Credit-dependent demand in a vendor-buyer model with a two-level delay-in-payments contract under a consignment-stock policy agreement

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Abstract

A consignment stock (CS) policy is a promising supply chain coordination mechanism. Delay-in-payments is a financing arrangement that facilitates purchases and increases sales by postposing a payment to some future time. It lowers costs and increases profitability, somewhat like what CS does. Zahran et al. [11] combined the two to reap more benefits by considering different lot sizing and payment scenarios. Buy now, pay later, technically delay-in-payments, is a business practice for increasing sales. This paper revisits the work in [11] by assuming demand increases with the length of the delay period. This increases sales and, subsequently, profits beyond what [11] reports.

Keywords: Supply chain coordination, consignment stock, credit-dependent demand, delay-in-payments, trade credit, supplier financing.

1. Introduction

Supply chain management (SCM) smooths the flow of raw materials and products along its stages, allowing for shared risks for better profitability and responsiveness in an ever-changing business environment [1]. It proposes two coordination decisions: centralised and decentralised. The first has a team of decision-makers representing all players decides the joint production and inventory policy that reduces costs (increases profitability). The second involves multiple decision-makers with conflicting objectives resulting in inefficient policies for some chain members. This practice is probably typical chains of complex structures where reaching reach a joint policy is not possible. Goyal [2] is the earliest work to model the joint economic lot size (JELS) problem and investigate its benefits, which became the building block of many production and inventory supply chain models. JELS minimises the sum of the total annual costs of a vendor and a buyer with the number of shipments and the lot size as decision variables. Glock [3] reviewed the JELS models in the literature and identified some research gaps and future research directions. JELS policies are either traditional [4] or a consignment stock (CS) agreement [5]. In the traditional or backward inventory stocking policy, the vendor produces and accumulates inventory up to a level and ships lots of equal sizes at equal intervals to the buyer, who pays the vendor upon receiving a shipment. The CS agreement is a forward inventory stocking policy where the vendor moves its inventory to the buyer's warehouse who only pays for the sold items.

The economic globalisation, the scarcity of capital in the wake of the financial crisis, especially for Small and Medium Enterprises (SMEs), and the development of advanced technologies drove companies in the same supply chain to coordinate their financial flows alongside those of materials and products [6,7]. In this context, the term supply chain finance (SCF) is "the inter-company optimization of financing as well as the integration of financing processes with customers, suppliers, and service providers to increase the value of all participating companies" [8]. The role of SCF is to optimise both the availability and cost of capital within a supply chain consisting of companies in different countries (economies) and different sizes, facing various uncertainties, not having the same bargaining powers over trading partners, and access to multiple capital markets. The supply chain management literature focuses mainly on operational decisions relating to material flow with little attention to that money. Very few studies considered the coordination of financial activities and their impact on supply chain performance. For instance, [9] presented a JELS model with cooperative financing between supply chain players where it considers a vendor-buyer supply chain and assumed that the vendor has the option to invest in increasing its production rate. Due to different access to capital, the vendor and the buyer may also share the investment and the uncertain outcome, benefiting both.

Financing arrangements between companies take many forms. Examples are quantity and price discounts and delay-in-payments, to name a few. They are a form of short-term financing, and trading partners use them regularly. Under a delay-in-payments financing scheme, the supplier allows the buyer to postpone payments after the invoice date leading to increased supply chain profitability. For this, it has received significant attention from researchers. The literature review proposed in [10] almost comprehensively analysed the literature on trade credit in inventory and supply chain management. They identified the motives behind offering trade credits and the emerging research streams and proposed a few research directions. Another relevant study to this paper [11] showed that delay-in-payments benefits players in a vendor-buyer supply chain with a CS agreement. They also presented a numerical experiment evaluating the effects of trade credit on economic performance and compared them with a supply chain under a traditional coordination policy. The models in the literature usually assume a supplier would offer a fixed credit period to the retailer, who does not offer a credit period to its customers. There is evidence that many retailers (buyers) offer their customers credit periods to boost sales and promote some products [12]. Trade credit, an alternative incentive policy to quantity discounts, is a powerful promotional tool for attracting new customers. It lowers purchasing costs and makes liquidity available, enabling final customers to make additional purchases.

In this regard, Heydari et al. [13] argued that credit periods affect demand; despite its importance and frequent mentioning by researchers, it received very little attention in the literature. In a way, they complemented the work of Seifert et al. [10], whose literature review of trade credit showed that none considered a credit-dependent demand function, despite that offering credit: e.g., delay-in-payments entices buyers to order more and pay later. Heydari et al. [13] identified only two studies, apart from theirs, that assume credit-dependent functions. In one study, even though demand theoretically increases for extended delay periods, there is a threshold value beyond which ordering more will have marginal benefits. Another one assumed that demand increases exponentially with delay-in-payment. Their credit-dependent demand function is close in form to the second. It worth noting that none of those previous studies investigated the benefits of a CS agreement in a credit-dependent demand context. They all assumed, including Heydari et al. [13], coordination follows a traditional JELS (e.g., [4]) with demand being dependent on the length of delay in payment. The literature, however, shows that a JELS with CS outperforms the former coordination mechanism. This study modifies the model in [11] to investigate the additional benefits that a CS policy with delay-in-payments [11] would bring when demand follows the form in [13]. The resultant model, which modifies [11], will be compared to the traditional one studied by [13], with the results discussed and managerial insights highlighted.

Table 1 lists the main features of the vendor–buyer models relevant to this paper, i.e., [4], [11], and [13], for better differentiation.

Table 1. Comparison of the characteristics of the relevant literature for this study

Reference	**Decision variables**	**SC structure**	**Decision making**	**Shortage cost**	**Inventory policy**	**Demand**	**Delay-in-payments**
Hill [4]	q, n, λ	SV – SB	C	-	Hill	F	-
Heydari et al. [13]	q, N, CP, n	SV – SB	C–D	✓	Hill	NF	✓
Zahran et al. [11]	q, n, m	SV – SB	C	-	CS	F	✓
This study	q, N, n, m	SV – SB	C–D	✓	CS	NF	✓

Legend: q represents the order lot size, n the number of shipments, λ the proportional increase in the size of successive shipments within a batch production run, N the length of the trade credit period granted by the buyer to the end customers, CP the length of the credit period offered by the vendor to the buyer and m the number of payments in one full cycle. SV = single vendor. SB = single buyer. C = centralized decisions. D = decentralised decisions. F= Fixed. NF = not fixed, credit dependent.

The remainder of the paper is structured as follows: Section 2 defines the problem and specifies the main assumptions and notations. Section 3 formulates the consignment stock models that describe different trade credit scenarios and compares its results with those in Table 1. The results of the numerical examples and the sensitivity analysis are in Section 4. Section 5 concludes the paper with some remarks.

2. Problem definition, assumptions, and notations

Consider a centralised vendor-buyer system with a CS agreement, similar in description to the one in [11]. The buyer sells items from the consigned inventory and pays the vendor for the withdrawn quantities, while the vendor emits invoices to the buyer at equal time intervals. The buyer can pay either when it receives an invoice or later. If the buyer decides to delay a payment, and the vendor agrees, then the buyer pays by the end of the permissible delay period. If the buyer wishes to extend the agreed-upon delay period, then it incurs additional costs. Unlike [11], the model of this paper assumes a Normally distributed demand whose mean annual value is $E(D)$ and depends on the length of the payment delay period offered by the buyer to the end customer. The variance of the demand is constant. Note that we will use the same notations in [11] and [13] to facilitate the comparison between the model in the paper and the other two.

The following straightforward assumptions were considered when developing the models:

1. $E(D) = be^{aN}$, where b is the market size, N is the length of the trade credit period (or delay in payment), and a is a model parameter [13], where long Ns spike purchases more than short ones. However, the demand will not increase indefinitely since higher credit period length increase also the cost that the buyer incurs and introduces a trade-off [14]. In this way, it is also in conformance with recent research in the field, and it makes possible the evaluation of the benefits introduced with the deployment of a CS agreement.
2. Time horizon is infinite and the lead time is zero.
3. The vendor's production rate is constant and is larger than the demand rate, $P > E(D)$.
4. Equal payments are made at equal time intervals. A payment is delayed to a future date [15].
5. The per unit time holding cost consists of two components: financial and physical.
6. The vendor and the buyer incur fixed setup and order costs that are independent of the produced and ordered quantities.
7. The cycle time, T, is common for the vendor and the buyer and it is a function of the lot size.

Parameters

α fraction of t, where the buyer makes a payment by time αt within the interest-free period after the timing of the invoice

A order cost of the buyer ($/order),

a positive real number representing the sensitivity of demand to the length of the credit period (year $^{-1}$),

β fraction of the payment time indicated on the invoice plus the permissible free period, in which the buyer settles its payment in the interest-charged scenario,

b expected market size for the case of no trade credit,

B_r unit shortage cost at the buyer's site ($/unit),

c_t transaction cost of the buyer ($/transaction),

c_v production cost of the vendor ($/unit),

δ length of interest-charged delay-in-payments period,

$E(D)$ expected annual demand rate (units/year),

γ number of units (components) needed to produce a finished item,

$h_{v,f}^{v}$ financial component of the holding cost of the vendor per item stocked at the vendor, warehouse and per period ($ unit/year), equals to $(c_v + \gamma r_v)i_v$,

$h_{v,f}^{b}$ financial component of the holding cost of the vendor per item stocked at the buyer, warehouse and per period ($ unit/year), equals to $p_v i_v$,

$h_{v,p}$ vendor's physical holding cost ($/unit/year),

$h_{b,f}$ buyer's financial holding cost ($/unit/year), and is equal to $p_v i_b$,

$h_{b,p}$ buyer's physical holding cost ($/unit/year),

i_b buyer's cost of capital (%/year),

i_v vendor's cost of capital (%/year),

k Safety factor,

N_{max} maximum length of the credit period,

P production rate (units/year),

p_b buyer's unit selling price ($/unit),

p_v vendor's unit selling price ($/unit), where $c_v + \gamma r_v + c_t < p_v < p_b$,

r_v vendor's unit cost of raw material ($/unit),

σ standard deviation of demand during lead time (unit),

S vendor's setup cost ($/set-up),

τ length of the permissible delay-in-payments,

t elapsed time between successive invoices, equals to T/m, and

T cycle time.

Decision variables

m number of payments made by the buyer to the vendor in one cycle,

N length of the trade credit period granted by the buyer to the end customer (year),

n number of shipments from the vendor to the buyer, and

q order lot size shipped to the buyer (unit).

3. Model formulation

This section presents the mathematics for three different scenarios of delay-in-payments, taken from [11], with a credit-dependent demand and a consignment stock agreement. The total system profit, the performance measure, is the sum of those of the vendor and the buyer.

The first scenario assumes the buyer pays the vendor once it receives an invoice since no delay-in-payments is allowed. In the second, the vendor invoices the buyer after every shipment and allows the buyer a permissible period to settle the payment at no cost (interest-free period). The third scenario assumes the buyer pays after the due date, incurring additional charges (interest-charge period). The buyer can also offer a delay-in-payment to the end customer since long trade credit periods stimulate end-customer demand significantly [13].

3.1. Scenario 1: Consignment stock without delay-in-payments

Scenario 1 studies the effects of the consignment stock agreement on the supply chain total profit when the buyer faces a credit-dependent demand and makes equal-sized payments to the vendor at equal intervals without delay-in-payment: $N = 0$ and $D = be^{aN} = b$. In cycle time T, the buyer settles m equal payments every $t = T/m$ unit of time to the vendor, who ships to the buyer's order in n shipments of size q each. Figure 2 and 2 show the inventory and the cash flow behaviour for the vendor and the buyer, respectively.

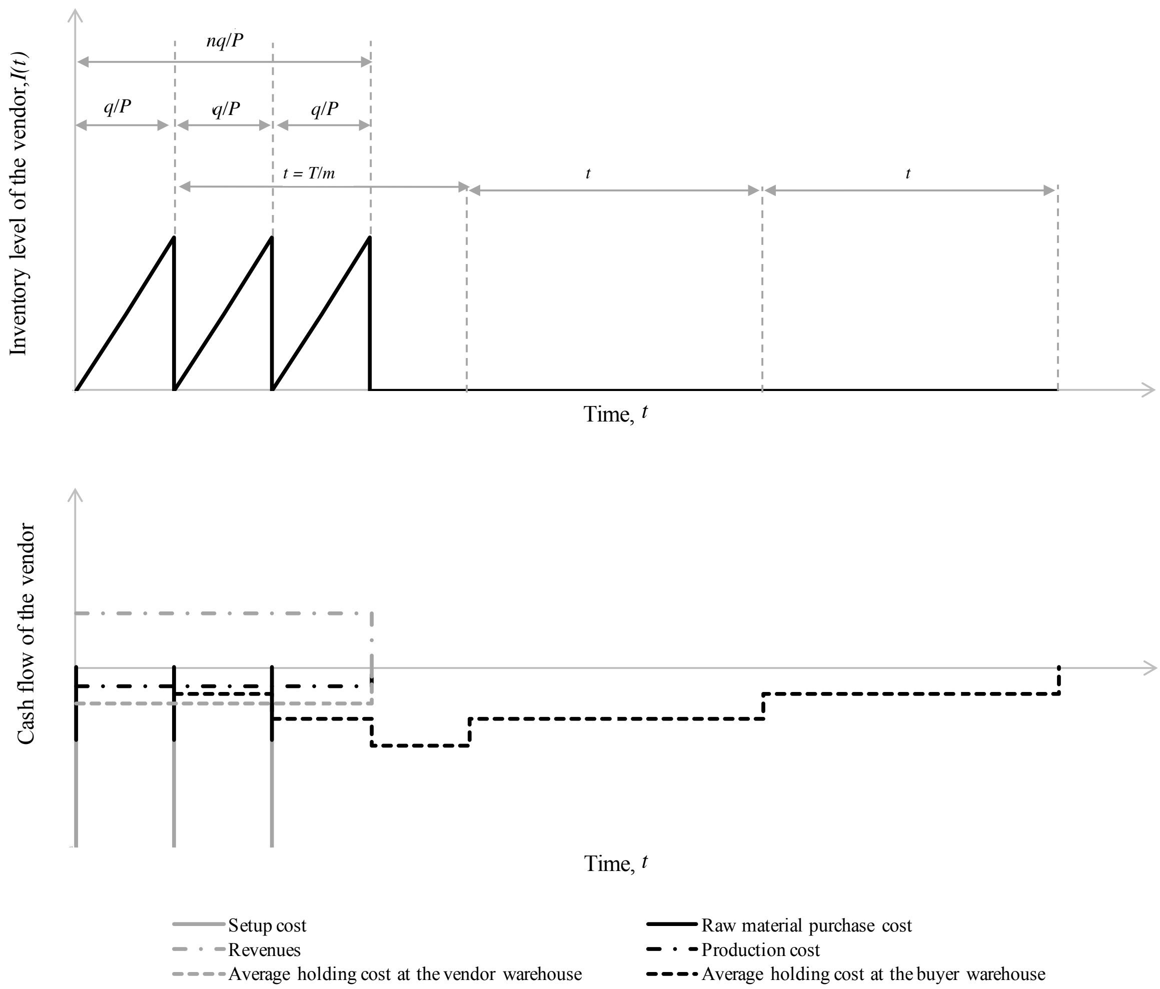

Figure 1. The behaviour of inventory and the cash flow for the vendor with consignment stock agreement and delay-in-payments.

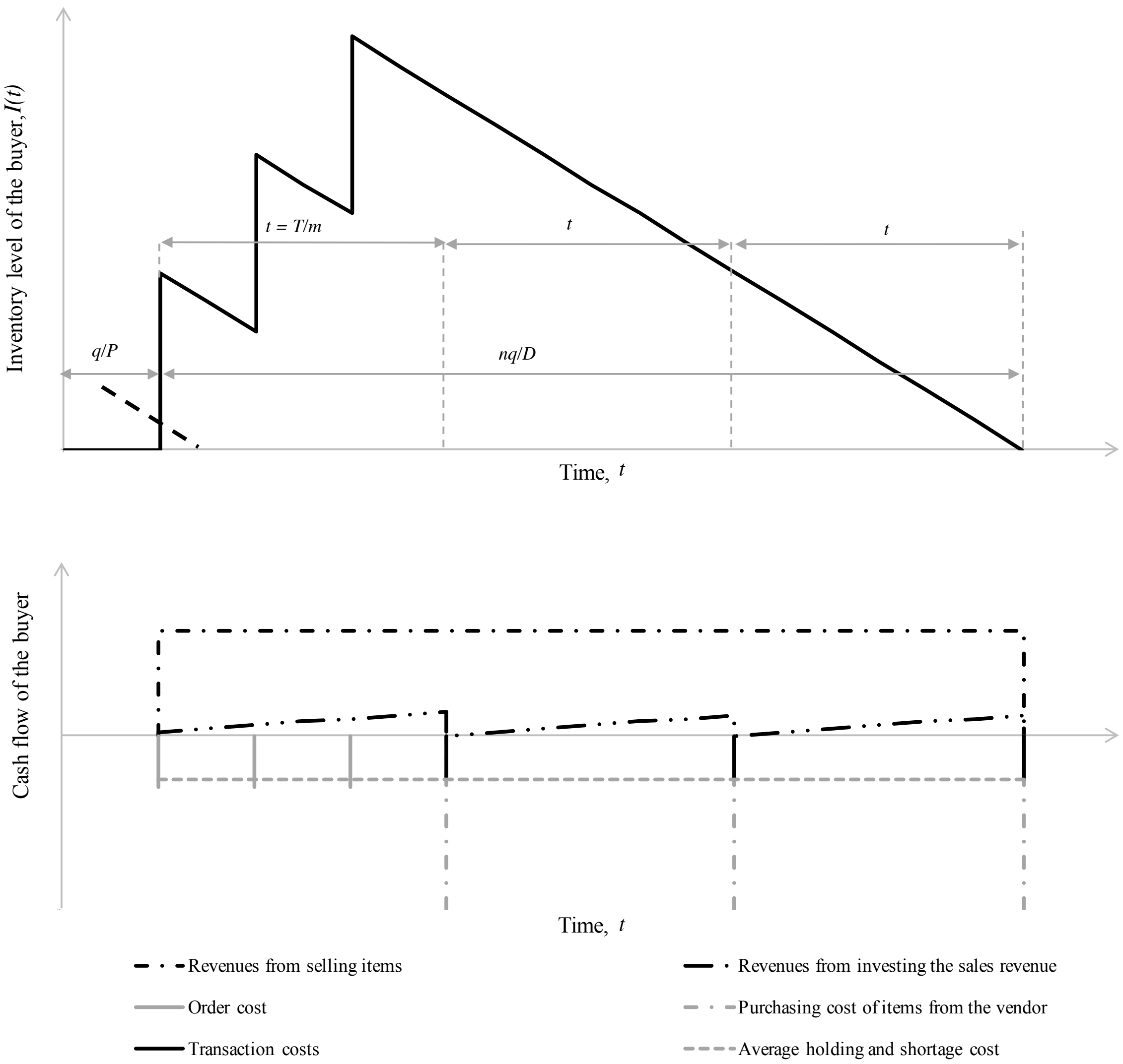

Figure 2. The behaviour of inventory and the cash flow of the buyer with consignment stock agreement and no delay-in-payments.

The vendor's total cost, $TC_{v,1}$, is given by:

$$TC_{v,1} = (\gamma r_v + c_v)b + S\frac{b}{nq} + h_{v,f}^{b}\frac{(m+1)nq}{2m} + \left[h_{v,p} + h_{v,f}^{v} - (n-1)h_{v,f}^{b}\right]\frac{qb}{2P} \tag{1}$$

$TC_{v,1}$ is the sum of those for purchasing raw materials and production (term 1), setup (term 2), and holding (financial, term 3, and physical, terms 4). Term 3 is an opportunity cost arising from money tied up in inventory at the buyer's side until the latter pays the invoices for the items sold during period t [11]. The buyer's total cost, $TC_{b,1}$, is given as:

$$TC_{b,1} = p_v b + (nA + mc_t)\frac{b}{nq} + h_{b,p}\left[\frac{nq}{2} - (n-1)\frac{qb}{2P}\right] + \left(h_{b,p} + h_{b,f}\right)k\sigma + \frac{B_r b\sigma}{q}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{2}$$

$TC_{b,1}$ is the sum of those of purchasing from the vendor (term 1), and the costs of ordering and order transactions (term 2), holding (only physical; terms 3 and 4), and shortages (term 5). The revenue functions for the vendor and the buyer are defined, respectively, in the following equations:

$$TR_{v,1} = p_v b \tag{3}$$

$$TR_{b,1} = p_b\left(b + i_b\frac{nq}{2m}\right) \tag{4}$$

The vendor's revenue is self-explanatory. The buyer generates its revenue from sales and investing the sales revenue at an annual rate of i_b for t units of time.

The supply chain total profit for Scenario 1, $TP_{s,1}$, is written from Eqs. (1)-(4), (3) + (4) – (1) – (2), as:

$$\begin{aligned} TP_{s,1}(q,n,m) &= TR_{v,1} + TR_{b,1} - TC_{v,1} - TC_{b,1} \\ &= (p_b - \gamma r_v - c_v)b - (S + nA + mc_t)\frac{b}{nq} \\ &\quad - \frac{nq}{2}\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{b}{P}\right) - \frac{qb}{2P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) \\ &\quad - \frac{nq}{2m}\left(h_{v,f}^{b} - p_b i_b\right) - \left(h_{b,p} + h_{b,f}\right)k\sigma - \frac{B_r b\sigma}{q}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \end{aligned} \tag{5}$$

Eq. (5) is concave, and its proof is given in Appendix A. The optimal values of the decision variables that maximize eq. (5) are:

$$q^* = \sqrt{\frac{\left[\frac{S + nA + mc_t}{n} + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]b}{\frac{n}{2}\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{b}{P}\right) + \frac{b}{2P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) + \frac{n}{2m}\left(h_{v,f}^{b} - p_b i_b\right)}} \tag{6}$$

$$n^* = \sqrt{\frac{(S + mc_t)\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right)b}{\left[A + B_r\sigma \int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right] P\left(\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{b}{P}\right) + \frac{\left(h_{v,f}^{b} - p_b i_b\right)}{m}\right)}} \tag{7}$$

Since the number of shipments should be an integer, the optimal value of $\tilde{n}$ is obtained by rounding to the nearest integer the value of n^*. By substituting q^* and $\tilde{n}$ in Eq. (5), it is then possible to find an expression for the optimal value of m. However, due to the complexity of the formulation, it is not possible to reach a closed-loop equation. The following solution algorithm was applied to find the optimal value of the decision variable.

<u>Step 1</u>. Set $q = q^*$, $\tilde{n} = \lfloor n^* \rfloor$, and $m = 1$.

<u>Step 2</u>. Calculate $TP_{s,1}(q^*, \tilde{n}, m)$ from eq. (5).

<u>Step 3</u>. Repeat Step 2 by increasing m by one unit until $TP_{s,1}(q^*, \tilde{n}, m+1) < TP_{s,1}(q^*, \tilde{n}, m)$ and $TP_{s,1}(q^*, \tilde{n}, m) > TP_{s,1}(q^*, \tilde{n}, m-1)$. The value of the decision variable m^* that maximizes the supply chain's total profit is determined and saved as the optimal solution.

3.2. Scenario 2: Consignment stock with interest-free delay-in-payment and credit-dependent demand.

In this scenario, the vendor offers the buyer a delay-in-payment, who settles it by the end of the permissible period, τ, given by the time of the invoice t plus an interest-free delay period $\alpha \cdot t$, where $\alpha > 0$ (i.e., $\tau = t + \alpha \cdot t$). Figure 4 show the inventory and cash flow behaviour for the vendor and the buyer, respectively.

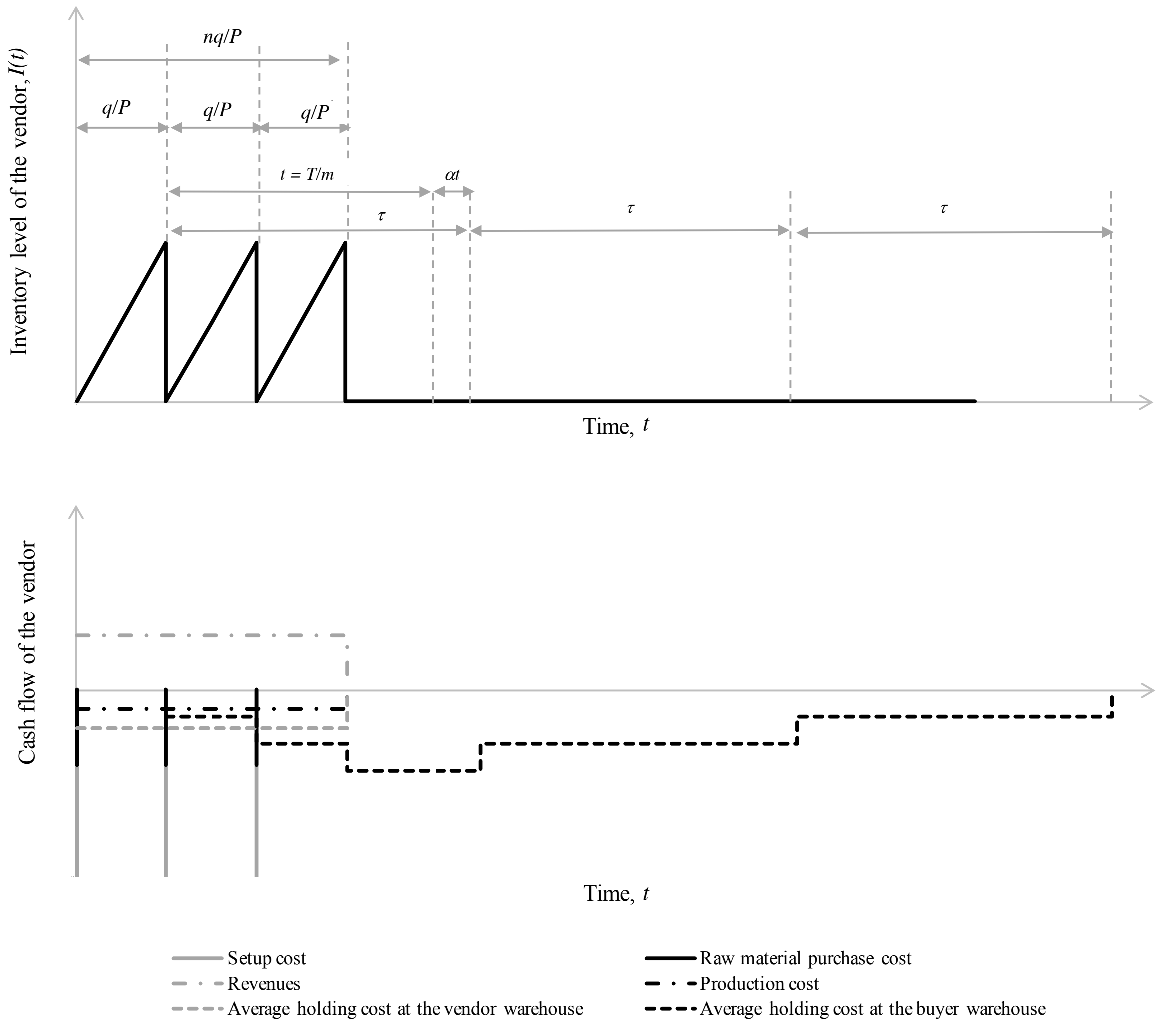

Figure 3. The behaviour of inventory and the cash flow for the vendor with consignment stock agreement and interest-free delay-in-payments periods.

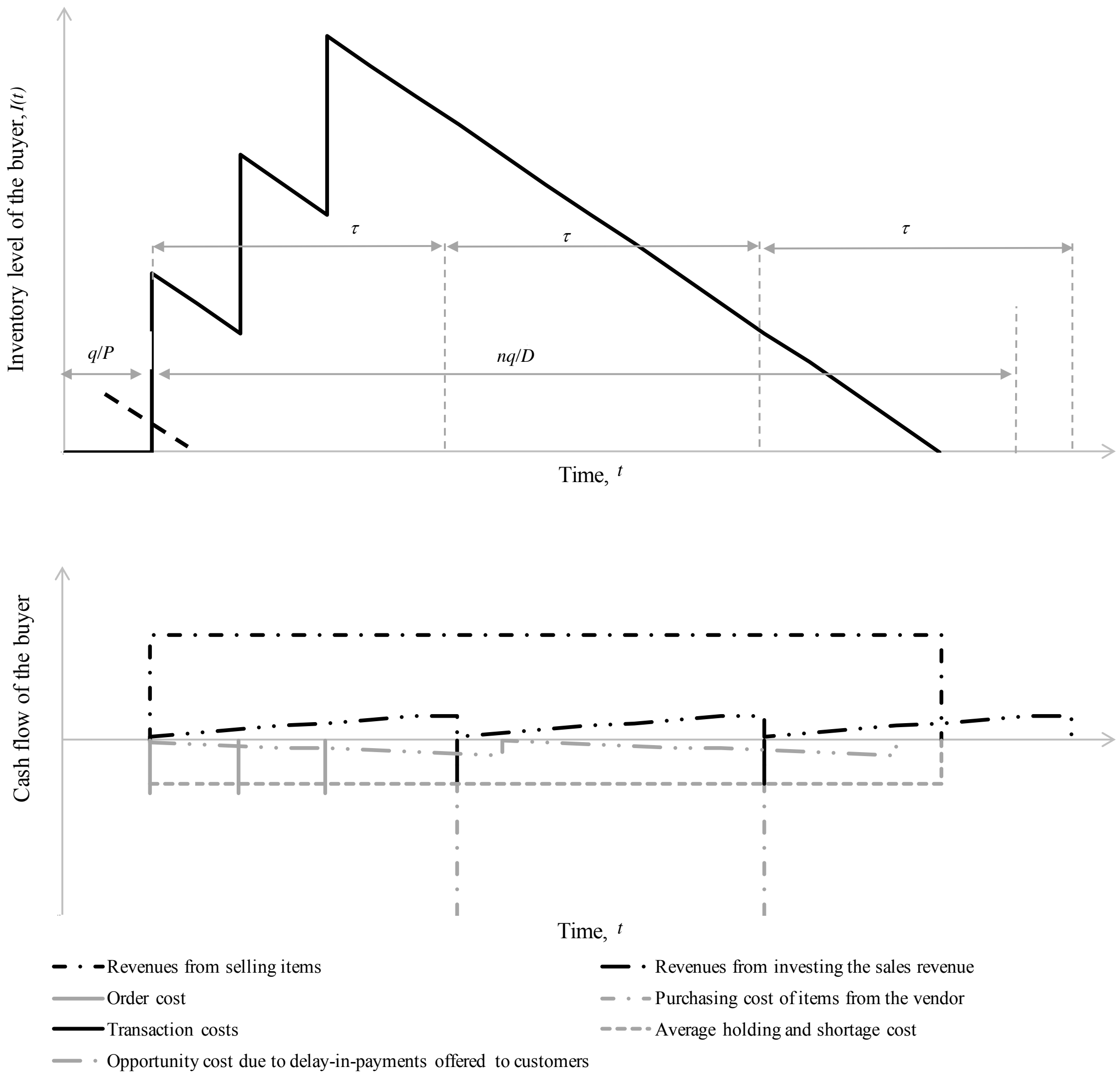

Figure 4. The behaviour of inventory and the cash flow for the buyer with consignment stock agreement and interest-free delay-in-payments period.

Scenario 2 offers delay-in-payments where a buyer settles its balance with the vendor at some time after receiving an invoice, $\tau > t$. It increases the vendor's costs and the buyer's profit. The first incurs additional opportunity cost while the second earns more interest from invested sales. The vendor's total cost, $TC_{v,2}$, and of the buyer's revenues, $TR_{b,2}$, become:

$$TC_{v,2} = (\gamma r_v + c_v)be^{aN} + S\frac{be^{aN}}{nq} + h_{v,f}^{b}\frac{(m+1+2\alpha)nq}{2m} + \left[h_{v,p} + h_{v,f}^{v} - (n-1)h_{v,f}^{b}\right]\frac{qbe^{aN}}{2P} \tag{8}$$

$$TR_{b,2} = p_b \left(be^{aN} + i_b \frac{(2\alpha + 1)nq}{2m} \right) \tag{9}$$

The buyer also offers its final customer a delay-in-payment of length N. Hence, the buyer's total costs, Eq. (10), are given by the same of Scenario 1 plus those costs associated with the trade credit scheme: i.e., the opportunity cost of capital. So, the buyer's total cost and revenue for Scenario 2 are given, respectively, by:

$$\begin{aligned} TC_{b,2} = {} & p_v b + (nA + mc_t) \frac{be^{aN}}{nq} + h_{b,p} \left[\frac{nq}{2} - (n-1) \frac{qbe^{aN}}{2P} \right] + \left(h_{b,p} + h_{b,f} \right) k\sigma \\ & + p_b i_b N be^{aN} + \frac{B_r be^{aN} \sigma}{q} \int_k^{\infty} (x - k) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \end{aligned} \tag{10}$$

$$TR_{v,2} = p_v be^{aN} \tag{11}$$

The supply chain total profit, $TP_{s,2}$, is then given by:

$$\begin{aligned} TP_{s,2}(q,n,m) = {} & (p_b - \gamma r_v - c_v) be^{aN} - (S + nA + mc_t) \frac{be^{aN}}{nq} \\ & - \frac{nq}{2} \left(h_{v,f}^b + h_{b,p} \right) \left(1 - \frac{be^{aN}}{P} \right) - \frac{qbe^{aN}}{2P} \left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p} \right) \\ & - \frac{nq}{2m} (2\alpha + 1) \left(h_{v,f}^b - p_b i_b \right) - \left(h_{b,p} + h_{b,f} \right) k\sigma - p_b i_b N be^{aN} \\ & - \frac{B_r be^{aN} \sigma}{q} \int_k^{\infty} (x - k) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \end{aligned} \tag{12}$$

Eq.

(12) is concave as shown in Appendix A. The optimal values of the decision variables that maximize the total profit of the supply chain, Eq.

(12), are:

$$q^* = \sqrt{\frac{\left[\frac{S + nA + mc_t}{n} + B_r\sigma \int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}}{\frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) + \frac{be^{aN}}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) + \frac{n}{2m}(2\alpha+1)\left(h_{v,f}^b - p_b i_b\right)}} \tag{13}$$

$$n^* = \sqrt{\frac{(S + mc_t)be^{aN}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right)}{\left[A + B_r\sigma \int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]P\left(\left(h_{v,f}^b + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) + \frac{(2\alpha+1)\left(h_{v,f}^b - p_b i_b\right)}{m}\right)}} \tag{14}$$

We propose the following solution algorithm to find the optimal value of the other decision variables (i.e., m and N).

Step 1. Set $q = q^*$, $\tilde{n} = \lfloor n^* \rfloor$, $m = 1$, and $N = 0$.

Step 2. Calculate $TP_{s,2}(q^*, \tilde{n}, m, N)$ from Eq.

(12).

Step 3. Repeat Step 2 by increasing m by one unit until $TP_{s,2}(q^*, \tilde{n}, m+1, N) < TP_{s,2}(q^*, \tilde{n}, m, N)$ and $TP_{s,2}(q^*, \tilde{n}, m, N) > TP_{s,2}(q^*, \tilde{n}, m-1, N)$.

Step 4. Repeat Step 2 and 3 for every value of N in the range $[0;\ N_{max}]$ incrementing by ε is a sufficiently small positive real number. The combination of the decision variables (m^*, N^*) that maximizes the supply chain total profit is determined and saved as the optimal solution.

3.3. Scenario 3: Consignment stock with interest-charge delay-in-payment and credit-dependent demand

In this scenario, as in the previous one, the vendor offers the buyer an interest-free period, αt, to settle the balance of the invoice received at time t. The buyer may postpone the payment by an additional period, δ, which corresponds to the permissible period τ, plus an interest-charge delay period $\beta\tau$, where $\beta > 0$. The buyer invests its revenues for a longer time than in the other scenarios but incurs additional interest charges. Here, the vendor experiences extra opportunity cost of capital. Figure 6 show the inventory and the cash flow behaviour for the vendor and the buyer, respectively.

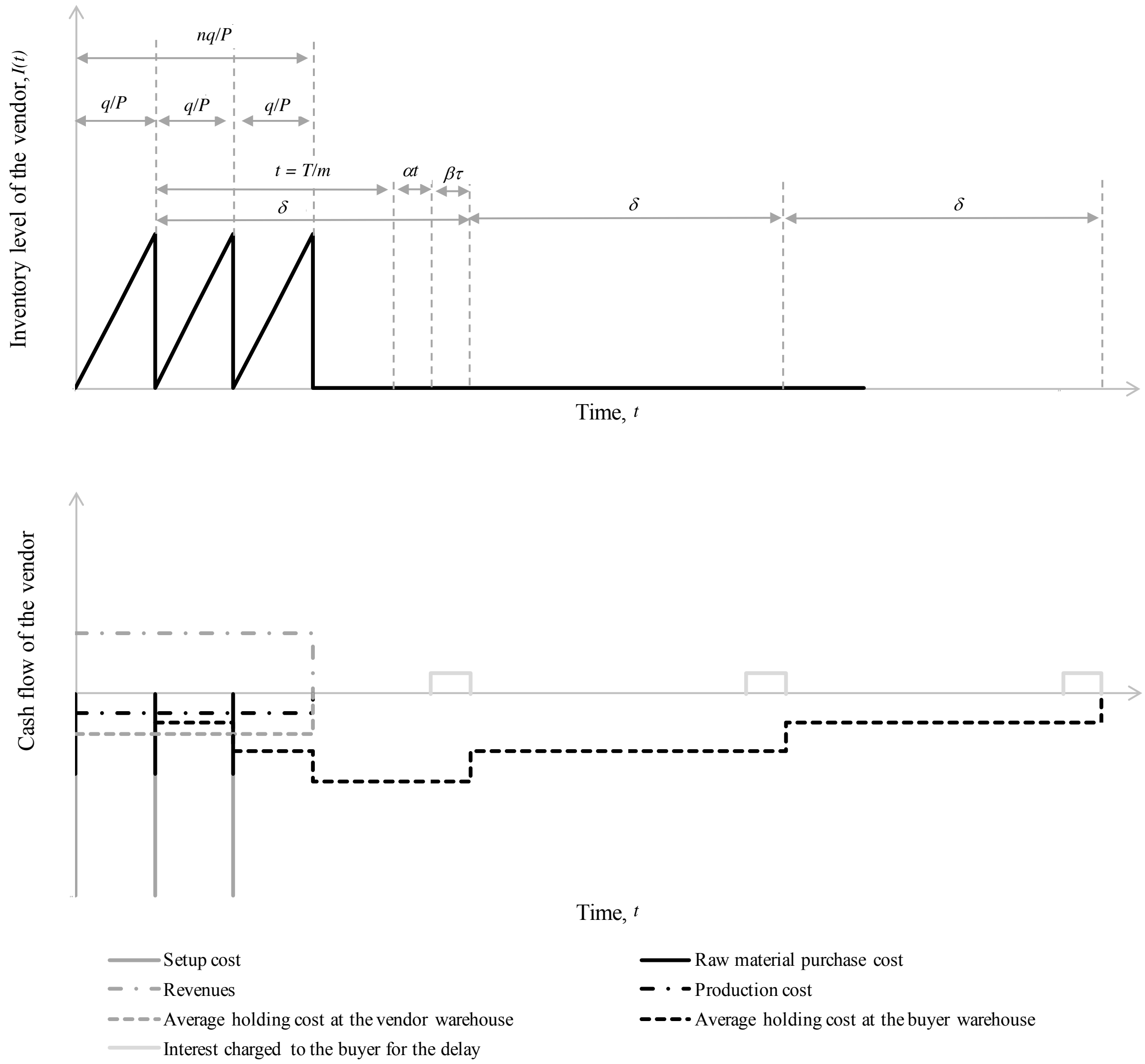

Figure 5. The behaviour of inventory and the cash flow for the vendor with consignment stock agreement and an interest-charged delay-in-payments period.

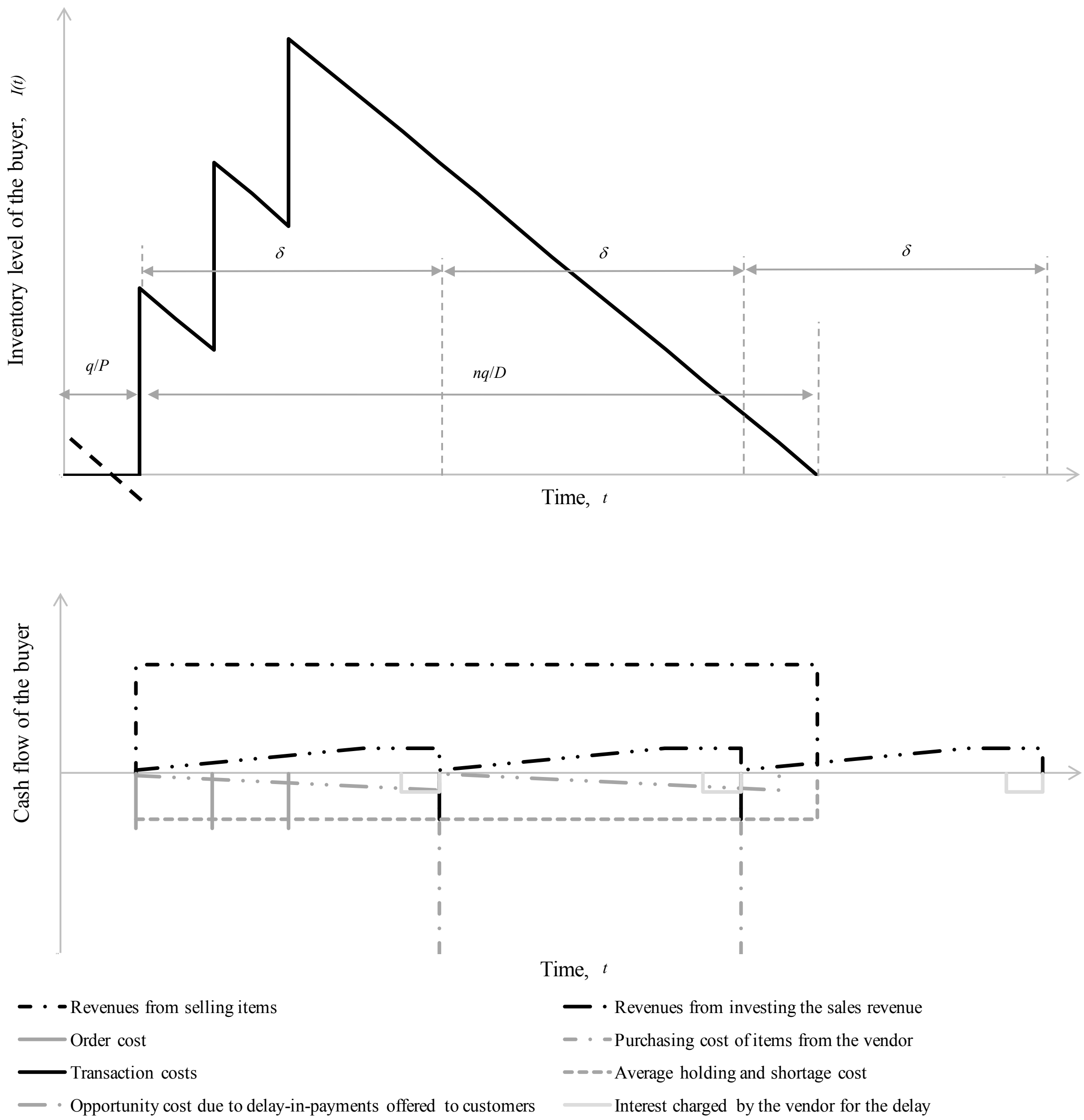

Figure 6. The behaviour of inventory and the cash flow for the buyer with consignment stock agreement and an interest-charged delay-in-payments period.

The vendor's, $TC_{v,3}$, and buyer's, $TC_{b,3}$, total costs become:

$$TC_{v,3} = (\gamma r_v + c_v)be^{aN} + S\frac{be^{aN}}{nq} + h_{v,f}^{b}\frac{\big(m+1+2\alpha+2\beta(1+\alpha)\big)nq}{2m} + \left[h_{v,p} + h_{v,f}^{v} - (n-1)h_{v,f}^{b}\right]\frac{qbe^{aN}}{2P} \tag{15}$$

$$TC_{b,3} = p_v be^{aN} + (nA + mc_t)\frac{be^{aN}}{nq} + h_{b,p}\left[\frac{nq}{2} - (n-1)\frac{qbe^{aN}}{2P}\right] + \left(h_{b,p} + h_{b,f}\right)k\sigma + h_{v,f}^{b}\frac{\beta(1+\alpha)nq}{m} + p_b i_b N be^{aN} + \frac{B_r be^{aN}\sigma}{q}\int_{k}^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{16}$$

The vendor generates revenues, $TR_{v,3}$, from selling items to the buyer and charging interest on outstanding balances. While, the buyer generates revenues, $TR_{b,3}$, from sales and earning interest on invested revenue. These two functions are given by:

$$TR_{v,3} = p_v\left(be^{aN} + i_v\frac{\beta(\alpha+1)nq}{m}\right) = p_v be^{aN} + h_{v,f}^{b}\frac{\beta(\alpha+1)nq}{m} \tag{17}$$

$$TR_{b,3} = p_b\left(be^{aN} + i_b\frac{\left(2\alpha + 1 + 2\beta(1+\alpha)\right)nq}{2m}\right) \tag{18}$$

The total profit of the supply chain, $TP_{s,3}$, is then given as:

$$TP_{s,3}(q,n,m) = (p_b - \gamma r_v - c_v)be^{aN} - (S + nA + mc_t)\frac{be^{aN}}{nq} - \frac{nq}{2}\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) - \frac{qbe^{aN}}{2P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) - \frac{nq}{2m}\left(1 + 2\alpha + 2\beta(1+\alpha)\right)\left(h_{v,f}^{b} - p_b i_b\right) - \left(h_{b,p} + h_{b,f}\right)k\sigma - p_b i_b N be^{aN} - \frac{B_r be^{aN}\sigma}{q}\int_{k}^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{19}$$

Eq. (19) is concave as shown in Appendix A. The optimal values of the decision variables that maximize Eq. (19(5) are:

$$q^* = \sqrt{\frac{\left[\frac{S+nA+mc_t}{n} + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}}{\frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) + \frac{be^{aN}}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) + \\ + \frac{n}{2m}\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^b - p_b i_b\right)}} \tag{20}$$

$$n^* = \sqrt{\frac{(S+mc_t)be^{aN}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right)}{\left[A + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]P\left(\begin{array}{c}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) + \\ + \frac{\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^b - p_b i_b\right)}{m}\end{array}\right)}} \tag{21}$$

We propose a solution algorithm to find the optimal values of the decision variables (i.e., m and N).

Step 1. Set $q = q^*$, $\tilde{n} = \lfloor n^* \rfloor$, $m = 1$, and $N = 0$.

Step 2. Calculate $TP_{s,3}(q^*, \tilde{n}, m, N)$ from eq. (19).

Step 3. Repeat Step 2 by increasing m by one unit until $TP_{s,3}(q^*, \tilde{n}, m+1, N) < TP_{s,3}(q^*, \tilde{n}, m, N)$ and $TP_{s,3}(q^*, \tilde{n}, m, N) > TP_{s,3}(q^*, \tilde{n}, m-1, N)$.

Step 4. Repeat Step 2 and 3 for every value of N in the range $[0;\ N_{max}]$ incrementing by ε is sufficiently small positive real number. The combination of the decision variable (m^*, N^*) that maximizes the supply chain's total profit is determined and saved as the optimal solution.

3.4. Comparison between the consignment stock and the traditional policy

This section presents versions of Hill's [4] model corresponding to the three scenarios developed above. The rationale for choosing Hill's model has been outlined in [16], who found it to perform better than other traditional models for delay-in-payments. The vendor issues in those models an invoice for every buyer's order and shipment (i.e., $m = n$). When the supplier does not offer delay-in-payments, the buyer pays the vendor for each shipment immediately upon receipt. The total profit is given by:

$$TP_{s,1}^{Hill}(q,n) = (p_b - \gamma r_v - c_v)b - (S + nA + nc_t)\frac{b}{nq} - h_v\left(\frac{qb}{P} + \frac{(P-b)nq}{2P}\right) \tag{22}$$

$$- (h_b - h_v)\frac{q}{2} - h_b k\sigma - \frac{B_r b\sigma}{q}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

where $h_v = h_{v,f}^v + h_{v,p}$ and $h_b = h_{b,f} + h_{b,p}$. The optimal values of the decision variables that maximize Eq. (22) are:

$$q^* = \sqrt{\frac{2\left(\frac{S}{n} + A + c_t + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)b}{h_v\left(\frac{2b}{P} + \frac{(P-b)n}{P}\right) + (h_b - h_v)}} \tag{23}$$

$$n^* = \sqrt{\frac{Sb\left(\frac{h_v b}{P} + \frac{(h_b - h_v)}{2}\right)}{\left(A + c_t + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)b\left(1 - \frac{b}{P}\right)}} \tag{24}$$

The second scenario under a traditional policy considers that the vendor guarantees the buyer a permissible period to settle its balance at no additional cost. The buyer offers a trade credit period of length N to its customers. The buyer pays the vendor by the end of the permissible period. The expected supply chain profit can be written as [13]:

$$\begin{aligned} TP_{s,2}^{Hill}(q,n,N) & \\ &= (p_b - \gamma r_v - c_v)be^{aN} - (S + nA + nc_t)\frac{be^{aN}}{nq} \\ &- h_v\left(\frac{qbe^{aN}}{P} + \frac{(P - be^{aN})nq}{2P}\right) - (h_b - h_v)\frac{q}{2} + \alpha(p_b i_b - p_v i_v)q \\ &- p_b i_b N be^{aN} - h_b k\sigma - \frac{B_r be^{aN}\sigma}{q}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \end{aligned} \tag{25}$$

The optimal values of the decision variables that maximize eq. (25) are:

$$q^* = \sqrt{\frac{2\left(\frac{S}{n} + A + c_t + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)be^{aN}}{h_v\left(\frac{2be^{aN}}{P} + \frac{(P-be^{aN})n}{P}\right) + (h_b - h_v) - 2\alpha(p_b i_b - p_v i_v)}} \tag{26}$$

$$n^* = \sqrt{\frac{Sbe^{aN}\left(\frac{h_v be^{aN}}{P} + \frac{(h_b - h_v)}{2} - 2\alpha(p_b i_b - p_v i_v)\right)}{\left(A + c_t + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)be^{aN}\left(1 - \frac{be^{aN}}{P}\right)}} \tag{27}$$

under the assumption $\frac{h_v be^{aN}}{P} + \frac{(h_b - h_v)}{2} - 2\alpha(p_b i_b - p_v i_v) > 0$.

The third scenario considers a delay-in-payment where the buyer pays the invoice post the interest-free delay period and incurs a cost from interest charged by the vendor. The expected supply chain total profit is written as:

$$\begin{aligned} TP_{s,3}^{Hill}(q,n,N) &= (p_b - \gamma r_v - c_v)be^{aN} - (S + nA + nc_t)\frac{be^{aN}}{nq} \\ &- h_v\left(\frac{qbe^{aN}}{P} + \frac{(P-be^{aN})nq}{2P}\right) - (h_b - h_v)\frac{q}{2} \\ &+ [\alpha + \beta(1+\alpha)](p_b i_b - p_v i_v)q - p_b i_b N be^{aN} - h_b k\sigma \\ &- \frac{B_r b\sigma}{q}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \end{aligned} \tag{28}$$

The optimal values of the decision variables that maximize eq. (28) are:

$$q^* = \sqrt{\frac{2\left(\frac{S}{n} + A + c_t + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)be^{aN}}{h_v\left(\frac{2be^{aN}}{P} + \frac{(P-be^{aN})n}{P}\right) + (h_b - h_v) - 2[\alpha + \beta(1+\alpha)](p_b i_b - p_v i_v)}} \tag{29}$$

$$n^* = \sqrt{\frac{Sbe^{aN}\left(\frac{h_v be^{aN}}{P} + \frac{(h_b - h_v)}{2} - 2[\alpha + \beta(1+\alpha)](p_b i_b - p_v i_v)\right)}{\left(A + c_t + B_r\sigma \int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right) be^{aN}\left(1 - \frac{be^{aN}}{P}\right)}} \tag{30}$$

, which are valid for $\frac{h_v be^{aN}}{P} + \frac{(h_b - h_v)}{2} - 2[\alpha + \beta(1+\alpha)](p_b i_b - p_v i_v) > 0$.

3.5. Comparison with the decentralized decision-making process

In a decentralized supply chain, each member of the supply chain makes decisions to optimize its profit without considering that of the supply chain. In the proposed model, the buyer has the opportunity to stimulate additional customer demand by granting a trade credit, which may increase its profit. In the decentralized setting, the retailer decides about the length of the trade credit period, N, and the order quantity, q.

When there is no delay in payments, the buyer pays the vendor for each shipment immediately upon receipt. The total profit of the buyer is then given as:

$$TP_{b,1}^{D}(q,m) = p_b\left(b + i_b\frac{nq}{2m}\right) - p_v b - (nA + mc_t)\frac{b}{nq} - h_{b,p}\left[\frac{nq}{2} - (n-1)\frac{qb}{2P}\right] - \left(h_{b,p} + h_{b,f}\right)k\sigma - \frac{B_r b\sigma}{q}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{31}$$

where $h_v = h_{v,f}^{v} + h_{v,p}$ and $h_b = h_{b,f} + h_{b,p}$. The optimal value of the decision variables that maximize eq. (31) is:

$$q^* = \sqrt{\frac{\left(A + \frac{mc_t}{n} + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)b}{h_{b,p}\left[\frac{n}{2} - (n-1)\frac{b}{2P}\right] - p_b i_b\frac{n}{2m}}} \tag{32}$$

Eq. (32) is valid when $h_{b,p}\left[\frac{n}{2} - (n-1)\frac{b}{2P}\right] - p_b i_b\frac{n}{2m} > 0$

The second scenario under a traditional policy considers that the vendor guarantees the buyer a permissible period to settle its balance at no additional cost, and the buyer offers a trade credit period of length N to its customers. The buyer pays the vendor by the end of the permissible period. The expected profit of the buyer can be written as:

$$TP_{b,2}^{D}(q,m,N) = p_b\left(be^{aN} + i_b\frac{(2\alpha+1)nq}{2m}\right) - p_v b - (nA + mc_t)\frac{be^{aN}}{nq} - h_{b,p}\left[\frac{nq}{2} - (n-1)\frac{qbe^{aN}}{2P}\right] - \left(h_{b,p} + h_{b,f}\right)k\sigma - p_b i_b N be^{aN} - \frac{B_r be^{aN}\sigma}{q}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{33}$$

The optimal values of the decision variables that maximize eq. (33) are:

$$q^* = \sqrt{\frac{\left(A + \frac{mc_t}{n} + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)be^{aN}}{h_{b,p}\left[\frac{n}{2} - (n-1)\frac{be^{aN}}{2P}\right] - p_b i_b\frac{(2\alpha+1)n}{2m}}} \tag{34}$$

Eq. (34) is valid when $h_{b,p}\left[\frac{n}{2} - (n-1)\frac{be^{aN}}{2P}\right] - p_b i_b\frac{(2\alpha+1)n}{2m} > 0$.

The third scenario considers a delay-in-payment where the buyer pays the invoice post the interest-free delay period and incurs a cost from interest charged by the vendor. The expected buyer total profit is given by:

$$TP_{b,3}^{D}(q,m,N) = p_b\left(be^{aN} + i_b\frac{\left(2\alpha+1+2\beta(1+\alpha)\right)nq}{2m}\right) - p_v be^{aN} - (nA + mc_t)\frac{be^{aN}}{nq} - h_{b,p}\left[\frac{nq}{2} - (n-1)\frac{qbe^{aN}}{2P}\right] - \left(h_{b,p} + h_{b,f}\right)k\sigma - h_{v,f}^{b}\frac{\beta(1+\alpha)nq}{m} - p_b i_b N be^{aN} - \frac{B_r be^{aN}\sigma}{q}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{35}$$

The optimal values of the decision variables that maximize eq. (35) are:

$$q^* = \sqrt{\frac{\left(A + \frac{mc_t}{n} + B_r\sigma \int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)be^{aN}}{h_{b,p}\left[\frac{n}{2} - (n-1)\frac{be^{aN}}{2P}\right] + h_{v,f}^{b}\frac{\beta(1+\alpha)n}{m} - p_b i_b \frac{\big(2\alpha + 1 + 2\beta(1+\alpha)\big)n}{2m}}} \quad (36)$$

Eq. (36) is valid when $h_{b,p}\left[\frac{n}{2} - (n-1)\frac{be^{aN}}{2P}\right] + h_{v,f}^{b}\frac{\beta(1+\alpha)n}{m} - p_b i_b \frac{(2\alpha+1+2\beta(1+\alpha))n}{2m} > 0$.

4. Numerical analysis

In this section, a numerical analysis is presented to illustrate the behaviour of the models developed for the three payment scenarios presented above. The input data used for this study comes from [4], [11], and [13] (see Table 1). The values for the parameters are: $\alpha = 0.1$, $a = 0.4$ year^{-1}, $A = 25$ \$/order, $\beta = 0.5$, $b = 1000$ unit/year, $B_r = 6$ \$/unit, $c_t = 0.5$ \$/transaction, $c_v = 1$ \$/unit, $\gamma = 1$, $h_{b,p} = 2.5$ \$/unit/year, $h_{v,p} = 4$ \$/unit/year, $i_b = 15$ %/year, $i_v = 10$ %/year, $k = 1.2816$, $P = 3200$ unit/year, $p_b = 7.29$ \$/unit, $p_v = 5.4$ \$/unit, $r_v = 3$ \$/unit, $\sigma = 1$ unit, $S = 100$ \$/setup. For scenario 2 and 3, the maximum delay period the vendor offers the buyer is 180 days.

The algorithms presented in the previous section (corresponding to each scenario) have been used to solve the numerical examples and were coded in Visual Basic for Applications in a Microsoft Excel environment; the results in Table 2 show that extending a payment period increases the supply chain expected profit. The supply chain profit increases by about 1% (from \$ 2382.73/year to \$ 2409.40/year) when Scenario 1 (no delay-in-payments) is adopted rather than Scenario 2 (interest-free delay period). Scenario 3 (charging interest over a delay period) increases the profit by about 7% (from \$ 2382.73/year to \$ 2551.57/year). Scenario 2 reduces the buyer's expected profit by about 4% (from \$ 1563.18/year to \$ 1500.86) due to increases in the opportunity cost. This increase is because the delay offered to the end customer are higher than the benefits from delays in payments to settle the vendor's invoice. The vendor can compensate the buyer for its loss by sharing some of its profits [9]. The economic performance of the vendor and the buyer improve with interest charges. The vendor benefits from charging interest charged on outstanding payments. Extended periods result in more interest charged. The buyer earns more interest in invested sales. This amount increases with *N* (increasing demand) and, subsequently, its profits, counterbalancing increasing costs. In this scenario, profit sharing is not necessary: interest charges act as a profit-sharing mechanism. The results suggest that longer delays are beneficial for supply chain performance. The production lot size (nq) increases when payments are delayed with (Scenario 3) or without (Scenario 2) interest charges. In the three scenarios, the optimal number of shipments is one ($m^* = 1$) since $h_{v,f}^{b} = 0.54 < p_b i_b = 1.094$.

Table 2. Optimal inventory and trade-credit policies for consignment stock agreement

Scenario		q (unit)	n (unit-less)	m (unit-less)	N (days)	TP_S (\$/year)	TP_V (\$/year)	TP_B (\$/year)
1	CS	167.29	2	1	0	2382.73	819.55	1563.18
2	CS	137.87	3	1	55	2409.40	908.54	1500.86
3	CS	144.56	4	1	105	2551.57	962.78	1588.79

The numerical examples in Table 2 were replicated for the backward policy, i.e., Hill's (1997) model, by using the formulae reported in section 3.4 with the results summarised in Table 3. By comparing the results in Tables 2 and 3, one can see that the supply chain and those of the players are higher for the CS agreement than for the backwards policy. The increases in profits are about 8% (from 2204.74 to 2382.73), 9% (from 2212.61 to 2409.4), and 12% (from 2286.97 to 2551.57) for the three scenarios, respectively. The vendor benefits the most by about 12% (from 734.93 to 819.55), 24% (from 735.83 to 908.54), and 30% (from 741.4 to 962.78), respectively. The buyer benefits the least by about 6% (from 1469.81 to 1563.18), 2% (from 1476.78 to 1500.86), and 3% (from 1545.56 to 1588.79), respectively. These results show that the vendor benefits more than the buyer when moving from Scenario 1, to 2, to 3. The results also show the delay periods, *N*, are significantly longer in the CS agreement than they are in the traditional for scenarios 2 and 3. The CS policy, which has larger lot sizes (nq) than the traditional one, increases supply chain profit by about 19% (from 280.42 to 334.58), 33% (from 310.2 to 413.61), and 109% (from 276.97 to 578.24), respectively. These results suggest that CS gives a supply chain a higher competitive advantage than the traditional policy. It also leaves a supply chain with more profits allowing it to invest a part to increase competitiveness. From an investor's point of view, higher profits mean higher dividends on shares. A CS with interest-charge delay-in-payments shows, for the conditions set in this paper, to be a win-win policy for all.

Table 3. Optimal inventory and trade-credit policies for the model of Hill (1997).

Scenario		q	n	m	N	TP_S	TP_V	TP_B
		(unit)	(unit-less)	(unit-less)	(days)	($/year)	($/year)	($/year)
1	Hill	140.21	2	2	0	2204.74	734.93	1469.81
2	Hill	155.10	2	2	7	2212.61	735.83	1476.78
3	Hill	276.97	1	1	0	2286.97	741.40	1545.56

The buyer's profitability decreases in the centralized model, compared to the decentralized model, as in the latter, its profit is higher. The comparison between the results shown in Table 4 and those in Table 2 illustrates that the proposed delay-in-payments model can resolve this issue of decreasing profit for the buyer while centralizing. Under coordinated decision making, this profitability is not always smaller than under decentralized decision making. Delay-in-payments represents a clear incentive to participate in coordinated decision making for one actor or both. Of course, its benefits are more realized as the permissible delay period becomes longer. Moreover, the supply chain expected profit in the centralized model is higher than the one in the decentralized model for about 2.81%, 0.79%, and 6.52% in scenario 1, 2, and 3, respectively, which means that the centralization introduced the higher benefits in a context with consignment stock agreement with interest-charge delay-in-payment and credit-dependent demand. It is also interesting to observe that in the centralized scenario with a CS agreement, the lot sizes are smaller and the days of delay are higher for both (decentralized and the centralized scenarios) with a traditional inventory policy.

Table 4. Optimal inventory and trade-credit policies when a decentralized consignment stock agreement is adopted.

Scenario		q	n	m	N	TP_S	TP_V	TP_B
		(unit)	(unit-less)	(unit-less)	(days)	($/year)	($/year)	($/year)
1	Decentralized	113.98	5	1	0	2317.25	876.88	1440.36
2	Decentralized	160.14	3	1	0.99	2390.28	824.62	1565.66
3	Decentralized	300.07	1	1	1.3	2395.07	683.44	1711.63

A sensitivity analysis is performed to study the effects of some relevant input parameters on the behaviour of the models and the selection of the best coordination policy, which are α, A,

S, b, P, i_b, i_v, $h_{v,p}$, and $h_{b,p}$. When varying one input parameter of the example in Table 3, the values of the others are kept unchanged.

Figure 7 shows the behaviour of the CS total profit, $TP_{s,i}$, for Scenarios 1, 2, and 3 (i =1,2, and 3) for changes in α, which is the fraction of the elapsed time, t (an interest–free period), between invoices for the buyer to make a payment. That is, it can pay immediately, α = 0, or by time αt. The results show that Scenario 3 is the dominant one ($TP_{s,3} > TP_{s,2} > TP_{s,1}$) for changes in α. This scenario favours a delay period that extends beyond what is permissible without interest charge, τ, benefiting the buyer and the vendor from interest amounts earned and charged, respectively. The results show that also TP_S for Scenario 2 increases as α increases. Figure 8 shows that $TP_{s,i}$ decreases steadily as the ratio of the buyer's order cost to the vendor's setup cost, A/S, increases. This finding suggests that the buyer and the vendor work together to eliminate necessary activities that add to the order cost. Keeping the order cost of the buyer at the lowest possible increases the profits for the three scenarios. Figure 9 shows that $TP_{s,i}$, with i =1,2, and 3, increases as the ratio of the market size to production rate, b/P, increases, where $TP_{s,2}$ is slightly larger than $TP_{s,1}$, and that Scenario 3 dominates ($TP_{s,3} > TP_{s,2} > TP_{s,1}$). The results also show that $TP_{s,3} - TP_{s,2}$ and $TP_{s,3} - TP_{s,1}$ increase as b/P ratio increases. A slower production rate P increases $TP_{s,i}$, mainly related to decreasing holdings costs. Scenario 3 has the highest reduction of the three. Figure 10 shows the behaviour of $TP_{s,i}$, i =1,2, and 3, for changes in the ratio of the interest earned by the buyer on sales to that charged by the vendor on an outstanding payment, i_b/i_v, where $TP_{s,3}$ increases exponentially as i_b/i_v increases. Similar behaviour can be found in finance or engineering economics textbooks. The exponential behaviour of an invested amount becomes visible in long investment periods coupled with high-interest-earnings. $TP_{s,2}$ on the other hand, decreases below $TP_{s,3}$ in the direction of $TP_{s,1}$ to a deflection point where its decrease slows, up to values close to $TP_{s,1}$. This behaviour has to do with a higher interest rate charged to the buyer and shorter delays periods to customers; hence, scenarios 2 and 3 with delay-in-payments approach scenario 1. The last analyses concern variations in the ratio of the vendor to the buyer's physical holding cost and the demand variance, whose behaviour is shown Figures 11 and 12.

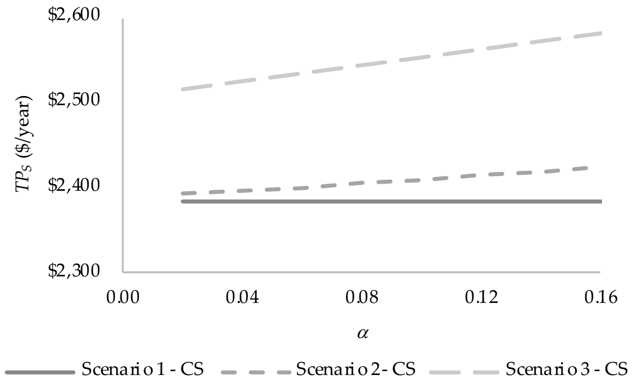

Figure 7. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the fraction of the invoice time given to the buyer to settle its payment (interest-free scenario).

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

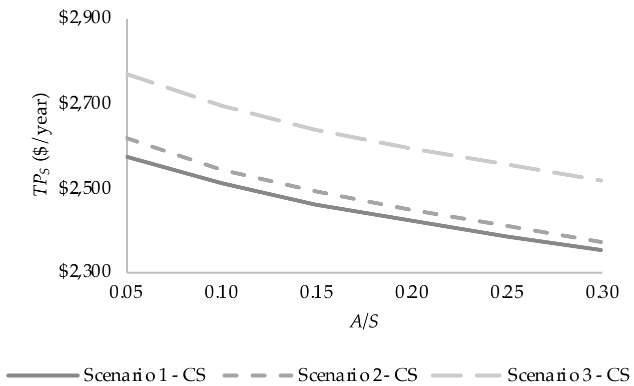

Figure 8. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the ratio of order to setup costs.

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

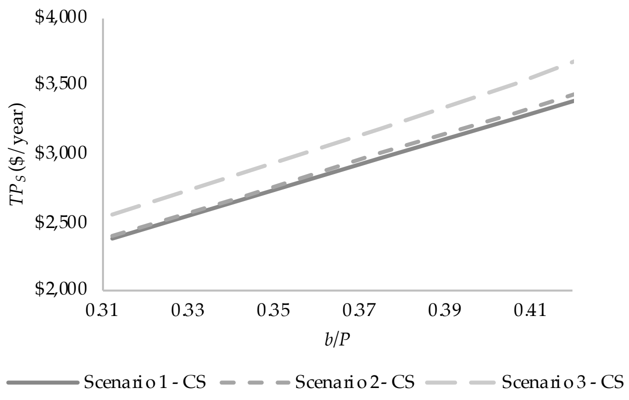

Figure 9. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the ratio of the expected demand without trade.

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

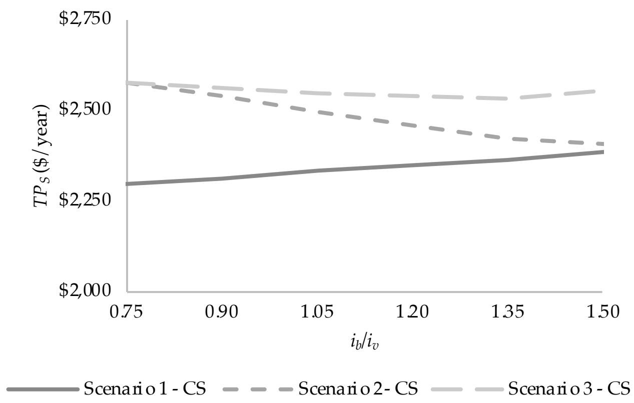

Figure 10. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the ratio of the buyer to the vendor capital costs.

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

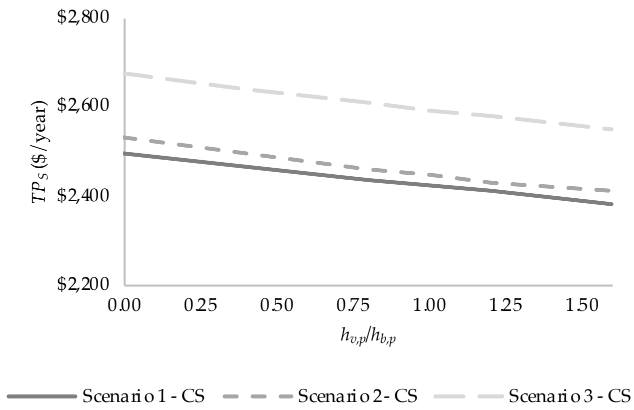

Figure 11. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the ratio of the buyer to the vendor physical holding costs.

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

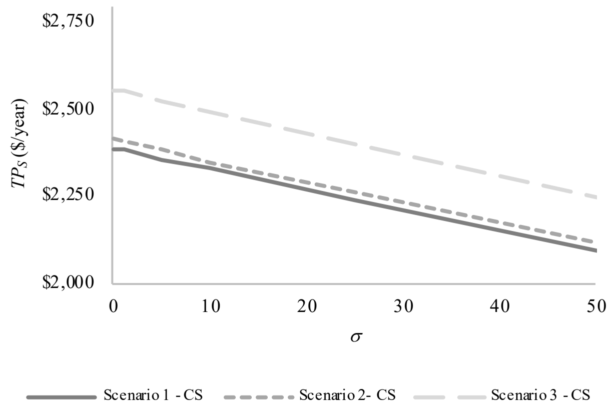

Figure 12. The behaviour of the supply chain expected total profit with consignment stock agreement for changes in the variance of the demand.

Notes: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

We define the variation in the total profit of the supply chain obtained from implementing the traditional agreement proposed by Hill, for the scenario with a consignment stock agreement as $\Delta TP_{S,CS-Hill} = \frac{TP_{s,i}^{Hill} - TP_{s,i}^{CS}}{TP_{s,i}^{CS}}$. Negative values of $\Delta TP_{S,CS-Hill}$ mean that the consignment stock agreement is more convenient than the traditional one in Hill's. Increasing the length of the interest-free period, the expected annual profit of the supply chain increases by about 1.3 % and 2.7 % (Figure 7), and the convenience of the consignment stock agreement over the Hill's model increases by about 0.76 % and 1.21 % in the considered range (Figure 13), for Scenario 2 and 3, respectively. Figures 8 and 14 shows that increasing the order cost lowers the total profit and the benefits that a consignment stock agreement brings. Scenario 3 is affected, while the others are not, where the reduction is 3.27% (Figure 14). An increase in the ratio of demand to the production rate increases the supply chain profit for the three (1, 2, and 3) by 49.0 %, 49.5 %, and 52 %, respectively (Figure 9). Unlike the first and the second, the third gives customers an extended delay period resulting in higher profits. For Scenario 3, the preference of the CS agreement over Hill's model increases exponentially but slightly decreases for the other two scenarios (Figure 15). Figure 10 shows the behaviour of the supply chain profit for different values of the ratio of the vendor to buyer capital costs. Scenario 3 performs better than 1and 2 for high i_b/i_v values, but not so for ones. The results also show that the consignment stock policy performs better than the traditional one of Hill for all scenarios when the ratio of the vendor to buyer capital costs is high. Noticeably, the profit difference (CS minus traditional) for scenario 3 (Figure 16) has, like the others, a decreasing trend but with a noticeable variation. This behaviour has to do with operational decisions affecting the number of shipments and the length of the delay period offered to customers. Figure 17 shows lower profits for higher $h_{v,p}/h_{b,p}$ values. The policy of Hill performs better than the CS one for Scenario 1 and when the ratios are more than 0.4 and 0.3, respectively. Figure 18 highlights that the CS policy performs better than the Hill's for all instances and that this convenience increases for higher variance, especially for Scenarios 2 and 3.

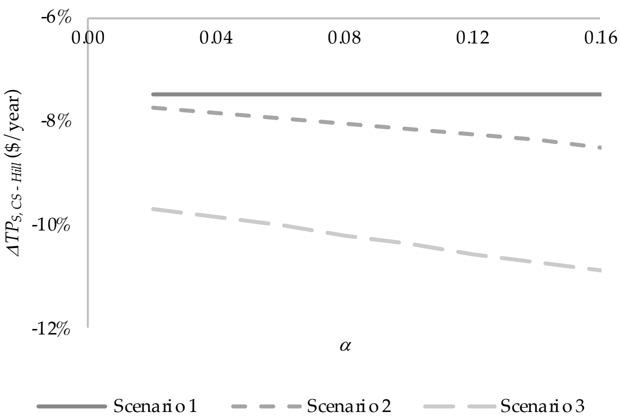

Figure 13. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the fraction of the invoice time given to the buyer to settle its payment (interest-free scenario).

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

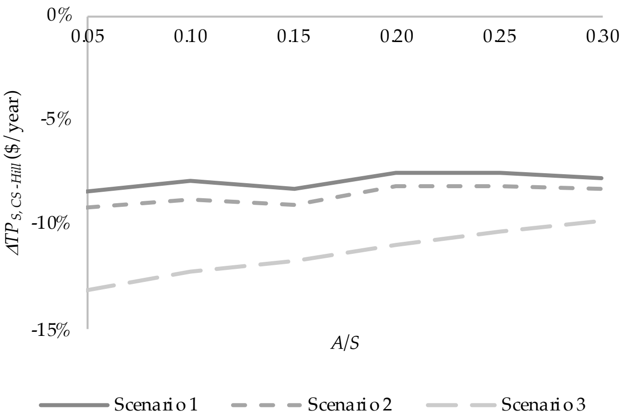

Figure 14. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the ratio of order to setup costs.

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

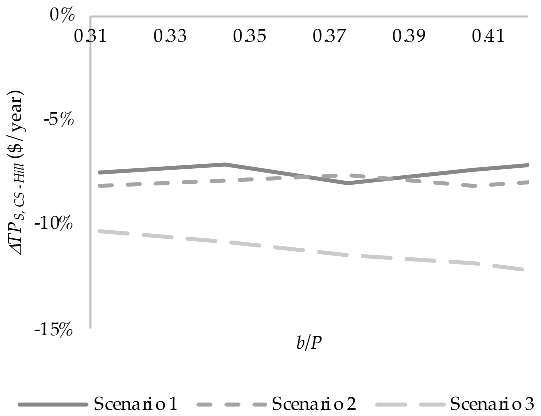

Figure 15. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the ratio of the expected demand without trade credit to that of the end customer.

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

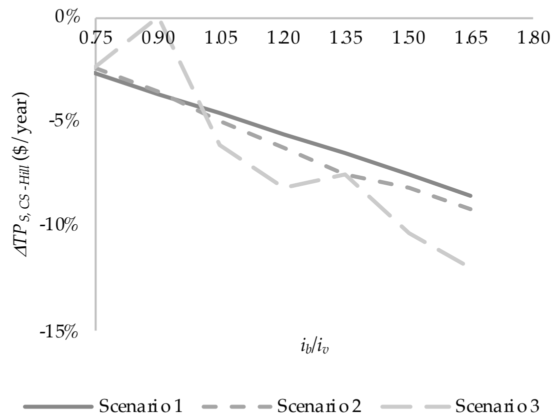

Figure 16. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the ratio of the buyer to the vendor capital costs.

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

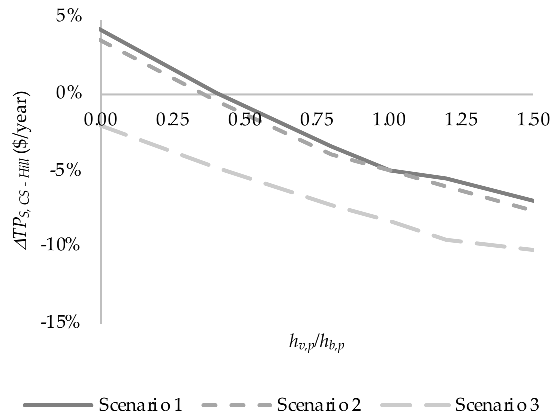

Figure 17. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the ratio of the buyer to the vendor physical holding costs.

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

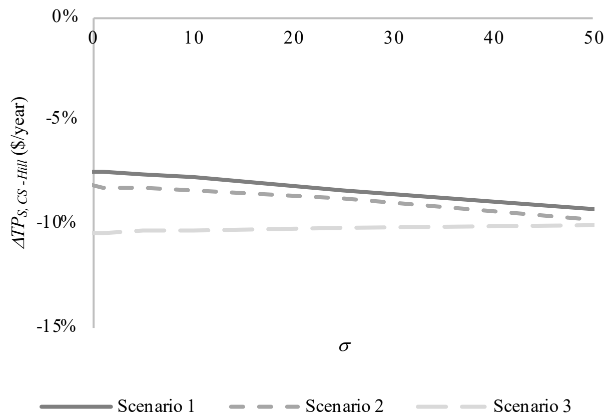

Figure 18. The behaviour of the difference in supply chain expected total profit, consignment stock versus the traditional policy of Hill for changes in the variance of the demand.

Note: **Scenario 1:** No delay-in-payments; **Scenario 2:** Delay in payments with no interest charges; **Scenario 3:** Delay in payment with interest charges

5. Conclusions

This paper investigated a vendor-buyer supply chain with a two-level (vendor-buyer) delay-in-payments with a credit-dependent demand showing that the length of credit periods and interest charges affect product demand. It assumed a consignment stock agreement between the players by coordinating the production quantity and the sizes and frequency of shipments. Delay-in-payments is a favoured business practice in inventory management with consignment stock agreements. The literature on the topic is scarce when it comes to demand being a function of the length of the trade credit period offered by the buyer to the end customer. Three trade-credit scenarios (models) were developed. The first considers a scheme where the buyer makes equal payments to the vendor with no delay-in-payments. The second and third models consider delay-in-payments. In the second scenario, the buyer settles its balance on or before the end of the interest-free delay period. The third scenario permits the buyer to pay the vendor after the interest-free period has passed, incurring interest charges in an outstanding balance. The models were solved by deriving closed-form expressions for the lot size and the number of shipments. The developed solution procedure optimizes the values of the number of payments made by the buyer to the vendor and the length of the trade credit periods given to the buyer and the end customers. The paper also extended the model of Zahran et al. [11] by assuming a credit-dependent demand as in Heydari et al. [13], who did not consider consignment stock coordination. The proposed model can benefit supply chain managers who wish to reap the benefits of consignment stock agreements coupled with delay-in-payments.

Numerical analyses were performed to understand the behaviour of the model and to draw some managerial insights. The results showed that a CS policy is more profitable to a supply chain, especially in the scenario where charging interest over a delay period is considered, leading to cost savings up to 12% with respect to the Hill's and up to 7% with respect to decentralized decision-making. They also showed that it gives managers better flexibility in deciding on the lot size quantity, which adds convenience. The CS policy performed much better with delay-in-payments, as reflected in the profit difference between CS and traditional. When the vendor grants the buyer extended delay-in-payments subject to interest charges, the supply chain profit for this scenario exceeds those of the other two and shows a significant preference of CS over traditional. So, offering interest-free delay-in-payments periods, the vendor's profit reduces because its capital is tied up in inventory, thus, losing the opportunity to invest its revenues. The results also showed that introducing interest charges on payments settled after the free-interest permissible delay period improves the profitability of the vendor and the buyer. In this case, extended delay periods play the role of a profit-sharing mechanism, where the vendor earns interest on the balance that the buyer owes, and the latter invests that balance and earns interest until the payment is due. The results also suggested that longer delays are beneficial for supply chain performance and increase production lot size. Moreover, the sensitivity analysis showed the robustness of the CS profit gains compared to the traditional policy for the three scenarios investigated.

The implementation of a consignment stock agreement between companies requires strong cooperation between the vendor and the buyer, pushing them towards a complete exchange of information and a consistent sharing of management risks. To make it feasible, it should lead to a win-win solution by being beneficial for both players. One could also conclude that delay-in-payments can act as a profit-sharing mechanism; however, they cannot always guarantee a win-win solution. This work could be, hence, developed further by incorporating other financial arrangements between supply chain partners, such as Reverse Factoring [17], or integrating third parties, e.g., financial institution, such as warehouse financing or the use of forward-contracts to mitigate the commodity risk and to increase the profit of the players [18]. One of the main limitations of this work is also the fixed variance of the demand. A direct extension of this paper could include the demand variance as a function of the credit period length offered to the customers. Another research stream consists of coordinating supply chain finances in energy-related decisions to stimulate investments in energy-efficient technologies, as recently investigated by [19].

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Appendix A

A.1 Scenario 1

This section shows that the expected supply chain total profit, TP_s, is concave in q (lot size) and n (number of shipments, and how their optimal values are determined. First, we calculate the first and second partial derivatives of Eq. (5) in q, and are given as:

$$\frac{\partial TP_{s,1}(q,n,m)}{\partial q} = \frac{(S+nA+mc_t)b}{nq^2} - \frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) - \frac{b}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) - \frac{n}{2m}\left(h_{v,f}^b - p_b i_b\right) + \frac{B_r b\sigma}{q^2}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{A1}$$

$$\frac{\partial^2 TP_{s,1}(q,n,m)}{\partial q^2} = -\frac{2(S+nA+mc_t)b}{nq^3} - \frac{B_r b\sigma}{q^3}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx < 0 \rightarrow \exists q^* \tag{A2}$$

The second partial derivative of Eq. (5), as seen from (A2), is concave in $q > 0$for given values of $n > 0$ and $m > 0$. The optimal lot size, q^*, is determined from (A1) by setting it equal to zero and solving for q to get:

$$q^* = \sqrt{\frac{\left[\frac{S+nA+mc_t}{n} + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]b}{\frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) + \frac{b}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) + \frac{n}{2m}\left(h_{v,f}^b - p_b i_b\right)}} \tag{A3}$$

By substituting q^* in $TP_{s,1}(q,n,m)$, it would then be possible to investigate the concavity of the total profit in n, following the same steps previously defined for the evaluation of the optimal lot size, which is given as:

$$TP_{s,1}(q^*,n,m)$$
$$= (p_b - \gamma r_v - c_v)b$$
$$- 2\sqrt{\frac{\left[\frac{S + nA + mc_t}{n} + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]}{b\left(\frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) + \frac{b}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) + \frac{n}{2m}\left(h_{v,f}^b - p_b i_b\right)\right)}}$$
$$- \left(h_{b,p} + h_{b,f}\right)k\sigma \tag{A4}$$

The value of *n* that maximizes Eq. (A4) is the same as the one that minimizes the second term since it represents the only cost affected by the number of shipments. The optimal value of n^* is determined by setting the first partial derivative to zero, after showing that the second derivative is positive for all values of *n*, and solving for *n* to get:

$$\frac{\partial TP_{s,1}(q^*,n,m)}{\partial n} = -\frac{S + mc_t}{n^2}\frac{b}{2P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right)$$
$$+ \left[A + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]\frac{1}{2}\left(\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) + \frac{\left(h_{v,f}^b - p_b i_b\right)}{m}\right) \tag{A5}$$

$$\frac{\partial^2 TP_{s,1}(q^*,n,m)}{\partial n^2} = \frac{S + mc_t}{n^3}\frac{b}{P}\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right) > 0 \rightarrow \exists n^* \tag{A6}$$

$$n^* = \sqrt{\frac{(S + mc_t)b\left(h_{v,p} + h_{v,f}^v + h_{v,f}^b + h_{b,p}\right)}{\left[A + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]P\left(\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) + \frac{\left(h_{v,f}^b - p_b i_b\right)}{m}\right)}} \tag{A7}$$

Eq. (A7) is valid when $\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{b}{P}\right) + \frac{\left(h_{v,f}^b - p_b i_b\right)}{m} > 0$.

A.2 Scenario 2

This section shows that the expected supply chain total profit, TP_s, is concave in q and n, and how their optimal values are determined. First, we calculate the first and second partial derivatives of Eq. (10) in q, and are given as:

$$\frac{\partial TP_{s,2}(q,n,m,N)}{\partial q} = \frac{(S+nA+mc_t)be^{aN}}{nq^2} - \frac{n}{2}\left(h_{v,f}^b + h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) - \frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^v+h_{v,f}^b+h_{b,p}\right) - \frac{n}{2m}(2\alpha+1)\left(h_{v,f}^b - p_b i_b\right) + \frac{B_r be^{aN}\sigma}{q^2}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{A8}$$

$$\frac{\partial^2 TP_{s,2}(q,n,m,N)}{\partial q^2} = -\frac{2(S+nA+mc_t)be^{aN}}{nq^3} - \frac{B_r be^{aN}\sigma}{q^3}\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx < 0 \rightarrow \exists q^* \tag{A9}$$

The second partial derivative of Eq. (10), as seen from (A9), is concave in q for given values of n, m, and N. The optimal lot size, q^*, is determined by setting (A8) equal to zero and solving for q to get:

$$q^* = \sqrt{\frac{\left[\frac{S+nA+mc_t}{n} + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}}{\frac{n}{2}\left(h_{v,f}^b+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) + \frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^v+h_{v,f}^b+h_{b,p}\right) + + \frac{n}{2m}(2\alpha+1)\left(h_{v,f}^b - p_b i_b\right)}} \tag{A10}$$

By substituting q^* in $TP_{s,2}(q,n,m,N)$, it would then be possible to investigate the concavity in n, following the same steps previously defined for the evaluation of the optimal lot size, which is given as:

$$TP_{s,2}(q^*,n,m,N) = (p_b - \gamma r_v - c_v)be^{aN} - 2\sqrt{\left[\frac{S+nA+mc_t}{n} + B_r\sigma\int_k^\infty (x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}\cdot \cdot\left\{\frac{n}{2}\left(h_{v,f}^b+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) + \frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^v+h_{v,f}^b+h_{b,p}\right) + + \frac{n}{2m}(2\alpha+1)\left(h_{v,f}^b - p_b i_b\right)\right\}} - \left(h_{b,p}+h_{b,f}\right)k\sigma - p_b i_b N be^{aN} \tag{A11}$$

The second term in Eq.(A11) is the only one dependent on n and minimizing it for n is like maximizing Eq. (A11). The optimal value of n, n^*, is determined by setting the first partial derivative of Eq. (A11) to zero and solving for n to get:

$$\frac{\partial TP_{s,2}(q^*,n,m,N)}{\partial n} = -\frac{S+mc_t}{n^2}\frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right) - \left[A+B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]\left\{\frac{n}{2}\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) + \frac{n}{2m}(2\alpha+1)\left(h_{v,f}^{b}-p_b i_b\right)\right\} \tag{A12}$$

$$\frac{\partial^2 TP_{s,2}(q^*,n,m,N)}{\partial n^2} = \frac{S+mc_t}{n^3}\frac{be^{aN}}{P}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right) > 0 \rightarrow \exists n^* \tag{A13}$$

$$n^* = \sqrt{\frac{(S+mc_t)be^{aN}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right)}{\left[A+B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]P\begin{pmatrix}\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right)+\\ +\frac{(2\alpha+1)\left(h_{v,f}^{b}-p_b i_b\right)}{m}\end{pmatrix}}} \tag{A14}$$

Eq.(A14) is valid when $\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right)+\frac{(2\alpha+1)\left(h_{v,f}^{b}-p_b i_b\right)}{m} > 0$.

A.3 Scenario 3

This section shows that the expected supply chain total profit, TP_s, is concave in q and in n, and how their optimal values are determined. First, we calculate the first and second partial derivatives of Eq. (19), and are given as:

$$\frac{\partial TP_{s,3}(q,n,m,N)}{\partial q} = \frac{(S+nA+mc_t)be^{aN}}{nq^2} - \frac{n}{2}\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right) - \frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right) - \frac{n}{2m}\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^{b}-p_b i_b\right) + \frac{B_r be^{aN}\sigma}{q^2}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \tag{A15}$$

$$\frac{\partial^2 TP_{s,3}(q,n,m,N)}{\partial q^2} = -\frac{2(S+nA+mc_t)be^{aN}}{nq^3} - \frac{B_r be^{aN}\sigma}{q^3}\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx < 0 \rightarrow \exists q^* \tag{A16}$$

Eq. (19), as seen from (A16), is concave in q for given values of n, m, and N. The optimal lot size, q^*, is determined by setting (A15) equal to zero and solving for q to get:

$$q^{*}=\sqrt{\frac{\left[\frac{S+nA+mc_t}{n}+B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}}{X}} \tag{A17}$$

where

$$X=\frac{n}{2}\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right)+\frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right)$$
$$+\frac{n}{2m}\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^{b}-p_b i_b\right)$$

by substituting q^{*} in $TP_{s,3}(q,n,m,N)$, it would then be possible to investigate the concavity of the total profit in n, following the same steps previously defined for the evaluation of the optimal lot size, which is given as:

$$\begin{aligned}TP_{s,3}(q^{*},n,m,N)&=(p_b-\gamma r_v-c_v)be^{aN}\\&-2\sqrt{\frac{\left[\frac{S+nA+mc_t}{n}+B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]be^{aN}}{\left\{\begin{array}{c}\frac{n}{2}\left(h_{v,f}^{b}+h_{b,p}\right)\left(1-\frac{be^{aN}}{P}\right)+\frac{be^{aN}}{2P}\left(h_{v,p}+h_{v,f}^{v}+h_{v,f}^{b}+h_{b,p}\right)+\\\frac{n}{2m}\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^{b}-p_b i_b\right)\end{array}\right\}}}\\&-\left(h_{b,p}+h_{b,f}\right)k\sigma-p_b i_b N be^{aN}\end{aligned} \tag{A18}$$

The n value that maximize Eq. (A18) is the same that minimize the second term since it represents the only cost affected by the number of shipments. The optimal value of n, n^{*}, is determined by setting the first partial derivative to zero, after showing that the second derivative is positive for all values of n, and solving for n to get:

$$\frac{\partial TP_{s,3}(q^*, n, m)}{\partial n} = -\frac{S + mc_t}{n^2}\frac{be^{aN}}{2P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) - \left[A + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]\left\{\begin{array}{c}\frac{n}{2}\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) + \frac{be^{aN}}{2P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) + \\ \frac{n}{2m}\left(1 + 2\alpha + 2\beta(1+\alpha)\right)\left(h_{v,f}^{b} - p_b i_b\right)\end{array}\right\} \quad \text{(A19)}$$

$$\frac{\partial^2 TP_{s,3}(q^*, n, m)}{\partial n^2} = \frac{S + mc_t}{n^3}\frac{be^{aN}}{P}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right) > 0 \rightarrow \exists n^* \quad \text{(A20)}$$

$$n^* = \sqrt{\frac{(S + mc_t)be^{aN}\left(h_{v,p} + h_{v,f}^{v} + h_{v,f}^{b} + h_{b,p}\right)}{\left[A + B_r\sigma\int_k^{\infty}(x-k)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right]P\left(\begin{array}{c}\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) + \\ + \frac{\left(1 + 2\alpha + 2\beta(1+\alpha)\right)\left(h_{v,f}^{b} - p_b i_b\right)}{m}\end{array}\right)}} \quad \text{(A21)}$$

Eq. (A21) is valid when $\left(h_{v,f}^{b} + h_{b,p}\right)\left(1 - \frac{be^{aN}}{P}\right) + \frac{\left(1+2\alpha+2\beta(1+\alpha)\right)\left(h_{v,f}^{b} - p_b i_b\right)}{m} > 0$.

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