European Journal of Mechanics / A Solids 102 (2023) 105108

Contents lists available at ScienceDirect

European Journal of Mechanics / A Solids

journal homepage: www.elsevier.com/locate/ejmsol

Time domain analytical bounds to the homogenized viscous kernels of linear viscoelastic composites

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ARTICLE INFO

Keywords:
Viscoelasticity
Variational principles
Composite materials
Bounds

ABSTRACT

New extremum principles in linear viscoelasticity are derived from general stationarity ones proposed in Carini and Mattei (2015), exploiting suitable selections of the admissible fields in the associated convolutive functionals. These new extremum principles have therefore a restricted validity. Analytical bounds to the homogenized viscous kernels of linear viscoelastic composites are derived in the time domain. In the restricted case of macroscopically isotropic composite materials, six new bounds are obtained from the new extremum principles. These bounds can be derived exploiting the choice of Representative Volume Elements (RVEs) loaded in a purely deviatoric way only. Two strict lower bounds to the homogenized viscous kernels, of the Reuss type, are also derived. One of these was already proposed in Huet (1995), and is valid for generic linear viscoelastic composites under general stress and strain states. The other Reuss-type strict lower bound is new, but has the same limited validity as the first six ones. The new upper bounds for isotropic composites, obtained both in terms of viscous kernels and of their time rates, as well as the strict lower bounds, are extensions to viscoelasticity of the Voigt and Reuss bounds for linear elastic composites. The performance of the obtained bounds is checked by comparison with numerical solutions. It is worth remarking that the use of the new extremum theorems for the purpose of deriving bounds is not possible in a general RVE stress case, i.e., deviatoric plus volumetric. As a consequence, the non-strict bounds holding for the case of deviatoric RVE loading are not valid for the volumetric case. The reason for this difference is not yet fully understood.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be the region occupied by a solid body made of a linear viscoelastic material, possibly heterogeneous. V denotes the volume of the region Ω and $\Gamma = \Gamma_u \cup \Gamma_p$ its external surface, with unit outward normals $n_i(x_r)$. The displacement, strain, and stress fields at point $x_r \in \Omega$, at time t, are denoted by $u_i(x_r,t)$, $\varepsilon_{ij}(x_r,t)$, and $\sigma_{ij}(x_r,t)$, respectively. The loading is given by a history of body forces $b_i(x_r,t)$, of surface tractions $p_i(x_r,t)$ acting on the loaded region Γ_p of the boundary, and of prescribed displacements $u_i^0(x_r,t)$ acting on the constrained boundary Γ_u. The body is undisturbed for $t<0$ and the whole loading history is supposed to be defined in the given time range $t \in [0,T]$, T being the end time of the loading process, an arbitrary value $T \in [0,\infty)$.

The direct viscoelastic constitutive law, that relates a known strain field $\varepsilon_{ij}(x_r,t)$ to the corresponding stress field $\sigma_{ij}(x_r,t)$, is written as follows:

$$\sigma_{ij}(x_r,t) = \int_{0^-}^{t} R_{ijhk}(x_r,t-\tau)\, d\varepsilon_{hk}(x_r,\tau) \tag{1.1}$$

where the integral has to be meant in the Stieltjes sense, and $R_{ijhk}(x_r,t)$, with $t>0$, is the relaxation tensor, or kernel. We assume that the constitutive law (1.1) is invertible, to give the inverse stress–strain equation written as:

$$\varepsilon_{ij}(x_r,t) = \int_{0^-}^{t} C_{ijhk}(x_r,t-\tau)\, d\sigma_{hk}(x_r,\tau) \tag{1.2}$$

in which $C_{ijhk}(x_r,t)$ denotes the creep tensor, or kernel.

The main interest of this work is in the homogenization problem for a heterogeneous viscoelastic solid. The studied body Ω is restricted to be a suitably defined Representative Volume Element (RVE), of volume V and external surface $\Gamma = \Gamma_u \cup \Gamma_p$. Henceforth, given any generic function $f(x_r,t)$, we adopt the notation $\langle f(t)\rangle$ to denote the volume average of $f(x_r,t)$ over Ω, i.e.,:

$$\langle f(t)\rangle = \frac{1}{V}\int_{\Omega} f(x_r,t)\, d\Omega \tag{1.3}$$

Unless strictly necessary, from now on we will omit indicating the dependence on the space coordinates x_r.

The homogenized counterparts of the viscous kernels, for a viscoelastic composite, allow one to relate the volume averages of both

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https://doi.org/10.1016/j.euromechsol.2023.105108
Received 13 April 2023; Received in revised form 6 July 2023; Accepted 14 August 2023
Available online 19 August 2023

stress and strain tensors. For the direct constitutive law this involves the tensor $R^h_{ijhk}(t)$, that will be assumed to be defined from

$$\langle \sigma_{ij}\rangle(t) := \int_{0^-}^{t} R^h_{ijhk}(t-\tau)\, d\langle \varepsilon_{hk}\rangle(\tau) \tag{1.4}$$

and for the inverse constitutive law the tensor $C^h_{ijhk}(t)$, assumed to be defined from:

$$\langle \varepsilon_{ij}\rangle(t) := \int_{0^-}^{t} C^h_{ijhk}(t-\tau)\, d\langle \sigma_{hk}\rangle(\tau) \tag{1.5}$$

An exact expression of the homogenized viscous kernels is not usually available, and it is customary to look either for estimates or for bounds.

In viscoelasticity, the lack of symmetry/positive definiteness of the constitutive operators with respect to standard bilinear forms (see for example Tonti, 1973, and Tonti, 1984) has severely hindered the development of bounding techniques for the homogenized constitutive operators, both direct and inverse. Some results have been obtained in the past making reference to *convolutive* bilinear forms. The viscoelastic problem was proved in Gurtin (1963) to be symmetric with respect to a convolutive bilinear form (see for example Tonti, 1973); nevertheless, even exploiting this symmetry, and despite a rather intense research activity (see for example Carini and Mattei, 2015, and references quoted therein), no usable extremum formulation has ever been obtained for linear viscoelasticity in the time domain,[1] and only few standard, explicit, analytical bounds in the time domain to the homogenized viscous kernels for viscoelastic composites have been obtained so far. To the best of our knowledge, the main contribution, in this sense, has been given by Huet (1995), who obtained a strict lower bound to the homogenized relaxation kernel, together with other bounds, both lower and upper, to the *rates* of both the homogenized creep and relaxation kernels. From these last, Huet (1995) obtained bounds also for total quantities which, however, appear of difficult practical application. Moreover, the same Huet (1995) points out, in his conclusions, that "there are still classical results of the elasticity theory that cannot be transferred through to the viscoelastic case (...) For this, true viscoelasticity minimum theorems (...) are still needed".

Carini and Mattei (2015) were able to exploit older ideas by Staverman and Schwarzl (1952), and by Mandel (1966), to obtain several new min-stat, or even minimum, formulations in viscoelasticity. None of these, unfortunately, could be adopted directly to obtain bounds to the homogenized viscous kernels.

Mattei and Milton (2016) proposed a new approach to derive time domain bounds to the response of a two-component viscoelastic composite under antiplane loading. The novelty, there, lies in the application of the so-called "analytical method", previously exploited for problems formulated in the frequency domain, to derive bounds in the time domain. The results obtained with this method turn out to be very accurate if sufficient information about the composite is available.

In the present work, new extremum principles for linear viscoelastic RVEs are derived from the theory presented in Carini and Mattei (2015), involving both finite and rate versions of the viscous kernels. These extremum theorems can be obtained by selecting in a very specific way the admissible fields in the functionals associated to the stationarity principles of Carini and Mattei (2015).

[1] Analytical minimum principles for linear viscoelasticity have been proposed in Rafalski (1969), Reiss and Haug (1978), and Carini et al. (1995). Nevertheless, none of these can be adopted to obtain bounds to the homogenized viscous kernels of viscoelastic composites in the time domain. The formulation proposed in Carini et al. (1995) can be adopted as a basis for time integration, but not for analytical developments leading to homogenization, whereas the other two can be adopted to obtain bounds only to the Laplace transforms of the homogenized viscous kernels. Cherkaev and Gibiansky (1994), and Milton (1990), formulated extremum principles for linear initial value problems, including the hereditary viscoelastic one, in the frequency domain. The work by Milton (1990) can be extended to the time domain, even though it was not explicitly formulated with this purpose.

From these new minimum theorems, upper and lower first-order bounds to the homogenized viscous kernels of macroscopically isotropic viscoelastic composites with any number of isotropic phases are derived. Two bounds are derived, following a similar strategy, also for the rates of the viscous kernels.

These new bounds can be obtained only exploiting (i) the assumed macroscopic isotropy of the viscoelastic composites and (ii) the loading typical of RVEs. These bounds, moreover, are valid only for the special case of RVEs subjected to a purely deviatoric loading, of both a purely kinematic and a purely static nature. In this special circumstance only, and under the other considered assumptions, in fact, it was possible to individuate the admissible fields that cancel the terms that destroy the minimum nature of the theorems of Carini and Mattei (2015). It is not yet clear why the same terms remain important under a volumetric type of loading of the RVE.

Two strict (non-optimal) lower bounds to the homogenized viscous kernels, both of the Reuss type, are finally presented. One of them, concerning the homogenized relaxation kernel, was already derived in Huet (1995), and is obtained following a strategy which does not require the availability of an extremum principle. This strict bound has therefore a general validity, i.e., it holds for generic anisotropic composites under both volumetric and deviatoric loading. The new strict Reuss-type lower bound to the homogenized creep kernel, instead, is valid only for macroscopically isotropic RVEs under deviatoric loading.

The performance of the bounds presented herein is finally checked by comparison with reference to numerical solutions, concerning RVEs, obtained by means of Finite Element analyses.

2. Stationarity theorems for linear viscoelastic RVEs

We start by recalling, also for future convenience, the original Gurtin convolutive functionals, of the Total Potential Energy and Total Complementary Energy type, respectively, which read as follows:

$$\mathrm{TPE}^G[u_i'] = \frac{1}{2}\int_\Omega R_{ijhk} \star d\varepsilon'_{hk} \star d\varepsilon'_{ij}\, d\Omega - \int_\Omega b_i \star du_i'\, d\Omega - \int_{\Gamma_p} p_i \star du_i'\, d\Gamma \tag{2.1}$$

$$\mathrm{TCE}^G[\sigma'_{ij}] = \frac{1}{2}\int_\Omega C_{ijhk} \star\ d\sigma'_{hk} \star\ d\sigma'_{ij}\, d\Omega - \int_{\Gamma_u} n_i u_j^0 \star\ d\sigma'_{ij}\, d\Gamma \tag{2.2}$$

in which u_i' must respect compatibility, σ'_{ij} must respect equilibrium, and which are both stationary in the solution of a viscoelastic problem. In these expressions use was made of the standard convolution symbol $\star$ between two functions $f(x_r,t)$ and $g(x_r,t)$ (for this notation see, for example, Gurtin and Sternberg, 1962), defined as follows:

$$f(x_r,t) \star\ dg(x_r,t) := \int_{0^-}^{t} f(x_r, t-\tau)\, dg(x_r,\tau) \tag{2.3}$$

Starting from these functionals, the relevant new ideas of Carini and Mattei (2015) are re-interpreted here considering the addition, to the time interval $[0,T]$, of the subsequent time interval $[T,2T]$. The unknowns are thus formally doubled, by adding the variables defined over the first sub-interval to those, fictitious, related to the second one.

This is motivated by an interesting observation, made independently by Staverman and Schwarzl (1952) and Mandel (1966). Both these authors remark that a more complete knowledge of the thermodynamic viscous properties in the time interval $[0,T]$ is obtained from information concerning the subsequent time interval $[T,2T]$. In their own words: "In simple relaxation or creep experiments the thermodynamic functions at time t depend on the relaxation or creep function at time $2t$" (Staverman and Schwarzl, 1952), and "... si la connaissance de la matrice f_{ij} (ou r_{ij}) entre 0 et t suffit pour definir le comportement purement mecanique d'un corps viscoelastique (a temperature constante) dans le meme intervalle de temps, elle ne definit pas son comportement energetique. Ce dernier ne serait defini dans l'intervalle

$0, t$ que lorsqu'on connait f_{ij} (ou r_{ij}) dans l'intervalle double $0, 2t$". (Mandel, 1966).

This observation helps indeed in extracting thermodynamically interesting — i.e., endowed with special convexity properties — quantities (typically, the free energy) from a viscoelastic system.

Accordingly, the strain and stress fields are written, respectively, as:

$$\varepsilon_{ij}(t) = \begin{cases} \varepsilon_{1ij}(t) & \text{for } t \in [0, T] \\ \varepsilon_{2ij}(t) & \text{for } t \in [T, 2T] \end{cases} \tag{2.4}$$

$$\sigma_{ij}(t) = \begin{cases} \sigma_{1ij}(t) & \text{for } t \in [0, T] \\ \sigma_{2ij}(t) & \text{for } t \in [T, 2T] \end{cases} \tag{2.5}$$

where the subscript 1 refers to the quantities defined over the time interval $[0, T]$, i.e., the actual time interval of interest, and the subscript 2 indicates quantities defined over $[T, 2T]$. The direct constitutive law (1.1), by virtue of (2.4) and of (2.5), and thanks to Boltzmann's superposition principle, can be split as follows:

$$\sigma_{1ij}(t) = \int_{0^-}^{t} R_{ijhk}(t-\tau)\, d\varepsilon_{1hk}(\tau) \quad \text{for } t \in [0, T] \tag{2.6}$$

$$\sigma_{2ij}(t) = \int_{0^-}^{T} R_{ijhk}(t-\tau)\, d\varepsilon_{1hk}(\tau) + \int_{T}^{t} R_{ijhk}(t-\tau)\, d\varepsilon_{2hk}(\tau) \quad \text{for } t \in [T, 2T] \tag{2.7}$$

which can be expressed in a compact matricial form as follows:

$$\boldsymbol{\sigma} = \mathbf{L}\,\boldsymbol{\varepsilon} \tag{2.8}$$

with

$$\boldsymbol{\varepsilon} := \begin{bmatrix} \varepsilon_{1ij}(t) \\ \varepsilon_{2ij}(t) \end{bmatrix} \tag{2.9}$$

$$\boldsymbol{\sigma} := \begin{bmatrix} \sigma_{2ij}(t) \\ \sigma_{1ij}(t) \end{bmatrix} \tag{2.10}$$

$$\mathbf{L} := \begin{bmatrix} A & B \\ \widetilde{B} & 0 \end{bmatrix} := \begin{bmatrix} \int_{0^-}^{T} R_{ijhk}(t-\tau)\, d(\cdot) & \int_{T}^{t} R_{ijhk}(t-\tau)\, d(\cdot) \\ \int_{0^-}^{t} R_{ijhk}(t-\tau)\, d(\cdot) & 0 \end{bmatrix} \tag{2.11}$$

so that one has

$$\sigma_{1ij} = \widetilde{B}\varepsilon_{1hk} \tag{2.12}$$

$$\sigma_{2ij} = A\varepsilon_{1hk} + B\varepsilon_{2hk} = \sigma^{A}_{2ij} + \sigma^{B}_{2ij} \tag{2.13}$$

Considering a given point x_r in Ω, the superimposed $\widetilde{\ }$ denotes an adjoint operator in the sense that

$$B\varepsilon'_{2hk} * \ d\varepsilon''_{1ij} = \widetilde{B}\varepsilon''_{1hk} * \ d\varepsilon'_{2ij} \tag{2.14}$$

The new symbol $*$ denotes a "partial convolution", different from the standard one of Eq. (2.3) because the functions of time appearing in the time integrals are defined over two different time intervals. More precisely, given two functions $f_1(t)$ and $g_2(t)$ defined, respectively, over the time intervals $[0^-, T]$ and $[T, 2T]$, in the sequel of this work we will adopt the following definition of the partial convolution product of f_1 by g_2:

$$f_1 * dg_2 := \int_{T}^{2T} f_1(2T-t)\, dg_2(t) \tag{2.15}$$

and the following definition of the partial convolution product of g_2 by f_1:

$$g_2 * df_1 := \int_{0^-}^{T} g_2(2T-t)\, df_1(t) \tag{2.16}$$

It is important to observe that, in general, one has

$$f_1 * dg_2 \neq g_2 * df_1 \tag{2.17}$$

Operator A is symmetric (self-adjoint) in the sense that

$$A\varepsilon'_{1hk} * \ d\varepsilon''_{1ij} = A\varepsilon''_{1hk} * \ d\varepsilon'_{1ij} \tag{2.18}$$

It is important to observe that, as a consequence of the non-commutativity expressed by Eq. (2.17), the following holds:

$$A\varepsilon'_{1hk} * \ d\varepsilon''_{1ij} \neq \varepsilon'_{1hk} * \ dA\varepsilon''_{1ij} \tag{2.19}$$

As a consequence of Eqs. (2.14) and (2.18), the operatorial formulation (2.8), equivalent to the constitutive laws (2.6)–(2.7), is symmetric in the sense that

$$\mathbf{L}\varepsilon' * \ d\varepsilon'' = \mathbf{L}\varepsilon'' * \ d\varepsilon' \tag{2.20}$$

Operator A is also, in general, positive semi-definite. In fact, the following quadratic form

$$\frac{1}{2}A\varepsilon_{1hk} * \ d\varepsilon_{1ij} = \frac{1}{2}\int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T-t-\tau)\, d\varepsilon_{1hk}(\tau)\, d\varepsilon_{1ij}(t) \tag{2.21}$$

has the physical meaning of the free energy per unit volume of the material (see for example Mandel, 1966), a non-negative quantity, which explains the positive semi-definiteness of operator A of Eq. (2.11). Note that the time integral in this operator has not the form of Eq. (1.1), i.e., of Eq. (2.3), owing to its upper integration limit. Therefore, even though we have assumed the constitutive operator (1.1) to be invertible, nothing can be said, in general, about the invertibility of operator A.

From these ideas, in Carini and Mattei (2015) the following stationarity principles, derived directly from Gurtin's approach, have been proved.

Stationarity principle of the Total Potential Energy type.

$$\text{TPE}[u_{1i}, u_{2i}] = \min_{u'_{1i}} \operatorname*{stat}_{u'_{2i}} \ \text{TPE}[u'_{1i}, u'_{2i}] \tag{2.22}$$

where

$$\begin{aligned} \text{TPE}[u'_{1i}, u'_{2i}] = {} & \frac{1}{2}\int_{\Omega}\left(A\varepsilon'_{1hk} * \ d\varepsilon'_{1ij} + 2\widetilde{B}\varepsilon'_{1hk} * \ d\varepsilon'_{2ij}\right) d\Omega - \\ & - \int_{\Omega} b_{2i} * \ du'_{1i}\, d\Omega - \int_{\Omega} b_{1i} * \ du'_{2i}\, d\Omega - \\ & - \int_{\Gamma_p} p_{2i} * \ du'_{1i}\, d\Gamma - \int_{\Gamma_p} p_{1i} * \ du'_{2i}\, d\Gamma \end{aligned} \tag{2.23}$$

Here, $u_{1i}(t)$ and $u_{2i}(t)$ are the exact solution of the problem, and $u'_{1i}(t)$, $u'_{2i}(t)$, $\varepsilon'_{1ij}(t)$, and $\varepsilon'_{2ij}(t)$ are arbitrary but compatible displacement and strain fields.[2]

Stationarity principle of the Total Complementary Energy type.

$$\text{TCE}[\sigma_{1ij}, \sigma_{2ij}] = \min_{\sigma'_{1ij}} \operatorname*{stat}_{\sigma'_{2ij}} \ \text{TCE}[\sigma'_{1ij}, \sigma'_{2ij}] \tag{2.24}$$

[2] Some comments about the adopted notation. The explicit form of the integrand in the first volume integral in Eq. (2.23) can be obtained, according to the definitions (2.15) and (2.16), and recalling the definition (2.11) of the adopted operators, through the following passages:

$$A\varepsilon'_{1hk} * \ d\varepsilon'_{1ij} + 2\widetilde{B}\varepsilon'_{1hk} * \ d\varepsilon'_{2ij} =$$

$$= \left(A\varepsilon'_{1hk} + B\varepsilon'_{2hk}\right) * \ d\varepsilon'_{1ij} + \widetilde{B}\varepsilon'_{1hk} * \ d\varepsilon'_{2ij} =$$

$$= \left[\int_{0^-}^{T} R_{ijhk}(t-\tau)\, d\varepsilon'_{1hk}(\tau) + \int_{T}^{t} R_{ijhk}(t-\tau)\, d\varepsilon'_{2hk}(\tau)\right] * d\varepsilon'_{1ij}(t) +$$

$$+ \left[\int_{0^-}^{t} R_{ijhk}(t-\tau)\, d\varepsilon'_{1hk}(\tau)\right] * \ d\varepsilon'_{2ij}(t) =$$

where

$$\mathrm{TCE}[\sigma'_{1ij},\sigma'_{2ij}] = \frac{1}{2}\int_{\Omega}\left(\mathcal{A}\sigma'_{1hk} * \ d\sigma'_{1ij} + 2\widetilde{\mathcal{B}}\sigma'_{1hk} * \ d\sigma'_{2ij}\right) d\Omega - \\ - \int_{\Gamma_u} n_i u^0_{2j} * \ d\sigma'_{1ij}\, d\Gamma - \int_{\Gamma_u} n_i u^0_{1j} * \ d\sigma'_{2ij}\, d\Gamma \tag{2.25}$$

in which

$$\mathcal{A} := \int_{0^-}^{T} C_{ijhk}(t-\tau)\, d(\cdot) = -B^{-1}A\widetilde{B}^{-1} \tag{2.26}$$

$$\widetilde{\mathcal{B}} := \int_{0^-}^{t} C_{ijhk}(t-\tau)\, d(\cdot) = \widetilde{B}^{-1} \tag{2.27}$$

$\sigma_{1ij}(t)$ and $\sigma_{2ij}(t)$ are the exact solution of the problem, and $\sigma'_{1ij}(t)$ and $\sigma'_{2ij}(t)$ are generic equilibrated stress fields.

From now on, the functionals and the stationarity principles presented in Carini and Mattei (2015) will be called "split Gurtin" functionals and stationarity principles.

2.1. Stationarity principles including rate viscous kernels

New stationarity principles in terms of rate viscous kernels can be obtained following a path quite similar to that adopted by Carini and Mattei (2015), starting now from the rate forms of the viscoelastic constitutive equations.

Consider the constitutive Eq. (1.1), and rewrite it in rate form:

$$\dot{\sigma}_{ij}(t) = R^0_{ijhk}\dot{\varepsilon}_{hk}(t) + \int_{0^-}^{t} \dot{R}_{ijhk}(t-\tau)\, d\varepsilon_{hk}(\tau) \tag{2.28}$$

where R^0_{ijhk} denotes the initial value of the relaxation kernel:

$$R^0_{ijhk} = R_{ijhk}(t=0) \tag{2.29}$$

and a superimposed dot indicates a derivative with respect to time.

One can follow the same path summarized previously and double the integration interval introducing, along with the variables with index 1, defined in the time interval of interest $t \in [0,T]$, auxiliary ones, with index 2, defined in the time interval $t \in [T,2T]$. Correspondingly, the rate problem (2.28) can be rewritten in the following split way:

$$\begin{bmatrix} \int_{0^-}^{T} \dot{R}_{ijhk}(t-\tau)\, d(\cdot) & R^0_{ijhk}\frac{\partial}{\partial t}(\cdot) + \int_{T}^{t} \dot{R}_{ijhk}(t-\tau)\, d(\cdot) \\ R^0_{ijhk}\frac{\partial}{\partial t}(\cdot) + \int_{0^-}^{t} \dot{R}_{ijhk}(t-\tau)\, d(\cdot) & 0 \end{bmatrix} \\ \times \begin{bmatrix} \varepsilon_{1hk} \\ \varepsilon_{2hk} \end{bmatrix} = \begin{bmatrix} \dot{\sigma}_{2ij} \\ \dot{\sigma}_{1ij} \end{bmatrix} \tag{2.30}$$

$$= \left[\int_{0^-}^{t} R_{ijhk}(t-\tau)\, d\varepsilon'_{hk}(\tau)\right] * \ d\varepsilon'_{1ij}(t) \\ + \left[\int_{0^-}^{t} R_{ijhk}(t-\tau)\, d\varepsilon'_{1hk}(\tau)\right] * \ d\varepsilon'_{2ij}(t) =$$

$$= \int_{0^-}^{T}\left[\int_{0^-}^{2T-t} R_{ijhk}(2T-t-\tau)\, d\varepsilon'_{hk}(\tau)\right] d\varepsilon'_{1ij}(t) \\ + \int_{T}^{2T}\left[\int_{0^-}^{2T-t} R_{ijhk}(2T-t-\tau)\, d\varepsilon'_{1hk}(\tau)\right] d\varepsilon'_{2ij}(t) =$$

$$= \int_{0^-}^{2T}\int_{0^-}^{2T-t} R_{ijhk}(2T-t-\tau)\, d\varepsilon'_{hk}(\tau)\, d\varepsilon'_{1ij}(t)$$

where the last passages have been introduced in order to express the result in the same terms as the unsplit Gurtin original formulation.

The meaning of the partial convolutive product $b_{2i} * \ du'_{1i}$ in the second integral of (2.23) (and of the analogous in all the other terms) is, according to the definition (2.16), the following:

$$b_{2i} * \ du'_{1i} = \int_{0^-}^{T} b_{2i}(2T-\tau)\, du'_{1i}(\tau).$$

or, in compact form,

$$\dot{\boldsymbol{\sigma}} = \begin{bmatrix} \dot{\sigma}_{2ij} \\ \dot{\sigma}_{1ij} \end{bmatrix} = \mathring{\mathbf{L}}\begin{bmatrix} \varepsilon_{1hk} \\ \varepsilon_{2hk} \end{bmatrix} = \mathring{\mathbf{L}}\boldsymbol{\varepsilon} \tag{2.31}$$

with

$$\mathring{\mathbf{L}} = \begin{bmatrix} \mathring{A} & \mathring{B} \\ \widetilde{\mathring{B}} & 0 \end{bmatrix} \tag{2.32}$$

The main result, thus obtained, is to have introduced operator $\mathring{A}$, which is symmetric and negative semi-definite, because the following holds (see, for example, Huet, 1995):

$$\mathring{A}\varepsilon'_{1hk} * \ d\varepsilon'_{1ij} = \int_{0^-}^{T} \mathring{A}\varepsilon'_{1hk}(2T-t)\, d\varepsilon'_{1ij}(t) \le 0 \quad \forall \varepsilon'_{1ij} \neq 0 \tag{2.33}$$

To better see this, one has to just write Eq. (2.33) in an explicit form, accounting for the definition (2.30) of operator $\mathring{A}$, as follows:

$$-\mathring{A}\varepsilon'_{1hk} * \ d\varepsilon'_{1ij} = -\int_{0^-}^{T}\int_{0^-}^{T} \dot{R}_{ijhk}(2T-t-\tau)\, d\varepsilon'_{1hk}(t)\, d\varepsilon'_{1ij}(\tau) \ge 0 \tag{2.34}$$

which (see Eq. (3.1) in Huet, 1995) is the dissipated power density per unit volume, clearly non-negative, when $\varepsilon'_{1ij}(t)$ coincides with the exact solution.

Following the same path taken in Carini and Mattei (2015), one obtains the following max-stat theorem:

$$\mathring{\mathrm{TPE}}[u_{1i},u_{2i}] = \max_{u'_{1i}}\operatorname*{stat}_{u'_{2i}}\ \mathring{\mathrm{TPE}}[u'_{1i},u'_{2i}] \tag{2.35}$$

where

$$\mathring{\mathrm{TPE}}[u'_{1i},u'_{2i}] = \frac{1}{2}\int_{\Omega}\left(\mathring{A}\varepsilon'_{1hk} * \ d\varepsilon'_{1ij} + 2\widetilde{\mathring{B}}\varepsilon'_{1hk} * \ d\varepsilon'_{2ij}\right) d\Omega - \\ - \int_{\Omega} \dot{b}_{2i} * \ du'_{1i}\, d\Omega - \int_{\Omega} \dot{b}_{1i} * \ du'_{2i}\, d\Omega - \\ - \int_{\Gamma_p} \dot{p}_{2i} * \ du'_{1i}\, d\Gamma - \int_{\Gamma_p} \dot{p}_{1i} * \ du'_{2i}\, d\Gamma \tag{2.36}$$

Operating now in terms of the rate inverse constitutive law, governed by the rate creep kernel $\dot{C}_{ijhk}$, it is possible to derive, in a way fully analogous to the previous one, the following stationarity formulation in terms of the rate kernel $\dot{C}_{ijhk}(t)$:

$$\mathring{\mathrm{TCE}}[\sigma_{1ij},\sigma_{2ij}] = \min_{\sigma'_{1ij}}\operatorname*{stat}_{\sigma'_{2ij}}\ \mathring{\mathrm{TCE}}[\sigma'_{1ij},\sigma'_{2ij}] \tag{2.37}$$

where

$$\mathring{\mathrm{TCE}}[\sigma'_{1ij},\sigma'_{2ij}] = \frac{1}{2}\int_{\Omega}\left(\mathring{\mathcal{A}}\sigma'_{1hk} * \ d\sigma'_{1ij} + 2\widetilde{\mathring{\mathcal{B}}}\sigma'_{1hk} * \ d\sigma'_{2ij}\right) d\Omega - \\ - \int_{\Gamma_u} n_i \dot{u}^0_{2j} * \ d\sigma'_{1ij}\, d\Gamma - \int_{\Gamma_u} n_i \dot{u}^0_{1j} * \ d\sigma'_{2ij}\, d\Gamma \tag{2.38}$$

in which

$$\mathring{\mathcal{A}} := \int_{0^-}^{T} \dot{C}_{ijhk}(t-\tau)\, d(\cdot) = -\mathring{B}^{-1}\mathring{A}\widetilde{\mathring{B}}^{-1} \tag{2.39}$$

$$\widetilde{\mathring{\mathcal{B}}} := \int_{0^-}^{t} \dot{C}_{ijhk}(t-\tau)\, d(\cdot) = \widetilde{\mathring{B}}^{-1} \tag{2.40}$$

$\sigma_{1ij}(t)$ and $\sigma_{2ij}(t)$ are the exact solution of the problem, and $\sigma'_{1ij}(t)$ and $\sigma'_{2ij}(t)$ are generic equilibrated stress fields.

3. New extremum principles deriving from specific choices of the admissible fields in the split Gurtin functionals

Consider the Total Potential Energy type stationarity theorem of Eqs. (2.22) and (2.23). It is well-known (and trivial to observe) that if one inserts into the functional (2.23) the true solution in the second time interval, u_{2i}, one immediately obtains a minimum principle of the type

$$\mathrm{TPE}[u_{1i}] = \min_{u'_{1i}} \mathrm{TPE}[u'_{1i}, u_{2i}] \tag{3.1}$$

where u'_{1i} must be admissible, i.e., fulfilling compatibility. The true solution $u_{2i}(t)$ in the second time interval is not available, in general, and the principle (3.1) seems useless. Nevertheless, it is possible to show that, by restricting the field of admissibility for $u'_{1i}(t)$ and $u'_{2i}(t)$, a fully usable minimum principle can be recovered.

As a starting point, observe that, given any two admissible (compatible) strains ε'_{1ij} and ε'_{2ij}, and denoting by $\sigma'_{1ij} = \widetilde{B}\varepsilon'_{1hk}$ the stress of Eq. (2.12), the following holds, as a consequence of the divergence theorem:

$$V\left\langle \widetilde{B}\varepsilon'_{1hk} * d\varepsilon'_{2ij} \right\rangle = \int_\Omega \sigma'_{1ij} * d\frac{\partial u'_{2j}}{\partial x_i}\, d\Omega =$$

$$= \int_\Omega \frac{\partial}{\partial x_i}\left(\sigma'_{1ij} * du'_{2j}\right) d\Omega - \int_\Omega \frac{\partial \sigma'_{1ij}}{\partial x_i} * du'_{2j}\, d\Omega =$$

$$= \int_\Gamma \sigma'_{1ij} n_i * du'_{2j}\, d\Gamma - \int_\Omega \frac{\partial \sigma'_{1ij}}{\partial x_i} * du'_{2j}\, d\Omega =$$

$$= \int_{\Gamma_p} p'_{1j} * du'_{2j}\, d\Gamma + \int_{\Gamma_u} \sigma'_{1ij} n_i * du^0_{2j}\, d\Gamma + \int_\Omega b'_{1j} * du'_{2j}\, d\Omega \tag{3.2}$$

where the symbols p'_{1j} and b'_{1j} denote the surface tractions and body forces, respectively, associated through equilibrium to the stress σ'_{1ij}. An obvious special case of this result is given by the real solution of any viscoelastic problem.

We now define as *strictly admissible* a pair of unknown displacements $u'_{1i}(t)$ and $u'_{2i}(t)$ that, beside being admissible in the customary sense, satisfy the following two conditions:

$$V\left\langle \widetilde{B}\varepsilon'_{1hk} * d\varepsilon_{2ij} \right\rangle = V\left\langle \widetilde{B}\varepsilon_{1hk} * d\varepsilon_{2ij} \right\rangle$$

$$V\left\langle \widetilde{B}\varepsilon'_{1hk} * d\varepsilon'_{2ij} \right\rangle = V\left\langle \widetilde{B}\varepsilon_{1hk} * d\varepsilon'_{2ij} \right\rangle \tag{3.3}$$

The next section proves that the identification of strictly admissible displacements in the sense of Eq. (3.3) is practically feasible in the analysis of RVEs within the homogenization theory for linear viscoelastic composites.

Upon replacing into the functional (2.23) the second of Eq. (3.3), exploiting the identity (3.2) after simple passages one arrives at the following result:

$$\overline{\mathrm{TPE}}[u'_{1i}] = \frac{1}{2}\int_\Omega A\varepsilon'_{1hk} * d\varepsilon'_{1ij}\, d\Omega -$$

$$-\int_\Omega b_{2i} * du'_{1i}\, d\Omega - \int_{\Gamma_u} \sigma_{1ij} n_i * du^0_{2j}\, d\Gamma - \int_{\Gamma_p} p_{2i} * du'_{1i}\, d\Gamma \tag{3.4}$$

in which the boundary integral over Γ_u contains known functions and can be neglected.

By adding and subtracting the two terms in the first of Eq. (3.3) to this result, and exploiting the identity of Eq. (3.2), one arrives precisely at reformulating the minimum principle of Eq. (3.1).

This argument allows one to establish the following

Minimum principle of the Total Potential Energy type:

$$\overline{\mathrm{TPE}}[u_{1i}] = \min_{u'_{1i}} \overline{\mathrm{TPE}}[u'_{1i}] \tag{3.5}$$

with functional $\overline{\mathrm{TPE}}[u'_{1i}]$ given by

$$\overline{\mathrm{TPE}}[u'_{1i}] = \frac{1}{2}\int_\Omega A\varepsilon'_{1hk} * d\varepsilon'_{1ij}\, d\Omega - \int_\Omega b_{2i} * du'_{1i}\, d\Omega - \int_{\Gamma_p} p_{2i} * du'_{1i}\, d\Gamma \tag{3.6}$$

in which the admissible displacement $u'_{1i}(t)$ must satisfy both requisites of Eq. (3.3).

It is worth recalling that the split Gurtin functional of Eq. (2.23) is fully equivalent to the original, unsplit Gurtin TPE-like convolutive functional of Eq. (2.1). This last, in general, furnishes a stationarity principle only; but if the admissible displacement, in it, satisfies the Eq. (3.3), it becomes a minimum theorem itself, of the type

$$\mathrm{TPE}^G[u_i] = \min_{u'_i} \mathrm{TPE}^G[u'_i] \tag{3.7}$$

under the strict admissibility condition (3.3) for a displacement $u'_i(t)$ defined in the whole time interval $t \in [0, 2T]$.

Starting from the stationarity principle of Eq. (2.24), a fully similar argument leads to establish the following

Minimum principle of the Total Complementary Energy type:

$$\overline{\mathrm{TCE}}[\sigma_{1ij}] = \min_{\sigma'_{1ij}} \overline{\mathrm{TCE}}[\sigma'_{1ij}] \tag{3.8}$$

where

$$\overline{\mathrm{TCE}}[\sigma'_{1ij}] = \frac{1}{2}\int_\Omega \mathcal{A}\sigma'_{1hk} * d\sigma'_{1ij}\, d\Omega - \int_{\Gamma_u} u^0_{2j} * d\sigma'_{1ij} n_i\, d\Gamma \tag{3.9}$$

for all the admissible (equilibrated) stress fields that also are strictly admissible in the following sense:

$$V\left\langle \widetilde{\mathcal{B}}\sigma'_{1hk} * d\sigma_{2ij} \right\rangle = V\left\langle \widetilde{\mathcal{B}}\sigma_{1hk} * d\sigma_{2ij} \right\rangle$$

$$V\left\langle \widetilde{\mathcal{B}}\sigma'_{1hk} * d\sigma'_{2ij} \right\rangle = V\left\langle \widetilde{\mathcal{B}}\sigma_{1hk} * d\sigma'_{2ij} \right\rangle \tag{3.10}$$

In a way analogous to Eq. (3.7), also in the TCE case of Eq. (2.2) the original Gurtin stationarity principle becomes a minimum principle of the type

$$\mathrm{TCE}^G[\sigma_{ij}] = \min_{\sigma'_{ij}} \mathrm{TCE}^G[\sigma'_{ij}] \tag{3.11}$$

under the strict admissibility conditions (3.10) for a stress $\sigma'_{ij}(t)$ defined in the whole time interval $t \in [0, 2T]$.

A similar reasoning allows one to obtain extremum principles, starting from the stationarity ones of Eqs. (2.35), (2.36), (2.37), and (2.38), containing the rates of the viscous kernels, and valid in the presence of strictly admissible displacements or stresses. In the case of these extremum theorems, containing the rates of the viscous kernels, trivial to be obtained and omitted for brevity, the strict admissibility of the main unknowns refers to operators different from $\widetilde{B}$ and $\widetilde{\mathcal{B}}$, and precisely to the operators $\mathring{\widetilde{B}}$ and $\mathring{\widetilde{\mathcal{B}}}$ of Eqs. (2.32) and (2.40), respectively.

4. Selection of strictly admissible fields in the special case of deviatoric loading applied to macroscopically isotropic RVEs in homogenization methods

In this section we show that, in the analysis of RVEs in the homogenization theory for linear viscoelastic composites, and for the restricted situation of a purely deviatoric macroscopic stress or strain state, it is possible to individuate strictly admissible displacement and stress fields in the sense of Eqs. (3.3) and (3.10), respectively.

Consider first the case in which one wants to make use of the Total Potential Energy minimum principle of Eqs. (3.5) and (3.6) to obtain bounds to the homogenized viscous kernels. Starting from the notation of Eqs. (2.8) and (2.11), the sought homogenized direct constitutive law is written in a split form as follows:

$$\left\langle \boldsymbol{\sigma} \right\rangle = \mathbf{L}^h \left\langle \boldsymbol{\varepsilon} \right\rangle \tag{4.1}$$

with

$$\mathbf{L}^h = \begin{bmatrix} A^h & B^h \\ \widetilde{B}^h & 0 \end{bmatrix} = \begin{bmatrix} \int_{0^-}^{T} R^h_{ijhk}(t-\tau)\, d(\cdot) & \int_{T}^{t} R^h_{ijhk}(t-\tau)\, d(\cdot) \\ \int_{0^-}^{t} R^h_{ijhk}(t-\tau)\, d(\cdot) & 0 \end{bmatrix} \tag{4.2}$$

which reduces the split homogenization problem to the determination of the homogenized viscous kernel $R^h_{ijhk}(t)$. Bounds to $R^h_{ijhk}(t)$ could be obtained exploiting the minimum principle of Eq. (3.5) if strictly admissible displacements could be precisely defined.

It is possible to show that

1. for macroscopically isotropic viscoelastic composites,
2. with the selection of the admissible displacements u'_{1i} and u'_{2i} as arbitrary functions of time but independent of space, and
3. *under a deviatoric loading* applied to the RVE,

the condition of strict admissibility of Eq. (3.3) always holds.

The following selection for the admissible displacements and strains will accordingly be made:

$$u'_{1i}(x_k,t) = \hat{\varepsilon}_{1ij}(t)x_j \;\rightarrow\; \varepsilon'_{1ij}(x_k,t) = \hat{\varepsilon}_{1ij}(t)$$

$$u'_{2i}(x_k,t) = \hat{\varepsilon}_{2ij}(t)x_j \;\rightarrow\; \varepsilon'_{2ij}(x_k,t) = \hat{\varepsilon}_{2ij}(t) \tag{4.3}$$

in which both the strains $\hat{\varepsilon}_{1ij}$ and $\hat{\varepsilon}_{2ij}$ are independent of space.

Consider the left hand side in the second of the Eq. (3.3), and insert into it the kinematic quantities of Eq. (4.3). The following passages can be followed, using the divergence theorem and denoting by $\hat{\sigma}_{1ij}$ the stress produced by the admissible strain $\hat{\varepsilon}_{1ij}(t)$ (see Eq. (2.12)):

$$V\left\langle \widetilde{B}\hat{\varepsilon}_{1hk} * \, d\hat{\varepsilon}_{2ij}\right\rangle = \int_\Omega \hat{\sigma}_{1ij} * \, d\frac{\partial}{\partial x_i}\left(\hat{\varepsilon}_{2hj}x_h\right)\, d\Omega =$$

$$= \int_\Omega \frac{\partial}{\partial x_i}\left(\hat{\sigma}_{1ij} * \, d\hat{\varepsilon}_{2hj}x_h\right) d\Omega - \int_\Omega \frac{\partial \hat{\sigma}_{1ij}}{\partial x_i} * \, d\hat{\varepsilon}_{2hj}x_h \, d\Omega =$$

$$= \int_\Gamma \hat{\sigma}_{1ij}n_i * \, d\hat{\varepsilon}_{2hj}x_h \, d\Gamma - \int_\Omega \frac{\partial \hat{\sigma}_{1ij}}{\partial x_i} * \, d\hat{\varepsilon}_{2hj}x_h \, d\Omega =$$

$$= \left[\int_\Gamma \hat{p}_{1j}x_h \, d\Gamma + \int_\Omega \hat{b}_{1j}x_h \, d\Omega\right] * \, d\hat{\varepsilon}_{2hj} \tag{4.4}$$

where the symbols $\hat{p}_{1j}$ and $\hat{b}_{1j}$ denote the non-zero surface tractions and body forces, respectively, associated to $\hat{\sigma}_{1ij}$. The last passage exploits the assumed and crucial space independence of the selected admissible strain.

This term can be proved to identically vanish if *the RVE is macroscopically isotropic and loaded in a purely deviatoric way*, and only in this case.

To examine in a complete way a purely deviatoric kinematic loading on the RVE it is sufficient to consider $\hat{\varepsilon}_{12} = \hat{\varepsilon}_{21}$ as the only non-zero admissible strain ($\hat{\varepsilon}_{2ij}$) components in the RVE. All the other possible cases could be approached in the same way. The expression in Eq. (4.4) reduces then to the following:

$$V\left\langle \widetilde{B}\hat{\varepsilon}_{1hk} * \, d\hat{\varepsilon}_{2ij}\right\rangle = \left[\int_\Gamma \hat{p}_{1,1}x_2 \, d\Gamma + \int_\Omega \hat{b}_{1,1}x_2 \, d\Omega\right] * \, d\hat{\varepsilon}_{21} +$$

$$+ \left[\int_\Gamma \hat{p}_{1,2}x_1 \, d\Gamma + \int_\Omega \hat{b}_{1,2}x_1 \, d\Omega\right] * \, d\hat{\varepsilon}_{12} \tag{4.5}$$

In this, both integrals in the square brackets have the meaning of couples, denoted hereafter as M_{12} and M_{21} respectively, so that one has:

$$V\left\langle \widetilde{B}\hat{\varepsilon}_{1hk} * \, d\hat{\varepsilon}_{2ij}\right\rangle = M_{12} * \, d\hat{\varepsilon}_{21} + M_{21} * \, d\hat{\varepsilon}_{12} \tag{4.6}$$

Upon a 90 degrees rotation of the reference system around the x_3 axis, which leaves the integrals in Eq. (4.6) unchanged, the couples M_{ij}, scalars, do not change, while the shear strain components simply change their signs, producing the following result:

$$V\left\langle \widetilde{B}\hat{\varepsilon}_{1hk} * \, d\hat{\varepsilon}_{2ij}\right\rangle = -\left[M_{12} * \, d\hat{\varepsilon}_{21} + M_{21} * \, d\hat{\varepsilon}_{12}\right] \tag{4.7}$$

The comparison of the last two results, in the case of a macroscopically homogeneous RVE, shows that, under the given conditions, the left hand side in the second of Eq. (3.3) is identically equal to zero.

One applies next this same reasoning to the right hand side in the first of the Eq. (3.3), containing the real solution. By Hill's lemma one can write:

$$\left\langle \widetilde{B}\varepsilon_{1hk} * \, d\varepsilon_{2ij}\right\rangle = \left\langle \widetilde{B}\varepsilon_{1hk}\right\rangle * \, d\left\langle \varepsilon_{2ij}\right\rangle =$$

$$= \left\langle \widetilde{B}\varepsilon_{1hk}\right\rangle * \, d\varepsilon^{ave}_{2ij} = \left\langle \widetilde{B}\varepsilon_{1hk} * \, d\varepsilon^{ave}_{2ij}\right\rangle \tag{4.8}$$

Since ε^{ave}_{2ij} does not depend on space, one can repeat the same passages of Eqs. (4.4) to (4.7). This shows that also the right hand side in the first of the Eq. (3.3) is identically equal to zero under the selection of Eq. (4.3).

The right hand side in the second of Eq. (3.3) is also identically equal to zero under the said assumptions and choices, because the same passages of Eqs. (4.4) to (4.7) can be repeated.

Finally, the left hand side of the first of Eq. (3.3) can be rewritten as follows:

$$\left\langle \widetilde{B}\varepsilon'_{1hk} * \, d\varepsilon_{2ij}\right\rangle = \left\langle B\varepsilon_{2hk} * \, d\varepsilon'_{1ij}\right\rangle \tag{4.9}$$

after which, exploiting the form of the selection (4.3), the same passages of Eqs. (4.4) to (4.7) lead to conclude that also this term, under the said assumption and choices, is identically equal to zero. This completes the proof that the selection of Eq. (4.3) is strictly admissible in the sense of Eq. (3.3).

Rather surprisingly, affine displacements of the type of Eq. (4.3) but possessing a volumetric component are not strictly admissible in the sense of Eq. (3.3). More comments on this point will be given later on and in Appendix.

The identification of a strictly admissible stress field for making use of the TCE stationarity theorem of Eq. (3.8) follows a path similar to the previous one. The following choice for the admissible stresses in the RVE can be adopted, and proved to be strictly admissible in the sense of Eq. (3.10):

$$\sigma'_{1ij}(x_k,t) = \hat{\sigma}_{1ij}(t)$$

$$\sigma'_{2ij}(x_k,t) = \hat{\sigma}_{2ij}(t) \tag{4.10}$$

The analytical passages illustrated here above, adopted now to prove the validity of the conditions (3.10), can be repeated in the same way by expressing the admissible stresses in the following way:

$$\hat{\sigma}_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \tag{4.11}$$

which is easily obtained by using the following definition:

$$w_i = \hat{\sigma}_{ik}x_k \tag{4.12}$$

Through this little stratagem, the procedure illustrated for the TPE functional can be applied identically to the TCE one. The details are omitted for brevity.

A similar reasoning may allow one to individuate strictly admissible displacement or stress fields in the functionals (2.36) and (2.38), in order to produce the corresponding extremum theorems containing the rates of the viscous kernels. The choices of Eqs. (4.3) and (4.10) turn out to be strictly admissible also in these cases. The relevant passages and extremum principles are omitted for brevity.

5. Derivation of analytical bounds to the homogenized viscoelastic kernels

When studying RVEs for deriving estimates of or bounds to the homogenized viscous kernels of viscoelastic composites, it is customary, and especially convenient, to load them either by so-called "affine" displacements, prescribed on the constrained boundary $\Gamma_u \equiv \Gamma$ of the RVE, or by surface tractions on $\Gamma_p \equiv \Gamma$ corresponding to uniform stresses in the homogenized solid. For both these loading conditions, in which there are no body forces, a unit-step time history is usually considered, as will always be done in the sequel of this work.

5.1. Voigt-type upper bounds derived from functionals (3.6) and (3.9)

Consider the Total Potential Energy type functional of Eq. (3.6). Consider the kinematic loading typical of a relaxation test, i.e., affine prescribed displacements on $\Gamma_u \equiv \Gamma$, of the type

$$u_i^0(x_r,t) = \langle \varepsilon_{ij}(x_r,t)\rangle x_j = \varepsilon_{ij}^0(t)x_j, \quad 0 \le t \le 2T \tag{5.1}$$

and zero body forces b_j in Ω. This loading produces a known strain $\varepsilon_{ij}^0(t)$, independent of space, in the homogenized RVE.

The definition of the loading terms is completed by assuming the following unit-step time history for the loading strain:

$$\langle \varepsilon_{ij}(x_r,t)\rangle = \varepsilon_{ij}^0(t) = \overline{\varepsilon}_{ij}\mathcal{H}(t), \quad 0 \le t \le 2T \tag{5.2}$$

$\overline{\varepsilon}_{ij}$ being a prescribed constant strain tensor, and $\mathcal{H}(t)$ the Heaviside function.

The $\overline{\mathrm{TPE}}$ functional of Eq. (3.6), reduced to the single first integral by the assumed loading condition, and evaluated in the solution of the RVE problem, is explicitly written as follows:

$$\overline{\mathrm{TPE}}[u_{1i}] = \overline{\mathrm{TPE}}_{\mathrm{sol}} = \frac{1}{2}\int_\Omega \int_{0^-}^T \int_{0^-}^T R_{ijhk}(2T-t-\tau)\, d\varepsilon_{1hk}(\tau)\, d\varepsilon_{1ij}(t)\, d\Omega \tag{5.3}$$

The argument of the volume integral, in Eq. (5.3), is identical with (2.21) and denotes the free energy density for this problem; exploiting Hill's lemma one can therefore write:

$$\frac{1}{V}\overline{\mathrm{TPE}}_{\mathrm{sol}} = \frac{1}{2}\overline{\varepsilon}_{ij}R_{ijhk}^h(2T)\overline{\varepsilon}_{hk} \tag{5.4}$$

Select, as a strictly admissible displacement in the RVE, the same choice of Eqs. (5.1) and (5.2), clearly a special case of Eq. (4.3), which satisfies the requisites of Eq. (3.3).

The minimum theorem of Eq. (3.5), accounting for the adopted loading condition, reduces to:

$$\overline{\varepsilon}_{ij}R_{ijhk}^h(2T)\overline{\varepsilon}_{hk} \le \left\langle A\varepsilon'_{1hk} * d\varepsilon'_{1ij}\right\rangle \quad \forall \varepsilon'_{1ij} \quad \text{strictly admissible} \tag{5.5}$$

From this, recalling the definition (2.11) of operator A and exploiting the choice (5.2) for $\varepsilon'_{1ij}(t)$, one obtains finally the desired upper bound to the homogenized relaxation kernel:

$$\overline{\varepsilon}_{ij}R_{ijhk}^h(2T)\overline{\varepsilon}_{hk} \le \overline{\varepsilon}_{ij}\left\langle R_{ijhk}(2T)\right\rangle\overline{\varepsilon}_{hk} \tag{5.6}$$

which, taking T as an arbitrary time, becomes valid for a generic time $T \in [0,\infty)$.

One must now recall that the result of Eq. (5.6), a consequence of the choice (5.2) for the strictly admissible strain field ε'_{1ij} in the RVE, holds only for the case of macroscopic isotropy and for a RVE loaded in a deviatoric way. Therefore, the obtained upper bound to the homogenized viscous relaxation tensor refers only to its scalar deviatoric (shear) component, hereafter simply denoted by $R^h(t)$.

In order to write the explicit result in this situation, consider a macroscopically isotropic composite material with N isotropic viscoelastic phases, denote by c_i the volume fraction of each phase (i), and denote by $R^{(i)}(t)$ and $C^{(i)}(t)$ the scalar shear relaxation and creep kernel component of phase (i) respectively. The explicit expression of bound (5.6) then reads as follows:

$$R^h(t) \le \sum_{i=1}^N c_i R^{(i)}(t) \quad \forall t \in [0,\infty) \tag{5.7}$$

One can produce an upper bound to the homogenized creep kernel $C_{ijhk}^h(t)$ in a similar way, and under the same restricting assumptions, starting from the functional $\overline{\mathrm{TCE}}$ of Eq. (3.9). The loading condition on the RVE is now defined in terms of prescribed surface tractions, of the following type

$$p_j(x_r,t) = \langle \sigma_{ij}(x_r,t)\rangle n_i = \sigma_{ij}^0(t)n_i \quad \text{in } \Gamma_p \equiv \Gamma \quad 0 \le t \le 2T \tag{5.8}$$

with

$$\sigma_{ij}^0(t) = \overline{\sigma}_{ij}\mathcal{H}(t) \tag{5.9}$$

and zero body forces in the RVE.

Next, one proceeds in a way analogous to the one illustrated for the TPE case. One chooses also for σ'_{1ij} in the whole RVE, in Eq. (3.9), the expression given by Eq. (5.9) on the RVE boundary, a special case of Eq. (4.10), therefore strictly admissible in the sense of Eq. (3.10). After passages similar to those leading to Eq. (5.6), omitted for brevity, the following upper bound to the homogenized creep function is obtained:

$$\overline{\sigma}_{ij}C_{ijhk}^h(t)\overline{\sigma}_{hk} \le \overline{\sigma}_{ij}\left\langle C_{ijhk}(t)\right\rangle\overline{\sigma}_{hk} \tag{5.10}$$

The explicit expression of this bound in terms of the scalar deviatoric component of the homogenized creep viscous kernel only, $C^h(t)$, reads as follows:

$$C^h(t) \le \sum_{i=1}^N c_i C^{(i)}(t) \quad \forall t \in [0,\infty) \tag{5.11}$$

5.2. Bounds to the homogenized rate viscous kernels, derived from functionals (2.36) and (2.38)

It is possible to exploit the rate counterparts of the extremum formulations of Eqs. (3.5) and (3.8) to obtain, respectively, a lower bound to the homogenized rate relaxation kernel, $\dot{R}_{ijhk}^h(t)$, and an upper bound to the homogenized rate creep kernel, $\dot{C}_{ijhk}^h(t)$.

We consider a homogenized rate viscous kernel, say for example the creep one, $\dot{C}_{ijhk}^h(t)$, as defined in Huet (1995), i.e., the time derivative of the homogenized creep kernel:

$$\dot{C}_{ijhk}^h(t) = \frac{dC_{ijhk}^h(t)}{dt} \tag{5.12}$$

The path to be followed to obtain bounds to the homogenized rate viscous kernels is the same illustrated in the previous subsection, just starting from the functionals containing the kernel rates instead of those containing the total kernels. Similar considerations apply, which allow one to write the following results:

- lower bound to $\dot{R}_{ijhk}^h(t)$:

$$\overline{\varepsilon}_{ij}\dot{R}_{ijhk}^h(t)\overline{\varepsilon}_{hk} \ge \overline{\varepsilon}_{ij}\left\langle \dot{R}_{ijhk}(t)\right\rangle\overline{\varepsilon}_{hk} \tag{5.13}$$

- upper bound to $\dot{C}_{ijhk}^h(t)$:

$$\overline{\sigma}_{ij}\dot{C}_{ijhk}^h(t)\overline{\sigma}_{hk} \le \overline{\sigma}_{ij}\left\langle \dot{C}_{ijhk}(t)\right\rangle\overline{\sigma}_{hk} \tag{5.14}$$

which both hold for macroscopically isotropic RVEs under deviatoric loading only.

Analogous results, but involving strict inequalities only, have been found in Huet (1995) following a different path, not based on the availability of extremum formulations. As a consequence, Huet's strict bounds have a fully general validity, i.e., they hold for possibly anisotropic viscoelastic composites subjected to both a deviatoric and a volumetric macroscopic stress or strain state.

5.3. Optimal lower bounds derived from functionals (3.6) and (3.9)

It is possible to obtain analytical lower bounds in the time domain for both the homogenized viscous kernels starting from the extremum theorems of Eqs. (3.5) or (3.8) but considering, in them, a loading dual than the one considered previously, i.e., prescribed surface tractions of the type (5.8) and (5.9) for the TPE case, and prescribed boundary displacements of the type (5.1) and (5.2) for the TCE case.

The validity of these lower bounds is limited, as for the previous upper bounds, to the case of macroscopically isotropic viscoelastic composites under deviatoric loading only.

The procedure adopted to obtain a lower bound to the homogenized creep kernel will be illustrated in detail; the one necessary to derive a lower bound to the homogenized relaxation kernel is fully analogous and will be omitted for brevity.

We remark that for this analysis we will start from the TPE functional of Eq. (3.6) in which the loading applied to the RVE is of the static type, given by Eqs. (5.8) and (5.9).

One observes first that, as is well known (Huet, 1995), the functional (3.6), with this type of loading and evaluated in the exact solution of the problem, yields the following result, an absolute minimum for the functional itself:

$$\overline{\mathrm{TPE}}[u_{1i}] = \frac{1}{2}\int_{\Omega} A\varepsilon_{1hk} * \ d\varepsilon_{1ij}\ d\Omega - \int_{\Gamma} p_{2i} * \ du_{1i}\ d\Gamma = -\frac{1}{2}\overline{\sigma}_{ij} C^h_{ijhk}(2T)\overline{\sigma}_{hk} \tag{5.15}$$

It is now necessary to evaluate the functional (3.6) in correspondence to a strictly admissible solution that makes it as small as possible, in order to obtain an optimal lower bound to the homogenized creep kernel.

As already indicated, for the present purposes one selects the loading condition of Eqs. (5.8) and (5.9), and chooses the strictly admissible displacement field of Eq. (4.3).

In order to proceed further, though, it is convenient to restart from the complete Total Potential Energy-type Gurtin functional of Eq. (2.1), that must be specialized to the considered case of a static loading on the boundary of the RVE. In Section 3 it was proved that if one inserts, in it, a strictly admissible displacement, such as the selected one of Eq. (4.3), the Gurtin functional becomes fully equivalent to the reduced one of Eq. (3.6), in the sense that the displacement which makes it stationary coincides with the displacement which makes stationary the reduced functional of Eq. (3.6) (see Eq. (3.7)).

Inserting into the complete TPE functional of Eq. (2.1) both the static loading of Eqs. (5.8) and (5.9) and the chosen strictly admissible strain fields of Eq. (4.3) in both time intervals, and writing it in an explicit form with the time t spanning the complete interval $t \in [0, 2T]$ (see also Eq. (2.1)), one can proceed in the following way:

$$\begin{aligned}\mathrm{TPE}^G[u_i'] &= V\left\{ \frac{1}{2}\int_{0^-}^{2T}\int_{0^-}^{2T-t}\left(\frac{1}{V}\int_{\Omega} R_{ijhk}(2T-t-\tau)\ d\Omega\right) d\varepsilon'_{hk}(t)\ d\varepsilon'_{ij}(\tau) - \right.\\ &\left. - \int_{0^-}^{2T}\frac{1}{V}\int_{\Gamma} p_i(2T-t) du_i'(t)\ d\Gamma \right\} =\\ &= V\left\{ \frac{1}{2}\int_{0^-}^{2T}\int_{0^-}^{2T-t}\left\langle R_{ijhk}(2T-t-\tau)\right\rangle d\hat{\varepsilon}_{hk}(t)\ d\hat{\varepsilon}_{ij}(\tau) - \right.\\ &\left. - \int_{0^-}^{2T}\overline{\sigma}_{ij}\mathcal{H}(2T-t)\left(\frac{1}{V}\int_{\Gamma} n_i x_k d\Gamma\right) d\hat{\varepsilon}_{kj}(t)\right\}\end{aligned} \tag{5.16}$$

which, recalling that the last surface integral in round brackets is equal to the Kronecker Delta δ_{ik}, can be rewritten as follows:

$$\begin{aligned}\mathrm{TPE}^G[u_i'] &= V\left\{ \frac{1}{2}\int_{0^-}^{2T}\int_{0^-}^{2T-t}\left\langle R_{ijhk}(2T-t-\tau)\right\rangle d\hat{\varepsilon}_{hk}(t)\ d\hat{\varepsilon}_{ij}(\tau) - \right.\\ &\left. - \int_{0^-}^{2T}\overline{\sigma}_{ij}\mathcal{H}(2T-t) d\hat{\varepsilon}_{ij}(t)\right\}\\ &= V\left\{ \frac{1}{2}\int_{0^-}^{2T}\int_{0^-}^{2T-t}\left\langle R_{ijhk}(2T-t-\tau)\right\rangle d\hat{\varepsilon}_{hk}(t)\ d\hat{\varepsilon}_{ij}(\tau) - \overline{\sigma}_{ij}\hat{\varepsilon}_{ij}(2T)\right\}\end{aligned} \tag{5.17}$$

The value of this functional, evaluated for any strictly admissible strain of the type of Eq. (4.3), is a minimum, in such a way as to provide the best possible lower bound to the homogenized creep kernel that appears in Eq. (5.15), if

$$\int_{0^-}^{2T-t}\left\langle R_{ijhk}(2T-t-\tau)\right\rangle d\hat{\varepsilon}_{hk}(\tau) = \overline{\sigma}_{ij} \tag{5.18}$$

a Volterra integral equation always possessing a unique solution that can be found, for example, by means of the Laplace transform technique.

Let us denote by

$$\hat{\varepsilon}^{opt}_{1ij}(t) = \overline{\sigma}_{ij} f(t) \tag{5.19}$$

the solution of Eq. (5.18) in the first time interval, i.e., the strictly admissible strain function $\hat{\varepsilon}_{1ij}(t)$ that minimizes both functionals (5.16) and (3.6). Applying the Laplace transform technique to solve Eq. (5.18), denoting by s the Laplace transformation parameter and by $\mathcal{L}(\cdot)$ and $\mathcal{L}^{-1}(\cdot)$ the direct and inverse Laplace transform, respectively, the function $f(t)$ can be easily shown to be equal to

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\left[\left\langle \mathcal{L}(R_{ijhk}(t))\right\rangle\right]^{-1}\right) \tag{5.20}$$

The functional (5.17), in correspondence to this solution which makes it a minimum with respect to the chosen strictly admissible strains, becomes

$$\frac{1}{V}\mathrm{TPE}^{G,opt} = \frac{1}{2}\int_{0^-}^{2T}\int_{0^-}^{2T-t}\left\langle R_{ijhk}(2T-t-\tau)\right\rangle d\hat{\varepsilon}^{opt}_{1hk}(t)\ d\hat{\varepsilon}^{opt}_{1ij}(\tau) - \overline{\sigma}_{ij}\hat{\varepsilon}^{opt}_{1ij}(2T) \tag{5.21}$$

which, recalling that the strain $\hat{\varepsilon}^{opt}_{1ij}$ solves Eq. (5.18), is rewritten as

$$\begin{aligned}\frac{1}{V}\mathrm{TPE}^{G,opt} &= \frac{1}{2}\int_{0^-}^{2T}\overline{\sigma}_{ij}\ d\hat{\varepsilon}^{opt}_{1ij}(t) - \overline{\sigma}_{ij}\hat{\varepsilon}^{opt}_{1ij}(2T)\\ &= -\frac{1}{2}\overline{\sigma}_{ij}\hat{\varepsilon}^{opt}_{1ij}(2T) = -\frac{1}{2}\overline{\sigma}_{ij} f(2T)\overline{\sigma}_{ij}\end{aligned} \tag{5.22}$$

The comparison of this result with Eq. (5.15), considering the time T as an arbitrary time, leads to the following lower bound to the homogenized creep kernel:

$$\overline{\sigma}_{ij} C^h_{ijhk}(t)\overline{\sigma}_{hk} \geq \overline{\sigma}_{ij} f(t)\overline{\sigma}_{ij} \tag{5.23}$$

The explicitation of result (5.23) depends on the possibility of calculating the inverse transform in Eq. (5.20). In any case, if needed, numerical results for the inverse Laplace transform could always be obtained. Recall also that the minimum principle of Eq. (3.5), at the basis of the preceding reasoning, for the strictly admissible strain fields selected in this work is valid only for the case of purely deviatoric loading on a RVE. Therefore, one needs to deal with the scalar shear components $R^{(i)}(t)$ and $C^{(i)}(t)$ of each phase (i) only, as was done to write Eqs. (5.7) and (5.11) previously.

In order to produce an explicit result that can be compared to the numerical solutions of the following Section 6, one may consider, as an example, a composite RVE with viscous kernels defined so that the volume average of the scalar shear relaxation kernel has the following form:

$$\left\langle R(t)\right\rangle = \sum_{i=1}^{N} c_i R^{(i)}(t) = a + b\exp(\alpha t) \tag{5.24}$$

which includes, among others, the case deriving from the study of a two-phase RVE having an elastic and a viscous phase, this last being governed by the expression of Eq. (6.1), a three-parameter Kelvin–Voigt, or Zener, viscoelastic solid.

Inserting this volume average expression into Eq. (5.20) and calculating the various transforms, one obtains the following explicit expression for the function $f(t)$ in result (5.19), for a viscoelastic composite with relaxation kernels satisfying Eq. (5.24):

$$f(t) = \left[\frac{1}{a} - \frac{b}{a(a+b)}\exp\left(\frac{\alpha a}{a+b}t\right)\right] \tag{5.25}$$

which, inserted into the result (5.23), gives the following explicit lower bound to the homogenized creep kernel:

$$C^h(t) \geq \left[\frac{1}{a} - \frac{b}{a(a+b)}\exp\left(\frac{\alpha a}{a+b}t\right)\right] \tag{5.26}$$

A similar path can be taken to obtain a lower bound to the homogenized relaxation kernel. One must restart from the TCE-type functional of Eq. (3.9) and consider, in it, a RVE loading of a purely kinematic type on the boundary $\Gamma_u \equiv \Gamma$, of the type defined by Eqs. (5.1) and (5.2). After passages similar to those illustrated for the TPE functional, and denoting by $\hat{\sigma}^{opt}_{1ij}(t) = \bar{\varepsilon}_{ij}g(t)$ the stress field that minimizes the TCE-type functional within the range of the strictly admissible stresses, one arrives at the following results:

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\left[\left\langle \mathcal{L}(C_{ijhk}(t))\right\rangle\right]^{-1}\right) \tag{5.27}$$

and

$$\bar{\varepsilon}_{ij}R^h_{ijhk}(t)\bar{\varepsilon}_{hk} \geq \bar{\varepsilon}_{ij}g(t)\bar{\varepsilon}_{ij} \tag{5.28}$$

An example of an explicit solution for the case of a two-phase composite having the viscous phase governed by the same three-parameter solid considered previously, i.e., having a scalar shear creep kernel given by Eq. (6.2) of Section 6, can be worked out as for the previous case. If the volume average of the creep kernels in the two-phase RVE can be written as

$$\left\langle C(t)\right\rangle = \sum_{i=1}^{N} c_i C^{(i)}(t) = v + w\exp(\beta t) \tag{5.29}$$

then the following lower bound to the homogenized shear kernel $R^h(t)$ can be arrived at:

$$R^h(t) \geq \left[\frac{1}{v} - \frac{w}{v(v+w)}\exp\left(\frac{\beta v}{v+w}t\right)\right] \tag{5.30}$$

5.4. Reuss-type general strict lower bound to $R^h_{ijhk}(t)$ (Huet, 1995) and Reuss-type strict lower bound to $C^h_{ijhk}(t)$

Two strict lower bounds to the viscous kernels, one having a general validity, the other restricted to the case of macroscopically isotropic composites under deviatoric loading, are derived next.

The general one has already been presented in Huet (1995), and we will here give a summary of the procedure followed to obtain it. Consider once again the same type of affine loading of Eqs. (5.1) and (5.2), and restart from Eq. (2.23) specialized to the considered loading type. Owing to the fact that the relaxation kernel is a monotonically decreasing function of time (Huet, 1995), the following holds:

$$\frac{1}{V}\overline{\text{TPE}}_{\text{sol}} = \frac{1}{2}\left\langle \int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T-t-\tau)\, d\varepsilon_{1ij}(\tau)\, d\varepsilon_{1hk}(t)\right\rangle >$$
$$> \frac{1}{2}\left\langle \int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T)\, d\varepsilon_{1ij}(\tau)\, d\varepsilon_{1hk}(t)\right\rangle =$$
$$= \frac{1}{2}\left\langle R_{ijhk}(2T)\varepsilon_{1ij}(T)\varepsilon_{1hk}(T)\right\rangle \tag{5.31}$$

in which the strain is the exact one.

The last term in Eq. (5.31) is now fully elastic, governed by the relaxation kernel evaluated at time $2T$. In it, the strain $\varepsilon_{1ij}(T)$ is the viscoelastic strain reached, during the actual time history, at time T, and it can therefore be considered as an admissible (compatible) strain field in a fully elastic theory. Therefore, the standard theorem of minimum of the Total Potential Energy in elasticity allows one to write also the following inequality:

$$\frac{1}{2}\left\langle R_{ijhk}(2T)\varepsilon_{1ij}(T)\varepsilon_{1hk}(T)\right\rangle \geq \frac{1}{V}\text{TPE}^{el}(\varepsilon^{el}_{ij}) \tag{5.32}$$

in which the symbol ε^{el}_{ij} denotes the real elastic solution in a homogenized RVE having constant elastic moduli defined by the values of $R^{h,el}_{ijhk}(2T)$.

One thus concludes that the following strict inequality can be established:

$$\overline{\text{TPE}}_{\text{sol}} > \text{TPE}^{el}(\varepsilon^{el}_{ij}) \tag{5.33}$$

This, rewritten exploiting the choice (5.2) for the loading history, becomes:

$$\bar{\varepsilon}_{ij}R^h_{ijhk}(2T)\bar{\varepsilon}_{hk} > \bar{\varepsilon}_{ij}R^{h,el}_{ijhk}(2T)\bar{\varepsilon}_{hk} \tag{5.34}$$

Finally, one can replace in the r.h.s. of Eq. (5.34), a purely elastic quantity, any elastic lower bound, i.e., a quantity not larger than the r.h.s. itself — let us call it ELBR (Elastic Lower Bound to R), arriving at the following bound to $R^h_{ijhk}(2T)$:

$$\bar{\varepsilon}_{ij}R^h_{ijhk}(2T)\bar{\varepsilon}_{hk} > \text{ELBR} \tag{5.35}$$

the same general strict inequality obtained in Huet (1995) (his eq. (5.7)) by adopting the same argument.

In the case of macroscopically isotropic composites, in order to write explicit and usable expressions of this bound for both the homogenized scalar deviatoric and volumetric components $R^{d,h}(t)$ and $R^{v,h}(t)$ only, one can choose in Eq. (5.35), for the quantity ELBR, Reuss' lower bound to the elastic moduli of the direct constitutive law, obtaining thus

$$R^{d,h}(t) > \frac{1}{\sum_{i=1}^{N}\frac{c_i}{R^{d,(i)}(t)}} \quad \forall t \in [0,\infty); \qquad R^{v,h}(t) > \frac{1}{\sum_{i=1}^{N}\frac{c_i}{R^{v,(i)}(t)}} \quad \forall t \in [0,\infty) \tag{5.36}$$

where $R^{d,(i)}(t)$ and $R^{v,(i)}(t)$ denote the deviatoric and volumetric relaxation kernels of the individual phases (i), respectively.

The derivation of an analogous lower bound to the homogenized creep kernel cannot start from Eq. (2.25), since the creep kernels are not monotonically decreasing functions of time. Therefore, considering instead that the creep *rate* kernel is a monotonically decreasing function of time, one is lead to start from the functional of (2.38) the same reasoning followed in the above paragraphs.

Assuming the same loading as in Eqs. (5.8) and (5.9), writing in an explicit form the functional $\overset{\circ}{\text{TCE}}$ of Eq. (2.38), and accounting for the said property of function $\dot{C}_{ijhk}(t)$, i.e., of being monotonically decreasing with time, one can reach the following result:

$$\frac{1}{V}\overset{\circ}{\text{TCE}}_{\text{sol}} = \frac{1}{2}\left\langle \int_{0^-}^{T}\int_{0^-}^{T} \dot{C}_{ijhk}(2T-t-\tau)\, d\sigma_{1ij}(\tau)\, d\sigma_{1hk}(t)\right\rangle >$$
$$> \frac{1}{2}\left\langle \int_{0^-}^{T}\int_{0^-}^{T} \dot{C}_{ijhk}(2T)\, d\sigma_{1ij}(\tau)\, d\sigma_{1hk}(t)\right\rangle =$$
$$= \frac{1}{2}\left\langle \dot{C}_{ijhk}(2T)\sigma_{1ij}(T)\sigma_{1hk}(T)\right\rangle \tag{5.37}$$

which, by the same reasoning explained for the relaxation rate kernel, and accounting for the theorem of the Total Complementary Energy in elasticity, leads to the following strict inequality:

$$\bar{\sigma}_{ij}\dot{C}^h_{ijhk}(2T)\bar{\sigma}_{hk} > \bar{\sigma}_{ij}\left(\dot{C}^{el}_{ijhk}(2T)\right)^h\bar{\sigma}_{hk} \tag{5.38}$$

where the symbol $\left(\dot{C}^{el}_{ijhk}(2T)\right)^h$ denotes a homogenized creep kernel computed operating on the fixed values taken by the individual creep rate kernels evaluated at time $2T$. This quantity is not a true rate of a homogenized creep kernel, of course. This result, a strict, general lower bound to the rate of the homogenized creep kernel, was also obtained in Huet (1995), Eq. (6.3). Unfortunately, the integration in time of this inequality, as done in Huet (1995), does not lead to a lower bound to the homogenized total creep kernel easy to be written in an explicit, usable form.

We could obtain a usable lower bound to the homogenized creep kernel without integrating Eq. (5.38), but only exploiting once again the reduced minimum principles of Section 3. Clearly, in this case the bound that can be produced returns to have only a restricted validity, the same holding for the theorems of Section 3.

In order to obtain this new lower bound, it is possible to exploit, in a way less general than done in Section 5.3, the reduced Total Potential

Energy theorem of Eq. (3.5), considering in it a static loading on the RVE boundary of the type defined by Eqs. (5.8) and (5.9). In this case, recalling Eqs. (3.6) and (5.15), one can write the following inequality:

$$\frac{1}{2}\overline{\sigma}_{ij}C^h_{ijhk}(2T)\overline{\sigma}_{hk} \geq -\frac{1}{2}\int_{0^-}^{T}\int_{0^-}^{T}\langle R_{ijhk}(2T-t-\tau)\rangle\, d\hat{\varepsilon}_{1hk}(t)\, d\hat{\varepsilon}_{1ij}(\tau)+\overline{\sigma}_{ij}\hat{\varepsilon}_{1ij}(T) \tag{5.39}$$

where $\hat{\varepsilon}_{1ij}(t)$ indicates a strictly admissible strain field of the type of Eq. (4.3). One can adopt the following specific choice for $\hat{\varepsilon}_{1ij}(t)$:

$$\hat{\varepsilon}_{1ij}(t) = \left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk}\mathcal{H}(t) \tag{5.40}$$

clearly less "optimal" than the one deriving from the solution of Eq. (5.18) but still belonging to the type of Eq. (4.3), strictly admissible. Inserting this expression into the inequality (5.39) and recalling the meaning of a Stieltjes integral, the following result is obtained:

$$\frac{1}{2}\overline{\sigma}_{ij}C^h_{ijhk}(2T)\overline{\sigma}_{hk} \geq -\frac{1}{2}\langle R_{ijhk}(2T)\rangle\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk}\left\langle C^{-1}_{hklm}(2T)\right\rangle^{-1}\overline{\sigma}_{lm}+ + \overline{\sigma}_{ij}\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk} \tag{5.41}$$

This already is a new, non-strict lower bound to the homogenized creep kernel, having the same reduced range of validity as the minimum principles of Section 3. If one wants to derive a lower bound involving only the individual creep kernels of the single phases of the composite, one can recall the following general property of the viscous kernels:

$$R_{ijhk}(t)C_{ijhk}(t) < 1 \quad \forall t \tag{5.42}$$

and then transform the r.h.s of the result (5.41) as follows:

$$\begin{aligned}&-\frac{1}{2}\langle R_{ijhk}(2T)\rangle\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk}\left\langle C^{-1}_{hklm}(2T)\right\rangle^{-1}\overline{\sigma}_{lm}+\overline{\sigma}_{ij}\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk} > \\ &> -\frac{1}{2}\left\langle C^{-1}_{ijhk}(2T)\right\rangle\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk}\left\langle C^{-1}_{hklm}(2T)\right\rangle^{-1}\overline{\sigma}_{lm}+\overline{\sigma}_{ij}\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk} = \\ &= \frac{1}{2}\overline{\sigma}_{ij}\left\langle C^{-1}_{ijhk}(2T)\right\rangle^{-1}\overline{\sigma}_{hk}\end{aligned} \tag{5.43}$$

Recalling the inequality (5.41) and considering the time $2T$ as a generic time, this allows one to produce the following new strict lower bound to the homogenized creep kernel:

$$\overline{\sigma}_{ij}C^h_{ijhk}(t)\overline{\sigma}_{hk} > \overline{\sigma}_{ij}\left\langle C^{-1}_{ijhk}(t)\right\rangle^{-1}\overline{\sigma}_{hk} \tag{5.44}$$

which holds only for a macroscopically isotropic composite under deviatoric loading.

The explicit expression of this bound for the shear (deviatoric) scalar component of the homogenized creep kernel, for a composite having N homogeneous viscoelastic phases, reads as follows:

$$C^h(t) > \frac{1}{\sum_{i=1}^{N}\frac{c_i}{C^{(i)}(t)}} \quad \forall t \in [0,\infty) \tag{5.45}$$

6. Numerical (FEM) checks

In this section we examine the performance of the obtained bounds by means of numerical tests concerning a variety of plane strain RVEs. The considered bounds are those given by Eqs. (5.7), (5.11), (5.26), (5.30), (5.36), and (5.45). In the case of the general, strict lower bound of Eq. (5.36), only the results for the deviatoric (shear) kernel components have been considered. Numerical calculations performed on 3D RVEs loaded in a volumetric way, not shown here for brevity, have confirmed the validity of the general strict lower bound (5.36) also for the volumetric case.

We recall that both the upper bounds (5.7) and (5.11), as well as both the non-strict (hereafter called "optimal") lower bounds of Eqs. (5.26) and (5.30), together with the strict lower bound (5.45), are valid only for the purely deviatoric behavior of macroscopically isotropic viscoelastic composites. Only the strict lower bound of Eq. (5.36), found by Huet (1995), has a general validity. Moreover, the optimal lower bounds of Eqs. (5.26) and (5.30) have been expressed in an explicit form for the special case, considered in the majority of the examples of this section, of a two-phase composite with a three-parameter Kelvin–Voigt (or Zener) viscous kernel.

It may be interesting to observe here that the bounds (5.7) and (5.11) become, in the limit case when all the phases are linear elastic, identical with the Voigt upper bounds in linear elasticity. Both lower bounds (5.36) and (5.45), adding to them the equality sign, become, in the same limit case, identical with the Reuss lower bounds in linear elasticity. Therefore, all these results can be considered extensions, albeit with a limited validity, to linear viscoelasticity of these two basic, first-order bounds in elasticity. For the linear viscoelastic case, both at the initial time $t = 0$ and for $t \to \infty$, they should therefore coincide with the elastic Voigt and Reuss bounds, and, for intermediate times, they can be expected to be affected by errors having the same order of magnitude as the Voigt and Reuss ones in elasticity.

In order to obtain numerical results, we consider viscoelastic phases governed by a standard three-parameter solid rheologic model of the Kelvin–Voigt (or Zener) type, for which, for a generic phase (i), the shear relaxation kernel is written as follows (Bland, 1960):

$$R^{(i)}(t) = (G^{(i)}_E + G^{(i)}_V) - G^{(i)}_V\left[1-\exp\left(-\frac{G^{(i)}_V t}{\eta^{(i)}_V}\right)\right] \tag{6.1}$$

where $G^{(i)}_E$ and $G^{(i)}_V$ are the shear moduli of the viscoelastic material, and $\eta^{(i)}_V$ is the viscosity coefficient.

The corresponding shear creep kernel reads as follows (Bland, 1960):

$$C^{(i)}(t) = \frac{1}{(G^{(i)}_E + G^{(i)}_V)} + \frac{G^{(i)}_V}{G^{(i)}_E(G^{(i)}_E + G^{(i)}_V)}\left[1-\exp\left(-\frac{G^{(i)}_E G^{(i)}_V t}{\eta^{(i)}_V(G^{(i)}_E + G^{(i)}_V)}\right)\right] \tag{6.2}$$

The performance of all the obtained bounds to the homogenized viscous kernels was checked against Finite Element results obtained by means of the commercial code ABAQUS (Hibbitt et al., 2018). ABAQUS allows the modeling of linear viscoelastic materials through the definition of material parameters associated to the Prony series for relaxation only; this, for a Kelvin–Voigt material and for each phase (i), is written in ABAQUS in the following form:

$$\tau^{(i)}(t) = G^{(i)}_0\int_0^t\left\{1-\frac{G^{(i)}_1}{G^{(i)}_0}\left[1-\exp\left(-\frac{G^{(i)}_1\tau}{\eta^{(i)}_1}\right)\right]\right\}\dot{\gamma}^{(i)}\, d\tau, \quad i = 1,\ldots,N \tag{6.3}$$

and requires in input the values of $G^{(i)}_0$, $G^{(i)}_1/G^{(i)}_0$, and $\eta^{(i)}_1/G^{(i)}_1$, $i = 1,\ldots,N$. A match between Eqs. (6.3) and (6.1) shows immediately that, in order to establish an equivalence between ABAQUS and the analytical results, one needs to set $G^{(i)}_0 = G^{(i)}_E + G^{(i)}_V$, $G^{(i)}_1 = G^{(i)}_V$, and $\eta^{(i)}_1 = \eta^{(i)}_V$.

Several square RVEs with unit sides have been constructed, with different microstructures, each possessing at least a 10 × 10 array of inclusions of different shapes. A plane strain simple shear problem was considered, applying, in the case of relaxation, boundary displacements corresponding to a constant value equal to 1 for the average in-plane shear strain, and, for creep, boundary tractions corresponding to a value equal to 1 for the average in-plane shear stress. This setup represents of course a transversely isotropic problem, not a fully isotropic one; nevertheless, the considered loading conditions allow one to obtain information about the homogenized shear viscous kernels of a macroscopically isotropic material under deviatoric loading.

Both loading conditions were applied as unit-steps at $t = 0$. Quadrilateral 8-noded plane strain elements with reduced integration (CPE8R in ABAQUS notation) were always adopted. At least 100 elements per

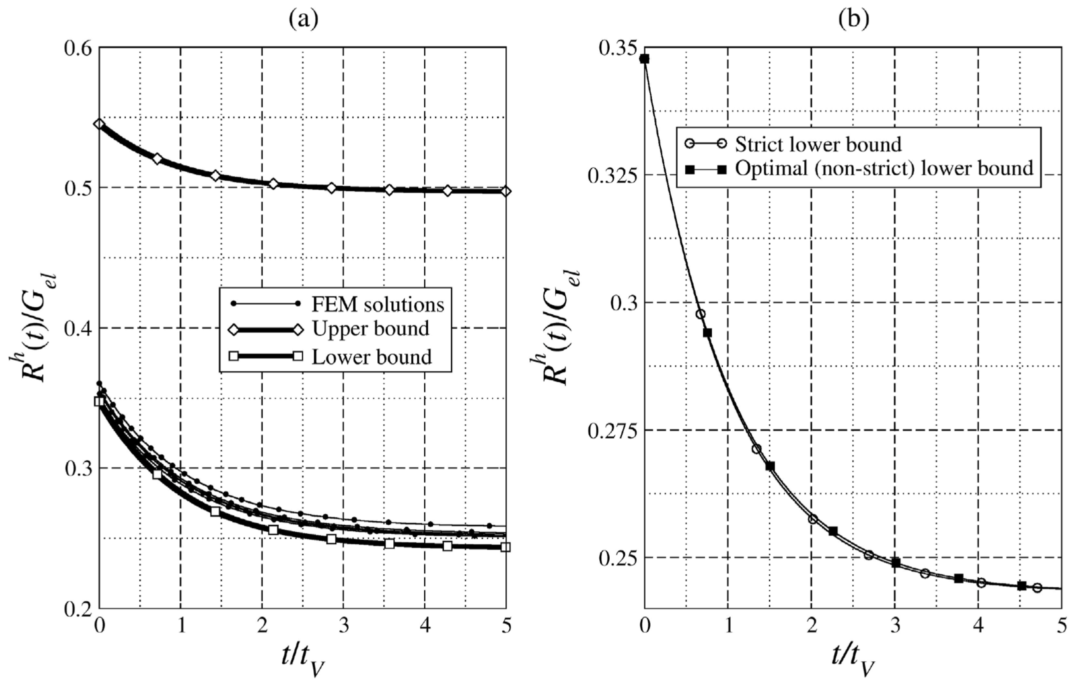

Fig. 1. Figure (a), left: normalized homogenized uniaxial shear relaxation kernel $R^h(t)/G_{el}$ as a function of the normalized time t/t_V for a two-phase composite. The solid thick curves with white symbols plot strict lower (squares, Eq. (5.36)) and upper (diamonds, Eq. (5.7)) bounds; the thin lines plot FEM solutions. Figure (b), right: comparison between the strict (Eq. (5.36)) and the optimal (Eq. (5.30)) lower bounds to the homogenized relaxation kernel.

each side of the RVE were adopted, arriving at 1000 per side in the case of the most complex microstructures.

The first set of analyses was run considering a two-phase material, the first ($i = 1$) linear elastic and the second ($i = 2$) linear viscoelastic. The material data have been taken equal to those adopted in Lahellec and Suquet (2007), i.e.,

$$c_1 = 0.4;\quad G^{(1)} = G_{el} = 166650 \text{ MPa}$$

$$c_2 = 0.6;\quad G_E^{(2)} = 26920 \text{ MPa};\quad G_V^{(2)} = G_V = 13460 \text{ MPa};\quad \eta_V^{(2)} = \eta_V = 10000 \text{ MPa s}$$

The bulk modulus of the elastic phase plays no role ($\nu^{(1)} = 0$ was always adopted), and that of the viscous phase has been set equal to zero.

We first checked the performance of the bounds of Section 5 with this set of data for five different RVE microstructures all referred to the same composite. This also allowed us to have an idea of the influence of the adopted RVE on the numerical solutions. Figs. 1 and 2 report the relevant results, the first for relaxation and the second for creep. In both Figures the time has been normalized by the relaxation time $t_V = \eta_V/G_V$; the curves of Fig. 1 plot the homogenized relaxation kernel normalized by the elastic shear modulus G_{el}, and those of Fig. 2 plot the homogenized creep kernel normalized once again by G_{el}. All the considered RVEs yield both families of curves lying within the respective bounds, and all tend to furnish similar relaxation and creep curves. In the rest of the examples we have considered just one of these RVEs, with the most (quasi) random microstructure.

Fig. 1(b) is meant to illustrate the difference between the strict and the optimal lower bounds to $R^h(t)$, given by Eqs. (5.36) and (5.30) respectively. Two facts are apparent: (i) the optimal lower bound never lies below the strict one, as it should be and (ii) the two curves are definitely very close to each other, the difference between them being of the order of — actually, definitely smaller than — the difference produced, in the FEM solutions, by choosing different RVEs. Both features (i) and (ii) were found to be valid for all the cases studied in this section.

On account of these negligible differences between the two lower bounds, in all the next figures only the strict ones, Eqs. (5.36) and (5.45), will be plotted, in order to keep the images reasonably readable.

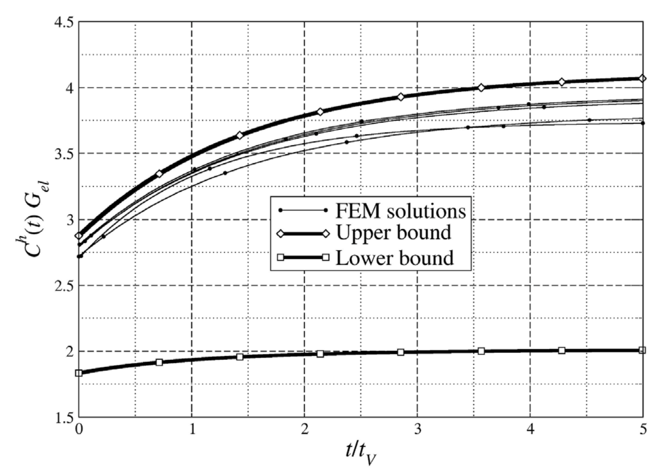

Fig. 2. Normalized homogenized uniaxial shear creep kernel $C^h(t)G_{el}$ as a function of the normalized time t/t_V for a two-phase composite. The solid thick curves with white symbols plot the optimal lower (squares, Eq. (5.26)) and upper (diamonds, Eq. (5.11)) bounds; the thin lines plot FEM solutions.

A second group of analyses keeps fixed the volume fractions of the two phases, $c_1 = 0.4$ and $c_2 = 0.6$, and explores the sensitivity to the contrast between the elastic shear moduli of the two phases. Denoting this contrast by $k = G_{el}/(G_E^{(2)} + G_V^{(2)})$, the starting data of the previous Figures have all $k = 4.1271$. Three more cases have been studied next, with $k = 0.1, k = 1$, and $k = 10$, respectively. Figs. 3 and 4 plot now, again as a function of the normalized time, the relative differences between the bounds to the homogenized viscous kernels and the FEM results. Fig. 3 refers to the relaxation, and Fig. 4 to the creep kernel. Here all the white symbols are lower bound differences, and all the black symbols refer to upper bound differences. Both figures show that all the bounds lie in their proper portion of the plane, and that the best results, as obvious, occur for $k = 1$. The differences remain for all

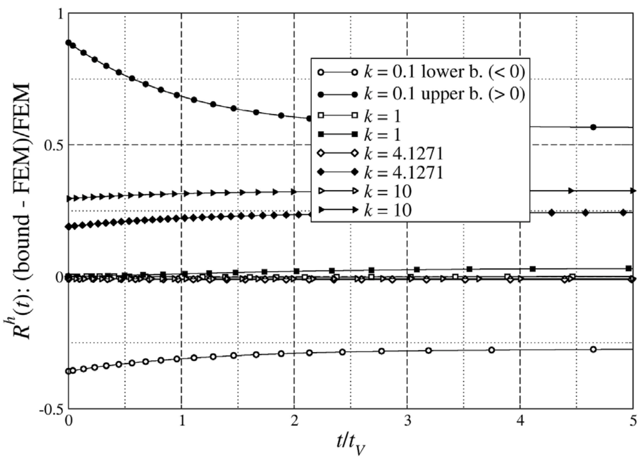

Fig. 3. Relative difference curves for the homogenized uniaxial shear relaxation kernel $R^h(t)$ as a function of the normalized time t/t_V for a two-phase composite, for various values of the contrast parameter $k = G_{el}/(G_E^{(2)} + G_V^{(2)})$. The white symbols denote lower bound differences (results of the strict bound (5.36) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

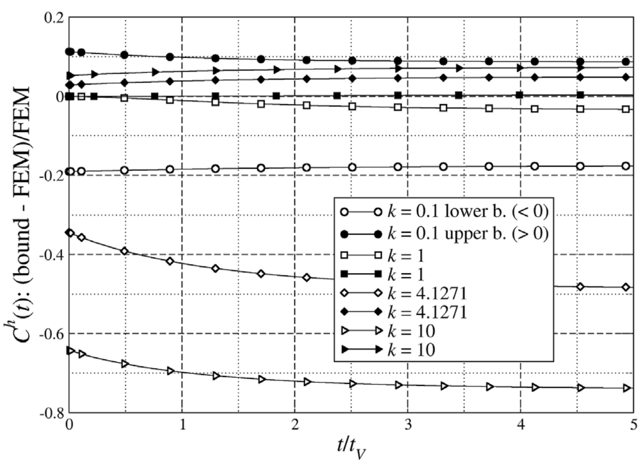

Fig. 4. Relative difference curves for the homogenized uniaxial shear creep kernel $C^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the contrast parameter $k = G_{el}/(G_E^{(2)} + G_V^{(2)})$. The white symbols denote lower bound differences (results of the strict bound (5.45) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

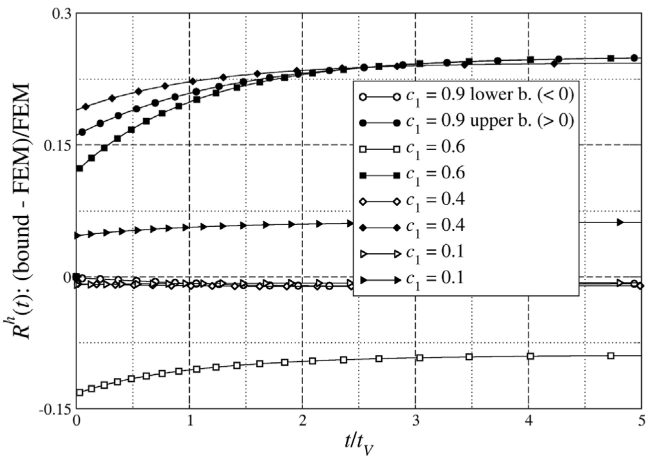

Fig. 5. Relative difference curves for the homogenized uniaxial shear relaxation kernel $R^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the volume fraction of the elastic phase c_1. The white symbols denote lower bound differences (results of the strict bound (5.36) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

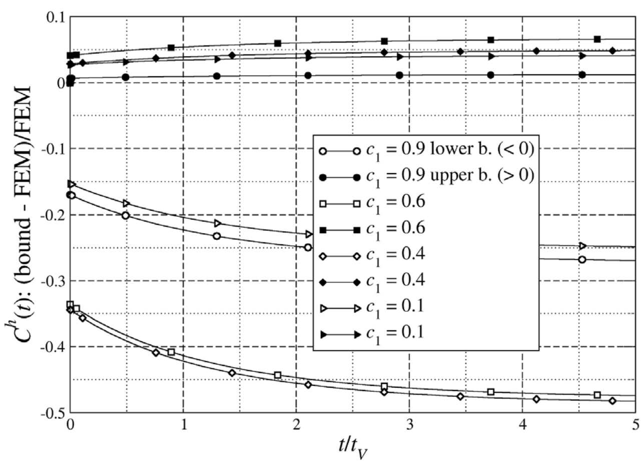

Fig. 6. Relative difference curves for the homogenized uniaxial shear creep kernel $C^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the volume fraction of the elastic phase c_1. The white symbols denote lower bound differences (results of the strict bound (5.45) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

times of the same order of magnitude (±80% in the worst cases) as the differences on the instantaneous, (elastic, time $t = 0$) moduli, and in some cases increase by a small amount for increasing time, in others decrease.

Figs. 5 and 6 show results concerning the sensitivity to the volume fractions, keeping the contrast equal to that of the first case, i.e., $k = 4.1271$. Four cases of volume fractions have been considered, namely $c_1 = 0.4$ as before, $c_1 = 0.1, c_1 = 0.6$, and $c_1 = 0.9$. This forced the adoption of different meshes for each case, with very dense ones for the case $c_1 = 0.9$. Figs. 5 and 6 plot, as a function of the normalized time, the relative differences between the bounds to the viscous kernels and the FEM results, Fig. 5 for the relaxation and Fig. 6 for the creep kernel. Both Figures show once again that all the bounds lie in their proper portion of the plane. The differences are now generally smaller than in the previous two figures, even for extreme values of the volume fractions; this is probably due to the value of the contrast k. The differences are always of the same order of magnitude as the initial, purely elastic ones. The next figures, 7 and 8, plot once again relative differences between analytical bounds and FEM results considering variations of the ratio $g_1 = G_V^{(2)}/(G_E^{(2)} + G_V^{(2)})$ in which the value of $G_V^{(2)}$ has been always kept fixed to its starting value $G_V^{(2)} = 13460$ MPa, so as to consider also limit cases of both absence of elasticity and very high elasticity in the viscous phase. Figs. 9 and 10, which still plot relative differences, refer to varying the relaxation time $t_V = \eta_V/G_V$ in the viscous phase, considering 4 different values $\eta_V = 10, 1000, 10000,$ and 1000000 MPa s, and keeping all the other parameters fixed at their basic values reported above. Note that the nondimensional time axes are now in log scale, since the adoption of relaxation times covering a wide range of values — from $t_V = 7.429 \times 10^{-4}$ s to $t_V = 74.2942$ s — produces viscous kernels so different from each other that they can be plotted superimposed only by using such a scale.

All these results confirm the effectiveness of the bounds derived in this work in their field of application.

Finally, four more analyses — two relaxation and two creep — have been run for the case of a material having one elastic and three viscoelastic phases, in order to check the performance of the bounds

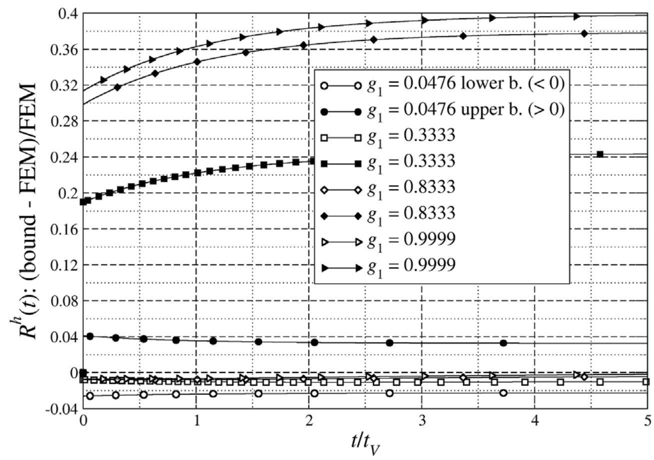

Fig. 7. Relative difference curves for the homogenized uniaxial shear relaxation kernel $R^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the ratio $g_1 = G_V^{(2)}/(G_E^{(2)} + G_V^{(2)})$ between the shear modulus of the viscous phase and the global one. The white symbols denote lower bound differences (results of the strict bound (5.36) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

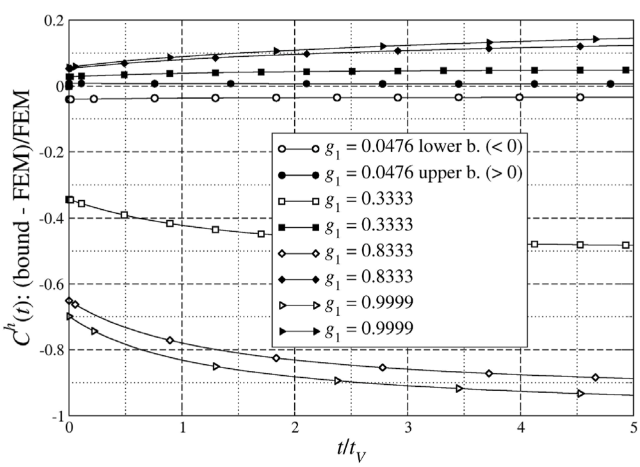

Fig. 8. Relative difference curves for the homogenized uniaxial shear creep kernel $C^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the ratio $g_1 = G_V^{(2)}/(G_E^{(2)} + G_V^{(2)})$ between the shear modulus of the viscous phase and the global one. The white symbols denote lower bound differences (results of the strict bound (5.45) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

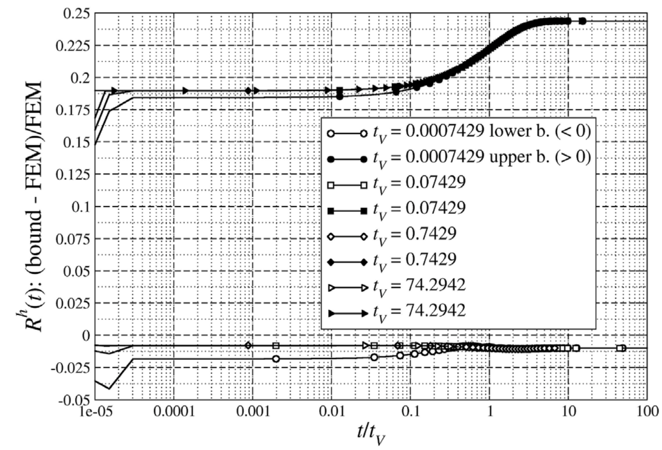

Fig. 9. Relative difference curves for the homogenized uniaxial shear relaxation kernel $R^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the relaxation time $t_V = \eta_V^{(2)}/G_V^{(2)}$. The white symbols denote lower bound differences (results of the strict bound (5.36) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

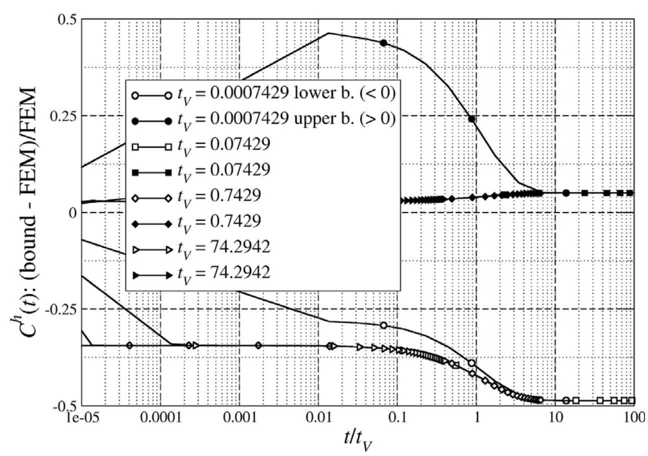

Fig. 10. Relative difference curves for the homogenized uniaxial shear creep kernel $C^h(t)$ as a function of the normalized time t/t_V, for a two-phase composite, for various values of the relaxation time $t_V = \eta_V^{(2)}/G_V^{(2)}$. The white symbols denote lower bound differences (results of the strict bound (5.45) minus FEM results divided by FEM results); the black symbols denote upper bound relative differences. Other parameters as indicated in the text.

also in the presence of a more complicated microstructure. A first group concerns a high volume fraction of the elastic phase, with $c_1 = 0.9$; a second group the opposite case, with $c_1 = 0.1$.

The adopted data are as follows:

- case with $c_1 = 0.9$:
  $G^{(1)} = G_{el} = 166650$ MPa
  $c_2 = 0.05$; $G_E^{(2)} = 26920$ MPa; $G_V^{(2)} = 13460$ MPa; $\eta_V^{(2)} = 20000$ MPa s
  $c_3 = 0.03333$; $G_E^{(3)} = 5384$ MPa; $G_V^{(3)} = 48456$ MPa; $\eta_V^{(3)} = 35998$ MPa s
  $c_4 = 0.01667$; $G_E^{(4)} = 13056.2$ MPa; $G_V^{(4)} = 403.8$ MPa; $\eta_V^{(4)} = 150.01$ MPa s
- case with $c_1 = 0.1$:
  $G^{(1)} = G_{el} = 166650$ MPa
  $c_2 = 0.5$; $G_E^{(2)} = 13056.2$ MPa; $G_V^{(2)} = 403.8$ MPa; $\eta_V^{(2)} = 150.01$ MPa s
  $c_3 = 0.3$; $G_E^{(3)} = 5384$ MPa; $G_V^{(3)} = 48456$ MPa; $\eta_V^{(3)} = 35998$ MPa s
  $c_4 = 0.1$; $G_E^{(4)} = 26920$ MPa; $G_V^{(4)} = 13460$ MPa; $\eta_V^{(4)} = 20000$ MPa s

These data cover a rather wide range of both relaxation times and of contrasts between elasticity of the elastic and the viscous phases, and are expected to provide a significantly severe test for the bounding equations. Figs. 11 and 12 plot directly the normalized relaxation and creep kernels, in the same plot, as functions of a time t normalized with respect to the relaxation time $t_{V,m}$ of the viscous phase with the highest volume fraction. Fig. 11 refers to the case $c_1 = 0.1$ (small elastic fraction), and Fig. 12 to the case $c_1 = 0.9$ (high elastic fraction). In Fig. 12 all the relaxation curves have been amplified by a factor of 10 in order to better show them, otherwise they would all appear as superimposed around a horizontal line at the zero value of the vertical axis.

All these results confirm once again the effectiveness of the newly obtained bounds, with differences of the same order of magnitude as the previous ones.

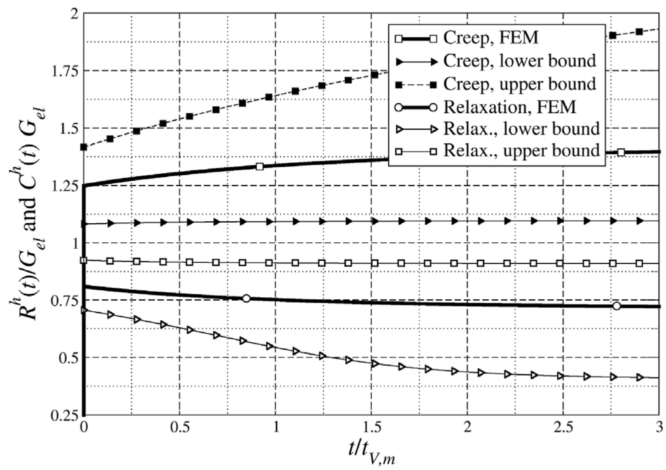

Fig. 11. Normalized homogenized uniaxial shear kernels as a function of the normalized time t/t_V for a four-phase composite, with $c_1 = 0.9$ (elastic), and $c_2 = 0.05, c_3 = 0.03333, c_4 = 0.01667$ (all viscoelastic). The thin curves with white symbols plot strict lower (right triangles, Eq. (5.45)) and upper (squares, Eq. (5.11)) bounds to the normalized homogenized creep kernel $C^h(t)G_{el}$; the thin curves with black symbols plot strict lower (right triangles, Eq. (5.36)) and upper (squares, Eq. (5.7)) bounds to the normalized homogenized relaxation kernel $R^h(t)/G_{el}$; the thick lines plot FEM solutions. Other parameters as indicated in the text.

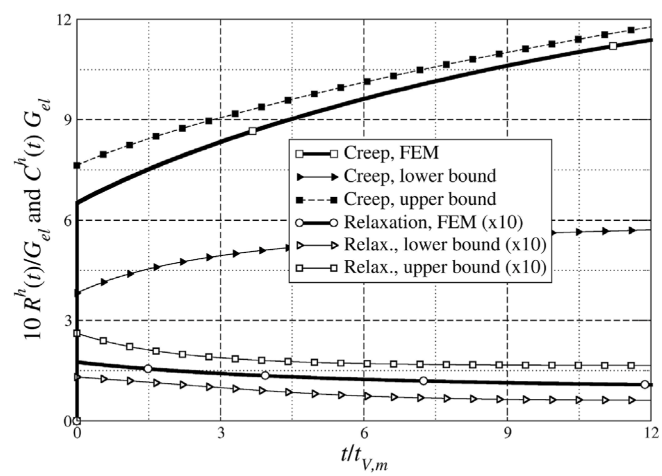

Fig. 12. Normalized homogenized uniaxial shear kernels as a function of the normalized time t/t_V for a four-phase composite, with $c_1 = 0.1$ (elastic), and $c_2 = 0.5, c_3 = 0.3, c_4 = 0.1$ (all viscoelastic). The thin curves with white symbols plot strict lower (right triangles, Eq. (5.45)) and upper (squares, Eq. (5.11)) bounds to the normalized homogenized creep kernel $C^h(t)G_{el}$; the thin curves with black symbols plot strict lower (right triangles, Eq. (5.36)) and upper (squares, Eq. (5.7)) bounds to the normalized homogenized relaxation kernel $R^h(t)/G_{el}$; the thick lines plot FEM solutions. Other parameters as indicated in the text.

7. Discussion and conclusions

Extremum principles for linear viscoelastic solids have been presented, valid for specific choices of the admissible functions in the associated functional, functions that were called strictly admissible. In the case of macroscopically isotropic, linear viscoelastic RVEs subjected to deviatoric loading, it was possible to identify some strictly admissible displacement and stress fields. Exploiting the new extremum theorems, analytical upper and lower bounds to the homogenized viscous kernels of macroscopically isotropic viscoelastic composites have been obtained in the time domain.

A similar theory based on extremum principles but holding for generic viscoelastic composites, i.e., non macroscopically isotropic under any type of stress or strain, could not be produced yet. The restriction to deviatoric loading only looks surprising enough, but it can be easily confirmed by means of counterexamples. Appendix illustrates analytical calculations for a simple uniaxial case, in which the new upper bounds are not valid. More FEM results for 3D RVEs loaded in a volumetric way, omitted here for brevity, also show that in this case the obtained upper bounds do not work.

The reasons for this basic difference between volumetric and deviatoric situations in macroscopically isotropic viscoelastic RVEs remain still to be understood. From the practical viewpoint, however, considering that often, when studying viscoelastic materials, viscosity is introduced only in the deviatoric part of the stress–strain equations, it is felt that even results holding only for the deviatoric components of the homogenized viscous kernels could be of interest.

We recall once again that all the obtained new bounds suffer from this limitation because they have all been derived from reduced extremum theorems holding in a restricted situation. The strict lower bound of Huet (1995) (Eq. (5.35) in the present work), instead, has a general validity. This is another fairly puzzling observation, that might deserve future attention in order to further develop this theory.

Both the strict lower bounds presented in Section 5.4 are not optimal. The new non-strict lower bounds, instead, are optimal within their range of underlying choices. In fact, as shown by Fig. 1b, they always stay above the strict ones, i.e., closer to the real solutions.

In the strict lower bound of Eq. (5.35) it is possible to adopt, for the elastic bound ELBR, more refined expressions, such as, for example, Hashin–Shtrikman's results. This might produce a tighter bound to the homogenized viscous kernels.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

Work done within a research project financed by the Italian Ministry of University and Research (MUR). The authors wish to thank the anonymous Reviewer of a previous version of this work for pointing out several critical issues that have been rectified here.

Appendix

Consider the problem of Fig. A.1, a fully fixed rod with constant unitary cross-section area, of total length l, with an elastic and a viscoelastic phase arranged in series. The rod is subjected to a displacement $u(t)$ prescribed to its right extremity, defined as:

$$u(t) = \bar{u}\ \mathcal{H}(t) \tag{A.1}$$

where $\bar{u}$ is a given constant and $\mathcal{H}(t)$ is the Heaviside function. Phase 1 of the composite is viscoelastic, with length l_1 and volume fraction c_1, governed by a standard two-parameter solid rheologic model of the Maxwell type. Its uniaxial relaxation kernel is written as follows (Bland, 1960):

$$R^{(1)}(t) = E_1 \exp\left(-\frac{E_1 t}{\eta_1}\right) \tag{A.2}$$

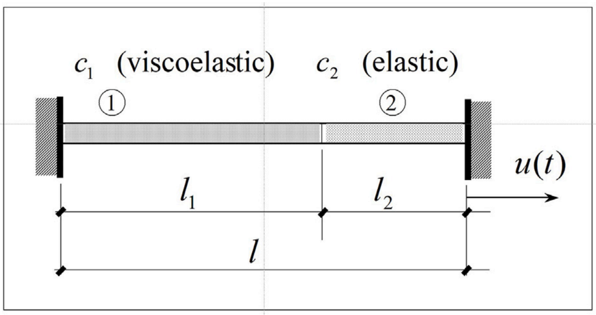

Fig. A.1. Uniaxial rod problem with elastic and viscoelastic phases arranged in series.

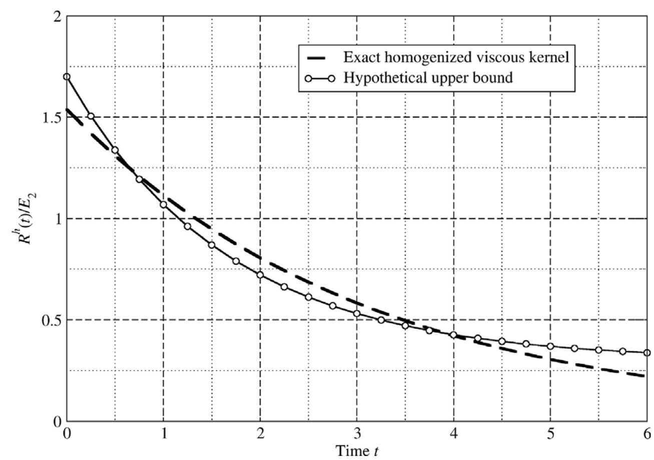

Fig. A.2. Normalized homogenized relaxation kernel $R^h(t)/E_2$ as a function of the time t (solid line); hypothetical upper bound of Eq. (A.4) (dot-dashed line).

Phase 2 of the composite is elastic, with volume fraction c_2 and Young modulus E_2. The numerical values used in this example are:

$c_1 = 0.7;\quad E_1 = 12000$ MPa; $\quad \eta_1 = 20000$ MPa s

$c_2 = 0.3;\quad E_2 = 6000$ MPa

The homogenized relaxation kernel for this example can be obtained in closed form using the Laplace transform $\mathcal{L}(\cdot)$ technique:

$$R^h(t) = \mathcal{L}^{-1}\left(\frac{1}{\frac{c_1}{\mathcal{L}(R^{(1)})} + \frac{c_2}{\mathcal{L}(E_2)}}\right) = \frac{E_1 E_2 \exp\left(-\frac{E_1 E_2 c_1 t}{E_1 \eta_1 c_2 + E_2 \eta_1 c_1}\right)}{E_1 c_2 + E_2 c_1} \tag{A.3}$$

The bounding term in the upper bound of Eq. (5.7), for this case, becomes:

$$\sum_{i=1}^{2} c_i R^{(i)}(t) = E_1 \exp\left(-\frac{E_1 t}{\eta_1}\right) c_1 + E_2 c_2 \tag{A.4}$$

The curves of Fig. A.2 plot the exact homogenized relaxation function of Eq. (A.3) superimposed to the curve of Eq. (A.4), both normalized by the elastic Young modulus E_2. It is apparent that the result of Eq. (5.7) is not valid for this example. This counterexample confirms that, in general, the contribution of the term in Eq. (4.4) to the functional (2.23) cannot be neglected. The same holds for most of the other bounds presented in Section 5 whose validity, therefore, must be restricted to the deviatoric loading case only. Other counterexamples were found numerically for 2D and 3D RVEs, which confirm the generality of this conclusion.

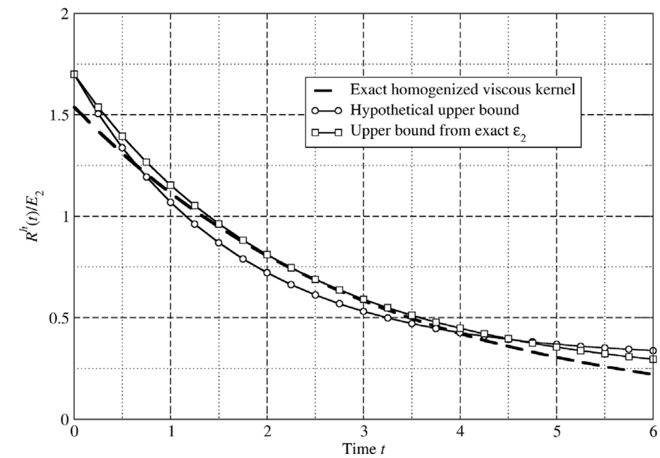

Fig. A.3. Normalized homogenized relaxation kernel $R^h(t)/E_2$ as a function of the time t. Comparison among the homogenized value, the invalid bounding expression (A.4), and the valid bounding expression of Eqs. (A.9) and (A.10).

On the other hand, it may be interesting to verify, for this simple rod example, that if one has available the exact solution for ε_{2ij} and inserts it into functional (2.23) one obtains a minimum principle in ε_{1ij}. In this case an upper bound analogous to (5.7), with suitable modifications, returns to be valid, as shown by the following calculations.

The functional TPE of Eq. (2.23), particularized to this example, reads as follows:

$$\begin{aligned}
\text{TPE} =& \frac{l_2}{2}\left[E_2 \varepsilon'^2_{1,2}(0) + 2E_2 \varepsilon'_{1,2}(0) \int_0^T \frac{\partial \varepsilon'_{1,2}}{\partial t} dt \right.\\
&\left. + E_2 \int_0^T \frac{\partial \varepsilon'_{1,2}}{\partial \tau} d\tau \int_0^T \frac{\partial \varepsilon'_{1,2}}{\partial t} dt \right] + \\
&+ \frac{l_1}{2}\left[R^{(1)}(2T)\varepsilon'^2_{1,1}(0) + 2\varepsilon'_{1,1}(0) \int_0^T R^{(1)}(2T-t) \frac{\partial \varepsilon'_{1,1}}{\partial t} dt + \right.\\
&\left. + \int_0^T \int_0^T R^{(1)}(2T-t-\tau) \frac{\partial \varepsilon'_{1,1}}{\partial \tau} \frac{\partial \varepsilon'_{1,1}}{\partial t} d\tau dt \right] + \\
&+ l_2 \left[E_2 \varepsilon'_{1,2}(0) \int_T^{2T} \frac{\partial \varepsilon'_{2,2}}{\partial t} dt + E_2 \int_T^{2T} \int_0^{2T-t} \frac{\partial \varepsilon'_{1,2}}{\partial \tau} \frac{\partial \varepsilon'_{2,2}}{\partial t} d\tau dt \right] + \\
&+ l_1 \left[\varepsilon'_{1,1}(0) \int_T^{2T} R^{(1)}(2T-t) \frac{\partial \varepsilon'_{2,1}}{\partial t} dt \right.\\
&\left. + \int_T^{2T} \int_0^{2T-t} R^{(1)}(2T-t-\tau) \frac{\partial \varepsilon'_{1,1}}{\partial \tau} \frac{\partial \varepsilon'_{2,1}}{\partial t} d\tau dt \right]
\end{aligned} \tag{A.5}$$

Now we adopt for the admissible strain ε'_1, in the first time subinterval and for both the rod phases, the following expression:

$$\varepsilon'_{1,i} = \bar{\varepsilon}\, \mathcal{H}(t) \qquad i = 1, 2 \tag{A.6}$$

with

$$\bar{\varepsilon} = \frac{\bar{u}}{l} \tag{A.7}$$

recalling that $\bar{u}$ is the given value of the right extremity displacement of the rod. In the second time subinterval we choose the exact solution of the problem, that can be computed using the Laplace transform technique. Denoting by u_M the exact viscoelastic displacement at the boundary between the two rod phases, one then has:

$$\varepsilon_{2,1} = \frac{u_M}{l_1}; \qquad \varepsilon_{2,2} = \frac{u(t) - u_M}{l_2} \tag{A.8}$$

Inserting these selections into the functional (A.5) it is possible to express the result as

$$\mathrm{TPE} = \frac{1}{2} l \, \bar{\varepsilon}^2 R^{\mathrm{TPE}}(2T) \tag{A.9}$$

where $R^{\mathrm{TPE}}(2T)$ denotes the factor of the constant term $\frac{1}{2}\bar{\varepsilon}^2 l$ at the end of the calculations. Fig. A.3 finally shows that, for this example, the following holds:

$$\frac{1}{2} R^h(2T)\bar{\varepsilon}^2 \le \frac{1}{2} R^{\mathrm{TPE}}(2T)\bar{\varepsilon}^2 \quad \Rightarrow \quad R^h(2T) \le R^{\mathrm{TPE}}(2T) \tag{A.10}$$

Finally, it is easy to check that for this example the general lower bound (5.36), given also in Huet (1995), remains valid, as confirmed also by numerical tests on 2D and 3D Finite Element RVEs loaded in a complete way.

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