Finite Fields and Their Applications 113 (2026) 102830

Contents lists available at ScienceDirect

Finite Fields and Their Applications

journal homepage: www.elsevier.com/locate/ffa

Linear codes arising from the point-hyperplane geometry — Part II: the twisted embedding

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ARTICLE INFO

Article history:
Received 23 July 2025
Received in revised form 9 February 2026
Accepted 11 March 2026
Available online xxxx
Communicated by Gary L. Mullen

MSC:
51E22
94B05
14M12

Keywords:
Twisted embedding
Point-hyperplane geometry
Projective code
Semilinear map

ABSTRACT

Let $\bar{\Gamma}$ be the point-hyperplane geometry of a projective space $\mathrm{PG}(V)$, where V is a $(n+1)$-dimensional vector space over a finite field $\mathbb{F}_q$ of order q. Suppose that σ is an automorphism of $\mathbb{F}_q$ and consider the projective embedding ε_σ of $\bar{\Gamma}$ into the projective space $\mathrm{PG}(V \otimes V^*)$ mapping the point $([x],[\xi]) \in \bar{\Gamma}$ to the projective point represented by the pure tensor $x^\sigma \otimes \xi$, with $\xi(x) = 0$. In [11], we focused on the case $\sigma = 1$ and we studied the projective code arising from the projective system $\Lambda_1 = \varepsilon_1(\bar{\Gamma})$. Here we focus on the case $\sigma \neq 1$ and we investigate the linear code $\mathscr{C}(\Lambda_\sigma)$ arising from the projective system $\Lambda_\sigma = \varepsilon_\sigma(\bar{\Gamma})$. In particular, after having verified that $\mathscr{C}(\Lambda_\sigma)$ is a minimal code, we determine its parameters, its minimum distance as well as its automorphism group. We also give a (geometrical) characterization of its minimum and second lowest weight codewords and determine its maximum weight when q and n are both odd.

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https://doi.org/10.1016/j.ffa.2026.102830
1071-5797/

1. Introduction

Let V be a vector space of dimension $k+1$ over a finite field $\mathbb{F}_q$ of order q. It is well known that given a set Ω of N points spanning $\mathrm{PG}(V)$ (a *projective system of* $\mathrm{PG}(V)$) it is possible to construct a projective code $\mathscr{C}(\Omega)$ of length N and dimension $k+1$ by taking as generator matrix for $\mathscr{C}(\Omega)$ a matrix G whose columns are vector representatives of the points of Ω; see [21]. This approach, pioneered by MacWilliams in her Ph.D. Thesis [16, Lemma 1.5], establishes a nice interplay between the properties of the code and the geometry of the projective system itself; for instance, the minimum distance of the code, its higher Hamming weights, its minimality as well as its automorphism group can all be studied by looking at the geometric properties of Ω.

In a series of papers, we have been interested in studying codes arising from embeddings of point-line geometries; see [8–12].

In particular, in [11] we considered the linear code arising from a special subvariety Λ_1 of the Segre variety Λ of $\mathrm{PG}(V \otimes V^*)$, where V^* is the dual space of V, consisting of all pure tensors $x \otimes \xi \in V \otimes V^*$ such that $\xi(x) = 0$.

The Segre variety is an important algebraic variety arising from the Segre embedding ε of the Segre geometry Γ of $\mathrm{PG}(V \otimes V^*)$. Accordingly, $\Lambda = \varepsilon(\Gamma)$. The variety Λ has been extensively studied, as it corresponds to the determinantal variety consisting of all matrices of rank 1. The code arising from the projective system of the $\mathbb{F}_q$-rational points of Λ has been investigated by Beleen, Ghorpade and Hasan in 2015 [4].

In [11] we introduced and studied a family of codes associated to the projective system Λ_1 defined by the image under the *Segre embedding* of the point-hyperplane geometry of $\mathrm{PG}(V)$. Keeping the same notation as in [11], we denote by $\bar{\Gamma}$ the point-hyperplane geometry of $\mathrm{PG}(V)$. Then $\bar{\Gamma}$ is a *geometric hyperplane* of Γ and the *Segre embedding* $\bar{\varepsilon}$ of $\bar{\Gamma}$ maps any point $([x],[\xi])$ of $\bar{\Gamma}$ into the point $[x \otimes \xi] \in \mathrm{PG}(M^0_{n+1}(q))$, where $M^0_{n+1}(q)$ is a hyperplane of $V \otimes V^*$. According to this notation, we have $\Lambda_1 := \bar{\varepsilon}(\bar{\Gamma})$.

This paper can be considered as the natural continuation of [11]: now we investigate the linear code arising from another relevant subvariety Λ_σ of the Segre variety defined again as the image of the point-hyperplane geometry $\bar{\Gamma}$ but under the so-called *twisted embedding* of $\bar{\Gamma}$, which turns out to be a projective embedding not isomorphic to the Segre embedding (if $\sigma \neq 1$).

The existence of the twisted embedding assumes the existence of non-trivial automorphisms of $\mathbb{F}_q$ (see Section 2.2) and its definition is a slight modification of the definition of the Segre embedding. Indeed, suppose $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$, $\sigma \neq 1$. Then the *twisted embedding* $\bar{\varepsilon}_\sigma$ of $\bar{\Gamma}$ maps any point $([x],[\xi])$ of $\bar{\Gamma}$ into the point $[x^\sigma \otimes \xi] \in \mathrm{PG}(V \otimes V^*)$. When $\sigma = 1$, ε_1 is just the Segre embedding. Accordingly,

$$\Lambda_\sigma := \bar{\varepsilon}_\sigma(\bar{\Gamma}) = \{[x^\sigma \otimes \xi] \colon [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*) \text{ and } [x] \in [\xi]\}$$

In the present paper we study the code $\mathscr{C}(\Lambda_\sigma)$ arising from Λ_σ for $\sigma \neq 1$.

We briefly recall that the twisted embedding $\bar{\varepsilon}_\sigma$ of $\bar{\Gamma}$ is a *homogeneous embedding* for any $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$, which means that the group $\mathrm{Aut}\,(\bar{\Gamma})$ can be lifted to act on Λ_σ by $\bar{\varepsilon}_\sigma$. More in detail, there exists a monomorphism $\pi : \mathrm{Aut}\,(\bar{\Gamma}) \to \mathrm{PGL}(V \otimes V^*)$ such that $\bar{\varepsilon}_\sigma(x^g) = (\bar{\varepsilon}_\sigma(x))^{\pi(g)}$, $\forall g \in \mathrm{Aut}\,(\bar{\Gamma})$.

When $\sigma \neq \sigma'$, the embeddings $\bar{\varepsilon}_\sigma$ and $\bar{\varepsilon}_{\sigma'}$ are not isomorphic (see [18, Corollary 1.13]). One striking difference between Λ_1 and Λ_σ for $\sigma \neq 1$ is that Λ_σ spans the whole of $\mathrm{PG}(V \otimes V^*)$ while Λ_1 spans a hyperplane of $\mathrm{PG}(V \otimes V^*)$.

1.1. Main results

In this paper, after introducing the family of codes $\mathscr{C}(\Lambda_\sigma)$ arising from the projective system Λ_σ defined in the Introduction, we will determine their parameters and their automorphism group. We will also prove that $\mathscr{C}(\Lambda_\sigma)$ is a minimal code and characterize its minimum and second lowest weight codewords. More in detail, the main results are the following.

Theorem 1.1. *Suppose V is a $(n+1)$-dimensional vector space over $\mathbb{F}_q$ and $1 \neq \sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$. Let Λ_σ be the projective system of $\mathrm{PG}(V \otimes V^*)$ whose points are represented by pure tensors $x^\sigma \otimes \xi$ such that $\xi(x) = 0$. The $[N_\sigma, k_\sigma, d_\sigma]$-linear code $\mathscr{C}(\Lambda_\sigma)$ associated to Λ_σ has parameters*

$$N_\sigma = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}, \quad k_\sigma = n^2 + 2n + 1,$$

$$d_\sigma = \begin{cases} q^3 - \sqrt{q}^3 & \text{if } \sigma^2 = 1 \text{ and } n = 2, \\ q^{2n-1} - q^{n-1} & \text{if } \sigma^2 \neq 1 \text{ or } n > 2. \end{cases}$$

Suppose $a \in \mathbb{F}_q$ and $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$. Denote by $\mathbb{K} = \mathbb{F}_s$ the fixed field of σ. When $\sigma \neq 1$ and $\sigma^2 = 1$ we put $s = \sqrt{q}$ and we define the norm of a as $N(a) := a^{s+1}$.

In the next theorem we characterize the minimum, the second lowest and some of the maximum weight codewords in terms of geometrical hyperplanes of the geometry $\bar{\Gamma}$.

We refer to Section 2.2 for the concept of *geometrical hyperplane of* $\bar{\Gamma}$ *arising from an embedding* and to Section 2.5 for the definition and description of the hyperplanes of $\bar{\Gamma}$ mentioned in the following theorem.

Theorem 1.2. *Let $\mathscr{C}(\Lambda_\sigma)$ be the code as defined in Theorem 1.1. The following hold.*

1. *The code $\mathscr{C}(\Lambda_\sigma)$ is minimal.*
2. *If $n = 2$ and $\sigma^2 = 1$, $\sigma \neq 1$ then the minimum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight $q^3 - \sqrt{q}^3$ and correspond to the hyperplanes of $\bar{\Gamma}$ associated to matrices M for which there exist three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in V^*$ and $\alpha, \beta, \gamma \in \mathbb{F}_q$ with $N(\alpha) = N(\beta) = N(\gamma)$ such that*

$$\xi_1 M = \alpha \xi_1^\sigma, \qquad \xi_2 M = \beta \xi_2^\sigma, \qquad \xi_3 M = \gamma \xi_3^\sigma.$$

3. *If $n > 2$ or $\sigma^2 \neq 1$ then the minimum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight $q^{2n-1} - q^{n-1}$ and correspond to quasi-singular, non-singular hyperplanes of $\bar{\Gamma}$. The second lowest weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight q^{2n-1} and correspond to singular hyperplanes of $\bar{\Gamma}$.*
4. *If both q and n are odd, then the maximum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight $q^{n-1}(q^{n+1}-1)/(q-1)$. Every semi-standard spread type hyperplane of $\bar{\Gamma}$ is associated to a maximum weight codeword of $\mathscr{C}(\Lambda_\sigma)$.*

We point out that *semi-standard spread type hyperplanes* may exist only in the case in which $\sigma^2 = 1$ (i.e. σ is involutory) and n is odd.

It is interesting to point out that the minimum and the second lowest weight codewords of $\mathscr{C}(\Lambda_\sigma)$ are associated to the same families of hyperplanes of $\bar{\Gamma}$ as the minimum and the second lowest weight codewords of the code $\mathscr{C}(\Lambda_1)$ studied in [11]; likewise the argument leading to the minimality of $\mathscr{C}(\Lambda_\sigma)$ depends only on $\bar{\Gamma}$. This suggests that the properties of the geometry $\bar{\Gamma}$ play a crucial role in giving information on the structure of the codes associated to it, regardless the way $\bar{\Gamma}$ is embedded.

We refer to the beginning of Section 3 for a description of the codewords c_M of $\mathscr{C}(\Lambda_\sigma)$.

Theorem 1.3. *The group* $\mathrm{GL}(n+1, q)$ *acts as an automorphism group of the code* $\mathscr{C}(\Lambda_\sigma)$ *via the action*

$$\varrho(g) : c_M \to c_{g^{-1} M g^\sigma}.$$

The kernel of this action is the subgroup of all scalar matrices of the form $K_\sigma := \{\alpha I : \alpha \in \mathbb{F}_s\}$.

1.2. Organization of the paper

Since we are considering codes arising from different embeddings of the same geometry $\bar{\Gamma}$, this work can be regarded as a continuation of [11]. Consequently, some of the set-up (as well as the notation) of the present paper overlaps with those of [11], even if the specific arguments and results about the code turn out to be quite different. In Section 2 we shall limit the presentation of the topics otherwise introduced in [11, Section 2], without providing proofs or extensive discussion (for which we refer the reader to the previous paper).

The organization of the paper is as follows. In Section 2.1 we set the notation to be used throughout the paper and recall the notion of *saturation form* for the space $M_{n+1}(q)$. In Section 2.2 we recall the basics about point-line geometries and their embeddings; in Section 2.3 we recall the basics about codes from projective systems and minimal codes; we refer to [6,11] for more details. Sections 2.4 and 2.5 are dedicated to the point-hyperplane geometry and its hyperplanes; we also discuss here its natural and its twisted embedding. In Section 3 we will prove our main theorems. Theorem 1.1 will be proved in Subsection 3.4, Theorem 1.2 will be proved in Subsection 3.5 and Theorem 1.3 will be proved in Subsection 3.6.

2. Notation and basics

2.1. Notation

Let $V = V(n+1, \mathbb{F}_q)$ be a $(n+1)$-dimensional vector space over $\mathbb{F}_q$ and V^* its dual. Henceforth we always assume that $E = (\mathbf{e}_i)_{i=1}^{n+1}$ is a given fixed basis of V and will always be regarded as expressed by their components with respect to $E^* = (\boldsymbol{\eta}_i)_{i=1}^{n+1}$ is its dual basis in V^* so that $\boldsymbol{\eta}_i(\mathbf{e}_j) = \delta_{ij}$. Vectors these bases. We also adopt the convention that the elements of V are denoted by roman letters and are column vectors, while the elements of V^* are row vectors denoted by greek letters.

Given an automorphism σ of $\mathbb{F}_q$ and $x = \sum_{i=1}^{n} \mathbf{e}_i x_i \in V$, $\xi = \sum_{i=1}^{n} \boldsymbol{\eta}_i \xi_i \in V^*$, we put $x^\sigma = \sum_{i=1}^{n} \mathbf{e}_i x_i^\sigma$ and $\xi^\sigma = \sum_{i=1}^{n} \boldsymbol{\eta}_i \xi_i^\sigma$.

It is well known that the space $V \otimes V^*$ can be canonically identified with the space $M_{n+1}(q)$ of all $(n+1) \times (n+1)$ matrices with entries in $\mathbb{F}_q$. More in detail, the elements $E \otimes E^* = \{\mathbf{e}_i \otimes \boldsymbol{\eta}_j\}_{1 \le i,j \le n+1}$ form a basis of $V \otimes V^*$. If we map $\mathbf{e}_i \otimes \boldsymbol{\eta}_j$ to the elementary matrix $\mathbf{e}_{ij}$ whose only non-zero entry is 1 in position (i, j), we see that this gives the isomorphism $\phi \colon V \otimes V^* \to M_{n+1}(q)$ described by

$$x \otimes \xi \in V \otimes V^* \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \begin{pmatrix} \xi_1 & \dots & \xi_{n+1} \end{pmatrix} \in M_{n+1}(q).$$

Thus, the tensor $x \otimes \xi \in V \otimes V^*$ can be regarded as the usual matrix product $x\xi$ between a column and a row vector and for a matrix $M \in M_{n+1}(q)$, the product $\xi M x$ is the scalar obtained as the product of the row ξ times M times the column x. Recall that a matrix M can be written as $x\xi$ (namely, it is a pure tensor of the form $x \otimes \xi$) if and only if it has rank 1.

If $M := \sum_{i,j=1}^{n+1} m_{ij}(\mathbf{e}_i \otimes \boldsymbol{\eta}_j)$ is any element of $V \otimes V^*$, then the *trace* of M is defined as $\mathrm{Tr}(M) := \sum_{i,j=1}^{n+1} m_{ij} \boldsymbol{\eta}_i(\mathbf{e}_j) = \sum_{i=1}^{n+1} m_{ii}$. It can be seen that the trace is independent from the choice of the bases E and E^*; indeed, for pure tensors $x \otimes \xi$ the trace of the corresponding matrix corresponds to the value of $\xi(x)$.

For any two matrices $X, Y \in M_{n+1}(q)$ we can consider *the saturation form* $f \colon M_{n+1}(q) \times M_{n+1}(q) \to \mathbb{F}_q$ of M_{n+1} given by

$$f(X, Y) = \mathrm{Tr}(XY), \ \ \forall\, X, Y \in M_{n+1}(q), \tag{1}$$

where XY is the usual row-times-column product. It is immediate to see that this form is bilinear, symmetric and non-degenerate; furthermore it does not depend on the choice of the basis of V; for more detail see [19].

Let $\perp$ be the orthogonality relation associated to f. Since f is non-degenerate, the hyperplanes of $M_{n+1}(q)$ are the orthogonal spaces

$$M^{\perp} = \{X \in M_{n+1}(q) \colon \mathrm{Tr}(XM) = 0\},$$

for $M \in M_{n+1}(q) \setminus \{O\}$ and, for two matrices $M, N \in M_{n+1}(q) \setminus \{O\}$, we have $M^{\perp} = N^{\perp}$ if and only if M and N are proportional.

The pure tensors in the orthogonal space of a matrix M with respect to the saturation form admit a simple description, as illustrated by the following proposition; see [11] for the proof.

Proposition 2.1. *Let $x \in V \setminus \{0\}$, $\xi \in V^* \setminus \{0\}$ and $M \in M_{n+1}(q)$. Then $x \otimes \xi \in M^{\perp}$ if and only if $\xi M x = 0$.*

Henceforth, we shall adopt either the matrix notation or the tensor product notation, silently switching between them, according to convenience.

Turning to projective spaces, let $\mathrm{PG}(n, q) = \mathrm{PG}(V)$ be the n-dimensional projective space defined by V. When we need to distinguish between a non-zero vector x of V and the point of $\mathrm{PG}(V)$ represented by it, we denote the latter by $[x]$. We extend this convention to subsets of V. If $X \subseteq V \setminus \{0\}$ then $[X] := \{[x] | x \in X\}$. The same conventions will be adopted for vectors and subsets of V^* and $V \otimes V^*$. In particular, if $\xi \in V^* \setminus \{0\}$ then $[\xi]$ is the point of $\mathrm{PG}(V^*)$ which corresponds to the hyperplane $[\ker(\xi)]$ of $\mathrm{PG}(V)$. In the sequel we shall freely take $[\xi]$ as a name for $[\ker(\xi)]$. Accordingly, if $0 \in V^*$ then $[0] := \mathrm{PG}(V)$.

Observe that in terms of projective spaces, Proposition 2.1 states that the point $[x \otimes \xi] \in \mathrm{PG}(V \otimes V^*)$ is contained in the space $[M^{\perp}]$ if and only if the point $[x] \in \mathrm{PG}(V)$ is contained in $[\xi M] \in \mathrm{PG}(V^*)$.

2.2. Embeddings and hyperplanes of point-line geometries

Let $\Gamma = (\mathscr{P}, \mathscr{L})$ and $\Gamma' = (\mathscr{P}', \mathscr{L}')$ be two point-line geometries with pointsets $\mathscr{P}$ respectively $\mathscr{P}'$ and linesets $\mathscr{L}$ respectively $\mathscr{L}'$ and incidence given by inclusion. An *embedding* of Γ in Γ' is an injective function $\iota : \mathscr{P} \to \mathscr{P}'$ which maps lines of $\mathscr{L}$ onto lines of $\mathscr{L}'$.

When $\Gamma' = \mathrm{PG}(V)$ is a projective geometry, we say that $\varepsilon : \Gamma \to \mathrm{PG}(V)$ is a *projective embedding* of Γ if ε is an embedding in the sense mentioned above and the image of ε spans $\mathrm{PG}(V)$. In this case we call *dimension of* ε the vector dimension of V.

A *subspace* of Γ is a nonempty subset $\mathscr{X}$ of $\mathscr{P}$ such that every line $\ell \in \mathscr{L}$ meeting $\mathscr{X}$ in at least 2 points is fully contained in $\mathscr{X}$. A subspace of Γ is *maximal* if it is not properly contained in any other proper subspace of Γ. A proper subspace $\mathscr{X}$ of Γ is a *geometric hyperplane* if it meets every line in at least 1 point.

Given a projective embedding $\varepsilon\colon \Gamma \to \mathrm{PG}(V)$ of Γ, for any projective hyperplane W of $\mathrm{PG}(V)$, the point set $\mathscr{W} := \varepsilon^{-1}(W)$ is a geometric hyperplane of Γ called *the geometric hyperplane arising from W via ε*. Clearly, $\varepsilon(\varepsilon^{-1}(W)) = W \cap \varepsilon(\mathscr{P})$.

2.3. Projective and minimal codes

We recall that given a projective system Ω consisting of N points in $\mathrm{PG}(k-1,q)$, the projective code $\mathscr{C}(\Omega)$ defined by Ω is a $[N,k,d]$-linear code whose generator matrix G has columns consisting of vector representatives of the points of Λ.

The minimum distance of $\mathscr{C}(\Omega)$ is related to the maximum possible intersection of the projective system Ω with any hyperplane of $\mathrm{PG}(k-1,q)$. In particular,

$$d = N - \max_{H \in \mathrm{PG}(k-1,q)^*} |H \cap \Omega|.$$

Another important property of codes which has a nice geometrical counterpart is that of *minimality*. We briefly recall the definition of minimal codes.

Let $\mathscr{C}$ be a projective $[N,k,d]$-code. For any $c = (c_1,\ldots,c_N) \in \mathscr{C}$ the *support* of c is the set $\mathrm{supp}(c) = \{i : c_i \neq 0\}$.

The notion of *minimal codewords* has been introduced by Massey in [17].

Definition 2.2. A codeword $c \in \mathscr{C}$ is *minimal* if

$$\forall c' \in \mathscr{C} : \mathrm{supp}(c') \subseteq \mathrm{supp}(c) \Rightarrow \exists \lambda \in \mathbb{F}_q : c' = \lambda c.$$

A code $\mathscr{C}$ is *minimal* if all its codewords are minimal.

Obviously, all codewords with minimum weight are minimal, however, to determine if *all* codewords satisfy Definition 2.2 might be a difficult problem in general.

Minimal codes have been extensively investigated by Ashikhmin and Barg [3] who also provided a necessary condition on the weights of the codewords for the code to be minimal.

For projective codes arising from a projective system Ω, being minimal is equivalent to ask that the projective system Ω is a so-called *cutting blocking set with respect to hyperplanes*, i.e. for any hyperplane H of $\mathrm{PG}(\langle\Omega\rangle)$, $\langle H \cap \Omega \rangle = H$; see [1,2,5,13,20].

Relying on the notion of cutting sets and on the properties of the geometric hyperplanes of a point-line geometry Γ, we have obtained in [6] a sufficient condition for a code to be minimal in terms of the maximality of the geometrical hyperplanes of Γ. We shall make use of the following characterization from [6].

Proposition 2.3. *Suppose that $\Gamma = (\mathscr{P}, \mathscr{L})$ is a point-line geometry where every geometric hyperplane is a maximal subspace. Then the projective code $\mathscr{C}(\varepsilon(\Gamma))$ is minimal, for any projective embedding ε of Γ.*

A *monomial transformation* of $\mathbb{F}_q^N$ is an invertible linear transformation $\mathbb{F}_q^N \to \mathbb{F}_q^N$ which is described with respect to the canonical basis by the product of a permutation matrix P by a diagonal matrix D. By the MacWilliams extension theorem [16, Theorem 1.10], the group of monomial transformations of $\mathbb{F}_q^N$ is the same as the group of isometries of $\mathbb{F}_q^N$ with respect to Hamming distance.

An *automorphism* of the code $\mathscr{C}(\Omega)$ (thus regarded as a subspace of $\mathbb{F}_q^N$) is an isometry of $\mathbb{F}_q^N$ which maps codewords into codewords; see [15]. It can be shown that if γ is an automorphism of the geometry Γ which lifts into a linear transformation of $\mathrm{PG}(\langle\varepsilon(\Gamma)\rangle)$ via an embedding ε, then γ induces an automorphism of $\mathscr{C}(\varepsilon(\Gamma))$.

2.4. The point-hyperplane geometry $\bar{\Gamma}$ of $\mathrm{PG}(V)$ and its embeddings

Following the notation of [11], denote by Γ the Segre geometry $\mathrm{PG}(V) \otimes \mathrm{PG}(V^*)$ whose points are all the ordered pairs $([p], [\xi])$, where $[p]$ and $[\xi]$ are respectively a point and a hyperplane of $\mathrm{PG}(V)$ and $\varepsilon \colon \Gamma \to \mathrm{PG}(V \otimes V^*)$ denote the Segre embedding of Γ, mapping (x, ξ) to $[x \otimes \xi]$.

As mentioned in Section 2.1, we shall always silently identify $V \otimes V^*$ with $M_{n+1}(q)$ by the isomorphism induced by $(\mathbf{e}_i \otimes \boldsymbol{\eta}_j) \to \mathbf{e}_i\boldsymbol{\eta}_j =: \mathbf{e}_{ij}$.

The *point-hyperplane geometry* of $\mathrm{PG}(V)$ is the geometry $\bar{\Gamma} = (\mathscr{P}, \mathscr{L})$ whose points are all the pairs $(p, H) \in \mathrm{PG}(V) \otimes \mathrm{PG}(V^*)$ such that $p \in H$ and whose lines are either of the form $\ell_{r,H} := \{(p, H) \in \mathscr{P} : p \in r\}$, where H is a given hyperplane of $\mathrm{PG}(V)$ and r is a given line contained in H, or $\ell_{p,S} := \{(p, H) \in \mathscr{P} : S \subseteq H\}$ where p is a given point of $\mathrm{PG}(V)$ and S is a given subspace of $\mathrm{PG}(V)$ of codimension 2 (i.e. a line of $\mathrm{PG}(V^*)$) with $p \in S$. Its linear automorphism group is $\mathrm{PGL}(n + 1, q)$ and it acts transitively on the points of $\bar{\Gamma}$.

The geometry $\bar{\Gamma}$ is also known as the *long root geometry for the special linear group* $\mathrm{SL}(n + 1, \mathbb{F}_q)$; see e.g. [7,14].

It follows from the definition of $\bar{\Gamma}$ that the identity map $\iota : \bar{\Gamma} \to \Gamma$ sending the point $(x, \xi) \in \bar{\Gamma}$ to the same point $(x, \xi) \in \Gamma$, is an embedding of geometries.

Consequently, the map $\bar{\varepsilon} := \varepsilon \circ \iota$ is a projective embedding of $\bar{\Gamma}$, called the *Segre* or *natural embedding* of $\bar{\Gamma}$. Note that the image of $\bar{\varepsilon}$ spans $\mathrm{PG}(M^0_{n+1}(q))$ where $M^0_{n+1}(q)$ is the hyperplane of the traceless matrices of $M_{n+1}(q)$. In [11] we studied the code $\mathscr{C}(\Lambda_1)$ where $\Lambda_1 := \bar{\varepsilon}(\bar{\Gamma})$.

The geometry $\bar{\Gamma}$ turns out to be a geometric hyperplane of the Segre geometry Γ which is called in [22] of *black type.* It is shown in [22] that any geometric hyperplane of Γ of black type corresponds to an embedding of $\bar{\Gamma}$ in Γ and that these hyperplanes might not lie in the same orbit with respect to the collineation group of the Segre geometry. Actually, $\bar{\Gamma} = \iota(\bar{\Gamma}) \subset \Gamma$ is one of these hyperplanes.

Let σ be a non-trivial automorphism of $\mathbb{F}_q$. Consider the *twisted map* of $\bar{\Gamma}$ defined as follows

$$\iota_\sigma : \bar{\Gamma} \to \Gamma,\ (x, \xi) \mapsto (x^\sigma, \xi).$$

Then, see [22], $\iota_\sigma(\bar{\Gamma}) \cong \bar{\Gamma}$ is again a hyperplane of Γ and different automorphisms of $\mathbb{F}_q$ correspond to different orbits of geometric hyperplanes of Γ under the automorphism group of Γ. Define the *twisted embedding* of $\bar{\Gamma}$ as

$$\bar{\varepsilon}_\sigma = \varepsilon \circ \iota_\sigma.$$

More explicitly, we have

$$\bar{\varepsilon}_\sigma \colon \bar{\Gamma} \to \mathrm{PG}(M_{n+1}(q)),\ \bar{\varepsilon}_\sigma(([x],[\xi])) = [x^\sigma \otimes \xi],$$

where $x^\sigma := ({x_i}^\sigma)_{i=1}^{n+1}$. The dimension of $\bar{\varepsilon}_1 := \bar{\varepsilon}$ is $(n+1)^2-1$, while the dimension of $\bar{\varepsilon}_\sigma$ is $(n+1)^2$, since the image of $\bar{\Gamma}$ by means of $\bar{\varepsilon}_\sigma$ spans $\mathrm{PG}(V \otimes V^*)$. We put

$$\Lambda_\sigma := \bar{\varepsilon}_\sigma(\bar{\Gamma}) = \{[x^\sigma \otimes \xi] \colon [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*) \text{ and } [x] \in [\xi]\} \tag{2}$$

For more information on the embeddings of $\bar{\Gamma}$, we refer the reader to [18]. In particular it is shown that embeddings related to different automorphisms of $\mathbb{F}_q$ are inequivalent. This being said, most of the properties of the code $\mathscr{C}(\Lambda_\sigma)$ which will be studied in the present paper depend only on whether $\sigma = 1$ (in which case we have the Segre embedding and we refer the reader to [11]) or $\sigma \neq 1$.

The group $\mathrm{GL}(n+1,q)$ acts on the geometry $\bar{\Gamma}$ as an automorphism group by the following action: given $([x],[\xi]) \in \bar{\Gamma}$ and $g \in \mathrm{GL}(n+1,q)$, then

$$([x],[\xi]) \to ([x],[\xi])^g := ([gx],[\xi g^{-1}]).$$

The kernel of the action consists exactly of the scalar matrices; so $\mathrm{PGL}(n+1,q) = \mathrm{GL}(n+1,q)/\{\alpha I : \alpha \in \mathbb{F}_q\}$ acts faithfully as a permutation group on $\bar{\Gamma}$. As $\mathrm{PGL}(n+1,q)$ is flag-transitive on $\mathrm{PG}(V)$, its action on $\bar{\Gamma}$ is transitive. As the embedding $\bar{\varepsilon}_\sigma$ is *homogeneous*, the action of $\mathrm{GL}(n+1,q)$ lifts through $\bar{\varepsilon}_\sigma$ to an automorphism subgroup of $\mathrm{PG}(M_{n+1}(q))$ as follows

$$\begin{aligned}\bar{\varepsilon}_\sigma(([x],[\xi])^g) = \bar{\varepsilon}_\sigma(([gx],[\xi g^{-1}])) = [g^\sigma x^\sigma \otimes \xi g^{-1}] = \\ [g^\sigma(x^\sigma\xi)g^{-1}] = g^\sigma \bar{\varepsilon}_\sigma(([x],[\xi])g^{-1}.\end{aligned} \tag{3}$$

In particular, extending the action to all of $M_{n+1}(q)$ we put

$$M^g := g^\sigma M g^{-1}, \qquad \forall M \in M_{n+1}(q).$$

Write $\Lambda_\sigma = \{[X_1],\dots,[X_N]\}$. By definition (3) of the action, $\Lambda_\sigma^g = \Lambda_\sigma$ for all $g \in \mathrm{GL}(n+1,q)$; so there is a group homomorphism $\pi : \mathrm{GL}(n+1,q) \to S_N$ (where S_N is the symmetric group on $\{1,\dots,N\}$) such that

$$[X_i]^g = [X_i^g] = [g^\sigma X_i g^{-1}] = [X_{\pi(g)(i)}]. \tag{4}$$

Observe that the action of $\mathrm{GL}(n+1,q)$ on $M_{n+1}(q)$ described above is matrix conjugation if and only if $\sigma = 1$.

2.5. Hyperplanes of $\bar{\Gamma}$

In this section we will briefly recall from [18] and [19] the most significant results related to the hyperplanes of $\bar{\Gamma}$ arising from the embeddings $\bar{\varepsilon}$ and $\bar{\varepsilon}_\sigma$. For the case of the embedding $\bar{\varepsilon}$ see also Section 2.7 of [11].

We begin with a general theorem about geometric hyperplanes.

Theorem 2.4 *(Theorem 1.5, [19]). All hyperplanes of $\bar{\Gamma}$ are maximal subspaces.*

Take now $M \in M_{n+1}(q) \setminus \langle I \rangle$ and let $\bar{\varepsilon}$ be the Segre embedding of $\bar{\Gamma}$. Then

$$\mathscr{H}_M := \bar{\varepsilon}^{-1}([M^\perp]) = \bar{\varepsilon}^{-1}(\{[X] \in \mathrm{PG}(M^0_{n+1}(q))\colon \mathrm{Tr}(XM) = 0\})$$

is a geometric hyperplane of $\bar{\Gamma}$ called a *hyperplane of plain type*, as defined in [19].

Proposition 2.5. *[19, Corollary 1.7] The hyperplanes of $\bar{\Gamma}$ which arise from the Segre embedding $\bar{\varepsilon}$ are precisely those of plain type.*

If $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$ then

$$\mathscr{H}_{M,\sigma} := \bar{\varepsilon}_\sigma^{-1}([M^\perp \cap \bar{\varepsilon}_\sigma(\bar{\Gamma})]) = \bar{\varepsilon}_\sigma^{-1}(\{[X] \in \mathrm{PG}(M_{n+1}(q))\colon \mathrm{Tr}(XM) = 0\})$$

is again a hyperplane of $\bar{\Gamma}$, but it is in general different from $\mathscr{H}_M$; see Proposition 2.7 for more details.

Take $p \in \mathrm{PG}(V)$, $A \in \mathrm{PG}(V^*)$. Put $\mathscr{M}_p := \{(p,H) : p \in H\}$ and $\mathscr{M}_A := \{(x,A) : x \in A\}$. Then,

$$\mathscr{H}_{p,A} := \{(x,H)\colon (x,H) \text{ collinear with a point of } \mathscr{M}_p \cup \mathscr{M}_A\} \tag{5}$$

is a geometric hyperplane of $\bar{\Gamma}$, called the *quasi-singular hyperplane* defined by (p,A). If $p \in A$, then $\mathscr{H}_{p,A}$ is called *singular hyperplane of deepest point* (p,A) and consists of all points of $\bar{\Gamma}$ not at maximal distance from (p,A) in the collinearity graph of $\bar{\Gamma}$.

The following theorem further describes the quasi-singular hyperplanes of $\bar{\Gamma}$.

Proposition 2.6 *(§1.3, [19]). Take $[x] \in \mathrm{PG}(V)$ and $[\xi] \in \mathrm{PG}(V^*)$. Then the quasi-singular hyperplane $\mathscr{H}_{[x],[\xi]}$ is the hyperplane of plain type $\mathscr{H}_M$ where $M = x\xi$.*

In particular, all quasi-singular hyperplanes are hyperplanes of plain type arising from matrices M of rank 1 and, conversely, for each matrix $M \in M_{n+1}(q)$ of rank 1 the hyperplane of plain type $\mathscr{H}_M$ is quasi-singular.

Proposition 2.7 *(Theorem 1.6, [18]). Let $\mathscr{H}$ be a geometric hyperplane of $\bar{\Gamma}$.*

1. *If $\mathscr{H}$ is quasi-singular, then $\mathscr{H}$ arises from $\bar{\varepsilon}_\sigma$ for all $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$.*
2. *If $\mathscr{H}$ is not quasi-singular, then $\mathscr{H}$ arises from $\bar{\varepsilon}_\sigma$ for at most one $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$.*

By Proposition 2.6, there is a one-to-one correspondence between quasi-singular hyperplanes of $\bar{\Gamma}$ and proportionality classes of matrices of rank 1; Proposition 2.7 shows that these hyperplanes are the only ones which arise from both the Segre embedding $\bar{\varepsilon}$ and also from all the twisted embeddings $\bar{\varepsilon}_\sigma$; as we will see in Subsection 3.1, they induce words with the same weight on the code $\mathscr{C}(\Lambda_\sigma)$ and on the code $\mathscr{C}(\Lambda_1)$ studied in [11].

We now consider a further family of hyperplanes. Suppose ϕ is a non-linear, fixed-point-free collineation of $\mathrm{PG}(V)$. The set $S_\phi := \{\langle p, \phi(p)\rangle : p \in \mathrm{PG}(V)\}$ is a line-spread of $\mathrm{PG}(V)$ if and only if ϕ is an involution; see [19, Lemma 2.9]. Such an involution may exist only if n is odd, since otherwise $\mathrm{PG}(V)$ does not admit line-spreads at all. We shall call a spread S_ϕ obtained in this way a *semi-standard line-spread* of $\mathrm{PG}(V)$. Under our assumptions ϕ is an involution, hence the semi-linear mapping $f : V \to V$ associated to ϕ is defined as $f(x) = Mx^\sigma$ for $M \in M_{n+1}(q)$ such that $M^\sigma = M^{-1}$ and $\sigma^2 = 1$.

We say that a line-spread S *admits a dual* if there exists a line spread S^* of $\mathrm{PG}(V^*)$ such that for every line $\ell^* \in S^*$ (i.e. for every 2-codimensional subspace of $\mathrm{PG}(V)$), the members of S contained in ℓ^*, form a line spread of ℓ^*; see [19].

If a dual spread exists, then it is unique.

Given a spread S admitting a dual S^* it is possible to construct a geometric hyperplane $\mathscr{H}_S$ of $\bar{\Gamma}$; see [19, Theorem 1.11]. In particular, if S_ϕ is a semi-standard line-spread of $\mathrm{PG}(V)$, then by [19, Proposition 2.10], S_ϕ admits a dual spread and the corresponding geometric hyperplane of $\bar{\Gamma}$ is defined as

$$\mathscr{H}_\phi := \{([x],[\xi]) \in \bar{\Gamma} \colon [\xi] \supset \ell_x\} = \{([x],[\xi]) \in \bar{\Gamma} \colon [x],[\phi(x)] \in [\xi]\} = \\ \{([x],[\xi]) \colon \xi x = 0, \xi M x^\sigma = 0\},$$

where $\ell_x := \langle x, \phi(x)\rangle$ is the unique element of S containing $[x]$ and M is the matrix appearing in the definition of ϕ; see [19, Theorem 1.11]. We call $\mathscr{H}_\phi$ a hyperplane of $\bar{\Gamma}$ of *semi-standard spread type.*

Let now $\bar{\varepsilon}_\sigma$ be the twisted embedding of $\bar{\Gamma}$ obtained using the same automorphism σ as above. The hyperplane $\mathscr{H}_\phi$ of semi-standard spread type arises from the twisted embedding $\bar{\varepsilon}_\sigma$. Indeed,

$$\bar{\varepsilon}_\sigma(\mathscr{H}_\phi) = \{\bar{\varepsilon}_\sigma(([x],[\xi])) \colon \xi x = 0, \xi M x^\sigma = 0\} = \\ \{[x^\sigma \otimes \xi] : \xi x = 0, \xi M x^\sigma = 0\} = \bar{\varepsilon}_\sigma(\bar{\Gamma}) \cap \{[x^\sigma \otimes \xi] \colon \xi M x^\sigma = 0\} = \\ \bar{\varepsilon}_\sigma(\bar{\Gamma}) \cap \{[x \otimes \xi] \colon \xi M x = 0\} = \bar{\varepsilon}_\sigma(\bar{\Gamma}) \cap [M^\perp].$$

Since $\mathscr{H}_\phi$ is not a (semi-)singular hyperplane, it arises *only* from $\bar{\varepsilon}_\sigma$; see [18, Theorem 1.12]. We summarize the above arguments in the following proposition.

Proposition 2.8. *If a semi-standard spread type hyperplane $\mathscr{H}_\phi$ of $\bar{\Gamma}$ arises from the twisted embedding $\bar{\varepsilon}_\sigma$ then $\sigma^2 = 1$, n is odd and $\phi\colon \mathrm{PG}(V) \to \mathrm{PG}(V)$ is an involutory fixed-point free semilinear collineation of the form $\phi([x]) = [Mx^\sigma]$ with $M^\sigma = M^{-1}$. Furthermore $\mathscr{H}_\phi = \bar{\varepsilon}_\sigma^{-1}(M^\perp)$.*

By the definition of collinearity in $\bar{\Gamma}$ and a direct counting argument in the case of semi-standard spread type hyperplanes, we have the following.

Proposition 2.9 *(Proposition 2.11, [11]). The following hold.*

1. *The cardinality of the singular hyperplanes of $\bar{\Gamma}$ is*

$$\frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \frac{q^n-1}{q-1}q^{n-1}. \tag{6}$$

2. *The cardinality of the quasi-singular but not singular hyperplanes of $\bar{\Gamma}$ is*

$$\frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + (\frac{q^n-1}{q-1}+1)q^{n-1}. \tag{7}$$

3. *The cardinality of a semi-standard spread type hyperplane of $\bar{\Gamma}$ is*

$$\frac{(q^{n+1}-1)}{q-1}\frac{(q^{n-1}-1)}{q-1}. \tag{8}$$

3. The code $\mathscr{C}(\Lambda_\sigma)$ from the twisted embedding

In this section we focus on the code $\mathscr{C}(\Lambda_\sigma)$. We first recall the construction of the codewords of $\mathscr{C}(\Lambda_\sigma)$. Then, we compute the cardinality of $[M^\perp]\cap\Lambda_\sigma$ where $M \in M_{n+1}(q)$ (Lemma 3.2), which is crucial to determine the weight spectrum of $\mathscr{C}(\Lambda_\sigma)$, since $\forall c_{\mathfrak{m}} \in \mathscr{C}(\Lambda_\sigma)$, $\mathrm{wt}\,(c_{\mathfrak{m}}) = N_\sigma - |[M^\perp] \cap \Lambda_\sigma|$. As we will see, $|[M^\perp] \cap \Lambda_\sigma|$ is a function of the number θ_M, to be computed as M varies in $M_{n+1}(q)$. To obtain the minimum distance θ_M needs to be maximized (Section 3.2) while to get the maximum distance θ_M needs to be minimized (Section 3.3).

Suppose $\Lambda_\sigma := \{[X_1], [X_2], \ldots, [X_N]\} \subseteq \mathrm{PG}(M_{n+1}(q))$ and denote by $M^*_{n+1}(q)$ the dual of the vector space $M_{n+1}(q)$. For any functional $\mathfrak{m} \in M^*_{n+1}(q)$, there exists a unique matrix $M \in M_{n+1}(q)$ such that

$$\mathfrak{m}\colon M_{n+1}(q) \to \mathbb{F}_q, \quad \mathfrak{m}(X) = \mathrm{Tr}(XM)$$

for all $X \in M_{n+1}(q)$. Define now $c_{\mathfrak{m}}$ as the N-uple

$$c_{\mathfrak{m}} = (\mathfrak{m}(X_1), \ldots, \mathfrak{m}(X_N)) \tag{9}$$

with

$$\mathfrak{m}(X_i) = \mathrm{Tr}(X_i M),\ 1 \le i \le N, \tag{10}$$

where $M \in M_{n+1}(q)$ is associated to $\mathfrak{m}$ as before. In this setting,

$$\mathscr{C}(\Lambda_\sigma) = \{c_{\mathfrak{m}} : \mathfrak{m} \in M^*_{n+1}(q)\}.$$

It is clear that the codeword $c_{\mathfrak{m}} \in \mathscr{C}(\Lambda_\sigma)$ is associated to the matrix $M \in M_{n+1}(q)$ defining the functional $\mathfrak{m}$, so we shall denote it also by c_M.

Lemma 3.1. *Let $X_1, \ldots, X_N$ be representatives of the points $[X_1], \ldots, [X_N]$ of Λ_σ. Then the map*

$$ev : \begin{cases} M_{n+1}(q) \to \mathscr{C}(\Lambda_\sigma) \\ M \to c_M := (\mathrm{Tr}(X_1 M), \mathrm{Tr}(X_2 M), \ldots, \mathrm{Tr}(X_N M)) \end{cases} \tag{11}$$

is a vector space isomorphism.

Proof. By the properties of the trace, ev is well defined, linear and surjective. Suppose now $M \in \ker(ev)$. Then $\mathrm{Tr}(X_i M) = 0$ for all $i = 1, \ldots, N$. On the other hand $\langle \Lambda_\sigma \rangle = \mathrm{PG}(M_{n+1}(q))$, so we have that the representatives of the points of Λ_σ contain a basis $(B_1, \ldots, B_{(n+1)^2})$ for $M_{n+1}(q)$. Since $\mathrm{Tr}(B_i M) = 0$ for all $i = 1, \ldots, (n+1)^2$, it follows that it must be $\mathrm{Tr}(XM) = 0$ identically for all $X \in M_{n+1}(q)$, whence $M = 0$. □

3.1. The weights of $\mathscr{C}(\Lambda_\sigma)$

To determine the weight of the codewords of $\mathscr{C}(\Lambda_\sigma)$ we need to compute the cardinality of $\Lambda_\sigma \cap [W]$ where $[W]$ is a hyperplane of $[\langle \Lambda_\sigma \rangle] = \mathrm{PG}(V \otimes V^*)$. Recall from Section 2 that any hyperplane of $\mathrm{PG}(V \otimes V^*)$ can be regarded as the orthogonal subspace $[M^\perp]$ of a $(n+1) \times (n+1)$-matrix M with respect to the saturation form f. The following lemma, an extension of [11, Lemma 3.2], is essential.

Lemma 3.2. *Let $[M^\perp]$ be a hyperplane of $\mathrm{PG}(V \otimes V^*)$ with $M \in M_{n+1}(q)$, and suppose $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$. Then*

$$|[M^\perp] \cap \Lambda_\sigma| = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \theta_M \cdot q^{n-1}, \tag{12}$$

where θ_M is the number of hyperplanes $[\xi]$ of $\mathrm{PG}(V)$ such that $[\xi]^\sigma \subseteq [\xi M]$.

Proof. First observe that $\Lambda_\sigma = \{\bar{\varepsilon}_\sigma(([x],[\xi]))\colon [x]\in[\xi]\}$ as a disjoint union

$$\Lambda_\sigma = \bigsqcup_{[\xi]\in \mathrm{PG}(V^*)} \{[x^\sigma\otimes\xi] : [x]\in[\xi]\}.$$

So,

$$[M^\perp]\cap\Lambda_\sigma = \bigsqcup_{[\xi]\in \mathrm{PG}(V^*)} (\{[x^\sigma\otimes\xi] : [x]\in[\xi]\}\cap[M^\perp]).$$

By Proposition 2.1, $[x^\sigma\otimes\xi]\in[M^\perp]$ if and only if $[x]^\sigma\in[\xi M]$. Hence, $[x^\sigma\otimes\xi]\in\Lambda_\sigma\cap[M^\perp]$ if and only if $[x]^\sigma\in[\xi^\sigma]=[\xi]^\sigma$ and $[x]^\sigma\in[\xi M]$, i.e.

$$\Lambda_\sigma\cap[M^\perp] = \bigsqcup_{[\xi]\in \mathrm{PG}(V^*)} (\{[x^\sigma\otimes\xi] : [x]^\sigma\in([\xi]^\sigma\cap[\xi M])\}).$$

Turning to cardinalities,

$$|\Lambda_\sigma\cap[M^\perp]| = \sum_{[\xi]\in \mathrm{PG}(V^*)} |[\xi]^\sigma\cap[\xi M]|. \tag{13}$$

Note that $[\xi]^\sigma$ is always a hyperplane of $\mathrm{PG}(V)$ and $[\xi M]$ is a hyperplane of $\mathrm{PG}(V)$ if ξM is not the null vector of V^*. The following cases may happen:

Case 1. $\boxed{\xi\notin\ker(M)\text{ and }[\xi M]\neq[\xi]^\sigma.}$ In this case $[\xi]^\sigma\cap[\xi M]$ is a subspace of codimension 2 of $\mathrm{PG}(V)$, so $|[\xi]^\sigma\cap[\xi M]| = \frac{q^{n-1}-1}{q-1}$.

Case 2. $\boxed{\xi\notin\ker(M)\text{ and }[\xi M]=[\xi]^\sigma.}$ In this case $|[\xi]^\sigma\cap[\xi M]| = \frac{q^n-1}{q-1}$.

Case 3. $\boxed{\xi\in\ker(M).}$ In this case ξM is the null functional, hence $[\xi M]=\mathrm{PG}(V)$. So, $[\xi]^\sigma\cap[\xi M]=[\xi]^\sigma$ is a hyperplane of $\mathrm{PG}(V)$ and again $|[\xi]^\sigma\cap[\xi M]| = \frac{q^n-1}{q-1}$.

Clearly, Cases 2 and 3 correspond to $[\xi]^\sigma\subseteq[\xi M]$ while Case 1 corresponds to $[\xi]^\sigma\not\subseteq[\xi M]$. By plugging the values of the corresponding cardinalities in (13) we get

$$|\Lambda_\sigma\cap[M^\perp]| = \theta_M\cdot\frac{q^n-1}{q-1} + \left(\frac{q^{n+1}-1}{q-1}-\theta_M\right)\cdot\frac{q^{n-1}-1}{q-1} = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \theta_M\cdot q^{n-1}. \qquad \square$$

Remark 1. If $\sigma=1$, then the condition of Lemma 3.2 becomes $[\xi]\subseteq[\xi M]$, that is ξ must be a *left* eigenvector of M.

The following is straightforward, considering that for any codeword $c_{\mathfrak{m}}\in\mathscr{C}(\Lambda_\sigma)$, the weight of $c_{\mathfrak{m}}$ is $wt(c_{\mathfrak{m}}) = N_\sigma - |[M^\perp]\cap\Lambda_\sigma|$, where M is the matrix associated to the codeword $c_{\mathfrak{m}}$ as defined at the beginning of Section 3.

Corollary 3.3. *Suppose $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$. The spectrum of the weights of $\mathscr{C}(\Lambda_\sigma)$ is*

$$\{q^{n-1}\frac{(q^{n+1}-1)}{(q-1)} - q^{n-1}\theta_M \colon M \in M_{n+1}(q)\}$$

where θ_M is defined as in Lemma 3.2.

Definition 3.4. Assume $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$ with $\sigma \neq 1$. Let $\mathrm{Fix}(\sigma) \cong \mathbb{F}_s$ be the subfield of $\mathbb{F}_q$ fixed by σ. So, $q = s^t$ for some $t > 1$ and there exists an index j such that for all $x \in \mathbb{F}_q$ we have $x^\sigma = x^{s^j}$. The *σ-fixed subgeometry* of $\mathrm{PG}(n, q)$ is the point-line geometry having as points, the points of $\mathrm{PG}(n, q)$ fixed by σ, and as lines, the lines of $\mathrm{PG}(n, q)$ stabilized by σ; incidence is given by inclusion.

The σ-fixed subgeometry of $\mathrm{PG}(n, q)$ is isomorphic to $\mathrm{PG}(n, s)$.

3.2. Minimum weight codewords

Recall that

$$\theta_M := |\{[\xi] \in \mathrm{PG}(V^*) \colon [\xi^\sigma] \subseteq [\xi M]\}|. \tag{14}$$

By Corollary 3.3, in order to determine the minimum weight of $\mathscr{C}(\Lambda_\sigma)$ we need to compute

$$\max\{\theta_M : M \in M_{n+1}(q)\}.$$

We distinguish two cases.

* **Case A: M is an invertible matrix.** Hence, $[\xi M]$ is a hyperplane of $\mathrm{PG}(V)$ for all $[\xi] \in \mathrm{PG}(V^*)$. According to (14), we need to count the number of points $[\xi] \in \mathrm{PG}(V^*)$ such that

$$[\xi^\sigma] = [\xi M]. \tag{15}$$

This is equivalent to consider the set of fixed points of the semilinear collineation $\vartheta_{M,\sigma}$ of $\mathrm{PG}(V^*)$, induced by σ and M, defined as

$$\vartheta_{M,\sigma} : \begin{cases} \mathrm{PG}(V^*) \to \mathrm{PG}(V^*) \\ [\zeta] \to [\zeta^{\sigma^{-1}} M]. \end{cases} \tag{16}$$

Indeed, put $\xi := \zeta^{\sigma^{-1}}$. Then, $[\zeta]$ is fixed by $\vartheta_{M,\sigma}$, i.e. $[\zeta] = [\zeta^{\sigma^{-1}} M]$, if and only if $[\xi^\sigma] = [\xi M]$. So, restricting to a subspace W of V^*, the following is straightforward.

Lemma 3.5. *Let* $W \subseteq V^*$ *and* $W^\sigma := \{\xi^\sigma : \xi \in W\}$. *Then the number of points of* $\mathrm{PG}(W^\sigma)$ *fixed by* $\vartheta_{M,\sigma}$ *is the same as the number of points of* $\mathrm{PG}(W)$ *satisfying* (15), *i.e.*

$$|\{[\xi] \in \mathrm{PG}(W) \colon [\xi^\sigma] = [\xi M]\}| = |\{[\xi^\sigma] \in \mathrm{PG}(W^\sigma) \colon \vartheta_{M,\sigma}([\xi^\sigma]) = [\xi^\sigma]\}|.$$

Lemma 3.6. *If* $M = I$, *then* $\theta_I = |\{[\xi] \in \mathrm{PG}(V^*) \colon [\xi^\sigma] = [\xi]\}| = \frac{s^{n+1}-1}{s-1}$.

Proof. By Lemma 3.5 we need to compute the number of fixed points of the semilinear collineation $\vartheta_{I,\sigma}$. A point $[\xi] \in \mathrm{PG}(V^*)$ is fixed by $\vartheta_{I,\sigma}$ if and only if it is a point of the σ-fixed subgeometry $\mathrm{Fix}(\sigma) \cong \mathrm{PG}(n,s)$. This completes the proof. □

Lemma 3.7. *Suppose that* $M \in M_{n+1}(q)$ *is invertible. The following hold.*

1. *If* W *is a vector subspace of* V^* *such that* $[\xi^\sigma] = [\xi M]$ $\forall \xi \in W$ *then* $\dim W \le 1$.
2. *If* U *is a subspace of* V^* *with* $\dim(U) = 2$ *then* $\mathrm{PG}(U)$ *contains at most* $s+1 = |\mathbb{F}_s|+1$ *points* $[\xi]$ *such that* $[\xi^\sigma] = [\xi M]$.

Proof. Part 1. Suppose by way of contradiction that $\dim W \ge 2$ and let ξ_1, ξ_2 be two linearly independent vectors of W such that $[\xi_i^\sigma] = [\xi_i M]$ for $i = 1, 2$. Consider the line $\ell = [\langle \xi_1, \xi_2 \rangle]$ spanned by $[\xi_1]$ and $[\xi_2]$. Then,

$$\xi_1^\sigma = \lambda_1 \xi_1 M, \qquad \xi_2^\sigma = \lambda_2 \xi_2 M \tag{17}$$

for some $\lambda_1, \lambda_2 \in \mathbb{F}_q \setminus \{0\}$.

We want to determine the number of points in ℓ satisfying (15) and different from $[\xi_1]$ and $[\xi_2]$, i.e. the number of elements $\gamma \in \mathbb{F}_q \setminus \{0\}$ for which there exists $\lambda_\gamma \ne 0$ such that

$$(\xi_1 + \gamma \xi_2)^\sigma = \lambda_\gamma (\xi_1 + \gamma \xi_2) M. \tag{18}$$

Observe that, by (18) and (17),

$$\lambda_\gamma (\xi_1 + \gamma \xi_2) M = (\xi_1 + \gamma \xi_2)^\sigma = \xi_1^\sigma + \gamma^\sigma \xi_2^\sigma = (\lambda_1 \xi_1 + \gamma^\sigma \lambda_2 \xi_2) M.$$

As $\xi_1 M$ and $\xi_2 M$ are linearly independent, it follows that $\lambda_\gamma = \lambda_1$ and $\gamma^\sigma \lambda_2 = \lambda_1 \gamma$. Hence

$$\gamma^{\sigma-1} = \lambda_1 \lambda_2^{-1}.$$

Since the kernel of the group homomorphism $\mathbb{F}_q^* \to \mathbb{F}_q^*$ given by $x \to x^{\sigma-1}$ is $\mathbb{F}_s^*$, it is not difficult to see that the above equation (in the unknown γ) admits either 0 or $s-1$ solutions. As $[\xi_1]$ and $[\xi_2]$ also satisfy (15), we have that the overall number of points in ℓ satisfying (15) is at most $s - 1 + 2 = s + 1$. Since $s < q$, this contradicts the hypothesis

that all $[\xi] \in W$ are fixed. Part 1. is proved.
Part 2. Note that $s+1 \geq 3$. If the number of points in U satisfying (15) is either 0 or 1 then the thesis immediately follows. If the number of points in U satisfying (15) is at least 2 then we can repeat the same proof as in Part 1 to have the thesis. □

Theorem 3.8. *Let W be a subspace of V^* with $\dim(W) = r$ and suppose M is invertible. Then the number of $[\xi] \in \mathrm{PG}(W)$ such that $[\xi^\sigma] = [\xi M]$ is at most $\frac{s^r-1}{s-1}$.*

Proof. Note that if $M = I$, then the thesis follows from Lemma 3.6.

We proceed by induction on r. If $r \leq 2$ the result follows from Lemma 3.7. Suppose $r > 2$. By Lemma 3.5, we will count the number of fixed points of $\vartheta_{M,\sigma}$ in $\mathrm{PG}(W^\sigma)$. Assume by induction that the number of fixed points of $\vartheta_{M,\sigma}$ in any subspace of dimension $r' < r$ is at most $\frac{s^{r'}-1}{s-1}$. We distinguish three cases:

1. No hyperplane of $\mathrm{PG}(W^\sigma)$ is stabilized by $\vartheta_{M,\sigma}$. Hence, the set of fixed points by $\vartheta_{M,\sigma}$ spans a subspace of $\mathrm{PG}(W^\sigma)$ of codimension at least 2. The thesis follows directly by the inductive hypothesis.
2. There is a hyperplane $[H]$ of $\mathrm{PG}(W^\sigma)$ stabilized by $\vartheta_{M,\sigma}$ and all fixed points of $\vartheta_{M,\sigma}$ are contained in $[H]$; then, by inductive hypothesis on $[H]$ the number of fixed points of W^σ is at most $\frac{s^{r-1}-1}{s-1} \leq \frac{s^r-1}{s-1}$.
3. There is a hyperplane $[H]$ stabilized by $\vartheta_{M,\sigma}$ and at least one point $[\xi^\sigma] \in \mathrm{PG}(W^\sigma) \setminus [H]$ fixed by $\vartheta_{M,\sigma}$. If $[\xi^\sigma]$ is the unique fixed point of $\mathrm{PG}(W^\sigma)$ then the thesis follows. Otherwise, suppose $[\zeta^\sigma] \neq [\xi^\sigma]$ is a point fixed by $\vartheta_{M,\sigma}$. Then the line $\ell := [\langle \zeta^\sigma, \xi^\sigma \rangle]$ is stabilized by $\vartheta_{M,\sigma}$ and not contained in $[H]$, so the point $\ell \cap [H]$ is fixed by $\vartheta_{M,\sigma}$. By inductive hypothesis, the number of fixed points in $[H]$ is at most $\frac{s^{r-1}-1}{s-1}$, so there exists at most $\frac{s^{r-1}-1}{s-1}$ such lines through $[\xi^\sigma]$. By Lemma 3.7, every line of $\mathrm{PG}(W^\sigma)$ through $[\xi^\sigma]$ contains at most s fixed points distinct from $[\xi^\sigma]$. So, the number of fixed points by $\vartheta_{M,\sigma}$ is at most

$$1 + s\frac{s^{r-1}-1}{s-1} = \frac{s^r - s + s - 1}{s-1} = \frac{s^r-1}{s-1}. \quad \square$$

* **Case B: M is an arbitrary matrix.**

Theorem 3.9. *Let $m : \mathbb{N} \to \mathbb{N}$ be the function*

$$m(r) := \frac{q^{n+1-r}-1}{q-1} + \frac{s^r-1}{s-1}. \tag{19}$$

Then, for any matrix $M \in M_{n+1}(q)$ and $r = 1, \dots, n+1$ we have

$$\max\{\theta_M : \mathrm{rank}\,(M) = r\} \leq m(r).$$

Proof. If $\xi \in \ker(M)$ then $[\xi M] = [0] = \mathrm{PG}(V^*)$, so the condition $[\xi^\sigma] \subseteq [\xi M]$ is always satisfied. Since rank $(M) = r$, clearly $|[\ker(M)]| = \frac{q^{n+1-r}-1}{q-1}$.

Consider the following subspace of V^*:

$$U := \langle \xi \in V^* \colon [\xi^\sigma] = [\xi M] \rangle = \\ \langle \xi \in V^* \colon \lambda_\xi \xi^\sigma = \xi M, \text{for some } 0 \neq \lambda_\xi \in \mathbb{F}_q \rangle.$$

We claim that $U \cap \ker(M) = \mathbf{0}$. Indeed, let $\mathfrak{B} := (\xi_1, \dots, \xi_l)$ be a basis of U consisting of vectors such that $[\xi_i^\sigma] = [\xi_i M]$. Then, $\mathfrak{B}^\sigma = (\xi_1^\sigma, \dots, \xi_l^\sigma)$ is a basis of U^σ and $\xi_i M = \lambda_i \xi_i^\sigma$ for all $i = 1, \dots, l$ and $\lambda_i \in \mathbb{F}_q \setminus \{0\}$. Suppose $\zeta \in U \cap \ker(M)$. Then $\zeta M = \mathbf{0}$ and $\zeta = \sum_{i=1}^l \alpha_i \xi_i$ for some $\alpha_i \in \mathbb{F}_q$. It follows that

$$\mathbf{0} = \zeta M = \left(\sum_{i=1}^l \alpha_i \xi_i \right) M = \sum_{i=1}^l \alpha_i (\xi_i M) = \sum_i \alpha_i \lambda_i \xi_i^\sigma.$$

Since the vectors ξ_i^σ are linearly independent, we have that $\alpha_1 \lambda_1 = \cdots = \alpha_l \lambda_l = 0$; as the λ_i's are non-zero, this implies $\alpha_1 = \cdots = \alpha_l = 0$, that is $\zeta = \mathbf{0}$ and so $U \cap \ker(M) = \{\mathbf{0}\}$. This implies $\dim(U) \leq r$. By Theorem 3.8 (clearly, M restricted to U is invertible) it follows that $\mathrm{PG}(U)$ contains at most $\frac{s^r-1}{s-1}$ points satisfying $[\xi^\sigma] = [\xi M]$. The thesis now follows. □

Theorem 3.10. *The maximum number of points $[\xi] \in \mathrm{PG}(V^*)$ such that $[\xi^\sigma] \subseteq [\xi M]$ is*

1. $m(3) = s^2 + s + 1$ *if* $\sigma^2 = 1$ *and* $n = 2$; *or*
2. $m(1) = \frac{q^n-1}{q-1} + 1 = q^{n-1} + \cdots + 2$, *if* $\sigma^2 \neq 1$ *or* $n > 2$.

Proof. Recall that $q = s^t$, $t > 1$

- If $n = 2$, then the only possibilities for the rank of a non-null matrix $M \in M_3(q)$ are $r \in \{1, 2, 3\}$. So, by (19), the possible values for $m(r)$ are

  $$m(1) = \frac{q^2-1}{q-1} + 1 = q + 2 = s^t + 2, \qquad m(2) = 1 + \frac{s^2-1}{s-1} = s + 2,$$
  $$m(3) = \frac{s^3-1}{s-1} = s^2 + s + 1.$$

  If $t = 2$ then $q = s^2$, i.e. $\sigma^2 = 1$ and $s^2 + s + 1 > q + 2 = s^2 + 2 > s + 2$. In this case $\theta_{\max} = \theta_I = m(3) = s^2 + s + 1$.

  If $t > 2$, then $q + 2 = s^t + 2 \geq s^3 + 2 > s^2 + s + 1 > s + 2$ and the maximum is $m(1) = q + 2$. We can now check that if $M = \mathbf{e}_{11}$, then $\theta_M = \theta_{\max} = m(1)$.
- Suppose now $n > 2$. We want to study the sequence

$$m(1), \dots, m(r), \dots, m(n+1) \tag{20}$$

of possible maxima as $r = \mathrm{rank}\,(M)$ ranges from 1 to $n+1$; as such, we compare two successive terms in (20) using (19), namely

$$m(r) := \frac{q^{n+1-r}-1}{q-1} + \frac{s^r-1}{s-1} = \frac{s^{(n+1-r)t}-1}{s^t-1} + \frac{s^r-1}{s-1}$$

and

$$m(r+1) := \frac{q^{n+1-r-1}-1}{q-1} + \frac{s^{r+1}-1}{s-1} = \frac{s^{(n-r)t}-1}{s^t-1} + \frac{s^{r+1}-1}{s-1}.$$

Subtracting the first equation from the second we get

$$m(r+1) - m(r) = s^r - s^{t(n-r)}.$$

This is non-positive if and only if $r \leq t(n-r)$, that is $r \leq \frac{tn}{t+1}$. This means that as r grows the value of $m(r)$ is first decreasing and then increasing; so, the sequence $m(r)$ with $r = 1, \dots, n+1$ is convex and its maximum must be attained by values on the boundary of its domain, namely either for $r = 1$ or $r = n+1$. In particular $m(1) = \frac{s^{tn}-1}{s^t-1} + 1 = s^{t(n-1)} + \cdots + 2$ and $m(n+1) = \frac{s^{n+1}-1}{s-1} = s^n + \cdots + 1$. Comparing these two latter quantities, we see that $s^{tn-t} > s^n$ if $n > \frac{t}{t-1}$. However, $\frac{t}{t-1} \leq 2$. So, if $n > 2$ the maximum of $m(r)$ is attained for $r = 1$. We also have $\theta_{\mathbf{e}_{11}} = m(1)$, so this is actually the maximum for $\theta_{\max}$. $\square$

Remark 2. As a consequence of Theorem 3.10 we see that for $r = 1$ (with $n > 2$ or $\sigma^2 \neq 1$) or $r = 3$ (with $n = 2$ and $\sigma^2 = 1$) we have the equality

$$\max\{\theta_M : \mathrm{rank}\,(M) = r\} = m(r).$$

We conjecture that this holds for all $r = 1, \dots, n+1$.

3.3. Maximum weight codewords

By Corollary 3.3, in order to determine the maximum weight of $\mathscr{C}(\Lambda_\sigma)$ we need to compute

$$\min\{\theta_M : M \in M_{n+1}(q)\}.$$

In this section we provide sufficient conditions for $\min\{\theta_M : M \in M_{n+1}(q)\}$ to be 0. When this happens, maximum weight codewords have weight $q^{n-1}\frac{q^{n+1}-1}{q-1}$.

Observe that if $\mathrm{rank}\,(M) < n+1$, then $\theta_M > 0$. So we shall assume throughout the section that M is an invertible matrix. As seen in Lemma 3.5 with $V = W$, we have $\theta_M = |\mathrm{Fix}(\vartheta_{M,\sigma})|$ where $\vartheta_{M,\sigma}$ is the collineation defined in (16) $\vartheta_{M,\sigma}([\zeta]) = [\zeta^{\sigma^{-1}}M]$. So, to get $\theta_M = 0$ we want to construct a σ^{-1}-semilinear collineation which is fixed-point free.

Lemma 3.11. *Let σ and j be as in Definition 3.4. Suppose* $\gcd(\frac{q^{n+1}-1}{q-1}, s^j-1) > 1$. *Then, there exists an invertible matrix $M \in M_{n+1}(q)$ such that $\theta_M = 0$.*

Proof. We need to show that there exists at least one σ^{-1}-semilinear collineation of $\mathrm{PG}(V^*)$ which is fixed-point-free. Observe that ψ is a σ^{-1}-semilinear collineation which is fixed-point-free if and only if ψ^{-1} is a σ-semilinear collineation which is also fixed-point-free. We shall now construct such ψ^{-1}.

The vector space V^* is isomorphic to the field $\mathbb{F}_{q^{n+1}}$, regarded as a vector space over $\mathbb{F}_q$. We identify the points $[\zeta]$ of $\mathrm{PG}(V^*)$ with elements of $\mathbb{F}_{q^{n+1}}/\mathbb{F}_q$. Let ω be a primitive element of $\mathbb{F}_{q^{n+1}}$ and define $\psi^{-1} : \mathrm{PG}(V^*) \to \mathrm{PG}(V^*)$ as the $\mathbb{F}_q$-semilinear collineation $[x] \to [\zeta^{s^j}\omega]$. A point $[\zeta] \in \mathbb{F}_{q^{n+1}}$ is fixed by ψ^{-1} if and only if $\exists \lambda \in \mathbb{F}_q$ with $\lambda\zeta = \zeta^{s^j}\omega$. This implies $\zeta^{s^j-1}\omega \in \mathbb{F}_q$, that is $(\zeta^{s^j-1}\omega)^{q-1} = 1$. In particular, the order of ω^{q-1} in $\mathbb{F}^*_{q^{n+1}}$ must be the same as the order of $\zeta^{(s^j-1)(q-1)}$. Since ω is a primitive element of $\mathbb{F}_{q^{n+1}}$, the order of ω^{q-1} is $(q^{n+1}-1)/(q-1)$. On the other hand, the order of $\zeta^{(s^j-1)(q-1)}$ divides

$$\frac{q^{n+1}-1}{\gcd((s^j-1)(q-1), q^{n+1}-1)} = \frac{q^{n+1}-1}{(q-1)\gcd(s^j-1, \frac{q^{n+1}-1}{q-1})}.$$

It follows that if $\gcd(s^j-1, \frac{q^{n+1}-1}{q-1}) > 1$, then ψ^{-1} does not have any fixed points. Since ψ^{-1} is a semilinear collineation, it can be written as $\psi^{-1}([\zeta]) = [\zeta^\sigma M^{-1}]$ for some invertible matrix M. It follows that also $\psi([\zeta]) = [\zeta^{\sigma^{-1}}M]$ is fixed-point free; consequently $\theta_M = 0$. This completes the proof. □

Corollary 3.12. *Suppose both q and n to be odd. Then there exists $M \in M_{n+1}(q)$ such that $\theta_M = 0$.*

Proof. Under these assumptions both $s^j - 1$ and $\frac{q^{n+1}-1}{q-1}$ are even. The previous lemma yields the result. □

As seen in Section 2.5, when n is odd and $\sigma^2 = 1$ the geometry $\bar{\Gamma}$ might admit semi-standard spread type hyperplanes $\mathscr{H}_\phi$ arising from ε_σ of the form $\varepsilon_\sigma^{-1}(M^\perp)$. As the following lemma shows, these hyperplanes correspond to words of maximum weight.

Lemma 3.13. *Suppose $\mathscr{H}_\phi = \varepsilon_\sigma^{-1}(M^\perp)$ is a semi-standard spread type hyperplane of $\bar{\Gamma}$ where ϕ is a fixed-point-free semilinear involutory collineation of* $\mathrm{PG}(V^*)$ *defined by the matrix M. Then c_M is a maximum weight codeword of $\mathscr{C}(\Lambda_\sigma)$.*

Proof. By Proposition 2.9, $|\mathscr{H}_\phi| = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2}$. So, the weight of a codeword associated with $M^\perp$ is $q^{n-1}\frac{q^{n+1}-1}{q-1}$. It follows by Corollary 3.3 that $\theta_M = 0$. Hence c_M is a maximum weight codeword of $\mathscr{C}(\Lambda_\sigma)$. □

Since $\theta_M \geq 0$ by definition, in light of Corollary 3.3 the following is immediate.

Corollary 3.14. *Suppose* $\gcd(\frac{q^{n+1}-1}{q-1}, s^j-1) > 1$*. Then the maximum weight of the codewords of* $\mathscr{C}(\Lambda_\sigma)$ *is* $w_{\max} = q^{n-1}\frac{q^{n+1}-1}{q-1}$*. In particular, this happens for all* $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$ *when both* n *and* q *are odd.*

3.4. Proof of Theorem 1.1

The computation of the length and dimension of $\mathscr{C}(\Lambda_\sigma)$ is straightforward from Section 2. Indeed, the length of $\mathscr{C}(\Lambda_\sigma)$ is the number of point-hyperplane pairs (p, H) of $\mathrm{PG}(n, q)$ with $p \in H$, that is

$$N_\sigma = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}.$$

The dimension of $\mathscr{C}(\Lambda_\sigma)$ is the dimension of the embedding $\bar{\varepsilon}_\sigma$; so

$$k_\sigma = (n+1)^2.$$

To determine the minimum distance, by Lemma 3.2 and Corollary 3.3, we need to find the maximum of θ_M as M varies in $M_{n+1}(q)$. Hence, the minimum distance of $\mathscr{C}(\Lambda_\sigma)$ follows from Theorem 3.10 and we have

$$d_\sigma = \begin{cases} q^3 - \sqrt{q}^3 & \text{if } \sigma^2 = 1 \text{ and } n = 2, \\ q^{2n-1} - q^{n-1} & \text{if } \sigma^2 \neq 1 \text{ or } n > 2. \end{cases} \quad \square$$

3.5. Proof of Theorem 1.2

Point 1 of Theorem 1.2 follows from Proposition 2.3 and Theorem 2.4. Point 4 of Theorem 1.2 follows from Corollary 3.12, Corollary 3.14 and Corollary 3.13.

We remind that for $\sigma^2 = 1$ and $\sigma \neq 1$, the σ-norm function is defined as follows: $N\colon \mathbb{F}_q \to \mathbb{F}_s,\ N(x) = x^{s+1}$, where $\mathrm{Fix}(\sigma) = \mathbb{F}_s,\ q = s^2$.

Theorem 3.15. *Suppose* $n = 2$*,* $\sigma^2 = 1$ *and* $\sigma \neq 1$ *and let* $M \in M_{n+1}(q) \setminus \{0\}$*. The matrix* M *defines a minimum weight codeword of* $\mathscr{C}(\Lambda_\sigma)$ *if and only if there exist three linearly independent vectors* $\xi_1, \xi_2, \xi_3 \in V^*$ *and three non-null scalars* $\alpha, \beta, \gamma \in \mathbb{F}_q \setminus \{0\}$ *such that*

$$\xi_1 M = \alpha \xi_1^\sigma, \qquad \xi_2 M = \beta \xi_2^\sigma, \qquad \xi_3 M = \gamma \xi_3^\sigma, \qquad N(\alpha) = N(\beta) = N(\gamma). \tag{21}$$

Proof. Suppose $[M^\perp]$ defines a minimum weight codeword of $\mathscr{C}(\Lambda_\sigma)$. By Theorem 3.10, M is an invertible 3×3-matrix with $\theta_M = m(3) = s^2+s+1$. Define $U := \{\xi \in V^*\colon [\xi^\sigma] = [\xi M]\}$. By the definition (14) of θ_M, we have $\theta_M = \frac{|U|-1}{q-1}$.

If $\dim(\langle U\rangle) \leq 2$, then by Theorem 3.8 we have $\frac{|U|-1}{q-1} \leq s+1$, which contradicts the maximality of θ_M by Theorem 3.10. Hence U spans V^*, i.e. there exist at least three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in U$ and three scalars $\alpha, \beta, \gamma \in \mathbb{F}_q \setminus \{0\}$ such that $\xi_1 M = \alpha\xi_1^\sigma$, $\xi_2 M = \beta\xi_2^\sigma$, $\xi_3 M = \gamma\xi_3^\sigma$. We now prove that $N(\alpha) = N(\beta) = N(\gamma)$.

If no vector of $\langle U\rangle$ different from elements of $[\xi_i]$, $i = 1, 2, 3$, is contained in U then $\frac{|U|-1}{q-1} = |\{[\xi_1], [\xi_2], [\xi_3]\}| = 3$, which contradicts again the maximality of θ_M by Theorem 3.10. Hence there exists at least one vector $\zeta \in U$ which is not a multiple of any ξ_i, $i = 1, 2, 3$. So there is vector $\zeta \in U$ such that $[\zeta^\sigma] = [\zeta M]$ and

$$\zeta = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$$

for some scalars a_1, a_2, a_3, at least two of which are not null. This implies that there exists $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that

$$\begin{cases} a_1^\sigma = \lambda a_1\alpha \\ a_2^\sigma = \lambda a_2\beta \\ a_3^\sigma = \lambda a_3\gamma. \end{cases} \tag{22}$$

Denote now by $\Delta := [\langle\xi_1, \xi_2\rangle] \cup [\langle\xi_1, \xi_3\rangle] \cup [\langle\xi_2, \xi_3\rangle]$ the triangle of $\mathrm{PG}(V^*)$ whose sides are the lines spanned by $[\xi_i]$ and $[\xi_j]$, $i \neq j$, $i, j \in \{1, 2, 3\}$.

Suppose by way of contradiction that all non-null vectors $\zeta \in U$ have the property that the associated projective points $[\zeta]$ are contained in Δ. Then, by Lemma 3.7, we have $\frac{|U|-1}{q-1} \leq 3s < s^2 + s + 1 = m(3)$, which contradicts again the maximality of $\theta_M = \frac{|U|-1}{q-1}$ by Theorem 3.10.

Hence there exists at least one non-null vector $\zeta \in U$ such that $[\zeta] \notin \Delta$, i.e. there exist three not null scalars a_1, a_2, a_3 and $\lambda \in \mathbb{F}_q \setminus \{0\}$ satisfying (22). This implies that $a_1a_2a_3 \neq 0$. We can assume without loss of generality $a_1 = 1$. Solving (22) in λ, we get

$$\alpha^{-1} = a_2^{\sigma-1}\beta^{-1} = a_3^{\sigma-1}\gamma^{-1}. \tag{23}$$

Let $\mathbb{S} = \{x^{\sigma-1} : x \in \mathbb{F}_q\}$ be the kernel of the norm function. If $\alpha^{-1}\beta \notin \mathbb{S}$ or $\alpha^{-1}\gamma \notin \mathbb{S}$ we immediately see that there are no possible solutions to (22) in the unknowns a_2, a_3 and this contradicts the existence of the scalars a_2, a_3 such that $\zeta \in U$. So, $\alpha^{-1}\beta \in \mathbb{S}$ and $\alpha^{-1}\gamma \in \mathbb{S}$, i.e., $N(\alpha^{-1}\beta) = 1$ and $N(\alpha^{-1}\gamma) = 1$ whence $N(\alpha) = N(\beta) = N(\gamma)$.

Conversely, suppose there exist three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in V^*$ and three non-null scalars $\alpha, \beta, \gamma \in \mathbb{F}_q \setminus \{0\}$ such that (21) hold. Then M is invertible. We will show that $\theta_M = s^2 + s + 1$, hence the thesis would follow from Theorem 3.10.

Take an arbitrary vector $\zeta \in V^*$ as $\zeta = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$ with $a_1, a_2, a_3 \in \mathbb{F}_q$. To compute θ_M, we need to count the number of points $[\zeta]$ such that $\zeta \in U$. This is equivalent to compute the number of solutions in the unknowns a_1, a_2, a_3, λ of the system (22). Put $\Delta := \bigcup_{1\leq i<j\leq 3}\langle[\xi_i], [\xi_j]\rangle$.

Suppose $[\zeta] \notin \Delta$. Then $a_1a_2a_3 \neq 0$. Without loss of generality, put $a_1 = 1$, hence $\lambda = \alpha^{-1}$ and the system (22) becomes

$$\begin{cases} a_2^{\sigma-1} = \alpha^{-1}\beta \\ a_3^{\sigma-1} = \alpha^{-1}\gamma. \end{cases} \tag{24}$$

Since by hypothesis $N(\alpha^{-1}\beta) = N(\alpha^{-1}\gamma) = 1$, the above system is solvable and each of the equations of (24) has exactly $s-1$ solutions. So, there exist precisely $(s-1)^2 = s^2 - 2s + 1$ points $[\zeta] \in \mathrm{PG}(V^*) \setminus \Delta$ with $\zeta \in U$.

Suppose $[\zeta] \in \Delta \setminus \{[\xi_1], [\xi_2], [\xi_3]\}$. Without loss of generality, assume $[\zeta] \in \langle [\xi_2], [\xi_3] \rangle$. i.e. $a_1 = 0$ and take $a_2 = 1$. Then the system (22) becomes

$$a_3^{\sigma-1} = \beta^{-1}\gamma. \tag{25}$$

Since $N(\beta^{-1}\gamma) = 1$, the above equation is solvable and has exactly $s-1$ solutions.

Thus, there exist precisely $(s-1)$ points $[\zeta] \notin \{[\xi_2], [\xi_3]\}$ on the line $[\langle \xi_2, \xi_3 \rangle]$ with $\zeta \in U$. The same argument applies to the lines $[\langle \xi_1, \xi_2 \rangle]$ and $[\langle \xi_1, \xi_3 \rangle]$. So,

$$\frac{|U|-1}{q-1} = \theta_M = (s-1)^2 + 3(s-1) + 3 = s^2 + s + 1.$$

By Theorem 3.10, $\theta_M = m(3)$. So, the theorem is proved. □

Part 2 of Theorem 1.2 is exactly Theorem 3.15 and Part 3 of Theorem 1.2 is the following theorem.

Theorem 3.16. *Suppose $n > 2$ or $\sigma^2 \neq 1$ and let $M \in M_{n+1}(q) \setminus \{O\}$. The matrix M defines a minimum weight codeword of $\mathscr{C}(\Lambda_\sigma)$ if and only if $\bar{\varepsilon}_\sigma^{-1}(M^\perp)$ is a quasi-singular but not a singular hyperplane of $\bar{\Gamma}$.*

The hyperplane $[M^\perp]$ defines a second minimum weight codeword of $\mathscr{C}(\Lambda_\sigma)$ if and only if $\bar{\varepsilon}_\sigma^{-1}(M^\perp)$ is a singular hyperplane of $\bar{\Gamma}$.

Proof. Suppose $\mathscr{H}$ is a quasi-singular hyperplane of $\bar{\Gamma}$. By Proposition 2.6, $\mathscr{H} = \bar{\varepsilon}_\sigma^{-1}(M^\perp)$, with M a non-null $(n+1)$-matrix of rank 1 and $|\mathscr{H}| = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \frac{q^n-1}{q-1}q^{n-1}$ if $\mathscr{H}$ is a singular hyperplane or $|\mathscr{H}| = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + (\frac{q^n-1}{q-1} + 1)q^{n-1}$ if $\mathscr{H}$ is a quasi-singular but not singular hyperplane of $\bar{\Gamma}$ by Proposition 2.9. By the injective property of the embedding $\bar{\varepsilon}_\sigma$, we have $|[M^\perp] \cap \Lambda_\sigma| = |\bar{\varepsilon}_\sigma^{-1}(M^\perp)|$. If we consider the codeword $c_M \in \mathscr{C}(\Lambda_\sigma)$ defined by the matrix M, we know by Corollary 3.2 that the weight of c_M is $wt(c_M) = N_\sigma - |[M^\perp] \cap \Lambda_\sigma| = N_\sigma - |\mathscr{H}|$.

By Theorem 3.10 and Corollary 3.3, comparing the cardinalities of a singular hyperplane and a quasi-singular non-singular hyperplane of $\bar{\Gamma}$, we immediately see that the minimum weight codewords are associated to a quasi-singular non-singular hyperplane $\varepsilon_\sigma^{-1}(M^\perp)$ of $\bar{\Gamma}$ with $\theta_M = \frac{q^n-1}{q-1} + 1$. As the singular hyperplanes of $\bar{\Gamma}$ yield codewords with $\theta_M = \frac{q^n-1}{q+1}$, they correspond to codewords with the second lowest weight.

Conversely, let $[M^\perp]$ define a minimum weight codeword of $\mathscr{C}(\Lambda_\sigma)$; then $\theta_M = m(1) = \frac{q^n-1}{q-1} + 1$ and, by Theorem 3.10, M is a rank 1 matrix. By Proposition 2.6, $\bar{\varepsilon}_\sigma^{-1}(M^\perp)$

is a quasi-singular hyperplane of $\bar{\Gamma}$. By Lemma 3.2, the weight of the minimum weight codewords corresponds to the maximal cardinality of the set $[M^{\perp}] \cap \Lambda_\sigma$. Since $|[M^{\perp}] \cap \Lambda_\sigma| = |\bar{\varepsilon}_\sigma^{-1}(M^{\perp})|$, the thesis now follows from Proposition 2.9.

Suppose now $[M^{\perp}]$ defines a second lowest weight codeword of $\mathscr{C}(\Lambda_\sigma)$. Then, by Theorem 3.10 and Corollary 3.3, its weight corresponds to $\theta_M = m(1) - 1 = \frac{q^n-1}{q-1}$. We first show that M has rank 1.

By way of contradiction suppose $\operatorname{rank}(M) \geq 2$. Then by Theorem 3.9, $\theta_M \leq m(2)$. By Theorem 3.9, $m(2) = \frac{q^{n-1}-1}{q-1} + \frac{s^2-1}{s-1}$ and $m(1) - 1 = \frac{q^n-1}{q-1}$, but this contradicts $(m(1)-1) = \theta_M \leq m(2)$. Hence $\operatorname{rank}(M) = 1$. The thesis now follows by Proposition 2.6 and Proposition 2.9. $\square$

As a consequence of Theorem 3.16 we point out that the minimum weight codewords and the second minimum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ depend on the properties of the hyperplanes of $\bar{\Gamma}$ and not on the way it is embedded into a projective space.

3.6. Proof of Theorem 1.3

Relying on the representation of the codewords of $\mathscr{C}(\Lambda_\sigma)$ given by (11) of Lemma 3.1, we see that $\varrho(g)$ maps an element of $\mathscr{C}(\Lambda_\sigma)$ to an element of $\mathscr{C}(\Lambda_\sigma)$, by the rule

$$c_M = (\operatorname{Tr}(X_1M), \dots, \operatorname{Tr}(X_NM)) \to \\ c_{g^{-1}Mg^\sigma} = (\operatorname{Tr}(X_1g^{-1}Mg^\sigma), \dots, \operatorname{Tr}(X_Ng^{-1}Mg^\sigma)).$$

Furthermore, $\varrho(g)$ is a linear transformation of $\mathscr{C}(\Lambda_\sigma)$ since

$$\begin{aligned}\varrho(g)(\alpha c_A + \beta c_B) &= \varrho(g)(c_{\alpha A+\beta B}) = \\ &= \varrho(g)(\operatorname{Tr}(X_1(\alpha A + \beta B)), \dots, \operatorname{Tr}(X_N(\alpha A + \beta B))) = \\ &= (\operatorname{Tr}(X_1g^{-1}(\alpha A + \beta B)g^\sigma), \dots, \operatorname{Tr}(X_Ng^{-1}(\alpha A + \beta B)g^\sigma)) = \\ &\alpha(\operatorname{Tr}(X_1g^{-1}Ag^\sigma), \dots, \operatorname{Tr}(X_Ng^{-1}Ag^\sigma))+ \\ &\beta(\operatorname{Tr}(X_1g^{-1}Bg^\sigma), \dots, \operatorname{Tr}(X_Ng^{-1}Bg^\sigma)) = \\ &= \alpha c_{g^{-1}Ag^\sigma} + \beta c_{g^{-1}Bg^\sigma} = \alpha\varrho(g)(c_A) + \beta\varrho(g)(c_B).\end{aligned}$$

There remains to prove that $\varrho(g)$ is an isometry. To this purpose observe that by the cyclic property of the trace,

$$\operatorname{Tr}(X_ig^{-1}Mg^\sigma) = \operatorname{Tr}(g^\sigma X_ig^{-1}M),$$

for all $i = 1, \dots, N$ and all $M \in M_{n+1}(q)$. On the other hand, $[g^\sigma X_ig^{-1}] = [X_i]^g = [X_{\pi(g)(i)}]$ by (4); consequently, there is a function $\lambda : \mathrm{GL}(n+1, q) \times \{1, \dots, N\} \to \mathbb{F}_q^\star$ depending on g and i such that $g^\sigma X_ig^{-1} = \lambda(g, i)X_{\pi(g)(i)}$. Thus, we can write

$$\mathrm{Tr}(X_i g^{-1} M g^{\sigma}) = \lambda(g,i)\mathrm{Tr}(X_{\pi(g)(i)} M),$$

for all $i = 1, \dots, N$. In particular, the components of c_M^g are obtained by permuting and multiplying by the scalars $\lambda(g,i)$ the components of c_M. Since $\lambda(g,i) \neq 0$, the number of null components in c_M^g, i.e. the number of i such that $\mathrm{Tr}(X_i g^{-1} M g^{\sigma}) = 0$, is the same as the number of null components in c_M. This proves that the weight of c_M^g is the same as that of c_M, i.e. that $\varrho(g)$ is a isometry.

To conclude, observe that $g^{-1} M g^{\sigma} = M$ for all M implies, as a special case when $M = I$, that $g^{-1} g^{\sigma} = 1$, i.e. $g^{\sigma} = g$. This happens if and only if g is a matrix with entries over $\mathbb{F}_s$. On the other hand, $Mg = g^{-1} M$ for all M implies that g must be a scalar matrix. It follows that the kernel of the action is K_{σ}. □

Remark 3. The action of $\mathrm{GL}(n+1,q)$ on $M_{n+1}(q)$ described in Theorem 1.3 is not the action of $\mathrm{GL}(n+1,q)$ on Λ_{σ} which lifts through $\bar{\varepsilon}_{\sigma}$ which is instead $X^g := g^{\sigma} X g^{-1}$. These two actions are adjoint with respect to the saturation form, in the sense that if f is the saturation form, $f(X^g, M) = \mathrm{Tr}(g^{\sigma} X g^{-1} M) = \mathrm{Tr}(X g^{-1} M g^{\sigma}) = f(X, M^g)$.

Acknowledgments

Both authors are affiliated with GNSAGA of INdAM (Italy) whose support they kindly acknowledge.

Data availability

No data was used for the research described in the article.

References

[1] G.N. Alfarano, M. Borello, A. Neri, A geometric characterization of minimal codes and their asymptotic performance, Adv. Math. Commun. 16 (1) (2022) 115–133, https://doi.org/10.3934/amc.2020104.

[2] G.N. Alfarano, M. Borello, A. Neri, A. Ravagnani, Three combinatorial perspectives on minimal codes, SIAM J. Discrete Math. 36 (1) (2022) 461–489, https://doi.org/10.1137/21M1391493.

[3] A. Ashikhmin, A. Barg, Minimal vectors in linear codes, IEEE Trans. Inf. Theory 44 (5) (1998) 2010–2017, https://doi.org/10.1109/18.705584.

[4] P. Beelen, S.R. Ghorpade, S.U. Hasan, Linear codes associated to determinantal varieties, Discrete Math. 338 (8) (2015) 1493–1500, https://doi.org/10.1016/j.disc.2015.03.009.

[5] M. Bonini, M. Borello, Minimal linear codes arising from blocking sets, J. Algebr. Comb. 53 (2) (2021) 327–341, https://doi.org/10.1007/s10801-019-00930-6.

[6] I. Cardinali, L. Giuzzi, On minimal codes arising from projective embeddings of point-line geometries, preprint, arXiv:2155.22747, https://doi.org/10.48550/ARXIV.2155.22747, Nov. 2025.

[7] I. Cardinali, L. Giuzzi, A. Pasini, The relatively universal cover of the natural embedding of the long root geometry for the group SL($n+1, \mathbb{K}$), Algebr. Comb. 9 (1) (2026) 231–259, https://doi.org/10.5802/alco.473.

[8] I. Cardinali, L. Giuzzi, Minimum distance of symplectic Grassmann codes, Linear Algebra Appl. 488 (2016) 124–134, https://doi.org/10.1016/j.laa.2015.09.031.

[9] I. Cardinali, L. Giuzzi, Line Hermitian Grassmann codes and their parameters, Finite Fields Appl. 51 (2018) 407–432, https://doi.org/10.1016/j.ffa.2018.02.006.

[10] I. Cardinali, L. Giuzzi, Minimum distance of orthogonal line-Grassmann codes in even characteristic, J. Pure Appl. Algebra 222 (10) (2018) 2975–2988, https://doi.org/10.1016/j.jpaa.2017.11.009.

[11] I. Cardinali, L. Giuzzi, Linear codes arising from the point-hyperplane geometry — part I: the Segre embedding, Finite Fields Appl. 111 (2026) 102766, https://doi.org/10.1016/j.ffa.2025.102766.

[12] I. Cardinali, L. Giuzzi, K.V. Kaipa, A. Pasini, Line polar Grassmann codes of orthogonal type, J. Pure Appl. Algebra 220 (5) (2016) 1924–1934, https://doi.org/10.1016/j.jpaa.2015.10.007.

[13] A.A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, Linear nonbinary covering codes and saturating sets in projective spaces, Adv. Math. Commun. 5 (1) (2011) 119–147, https://doi.org/10.3934/amc.2011.5.119.

[14] W.M. Kantor, Subgroups of classical groups generated by long root elements, Trans. Am. Math. Soc. 248 (2) (1979) 347–379, https://doi.org/10.2307/1998972.

[15] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes, North-Holland Mathematical Library, vol. 16, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[16] F.J. MacWilliams, Combinatorial properties of elementary abelian groups, thesis (Ph.D.)–Radcliffe College, Harvard University, Cambridge, MA, 1962.

[17] J. Massey, Minimal codewords and secret sharing, in: Proceedings of the 6th Joint Swedish-Russian International Workshop on Information Theory, 1993, pp. 276–279.

[18] A. Pasini, Embeddings and hyperplanes of the Lie geometry $A_{n,\{1,n\}}(\mathbb{F})$, Comb. Theory 4 (2) (2024) 5, 25.

[19] A. Pasini, Geometric hyperplanes of the Lie geometry $A_{n,\{1,n\}}(\mathbb{F})$, Ric. Mat. (Apr. 2024), https://doi.org/10.1007/s11587-024-00859-4.

[20] C. Tang, Y. Qiu, Q. Liao, Z. Zhou, Full characterization of minimal linear codes as cutting blocking sets, IEEE Trans. Inf. Theory 67 (6) (2021) 3690–3700, https://doi.org/10.1109/TIT.2021.3070377.

[21] M. Tsfasman, S. Vlăduţ, D. Nogin, Algebraic Geometric Codes: Basic Notions, vol. 139, American Mathematical Society, 2007.

[22] H. Van Maldeghem, Hyperplanes of Segre geometries, Ars Comb. 160 (2024) 59–71, https://doi.org/10.61091/ars-160-07.