF}(q^2)\setminus\{0\}$ such that $\mathcal{M}_{\alpha,\beta}$ is projectively equivalent to $\mathcal{M}_{\alpha',\varepsilon}$.*

Proof. Write $\beta = \beta_0 + \varepsilon\beta_1$, with $\beta_0, \beta_1 \in \mathrm{GF}(q)$ and $\beta_1 \neq 0$. Then, there exists $b \in \mathrm{GF}(q^2)\setminus\{0\}$, such that $\beta_1/b^{q+1} = 1$. Therefore
$(\alpha, \beta) \sim (\alpha/b^2, \beta/b^{q+1} - \beta_0/b^{q+1}) = (\alpha/b^2, \varepsilon)$. □

In light of the previous lemma, to determine the equivalence classes of BM quasi-Hermitian varieties it is enough to determine when two varieties $\mathcal{M}_{\alpha,\varepsilon}$ and $\mathcal{M}_{\alpha',\varepsilon}$ are equivalent. This is done in the following.

Lemma 5.6. *Let $q = p^n$ be an odd prime, ε be a primitive element of $\mathrm{GF}(q^2)$, $\mathcal{M}_{\alpha,\varepsilon}$, and $\mathcal{M}_{\alpha',\varepsilon}$ be two BM quasi-Hermitian varieties of $\mathrm{PG}(3, q^2)$. Put*

$$\delta(\alpha) := \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^{q+1}}.$$

Then, $\mathcal{M}_{\alpha,\varepsilon}$ is projectively equivalent to $\mathcal{M}_{\alpha',\varepsilon}$ if and only if there exist $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$ such that

$$\delta(\alpha') = \delta(\alpha)^\sigma.$$

Proof. First we observe that for all $\alpha \in \mathrm{GF}(q^2)\setminus\{0\}$ such that $4\alpha^{q+1} + (\varepsilon^q - \varepsilon)^2 \neq 0$

$$\delta(\alpha) := \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^{q+1}}$$

belongs to $\mathrm{GF}(q)\setminus\{0, -1\}$. Conversely, given any $\delta \in \mathrm{GF}(q)\setminus\{0, -1\}$ we can generate some BM quasi-Hermitian varieties $\mathcal{M}_{\alpha,\varepsilon}$, by choosing α to be any solution of $4\delta x^{q+1} = (\varepsilon^q - \varepsilon)^2$. In fact, it turns out that $(\varepsilon^q - \varepsilon)^2 + 4\alpha^{q+1} \neq 0$. Furthermore, let α_1 and α_2 be any two such solutions. Then there exists k such that $\alpha_2 = \varepsilon^{k(q-1)}\alpha_1$. On the other hand, $(\alpha_1, \varepsilon) \sim (\alpha_1\varepsilon^{-2}\varepsilon^{q+1}, \varepsilon\varepsilon^{-(q+1)}\varepsilon^{q+1}) = (\alpha_1\varepsilon^{q-1}, \varepsilon)$. By repeating this process k times, we see

$$(\alpha_1, \varepsilon) \sim (\alpha_1 \varepsilon^{k(q-1)}, \varepsilon) = (\alpha_2, \varepsilon).$$

Thus $\delta(\alpha_1) = \delta(\alpha_2)$ implies that $\mathcal{M}_{\alpha_1,\varepsilon}$ is projectively equivalent to $\mathcal{M}_{\alpha_2,\varepsilon}$. Hence, to determine the number N of projectively inequivalent BM quasi-Hermitian varieties we need to count the number of “inequivalent” $\delta \in \mathrm{GF}(q)\setminus\{0, -1\}$.

Now, given two BM quasi-Hermitian varieties $\mathcal{M}_{\alpha,\varepsilon}$ and $\mathcal{M}_{\alpha',\varepsilon}$ and setting

$$\delta = \delta(\alpha) = \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha^{q+1}}, \quad \delta' = \delta(\alpha') = \frac{(\varepsilon^q - \varepsilon)^2}{4\alpha'^{q+1}},$$

we have to show that $\mathcal{M}_{\alpha,\varepsilon} \sim \mathcal{M}_{\alpha',\varepsilon}$ if and only if $\delta' = \delta^\sigma$ for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$.

First, suppose that $\mathcal{M}_{\alpha,\varepsilon}$ and $\mathcal{M}_{\alpha',\varepsilon}$ are equivalent, that is, $(\alpha', \varepsilon) \sim (\alpha, \varepsilon)$. This is true if and only if

$$\alpha' = \frac{\alpha^\sigma a}{b^2 + c^2}, \quad \varepsilon = \frac{a\varepsilon^\sigma}{b^{q+1} + c^{q+1}} + u,$$

for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$, $a \in \mathrm{GF}(q)\setminus\{0\}$, $u \in \mathrm{GF}(q)$, $b, c \in \mathrm{GF}(q^2)$ such that the conditions in the thesis of Lemma 5.4 hold.

Then

$$\delta' = (b^2 + c^2)^{q+1} \frac{(\varepsilon^q - \varepsilon)^2}{4a^2(\alpha^\sigma)^{q+1}}, \quad \delta^\sigma = (b^{q+1} + c^{q+1})^2 \frac{(\varepsilon^q - \varepsilon)^2}{4a^2(\alpha^\sigma)^{q+1}}.$$

We observe that

$$(b^2 + c^2)^{q+1} = (b^{q+1} + c^{q+1})^2. \tag{21}$$

In fact, if either $b = 0$ or $c = 0$, then (21) is trivially satisfied and there is nothing further to prove. Otherwise, a direct manipulation yields that (21) is equivalent to

$$\frac{b^{q-1}}{c^{q-1}} + \frac{c^{q-1}}{b^{q-1}} = 2.$$

This gives $\frac{b^{q-1}}{c^{q-1}} = 1$, which is always true, since (19) holds. Because of (21) then $\delta' = \delta^\sigma$.

Conversely, suppose that $\delta' = \delta^\sigma$ for some σ. Then we observe that $(\alpha, \varepsilon) \sim (\alpha^\sigma, \varepsilon^\sigma)$. Furthermore $(\alpha^\sigma, \varepsilon^\sigma) \sim (\alpha^\sigma/b^2, \varepsilon)$ where $\varepsilon^\sigma = b_1\varepsilon + b_0$ with $b_1/b^{q+1} = 1$ for a suitable $b \in \mathrm{GF}(q^2)\setminus\{0\}$, as seen in the proof of Lemma 5.5.

Thus we have that

$$\begin{aligned}\delta(\alpha^\sigma/b^2) &= (\varepsilon^q - \varepsilon)^2(b^2)^{q+1}/4(\alpha^\sigma)^{q+1}\\ &= (b^2)^{q+1}\{[(\varepsilon^\sigma)^q - b_0] - (\varepsilon^\sigma - b_0)\}^2/(4(\alpha^\sigma)^{q+1}(b^{q+1})^2)\\ &= [(\varepsilon^q - \varepsilon)^2]^\sigma/4(\alpha)^{(q+1)\sigma} = \delta^\sigma = \delta'.\end{aligned}$$

Hence,

$$(\alpha', \varepsilon) \sim (\alpha^\sigma/b^2, \varepsilon) \sim (\alpha^\sigma, \varepsilon^\sigma) \sim (\alpha, \varepsilon)$$

□

Conjecture 5.7. *We conjecture that Lemma 5.6 holds for all odd $r \geq 3$, as the conditions on the coefficients α, β are the same and the block structure of the matrices representing the classes should be analogous to that of Lemma 5.4. For r even the algebraic conditions on α and β to construct quasi-Hermitian varieties are different, see [3].*

Theorem 5.8. *Let $q = p^n$ with p an odd prime. Then the number N of projectively inequivalent BM quasi-Hermitian varieties $\mathcal{M}_{\alpha,\beta}$ of $\mathrm{PG}(3, q^2)$ is*

$$N = \frac{1}{n}\left(\sum_{k|n} \Phi\left(\frac{n}{k}\right) p^k\right) - 2,$$

where Φ is the Euler Φ-function.

Proof. For all $\delta, \delta' \in \mathrm{GF}(q)\backslash\{0, -1\}$ write $\delta \sim \delta'$ if and only if $\delta' = \delta^\sigma$ for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$. By Lemma 5.6, N is the number of inequivalent classes $[\delta]$ under ~. Let $N_e = |\{\delta \in \mathrm{GF}(p^e)\backslash\{0, -1\} : \delta$ is not contained in any smaller subfield of $\mathrm{GF}(q)\}|$. We have

$$N = \sum_{e|n} \frac{N_e}{e}.$$

Observing that

$$\sum_{e'|e} N_{e'} = p^e - 2,$$

denote by $\mu(x)$ the Möbius function [12]. Then, Möbius inversion gives

$$N_e = \sum_{e'|e} \mu(e') p^{e/e'} - 2\sum_{e'|e} \mu(e').$$

It follows that

$$N = \left(\sum_{e|n} \frac{1}{e} \sum_{e'|e} \mu(e') p^{e/e'}\right) - 2.$$

Let $m = e/e'$ be a divisor of n, then the coefficient of p^m is

$$\frac{1}{n} \sum_{(e/m)|(n/m)} \mu\left(\frac{e}{m}\right) \frac{n/m}{e/m} = \frac{1}{n}\Phi\left(\frac{n}{m}\right)$$

and finally

$$N = \frac{1}{n}\left(\sum_{k/n} \Phi\left(\frac{n}{k}\right)p^k\right) - 2.$$

□

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