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Introduction to electromagnetic scattering: tutorial

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In this paper, an introduction to electromagnetic scattering is presented. We introduce the basic concepts needed to face a scattering problem, including the scattering, absorption, and extinction cross sections. We define the vector harmonics and we present some of their properties. Finally, we tackle the two canonical problems of the scattering by an infinitely long circular cylinder, and by a sphere, showing that the introduction of the vector wave function makes the imposition and solution of the boundary conditions particularly simple.

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1. INTRODUCTION

The aim of the present tutorial is to give an introduction to electromagnetic scattering. Electromagnetic scattering is a wide topic that has been researched for more than a century. We split the tutorial into two parts: in this first part, we present the mathematical background needed to face the scattering problems and we give the solution for two canonical problems: the scattering by an infinite circular cylinder and by a sphere; in the second part, we will present more general scattering problems and some applications. In particular, in the present part of the tutorial we use the boundary conditions method in order to solve the electromagnetic scattering problem using vector harmonics.

The historic origin of the scattering problem is connected to the diffraction by either circular cylinders or spheres, which are classical problems in both electromagnetism and acoustics. According to [1], the scattering by a dielectric circular cylinder was first solved by Rayleigh [2], while the scattering by a dielectric sphere was first solved by Lorenz [3]. On the other hand, the diffraction of waves by conducting bodies, both circular cylinder and sphere, was presented by Thomson [4]. Nevertheless, the theoretical solution of the scattering problem by a sphere is widely known as Mie scattering, because of the well-known solution published by Mie [5]. These canonical scattering problems have been solved, in later years, with a much simpler and more compact mathematical formulation, and they are part of the topics presented by several textbooks, for example, [1,6–10].

The formulation of a scattering problem requires two elements: the incident radiation and the object with which the radiation interacts, called the scatterer. To solve the electromagnetic problem, the boundary conditions on the scatterer's surface can be imposed. They require the continuity of the tangential components of the electric and magnetic fields on the object's surface, having assumed the surface currents to be zero. For an object of arbitrary shape, closed-form solutions of the fields are not available, and the fields are usually represented in an integral form [10]. On the other hand, for some of the simplest shapes, it is possible to compute analytical expressions for both the scattered and the internal fields. The theoretical formulation in such cases involves the decomposition of the total field in a superposition of different contributions. The first contribution is given by the incident field, defined in each point of the space as the field that there would be if the obstacle was not there. It is usually a plane wave, but it can also be a more complicated wave, such as a cylindrical wave or a Gaussian beam. The second contribution is given by the scattered field, with the value of the scattered field in each external point to the obstacle being of particular interest. Finally, the internal field to the obstacle must be considered. These fields are represented in series expansions of suitable vector harmonics.

The crucial mathematical tools that allow one to simplify the solution of any scattering problems are the vector harmonics (or vector wave functions) [6–10]. These vector functions are solutions of the vector wave equation, and any electromagnetic field or potential can be expanded in series of these harmonics with suitable coefficients. The particular properties of the vector harmonics allow one to easily compute the different fields and, perhaps more important, to easily impose the boundary conditions. As we will see later, using vector harmonics we can express the electromagnetic fields in the same form independently of the scatterer shape. Therefore, it will be possible to couple the spherical wave with the cylindrical one [11].

This theoretical approach allows us to solve the scattering problem in several scenarios. In addition to the scattering by a cylinder or a sphere in free space, several analytical solutions for more complicated problems are presented in the literature. An important classical example is the diffraction by planar surfaces or by apertures on a screen [9,12]. Also interesting are the canonical scattering problems by other simple geometries in free space, such as cylindrical wedges, elliptic cylinders, conical structures, or spheroids [9,13]. Another widely studied case is the scattering by objects placed next to planar interfaces between two or more media, i.e., stratified materials. Several solutions are available in the literature, for both the circular cylinder and the sphere, in the case of both lossless and lossy materials, or for rough surfaces [14–20]. The presence of the planar interface is taken into account with the plane-wave spectrum of the vector wave functions, available for several geometries [21–23]. Another important improvement is given by considering an alignment of objects with the same geometry, i.e., the scattering by a certain number of cylinders or spheres with different radii in an arbitrary arrangement [24–26]. Contrarily, the scattering by several objects with different geometries is not so widely addressed in the literature, at least with closed-form solutions. Recently, a closed-form solution for the scattering by a dielectric sphere embedded in a circular cylinder has been proposed [11]. Finally, the scattering by cylinders and spheres with anisotropic permittivity or permeability has been tacked on, too [27–32].

In Section 2, we introduce the concepts of extinction power and of the scattering, absorption, and extinction cross sections: quantities that are extremely important for the characterization of a scatterer and for the comparison of the behaviors of different scatterers. In Section 3, we introduce the concept of vector harmonics, and derive their principal properties. In Section 4, we derive the expressions of the cylindrical vector harmonics, and apply them to the solution of the scattering of an elliptically polarized plane wave obliquely incident on an infinitely long circular cylinder. In Section 5, we compute the expressions of the vector spherical harmonics, and apply them to the solution of the scattering of an elliptically polarized plane wave by a sphere. Finally, in Section 6, the conclusions are drawn.

2. SCATTERING, EXTINCTION, AND ABSORPTION CROSS SECTIONS

An important concept in scattering problems is the *cross section* [8,23]. This term indicates a quantity with the dimensions of an area related to the electromagnetic power scattered or absorbed by the object interacting with the incident wave, i.e., the scatterer. Let us call $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$ the electric and magnetic fields of the incident wave, $\boldsymbol{E}_s$ and $\boldsymbol{H}_s$ the electric and magnetic fields of the scattered wave, and $\boldsymbol{E} = \boldsymbol{E}_i + \boldsymbol{E}_s$ and $\boldsymbol{H} = \boldsymbol{H}_i + \boldsymbol{H}_s$ the total electric and magnetic fields outside the scatterer, where

$$\boldsymbol{E}_i = \boldsymbol{E}_0 e^{(i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t)}, \tag{1}$$

$$\boldsymbol{H}_i = \boldsymbol{H}_0 e^{(i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t)}, \tag{2}$$

$\boldsymbol{k}$ is the wave vector appropriate to the surrounding medium, and $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are the electric and magnetic polarization vectors, respectively. Furthermore, as will be seen below, the form of the scattered fields will depend on the scatterer shape. A time dependence $\exp(-i\omega t)$ will be omitted throughout the paper.

We consider the Poynting vector of the total field:

$$\boldsymbol{S} = \frac{1}{2}\,\mathrm{Re}[\boldsymbol{E} \times \boldsymbol{H}^*]. \tag{3}$$

It can be written as a function of the fields of the incident and scattered waves:

$$\boldsymbol{S} = \frac{1}{2}\,\mathrm{Re}[\boldsymbol{E}_i \times \boldsymbol{H}_i^*] + \frac{1}{2}\,\mathrm{Re}[\boldsymbol{E}_s \times \boldsymbol{H}_s^*] + \frac{1}{2}\,\mathrm{Re}[\boldsymbol{E}_i \times \boldsymbol{H}_s^* + \boldsymbol{E}_s \times \boldsymbol{H}_i^*] = \boldsymbol{S}_i + \boldsymbol{S}_s + \boldsymbol{S}_e, \tag{4}$$

where $\boldsymbol{S}_i$ is the Poynting vector of the incident wave, $\boldsymbol{S}_s$ is the Poynting vector of the scattered wave, and $\boldsymbol{S}_e$ is the term that arises from the interaction between the incident and the scattered fields. Another wave that we should consider is the one inside the scatterer; let $\boldsymbol{E}_p$ and $\boldsymbol{H}_p$ be its electric and magnetic fields. Then, if we consider the scatterer's surface S_s, the amount of energy absorbed by the object can be computed as

$$W_a = -\int_{S_s} \hat{\boldsymbol{n}} \cdot \boldsymbol{S}\,\mathrm{d}S, \tag{5}$$

where $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the surface. The equality is due to the continuity of the tangential components of the electric and magnetic fields on the scatterer's surface. It can be proved, applying the Green second vector theorem, that the previous integral on the total field is equal to the integral extended to a spherical surface, S, centered on the scatterer and with an arbitrarily large radius [23]. As a consequence, we can obtain an expression of the amount of the electromagnetic energy absorbed by the scatterer as the following integral:

$$W_a = -\int_{S} \hat{\boldsymbol{n}} \cdot \boldsymbol{S}\,\mathrm{d}S = W_i - W_s + W_e, \tag{6}$$

where

$$W_i = -\int_{S} \hat{\boldsymbol{n}} \cdot \boldsymbol{S}_i\,\mathrm{d}S, \quad W_s = \int_{S} \hat{\boldsymbol{n}} \cdot \boldsymbol{S}_s\,\mathrm{d}S, \quad W_e = -\int_{S} \hat{\boldsymbol{n}} \cdot \boldsymbol{S}_e\,\mathrm{d}S. \tag{7}$$

The power associated with the incident wave is zero: in fact, all the incident power is both incoming on and outgoing from the surface; then W_i vanishes identically for a non-absorbing medium (for simplicity). Hence,

$$W_e = W_a + W_s. \tag{8}$$

At this point, we can give a physical interpretation to the term W_e. It is the sum of the energy absorbed by the scatterer and of the scattered energy, i.e., it is the amount of energy subtracted from the incident wave because of the interaction with the scatterer. For this reason, W_e is called extinction power.

Now, we can introduce the concept of the cross section. The scattering, absorption, and extinction cross sections are defined as follows:

$$\sigma_s = \frac{W_s}{|\mathbf{S}_i|} \qquad \sigma_a = \frac{W_a}{|\mathbf{S}_i|} \qquad \sigma_e = \frac{W_e}{|\mathbf{S}_i|}. \tag{9}$$

From such definitions it is clear that the dimension of the cross section is that of an area. Moreover, the relation between the cross sections is the same as the one between the powers:

$$\sigma_e = \sigma_a + \sigma_s. \tag{10}$$

The scattering cross section represents the amount of power scattered by the object over the amount of power per unit area carried by the incident wave. Similarly, the absorption cross section represents the amount of power absorbed by the scatterer over the amount of power per unit area carried by the incident wave. Finally, the extinction cross section represents the amount of overall power subtracted from the incident wave over the amount of power per unit area carried by the incident wave.

Finally, we want to introduce the so-called efficiencies for the scattering, absorption, and extinction:

$$Q_s = \frac{\sigma_s}{G} \qquad Q_a = \frac{\sigma_a}{G} \qquad Q_e = \frac{\sigma_e}{G}, \tag{11}$$

where G is the particle cross-sectional area projected onto a plane perpendicular to the incident wave. Essentially, the efficiencies normalize the cross sections with respect to the area viewed by the incident field.

3. VECTOR WAVE FUNCTIONS FORMALISM

In this section, we present a procedure to solve the vector wave equation. This procedure was first proposed by Hansen [6,33].

He showed that starting from Maxwell's equations,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{12}$$

$$\nabla \times \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{13}$$

it was possible to describe any field or electromagnetic potential through the Helmholtz equation. Indeed, if we consider the generic field $\boldsymbol{C}$, equal to either the electric field, magnetic field, electric flux density, magnetic flux density, electric potential, magnetic potential, or electric and magnetic Hertz–Debye potentials, i.e., $\mathbf{C} = \{\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{F}, \mathbf{A}, \Pi_e, \Pi_m\}$ [6], then such a field, in the absence of sources and in a linear, isotropic, homogeneous medium, must satisfy the general homogeneous wave equation in the time domain:

$$\nabla^2 \mathbf{C} - \mu\varepsilon \frac{\partial^2 \mathbf{C}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{C}}{\partial t} = 0, \tag{14}$$

where ε is the electric permittivity, μ is the magnetic permeability, and σ is the conductivity of the considered medium. The equation can also be expressed in the frequency domain, i.e., the Helmholtz equation:

$$\nabla^2 \mathbf{C} + k^2 \mathbf{C} = 0, \tag{15}$$

where it is placed $k^2 = \omega^2 \varepsilon_c \mu + i\sigma\omega$, remembering that a time dependence $\exp(-i\omega t)$ is omitted here and throughout the paper. This vector differential equation can be projected along the unit vectors, $\hat{\mathbf{q}}_i$, $i = 1, 2, 3$, of a generic reference system, becoming a system of three scalar differential equations. However, such a system is not easy to solve in most of the coordinate systems. Therefore, it can be very useful to get hold of a general procedure in order to solve the vector equation.

While the solution of the vector Helmholtz equation is not a simple task in several coordinate systems, it is easy to solve the scalar Helmholtz equation:

$$\nabla^2 \psi + k^2 \psi = 0. \tag{16}$$

This simple scalar differential equation has solutions in most canonical coordinate systems. There are 11 coordinate systems in which the scalar solution to the Helmholtz equation is known [13,34]: orthogonal Cartesian, circular cylindrical, elliptic cylindrical, parabolic cylindrical, spherical, prolate spheroidal, oblate spheroidal, conical, ellipsoidal, paraboloidal, and parabolic rotation.

At this point we define the vector harmonics as follows:

$$\mathbf{L} = \nabla\psi; \qquad \mathbf{M} = \nabla \times (\hat{\mathbf{a}}\psi); \qquad \mathbf{N} = \frac{1}{k} \nabla \times \mathbf{M}, \tag{17}$$

where $\hat{\mathbf{a}}$ is a typically constant unit vector, sometimes called the pilot vector [6,8] (in general, it must be $\nabla \times \hat{\mathbf{a}} = 0$). It is easy to demonstrate that these vectors satisfy the vector Helmholtz equation. In fact, taking into account the previous definition $\mathbf{M} = \nabla \times (\hat{\mathbf{a}}\psi)$, that the divergence of the curl of any vector function vanishes, $\nabla \cdot \mathbf{M} = 0$, and considering the use of the vector identities, we obtain

$$\nabla^2 \mathbf{M} + k^2 \mathbf{M} = \nabla \times [\hat{\mathbf{a}}(\nabla^2 \psi + k^2 \psi)]. \tag{18}$$

Then, $\mathbf{M}$ satisfies the vector wave equation if ψ is a solution to the scalar wave equation: $\nabla^2 \psi + k^2 \psi = 0$. Similarly, the other vectors, $\mathbf{N}$ and $\mathbf{L}$, can be considered. Thus, the vectors $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ are suitable to represent the generic electromagnetic field in a given coordinate system. These vectors have other valuable properties [6,8], for example,

$$\mathbf{M} = \frac{1}{k} \nabla \times \mathbf{N}; \qquad \nabla \cdot \mathbf{L} = \nabla^2 \psi = -k^2 \psi. \tag{19}$$

Moreover, the following relations can easily be proved: $\nabla \times \mathbf{L} = 0$, $\nabla \cdot \mathbf{M} = 0$, and $\nabla \cdot \mathbf{N} = 0$; then, $\mathbf{L}$ is irrotational, while $\mathbf{M}$ and $\mathbf{N}$ are solenoidal. As a consequence, an electromagnetic field in the absence of sources can be expressed only by a superposition of $\mathbf{M}$ and $\mathbf{N}$, while $\mathbf{L}$ must necessarily be taken into account only when electromagnetic sources are considered. Finally, we note that the generic solution $\psi(\mathbf{r})$ of the Helmholtz equation is a set of solutions ψ_n (eigenvectors) that will form a vector space basis L^2 of square-summable functions (Hilbert space). A set of vectors $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$ is associated with this set of scalar solutions. If we consider the magnetic vector potential $\mathbf{A}$ to be a solution of the vector Helmholtz equation, it can be expressed as a linear combination of the set of vectors $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$:

$$\mathbf{A} = \frac{1}{i\omega}\sum_{n=0}^{+\infty}(a_n\mathbf{M}_n + b_n\mathbf{N}_n + c_n\mathbf{L}_n). \quad (20)$$

Recalling the relation between the magnetic potential and the electric field and applying the Maxwell equations and the properties of $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$, we can write the following expressions of the electric and magnetic fields:

$$\mathbf{E} = \sum_{n=0}^{+\infty}(a_n\mathbf{M}_n + b_n\mathbf{N}_n), \quad (21)$$

$$\mathbf{H} = \frac{k}{i\omega\mu}\sum_{n=0}^{+\infty}(a_n\mathbf{N}_n + b_n\mathbf{M}_n). \quad (22)$$

As we can see, the properties of $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$ make the calculation of the fields extremely simple. Finally, we want to show another important property of this formalism. In Cartesian rectangular coordinates, as is well known, the solution of the Helmholtz scalar equation can be written, assuming unitary amplitude, as follows:

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (23)$$

where $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$. Then, considering Eq. (17), **L**, **M**, and **N** can be written as follows [6,8,12]:

$$\mathbf{L} = i\psi\mathbf{k}; \qquad \mathbf{M} = i\psi\mathbf{k}\times\hat{\mathbf{a}}; \qquad \mathbf{N} = \frac{1}{k}\psi(\mathbf{k}\times\mathbf{a}_0)\times\mathbf{k}. \quad (24)$$

These expressions allow us to obtain the plane-wave expansion of a generic vector electromagnetic field. In fact, if we consider the plane-wave expansion of a generic solution of the wave equation [6,12],

$$\psi(\boldsymbol{r}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(k_x, k_y)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}k_x\mathrm{d}k_y = \iint g(\alpha,\beta)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\alpha\mathrm{d}\beta, \quad (25)$$

where α and β are the angles, in spherical coordinates, that the vector $\boldsymbol{k}$ of the elementary plane wave forms with the coordinate axes, and $g(\alpha,\beta)$ is the plane-wave angular spectrum of the function $\psi(\boldsymbol{r})$. In order to obtain the plane-wave expansion of the vectors **L**, **M**, and **N**, we have only to insert Eq. (25) into Eq. (24), obtaining the following expansions:

$$\mathbf{L} = i\iint g(\alpha,\beta)\boldsymbol{k}(\alpha,\beta)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\beta\mathrm{d}\alpha,$$

$$\mathbf{M} = i\iint g(\alpha,\beta)\boldsymbol{k}(\alpha,\beta)\times\hat{\boldsymbol{a}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\beta\mathrm{d}\alpha,$$

$$\mathbf{N} = \frac{1}{k}\iint g(\alpha,\beta)[\boldsymbol{k}(\alpha,\beta)\times\hat{\boldsymbol{a}}]\times\boldsymbol{k}(\alpha,\beta)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\beta\mathrm{d}\alpha. \quad (26)$$

As is well known, the plane-wave expansion is an important tool for the solution of many electromagnetic problems. However, the vectors **L**, **M**, and **N** allow us to obtain other kinds of transformation between different coordinate systems. As an example, in [35] the transformations between the vectors in circular cylindrical and in spherical coordinate systems are presented.

As we have seen, any solution of the vector Helmholtz equation can be written as a linear combination of the three vectors **L**, **M**, and **N**, once the solution of the scalar differential equation is known. Such vectors have several useful properties that make calculations involving the electromagnetic field and the imposition of boundary conditions very simple. In the following sections, we will apply these vector functions in the solution of two canonical scattering problems.

4. SCATTERING BY A CYLINDRICAL OBJECT: 2D CASE

In this section, we determine the scattered electromagnetic field by an infinitely long circular cylinder. The incident wave is an elliptically polarized plane wave at oblique incidence with respect to the cylinder's axis. The electromagnetic field will be expressed as a superposition of infinite vector cylindrical harmonics (or cylindrical vector wave functions), solutions of the vector Helmholtz equation in cylindrical coordinates. In the present formulation an infinite cylinder with circular cross section and axis coincident with the z-axis, with radius a, characterized by a relative dielectric constant equal to ε_2 (wavenumber k_2) is considered. The cylinder is immersed in a free space characterized by a relative permittivity ε_1 (wavenumber k_1); see Fig. 1. We consider the medium 1 homogeneous and isotropic, without any source.

The first step is to determine the scalar function ψ, which is the solution of the scalar Helmholtz equation in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + (k^2 - k_z^2)\psi = 0. \quad (27)$$

The solution of the differential equation can easily be obtained applying the method of separation of variables, giving [6]

$$\psi_m(\rho,\varphi,z) = Ae^{im\varphi}Z_m(k_\rho\rho)e^{ik_z z}, \quad (28)$$

where A is a complex constant, $k_\rho^2 + k_z^2 = k^2$, with $k_\rho^2 = k_x^2 + k_y^2$, and k_z^2 are the squared transverse and longitudinal components of the wave vector, respectively. The function $Z_m(k_\rho\rho)$ represents the generic Bessel function of the first, second, third, or fourth kind, i.e., $Z_m(k_\rho\rho) = \{J_m(k_\rho\rho), Y_m(k_\rho\rho), H_m^{(1)}(k_\rho\rho), H_m^{(2)}(k_\rho\rho)\}$; in particular, the two last functions are also known as the Hankel function of the first and second kind, respectively [36]. Inserting Eq. (28) into

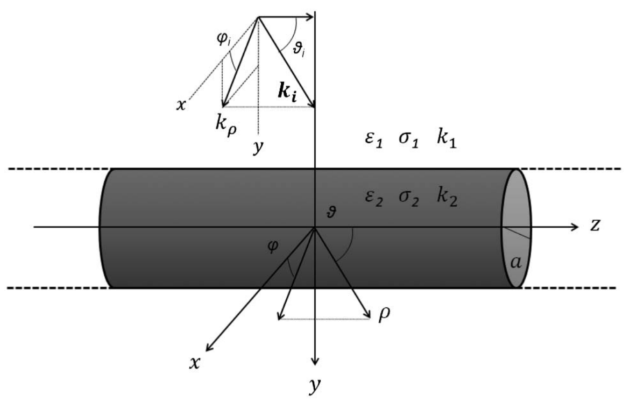

Fig. 1. Representation of an obliquely incident plane wave on a dielectric infinite circular cylinder.

Eq. (24), and taking as constant unit vector $\hat{\boldsymbol{a}} = \hat{\mathbf{z}}$, the following expressions for the vector cylindrical harmonics can be obtained:

$$\mathbf{L}_m(\mathbf{r}) = \mathbf{l}_m(k_\rho \rho) e^{im\varphi} e^{ik_z z}, \tag{29}$$

$$\mathbf{M}_m(\mathbf{r}) = \mathbf{m}_m(k_\rho \rho) e^{im\varphi} e^{ik_z z}, \tag{30}$$

$$\mathbf{N}_m(\mathbf{r}) = \mathbf{n}_m(k_\rho \rho) e^{im\varphi} e^{ik_z z}, \tag{31}$$

where the radial-dependent vectors can be written as follows:

$$\mathbf{l}_m(k_\rho \rho) = \frac{\partial Z_m(k_\rho \rho)}{\partial \rho} \hat{\boldsymbol{\rho}} \mp i \frac{m}{\rho} Z_m(k_\rho \rho) \hat{\boldsymbol{\varphi}} + i k_z Z_m(k_\rho \rho) \hat{\mathbf{z}}, \tag{32}$$

$$\mathbf{m}_m(k_\rho \rho) = im \frac{Z_m(k_\rho \rho)}{\rho} \hat{\boldsymbol{\rho}} - k_\rho \frac{\partial Z_m(k_\rho \rho)}{\partial \rho} \hat{\boldsymbol{\varphi}}, \tag{33}$$

$$\mathbf{n}_m(k_\rho \rho) = i \frac{k_z k_\rho}{k} \frac{\partial Z_m(k_\rho \rho)}{\partial \rho} \hat{\boldsymbol{\rho}} - \frac{m k_z}{k} \frac{Z_m(k_\rho \rho)}{\rho} \hat{\boldsymbol{\varphi}} + \frac{k_\rho^2}{k} Z_m(k_\rho \rho) \hat{\mathbf{z}}. \tag{34}$$

It is convenient to write the radial-dependent vectors as a function of their three components:

$$\mathbf{l}_m(k_\rho \rho) = l_\rho(k_\rho \rho) \hat{\boldsymbol{\rho}} + l_\varphi(k_\rho \rho) \hat{\boldsymbol{\varphi}} + l_z(k_\rho \rho) \hat{\mathbf{z}}, \tag{35}$$

$$\mathbf{m}_m(k_\rho \rho) = m_\rho(k_\rho \rho) \hat{\boldsymbol{\rho}} + m_\varphi(k_\rho \rho) \hat{\boldsymbol{\varphi}}, \tag{36}$$

$$\mathbf{n}_m(k_\rho \rho) = n_\rho(k_\rho \rho) \hat{\boldsymbol{\rho}} + n_\varphi(k_\rho \rho) \hat{\boldsymbol{\varphi}} + n_z(k_\rho \rho) \hat{\mathbf{z}}. \tag{37}$$

An elliptically polarized plane wave, incident obliquely to the cylinder's axis, can be represented as a linear combination of two components, with respect to the surface of the cylinder: a vertical one and a horizontal one [10]. Hence, the incident electric field can be written as follows:

$$\mathbf{E}_i(\mathbf{r}) = [E_{vi} \hat{\mathbf{v}}(\vartheta_i, \varphi_i) + E_{hi} \hat{\mathbf{h}}(\vartheta_i, \varphi_i)] e^{i\mathbf{k}_1 \cdot \mathbf{r}}, \tag{38}$$

where ϑ_i is the angle between the z-axis and the wave vector $\mathbf{k}_1$, φ_i is the angle between the x-axis and the projection of vector $\mathbf{k}_1$ on the (x, y) plane, and $\hat{\mathbf{h}}$ and $\hat{\mathbf{v}}$ are two unit vectors defined as $\hat{\mathbf{h}} = \hat{\mathbf{z}} \times \hat{\mathbf{k}}_{1\rho}$ and $\hat{\mathbf{v}} = \hat{\mathbf{k}}_1 \times \hat{\mathbf{h}}$. Here, we are indicating with $\hat{\mathbf{k}}_1$ the normalized wave vector with respect to the wavenumber, $\hat{\mathbf{k}}_1 = \mathbf{k}_1 / k_1$, and with $\hat{\mathbf{k}}_{1\rho}$ the radial unit vector defined as $\hat{\mathbf{k}}_{1\rho} = \cos\varphi_i \hat{\mathbf{x}} + \sin\varphi_i \hat{\mathbf{y}}$. Finally, the quantities E_{vi} and E_{hi} are complex constants. As we have seen in Section 3, the electric field Eq. (38) can be expressed as a superposition of cylindrical vector harmonics [37,38]:

$$\mathbf{E}_i(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} [a_m \mathbf{M}^{(1)}(\mathbf{r}) + b_m \mathbf{N}^{(1)}(\mathbf{r})], \tag{39}$$

where

$$a_m = \frac{E_{hi}}{k_{1\rho}} i^{m+1} e^{-im\varphi_i} \qquad b_m = -\frac{E_{vi}}{k_{1\rho}} i^m e^{-im\varphi_i}, \tag{40}$$

$$k_{1z} = k_1 \cos\vartheta_i \qquad k_{1\rho} = k_1 \sin\vartheta_i. \tag{41}$$

It is important to emphasize the particular type of harmonic that has been chosen. In particular, for the incident field, we have chosen a radial dependence as a Bessel function of the first kind. This choice can be explained in different manners. First of all, we can note that the incident field is the field that there would be if the obstacle was not there; then it must exist in the origin, and the Bessel function of the first kind is the only one that is continuous when ρ approaches zero. From another perspective, an analogy can be drawn between the four Bessel functions and with the sine and cosine functions. Looking at their behavior for large values of the argument, the functions of the first and the second kind are analogous to a cosine and a sine function, respectively, while the functions of the third and the fourth kind are analogous to imaginary exponential functions, progressive and regressive, respectively. As a consequence, it is natural to associate a plane wave in free space to a Bessel function of the first kind.

The scattered field ($\mathbf{E}_{sc}$) and the field inside the cylinder ($\mathbf{E}_{cy}$) can be expanded in cylindrical vector wave functions, as well, obtaining

$$\mathbf{E}_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} [c_m \mathbf{M}_m^{(3)}(\mathbf{r}) + d_m \mathbf{N}_m^{(3)}(\mathbf{r})], \tag{42}$$

$$\mathbf{E}_{cy}(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} [e_m \mathbf{M}_m^{(1)}(\mathbf{r}) + f_m \mathbf{N}_m^{(1)}(\mathbf{r})]. \tag{43}$$

Again, the internal field includes the Bessel function of the first kind, because it must be continuous in the origin, while the scattered field includes the Bessel function of the third kind because being the field emerging from the cylinder, it is natural to consider it as a progressive wave, instead of a regressive one. The fields (42) and (43) are unknown. The expansion in vector harmonics makes the only unknowns of the problem the coefficients of the expansions: c_m, d_m, e_m, and f_m.

To solve the scattering problem, we can impose the boundary conditions on the cylinder's surface determining the unknowns of the problem. The boundary conditions consist in the continuity of the tangential components of the electric and magnetic fields on the cylinder's surface, and they can be expressed as follows:

$$(\mathbf{E}_i + \mathbf{E}_{sc} - \mathbf{E}_{cy}) \times \hat{\boldsymbol{\rho}} = 0 \quad \text{for } \rho = a, \tag{44}$$

$$[\nabla \times (\mathbf{E}_i + \mathbf{E}_{sc} - \mathbf{E}_{cy})] \times \hat{\boldsymbol{\rho}} = 0 \quad \text{for } \rho = a. \tag{45}$$

From Eqs. (33) and (34), considering Eqs. (32)–(37) and Eqs. (17) and (19), the following equalities can be proved:

$$\mathbf{m}_m(k_\rho \rho) \times \hat{\boldsymbol{\rho}} = m_{\varphi m}(k_\rho \rho) \hat{\mathbf{z}}, \tag{46}$$

$$\mathbf{n}_m(k_\rho \rho) \times \hat{\boldsymbol{\rho}} = n_{\varphi m}(k_\rho \rho) \hat{\mathbf{z}} - n_{zm}(k_\rho \rho) \hat{\boldsymbol{\varphi}}, \tag{47}$$

$$[\nabla \times \mathbf{m}_m(k_\rho \rho)] \times \hat{\boldsymbol{\rho}} = k n_{\varphi m}(k_\rho \rho) \hat{\mathbf{z}} - k n_{zm}(k_\rho \rho) \hat{\boldsymbol{\varphi}}, \tag{48}$$

$$[\nabla \times \mathbf{n}_m(k_\rho \rho)] \times \hat{\boldsymbol{\rho}} = -k m_{\varphi m}(k_\rho \rho) \hat{\mathbf{z}}. \tag{49}$$

Inserting these expressions into Eqs. (44) and (45), by exploiting the orthogonality properties of the exponential functions, and by noting that the z dependence is common to all the factors, and hence it can be elided, we obtain, for each value of the index m, the following four scalar equations:

$$\begin{cases} c_m m_{\varphi m}^{(3)}(k_{1\rho}a) + d_m n_{\varphi m}^{(3)}(k_{1\rho}a) - e_m m_{\varphi m}^{(1)}(k_{2\rho}a) - \\ f_m n_{\varphi m}^{(1)}(k_{2\rho}a) = -a_m m_{\varphi m}^{(1)}(k_{1\rho}a) - b_m n_{\varphi m}^{(1)}(k_{1\rho}a) \\ c_m k_1 n_{\varphi m}^{(3)}(k_{1\rho}a) - d_m k_1 m_{\varphi m}^{(3)}(k_{1\rho}a) - e_m k_2 n_{\varphi m}^{(1)}(k_{2\rho}a) + \\ f_m k_2 m_{\varphi m}^{(1)}(k_{2\rho}a) = -a_m k_1 n_{\varphi m}^{(1)}(k_{1\rho}a) + b_m k_1 m_{\varphi m}^{(1)}(k_{1\rho}a) \\ d_m n_{zm}^{(3)}(k_{1\rho}a) - f_m n_{zm}^{(1)}(k_{2\rho}a) = -b_m n_{zm}^{(1)}(k_{1\rho}a) \\ c_m k_1 m_{zm}^{(3)}(k_{1\rho}a) - e_m k_2 m_{zm}^{(1)}(k_{2\rho}a) = -a_m k_1 m_{zm}^{(1)}(k_{1\rho}a). \end{cases}$$

The solution of this linear system of equations gives the coefficients of the expansions (42) and (43). We avoid giving here the explicit expressions of the coefficients because they cannot be written in a compact form, but they can be found in many textbooks, such as [1].

A compact expression of the coefficients can be obtained in the simple case of a perfect electric conductor (PEC) cylinder. In this particular situation, the internal field to the cylinder is not present, and the boundary conditions reduce to

$$(\mathbf{E}_i + \mathbf{E}_{sc}) \times \hat{\boldsymbol{\rho}} = 0 \quad \text{for } \rho = a. \tag{50}$$

In this case, solving the linear system, simple expressions of the two unknowns can be obtained:

$$\begin{cases} c_m = -a_m \frac{m_{\varphi m}^{(1)}(k_{1\rho}a)}{m_{\varphi m}^{(3)}(k_{1\rho}a)} - b_m \left[\frac{n_{\varphi m}^{(1)}(k_{1\rho}a)}{m_{\varphi m}^{(3)}(k_{1\rho}a)} - \frac{n_{zm}^{(1)}(k_{1\rho}a) n_{\varphi m}^{(3)}(k_{1\rho}a)}{n_{zm}^{(3)}(k_{1\rho}a) m_{\varphi m}^{(3)}(k_{1\rho}a)} \right] \\ d_m = -b_m \frac{n_{zm}^{(1)}(k_{1\rho}a)}{n_{zm}^{(3)}(k_{1\rho}a)}. \end{cases} \tag{51}$$

If now we consider the simplest case of normal incidence ($\vartheta_i = \pi/2, \varphi_i = 0$), we see, from Eqs. (34) and (41), that $n_\varphi = 0$; hence,

$$\begin{cases} c_m = -a_m \frac{M_{\varphi m}^{(1)}(k_{1\rho}a)}{M_{\varphi m}^{(3)}(k_{1\rho}a)} \\ d_m = -b_m \frac{N_{zm}^{(1)}(k_{1\rho}a)}{N_{zm}^{(3)}(k_{1\rho}a)}. \end{cases} \tag{52}$$

Making explicit the dependence on the Bessel functions, the following well-known expressions can be obtained:

$$\begin{cases} c_m = -a_m \frac{\dot{J}_m(k_{1\rho}a)}{\dot{H}_m^{(1)}(k_{1\rho}a)} \\ d_m = -b_m \frac{J_m(k_{1\rho}a)}{H_m^{(1)}(k_{1\rho}a)}. \end{cases} \tag{53}$$

These are the scattering coefficients in the case of normal incidence on a PEC cylinder, very well known in the literature [39]. In particular, we consider, instead of an elliptically polarized wave, two linearly polarized waves, one in H (TE) polarization, and the other one in E (TM) polarization, i.e., with either the magnetic or the electric field directed along the cylinder's axis. In the former case, $b_m = 0$, and the only non-zero scattering coefficient is c_m, while in the latter case, $a_m = 0$, and the only non-zero scattering coefficient is d_m.

Now, we can compute the cross sections of a cylinder as a function of the scattering coefficients. Because of the two-dimensional nature of the cylinder, we must change the definition of the cross section, computing it not on a sphere of arbitrary radius, but on a circumference, on a plane perpendicular to the cylinder's axis, of arbitrary radius. For this reason, this quantity is often called scattering width instead of cross section [40]. The Poynting vectors for the incident and scattered waves can be computed and integrated from their analytical expressions. We omit the analytical procedure here for the sake of brevity, but, thanks to an important property of the Bessel functions reported in Appendix A, the scattering cross section takes the following simple form [41]:

$$C_s = \frac{4\pi}{k_1} \sum_{m=1}^{+\infty} \left(\left| \frac{c_m}{a_m} \right|^2 + \left| \frac{d_m}{b_m} \right|^2 \right). \tag{54}$$

Similarly, the extinction cross section is

$$C_e = \frac{4\pi}{k_1} \sum_{m=1}^{+\infty} \mathrm{Re} \left[\left| \frac{c_m}{a_m} \right|^2 + \left| \frac{d_m}{b_m} \right|^2 \right]. \tag{55}$$

In Fig. 2, the scattering cross section for a PEC cylinder with radius 0.25 m is shown; the plane wave, at normal incidence, is linearly TM-polarized, in the range of frequencies from 0.1 MHz to 8.0 GHz. The cross section is computed both

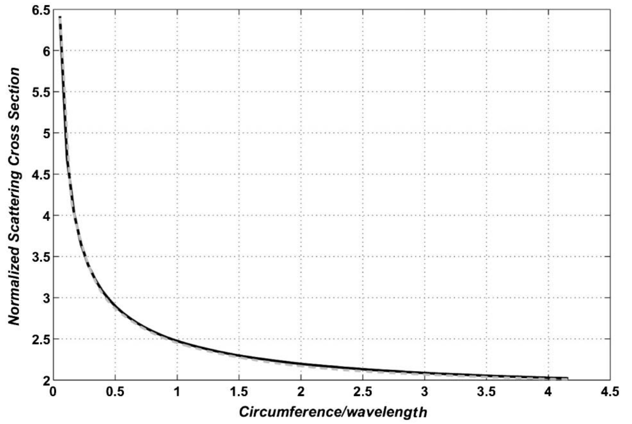

Fig. 2. Scattering cross section of a PEC cylinder with radius 0.25 m, when the plane wave at normal incidence is linearly TM-polarized in the range of frequencies from 0.1 MHz to 8.0 GHz. The cross section is computed (solid line) implementing on MATLAB the formula (54) and (dashed line) simulating the scattering problem on a software based on the finite-element method.

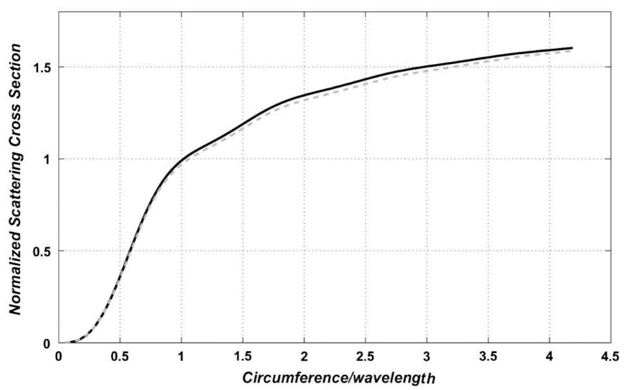

Fig. 3. Scattering cross section of a PEC cylinder with radius 0.25 m, when the plane wave at normal incidence is linearly TE-polarized in the range of frequencies from 0.1 MHz to 8.0 GHz. The cross section is computed (solid line) implementing on MATLAB the formula (54) and (dashed line) simulating the scattering problem on a software based on the finite-element method.

implementing Eq. (54) on MATLAB and computing the scattering scenario on a software based on the finite-element method. In Fig. 3, the scattering cross section is shown, in the same scenario of Fig. 2, but with a TE-polarized incident wave.

5. SCATTERING BY A SPHERICAL OBJECT: 3D CASE

In this section, we address the electromagnetic scattering by a sphere. The first step is to determine the expressions of the spherical harmonics (or spherical vector wave functions) $\mathbf{M}_{mn}(\mathbf{r})$ and $\mathbf{N}_{mn}(\mathbf{r})$. The scalar Helmholtz equation in spherical coordinates can be written as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right)+ \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2}+k^2\psi=0. \tag{56}$$

The solution of this differential equation is well known in the literature [6]:

$$\psi(r)=Az_n(kr)P_n^m(\cos\vartheta)e^{im\varphi}, \tag{57}$$

where A is a complex constant; $z_n(kr)$ represents a spherical Bessel function of the first, second, third, or fourth kind, i.e., $z_n(k_\rho\rho)=\{j_n(k_\rho\rho),y_n(k_\rho\rho),h_n^{(1)}(k_\rho\rho),h_n^{(2)}(k_\rho\rho)\}$ [36]; and $P_n^m(\cos\vartheta)$ represents the associated Legendre function [36]. At this point, we should apply the definition of the harmonics in Eq. (17) in order to obtain the expressions of the spherical vector wave functions. However, unlike the case of cylindrical coordinates, it is not easy to recognize a pilot vector $\hat{\boldsymbol{a}}$ to insert into such definitions. In fact, the three coordinate unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\vartheta}}$, and $\hat{\boldsymbol{\varphi}}$ are not constant. This difficulty can be avoided by considering, instead of the constant unit vector $\hat{\boldsymbol{a}}$, the vector $r\hat{\boldsymbol{r}}$. Most often the choice of pilot vector is dictated by whatever symmetry may exist in the problem. However, in other cases, the choice of pilot vector is somewhat less obvious. It can be proved that the vector harmonic [6,8,10],

$$\mathbf{M}_{mn}(\boldsymbol{r})=\nabla\times[r\hat{\boldsymbol{r}}\psi_{mn}(\boldsymbol{r})], \tag{58}$$

is a solution of the vector Helmholtz equation, and then this definition can be implemented instead of the definition given in Eq. (17).

Thanks to the new definition of $\mathbf{M}_{mn}$, in Eq. (58), and with the standard definition of $\mathbf{N}_{mn}$, in Eq. (17), we are able to calculate the following expressions:

$$\mathbf{M}_{mn}(\boldsymbol{r})=\frac{im}{\sin\vartheta}z_n(kr)P_n^m(\cos\vartheta)e^{im\varphi}\hat{\boldsymbol{\vartheta}} - z_n(kr)\frac{\partial P_n^m(\cos\vartheta)}{\partial\vartheta}e^{im\varphi}\hat{\boldsymbol{\varphi}}, \tag{59}$$

$$\mathbf{N}_{mn}(\boldsymbol{r})=\frac{z_n(kr)}{kr}n(n+1)P_n^m(\cos\vartheta)e^{im\varphi}\hat{\boldsymbol{r}} +\frac{1}{kr}\frac{\partial[rz_n(kr)]}{\partial r}\frac{\partial P_n^m(\cos\vartheta)}{\partial\vartheta}e^{im\varphi}\hat{\boldsymbol{\vartheta}} +\frac{1}{kr}\frac{\partial[rz_n(kr)]}{\partial r}\frac{im}{\sin\vartheta}P_n^m(\cos\vartheta)e^{im\varphi}\hat{\boldsymbol{\varphi}}. \tag{60}$$

Expressions (59) and (60) can be considerably simplified by introducing two scalar functions, widespread in the literature and called scalar tesseral functions, related to the associated Legendre function [6,8,23,42,43]:

$$\pi_{mn}(\vartheta)=m\frac{P_n^m(\cos\vartheta)}{\sin\vartheta}, \tag{61}$$

$$\tau_{mn}(\vartheta)=\frac{dP_n^m(\cos\vartheta)}{d\vartheta}. \tag{62}$$

Inserting Eqs. (61) and (62) into Eqs. (59) and (60), the following expressions can be obtained:

$$\mathbf{M}_{mn}=z_n(kr)[i\pi_{mn}(\cos\vartheta)\hat{\boldsymbol{\vartheta}}-\tau_{mn}(\cos\vartheta)\hat{\boldsymbol{\varphi}}]e^{im\varphi}, \tag{63}$$

$$\mathbf{N}_{mn}=\left\{n(n+1)\frac{z_n(kr)}{r}P_n^m(\cos\vartheta)\hat{\mathbf{r}}+ \frac{1}{kr}\frac{\partial[rz_n(kr)]}{\partial r}[\tau_{mn}(\cos\vartheta)\hat{\boldsymbol{\vartheta}}+i\pi_{mn}(\cos\vartheta)\hat{\boldsymbol{\varphi}}]\right\}e^{im\varphi}. \tag{64}$$

At this point, three important vector functions can be introduced:

$$\mathbf{m}_{mn}(\vartheta,\varphi)=e^{im\varphi}[i\pi_{mn}(\cos\vartheta)\hat{\boldsymbol{\vartheta}}-\tau_{mn}(\cos\vartheta)\hat{\boldsymbol{\varphi}}], \tag{65}$$

$$\mathbf{n}_{mn}(\vartheta,\varphi)=e^{im\varphi}[\tau_{mn}(\cos\vartheta)\hat{\boldsymbol{\vartheta}}+i\pi_{mn}(\cos\vartheta)\hat{\boldsymbol{\varphi}}], \tag{66}$$

$$\mathbf{p}_{mn}(\vartheta,\varphi)=e^{im\varphi}n(n+1)P_n^m(\cos\vartheta)\hat{\mathbf{r}}. \tag{67}$$

These functions have several important properties presented in Appendix A, including the orthogonality relations. They are known in the literature as vector tesseral (or sectorial) harmonics and they depend only on the angular variables. These harmonics are so called because the curves on which they vanish are the parallels and the meridians of a sphere of arbitrary radius, splitting the surface of the sphere into quadrangles [44]. As a consequence, the vector tesseral harmonics are independent of the wave type: stationary, progressive, or regressive, i.e., they are independent of the particular spherical Bessel function involved in the vector harmonics.

By comparing Eqs. (63), (64), and (65)–(67), the spherical vector wave functions assume their final and simpler expressions:

$$\mathbf{M}_{mn}(r,\vartheta,\varphi)=z_n(r)\mathbf{m}_{mn}(\vartheta,\varphi), \tag{68}$$

$$\mathbf{N}_{mn}(r,\vartheta,\varphi)=\frac{z_n(r)}{r}\mathbf{p}_{mn}(\vartheta,\varphi)+\frac{1}{r}\frac{\partial[rz_n(r)]}{\partial r}\mathbf{n}_{mn}(\vartheta,\varphi). \tag{69}$$

At this point, all the functions needed to solve the scattering problem have been introduced and the fields can be expressed as in Eq. (21). In the three-dimensional case, we prefer to start the analysis with the simplest case of a PEC sphere, with radius a, placed in the origin of a reference frame and immersed in a linear, homogeneous, and isotropic medium with electric relative permittivity ε_1, with wavenumber k_1; see Fig. 4. We consider an elliptically polarized plane wave incident on the sphere [16,23,45]:

$$\mathbf{E}_i(\mathbf{r})=\mathbf{e}_{\text{pol}}e^{i\mathbf{k}_i\cdot\mathbf{r}}=(E_{\vartheta i}\hat{\boldsymbol{\vartheta}}_i+E_{\varphi i}\hat{\boldsymbol{\varphi}}_i)e^{i\mathbf{k}_i\cdot\mathbf{r}}, \tag{70}$$

where $\mathbf{e}_{\text{pol}}$ is the polarization vector of the plane wave. The vectors $\hat{\boldsymbol{\vartheta}}_i$ and $\hat{\boldsymbol{\varphi}}_i$ are the unit vectors of the local spherical

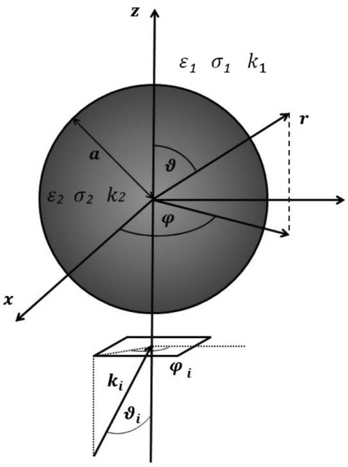

Fig. 4. Representation of a plane wave incident on a sphere.

coordinate frame with respect to the wave vector of the plane wave:

$$\mathbf{k}_i = k_1\hat{\boldsymbol{k}}_i = k_1(\sin\vartheta_i\cos\varphi_i\hat{\mathbf{x}} + \sin\vartheta_i\sin\varphi_i\hat{\mathbf{y}} + \cos\vartheta_i\hat{\mathbf{z}}), \tag{71}$$

$$\hat{\boldsymbol{\varphi}}_i = \frac{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_i}{|\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_i|} = -\sin\varphi_i\hat{\mathbf{x}} + \cos\varphi_i\hat{\mathbf{y}}, \tag{72}$$

$$\hat{\boldsymbol{\vartheta}}_i = \hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{k}}_i = \cos\vartheta_i\cos\varphi_i\hat{\mathbf{x}} + \cos\vartheta_i\sin\varphi_i\hat{\mathbf{y}} - \sin\vartheta_i\hat{\mathbf{z}}. \tag{73}$$

The azimuthal angle ϑ_i is the angle between the wave vector and the z-axis, while the equatorial angle φ_i is the angle between the projection of the wave vector on the (x, y) plane and the x-axis. The incident plane wave can be expanded in spherical harmonics as follows:

$$\mathbf{E}_i(\mathbf{r}) = \sum_{n=1}^{+\infty}\sum_{m=-n}^{n}[a_{mn}\mathbf{M}_{mn}^{(1)}(\mathbf{r}) + b_{mn}\mathbf{N}_{mn}^{(1)}(\mathbf{r})], \tag{74}$$

with [45]

$$a_{mn} = (-1)^m i^n \frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\mathbf{e}_{\text{pol}}\cdot\mathbf{m}_{mn}^*(\vartheta_i,\varphi_i), \tag{75}$$

$$b_{mn} = (-1)^m i^{n-1} \frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\mathbf{e}_{\text{pol}}\cdot\mathbf{n}_{mn}^*(\vartheta_i,\varphi_i). \tag{76}$$

As in the cylindrical case the superscript (1) in the vector harmonics indicates that the radial dependence follows the spherical Bessel function of the first kind, typical for stationary waves. The scattered field can be expanded in spherical vector wave functions as well:

$$\mathbf{E}_s(\mathbf{r}) = \sum_{n=1}^{+\infty}\sum_{m=-n}^{n}[c_{mn}\mathbf{M}_{mn}^{(3)}(\mathbf{r}) + d_{mn}\mathbf{N}_{mn}^{(3)}(\mathbf{r})]. \tag{77}$$

The superscript (3) in the vector harmonics indicates that the radial dependence follows the spherical Bessel function of the third kind, i.e., the spherical Hankel function of the first type, typical for progressive waves. The coefficients c_{mn} and d_{mn} in Eq. (77) are the unknowns of the problem and they would be determined applying the boundary conditions, i.e., the cancellation of the tangential components of the electric fields on the sphere's surface:

$$(\mathbf{E}_i + \mathbf{E}_s)\times\hat{\mathbf{r}} = 0 \quad \text{for } r = a. \tag{78}$$

Applying Eqs. (65)–(67), the dependence on the coordinates in Eqs. (74)–(77) can be made explicit:

$$\mathbf{E}_i(\mathbf{r}) = \sum_{n=1}^{+\infty}\sum_{m=-n}^{n}\Big[a_{mn}\mathbf{m}_{mn}(\vartheta,\varphi)j_n(k_1r) + b_{mn}\mathbf{n}_{mn}(\vartheta,\varphi)\dot{j}_n(k_1r) + b_{mn}\mathbf{p}_{mn}(\vartheta,\varphi)\frac{j_n(k_1r)}{k_1r}\Big], \tag{79}$$

$$\mathbf{E}_s(\mathbf{r}) = \sum_{n=1}^{+\infty}\sum_{m=-n}^{n}\Big[c_{mn}\mathbf{m}_{mn}(\vartheta,\varphi)h_n^{(1)}(k_1r) + d_{mn}\mathbf{n}_{mn}(\vartheta,\varphi)\dot{h}_n^{(1)}(k_1r) + d_{mn}\mathbf{p}_{mn}(\vartheta,\varphi)\frac{h_n^{(1)}(k_1r)}{k_1r}\Big], \tag{80}$$

where

$$\dot{z}_n(kr) = \frac{1}{kr}\frac{\partial[xz_n(kx)]}{\partial x}\Big|_{x=r} \tag{81}$$

having indicated with z_n a generic Bessel function: j_n or y_n or $h_n^{(1),(2)}$. From Eqs. (65)–(67), the following relations can be deduced:

$$\mathbf{m}_{mn}(\vartheta,\varphi)\times\hat{\mathbf{r}} = -\mathbf{n}_{mn}(\vartheta,\varphi), \tag{82}$$

$$\mathbf{n}_{mn}(\vartheta,\varphi)\times\hat{\mathbf{r}} = \mathbf{m}_{mn}(\vartheta,\varphi), \tag{83}$$

$$\mathbf{p}_{mn}(\vartheta,\varphi)\times\hat{\mathbf{r}} = 0. \tag{84}$$

Hence, substituting Eqs. (74)–(77) in Eq. (78), and applying Eqs. (82)–(84), the following equation can be obtained:

$$\sum_{n=1}^{+\infty}\sum_{m=-n}^{n}\{\mathbf{m}_{mn}(\vartheta,\varphi)[b_{mn}\dot{j}_n(k_1a) + d_{mn}\dot{h}_n^{(1)}(k_1a)] - \mathbf{n}_{mn}(\vartheta,\varphi)[a_{mn}j_n(k_1a) + c_{mn}h_n^{(1)}(k_1a)]\} = 0. \tag{85}$$

We can apply now the orthogonality properties of the vector sectorial harmonics. Two scalar equations can be obtained: the former by dot multiplying Eq. (85) times $\mathbf{m}_{m'n'}^*\sin\vartheta$ and integrating in ϑ and φ between $[0,\pi]$ and $[0,2\pi]$, respectively, and the latter by dot multiplying times $\mathbf{n}_{m'n'}^*\sin\vartheta$ and integrating again in ϑ and φ, in the same intervals. As a result, the following equations can be derived:

$$\begin{cases} a_{mn}j_n(k_1a) + c_{mn}h_n^{(1)}(k_1a) = 0 \\ b_{mn}\dot{j}_n(k_1a) + d_{mn}\dot{h}_n^{(1)}(k_1a) = 0 \end{cases}, \tag{86}$$

obtaining two simple expressions for the scattering coefficients:

$$\begin{cases} c_{mn} = -a_{mn}\frac{j_n(k_1a)}{h_n^{(1)}(k_1a)} \\ d_{mn} = -b_{mn}\frac{\dot{j}_n(k_1a)}{\dot{h}_n^{(1)}(k_1a)}. \end{cases} \tag{87}$$

These coefficients are known as Mie scattering coefficients for a PEC sphere. It is interesting to emphasize how the properties of the vector tesseral functions make simple the imposition of the boundary conditions.

Let us now consider the scattering by a dielectric sphere with electric permittivity ε_2 and wavenumber k_2. Similarly to the case of the dielectric cylinder, we have to consider an internal field to the sphere:

$$\mathbf{E}_p(\mathbf{r}) = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} [e_{mn}\mathbf{M}_{mn}^{(1)}(\mathbf{r}) + f_{mn}\mathbf{N}_{mn}^{(1)}(\mathbf{r})]. \tag{88}$$

As in the case of the scattering by a dielectric cylinder, the superscript of the spherical harmonics is (1), indicating a radial dependence connected to the spherical Bessel function of the first kind. In this case, the boundary conditions can be written as follows:

$$(\mathbf{E}_i + \mathbf{E}_s - \mathbf{E}_p) \times \hat{\mathbf{r}} = 0 \quad \text{for } r = a, \tag{89}$$

$$[\nabla \times (\mathbf{E}_i + \mathbf{E}_s - \mathbf{E}_p)] \times \hat{\mathbf{r}} = 0 \quad \text{for } r = a. \tag{90}$$

Applying the properties of the harmonics given in Eqs. (17) and (19), and recalling the properties of the vector tesseral functions (82)–(84), the two vector equations can be written as follows:

$$\begin{aligned}\sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \{&\mathbf{m}_{mn}(\vartheta,\varphi)[b_{mn}\dot{j}_n(k_1a) + d_{mn}\dot{h}_n^{(1)}(k_1a) \\ &- f_{mn}\dot{j}_n(k_2a)] - \mathbf{n}_{mn}(\vartheta,\varphi)[a_{mn}j_n(k_1a) \\ &+ c_{mn}h_n^{(1)}(k_1a) - e_{mn}j_n(k_2a)]\} = 0,\end{aligned} \tag{91}$$

$$\begin{aligned}\sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \{&\mathbf{m}_{mn}(\vartheta,\varphi)[k_1a_{mn}\dot{j}_n(k_1a) + k_1c_{mn}\dot{h}_n^{(1)}(k_1a) \\ &- k_2e_{mn}\dot{j}_n(k_2a)] - \mathbf{n}_{mn}(\vartheta,\varphi)[k_1b_{mn}j_n(k_1a) \\ &+ k_1d_{mn}h_n^{(1)}(k_1a) - k_2f_{mn}j_n(k_2a)]\} = 0.\end{aligned} \tag{92}$$

By a dot product of Eqs. (91) and (92) times $\mathbf{m}^*_{m'n'}$ sin ϑ and a double integration in ϑ and φ between $[0, \pi]$ and $[0, 2\pi]$, respectively, a first set of two scalar equations can be obtained, whereas, by a dot multiplication of Eqs. (91) and (92) times $\mathbf{n}^*_{m'n'}$ sin ϑ and a double integration in ϑ and φ between $[0, \pi]$ and $[0, 2\pi]$, respectively, a second set of two scalar equations can be obtained. As a result, the following system of equations can be computed:

$$\begin{cases} a_{mn}j_n(k_1a) + c_{mn}h_n^{(1)}(k_1a) - e_{mn}j_n(k_2a) = 0 \\ b_{mn}\dot{j}_n(k_1a) + d_{mn}\dot{h}_n^{(1)}(k_1a) - f_{mn}\dot{j}_n(k_2a) = 0 \\ k_1a_{mn}\dot{j}_n(k_1a) + k_1c_{mn}\dot{h}_n^{(1)}(k_1a) - k_2e_{mn}\dot{j}_n(k_2a) = 0 \\ k_1b_{mn}j_n(k_1a) + k_1d_{mn}h_n^{(1)}(k_1a) - k_2f_{mn}j_n(k_2a) = 0. \end{cases} \tag{93}$$

Introducing the dielectric contrast $\chi = \frac{k_2}{k_1}$ and solving the linear system of equations, we get

$$c_{mn} = -a_{mn}\frac{\dot{j}_n(k_1a)j_n(k_2a) - \chi j_n(k_1a)\dot{j}_n(k_2a)}{\dot{h}_n^{(1)}(k_1a)j_n(k_2a) - \chi h_n^{(1)}(k_1a)\dot{j}_n(k_2a)}, \tag{94}$$

$$d_{mn} = -b_{mn}\frac{j_n(k_1a)\dot{j}_n(k_2a) - \chi\dot{j}_n(k_1a)j_n(k_2a)}{h_n^{(1)}(k_1a)\dot{j}_n(k_2a) - \chi\dot{h}_n^{(1)}(k_1a)j_n(k_2a)}, \tag{95}$$

which are the most general expressions of the Mie scattering coefficients for the scattering by a dielectric sphere.

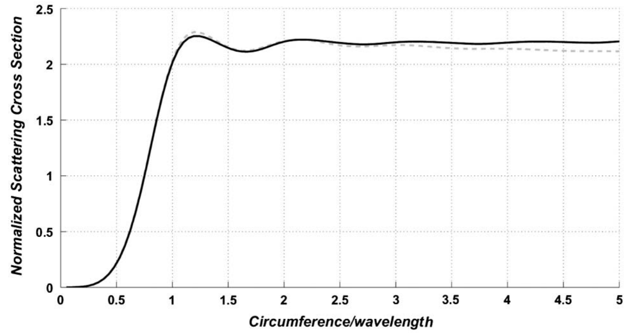

Fig. 5. Scattering cross section of a PEC sphere with radius 0.25 m for an incident plane wave in the range of frequencies from 0.1 MHz to 10.0 GHz. The cross section is computed (solid line) implementing on MATLAB the formula (96) and (dashed line) simulating the scattering problem on a software based on the finite-element method. As we can see, the asymptotic limit of the graph is 2, meaning that in the large particle limit, twice as much energy is removed as expected based on the geometric cross section. This is contrary to intuition and is referred to as the extinction paradox.

Now, we can compute the cross sections of a sphere as a function of the scattering coefficients. Computing the scattered power as in Eq. (7), we must cross multiply the electric and magnetic fields. On physical grounds, we know that the scattered power is independent of the polarization; then we can suppose that the incident wave is linearly polarized along x, i.e., $\vartheta_i = 0$ and $\varphi_i = \pi/2$. In this case, as we can see from Eqs. (75) and (76), only the terms with $m = 1$ are different from zero; see Appendix A. At this point, thanks to the orthogonality properties of the vector tesseral functions and to an important property of the spherical Bessel functions reported in Appendix A, the scattering cross section takes the following simple form [8]:

$$C_s = \frac{2\pi}{k_1^2}\sum_{n=1}^{+\infty}(2n+1)\left(\left|\frac{c_{1n}}{a_{1n}}\right|^2 + \left|\frac{d_{1n}}{b_{1n}}\right|^2\right). \tag{96}$$

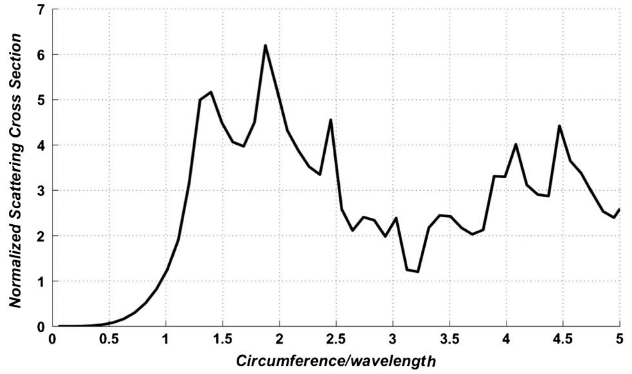

Fig. 6. Scattering cross section of a dielectric sphere with radius 0.25 m and relative permittivity $\varepsilon_2 = 4$ for an incident plane wave in the range of frequencies from 0.1 MHz to 10.0 GHz.

Similarly, the extinction cross section is

$$C_e = \frac{2\pi}{k_1^2}\sum_{n=1}^{+\infty}(2n+1)\mathrm{Re}\left[\left|\frac{c_{1n}}{a_{1n}}\right|^2 + \left|\frac{d_{1n}}{b_{1n}}\right|^2\right]. \quad \text{(97)}$$

In Fig. 5, the scattering cross section for a PEC sphere with radius 0.25 m is shown, in the range of frequencies from 0.1 MHz to 10.0 GHz, computed by implementing Eq. (96) on MATLAB and by an electromagnetic simulation obtained from a software implementing a finite-element method.

In Fig. 6, the scattering cross section of a dielectric sphere with radius 0.25 m and relative permittivity $\varepsilon_2 = 4$ is shown, in the range of frequencies from 0.1 MHz to 10.0 GHz, computed by a software implementing a finite-element method.

6. CONCLUSIONS

In this first part of the tutorial, the basic theoretical formulation to approach any scattering problem has been presented. The concept of cross section has been defined and its relation with the electromagnetic power has been clarified. The vector harmonics have been defined, their physical meaning has been pointed out, and their application to the representation of the electromagnetic field in different reference frames has been presented. The solutions to the two canonical problems of scattering by an infinitely long circular cylinder and by a sphere have been illustrated. The analytical expressions of all the involved fields have been given and the expressions of the scattering coefficients have been presented. Finally, the scattering and extinction cross sections for both the cylinder and the sphere have been shown as a function of the scattering coefficients. Moreover, some comparisons between such expressions and electromagnetic full-wave simulations have been reported. In the following part of the tutorial, some practical aspects of the scattering problems will be faced, and the theoretical formulation of more complicated scenarios will be presented. In particular, we will compare the scattering by N spheres and N cylinders. By doing this, we can model any three-dimensional and two-dimensional geometric shape with a cluster of such canonical shapes.

APPENDIX A

1. Properties of the Bessel Functions

The Wronskian of the Bessel functions of the first and second kind assumes the following expression [36]:

$$J_n(z)Y_n'(z) - J_n'(z)Y_n(z) = \frac{2}{\pi z}. \quad \text{(A1)}$$

2. Properties of Tesseral Vector Functions

Orthogonality properties of the vector tesseral functions [6,46,47] are as follows:

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{m}_{mn}\cdot\mathbf{n}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 0, \quad \text{(A2)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{p}_{mn}\cdot\mathbf{m}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 0, \quad \text{(A3)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{p}_{mn}\cdot\mathbf{n}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 0, \quad \text{(A4)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{m}_{mn}\cdot\mathbf{m}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{mm'}\delta_{nn'}, \quad \text{(A5)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{n}_{mn}\cdot\mathbf{n}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{mm'}\delta_{nn'}, \quad \text{(A6)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{p}_{mn}\cdot\mathbf{p}^*_{m'n'} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 4\pi\frac{[n(n+1)]^2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{mm'}\delta_{nn'}. \quad \text{(A7)}$$

The vectors $\boldsymbol{m}_{mn}(0,\varphi)$ and $\boldsymbol{n}_{mn}(0,\varphi)$ are zero for any $m \neq 1$ and φ. In fact,

$$\lim_{\vartheta\to 0}\pi_{mn}(\cos\vartheta) = \lim_{\vartheta\to 0} m\frac{P_n^m(\cos\vartheta)}{\sin\vartheta}. \quad \text{(A8)}$$

The limit is zero when $m = 0$. For $m \neq 0$, we recall that

$$P_n^m(\cos\vartheta) = (-1)^m \sin^m\vartheta\frac{\mathrm{d}^m P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)^m}. \quad \text{(A9)}$$

For $m = 1$ the sine function is simplified, while for $m > 1$, it remains a multiplication by the sine function that is zero when $\vartheta \to 0$. Now, we can recall the following relations [36]:

$$\lim_{\vartheta\to 0}\frac{\mathrm{d}^m P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)^m} = \lim_{x\to 1}\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m} = 1\cdot 3\cdot\ldots\cdot(2m-1)\lim_{x\to 1}C_{n-m}^{(m-\frac{1}{2})}(x) = \frac{(n+m)!}{(n-m)!}, \quad \text{(A10)}$$

which is a quantity always different from zero. Here, the function $C_n^{(\alpha)}(x)$ is the Gegenbauer polynomial.

On the other hand,

$$\lim_{\vartheta\to 0}\tau_{mn}(\cos\vartheta) = \lim_{\vartheta\to 0}\frac{\mathrm{d}P_n^m(\cos\vartheta)}{\mathrm{d}\vartheta}. \quad \text{(A11)}$$

By the definition of the associate Legendre function in Eq. (A9), when $m = 0$, we obtain

$$\lim_{\vartheta\to 0}\tau_{mn}(\cos\vartheta) = \lim_{\vartheta\to 0} -\sin\vartheta\frac{\mathrm{d}P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)} = 0, \quad \text{(A12)}$$

because the sine is zero and Eq. (A10) holds. When $m = 1$, we get

$$\lim_{\vartheta\to 0}\tau_{mn}(\cos\vartheta) = -\lim_{\vartheta\to 0}\left[\cos\vartheta\frac{\mathrm{d}P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)} - \sin^2\vartheta\frac{\mathrm{d}^2 P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)^2}\right] = n(n+1). \quad \text{(A13)}$$

When $m > 1$, taking into account Eq. (A10), we obtain

$$\lim_{\vartheta\to 0}\tau_{mn}(\cos\vartheta) = (-1)^m\lim_{\vartheta\to 0}\left[m\sin^{m-1}\vartheta\cos\vartheta\frac{\mathrm{d}^m P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)^m} - \sin^{m+1}\vartheta\frac{\mathrm{d}^{m+1}P_n(\cos\vartheta)}{\mathrm{d}(\cos\vartheta)^{m+1}}\right] = 0. \quad \text{(A14)}$$

As a consequence, the following equalities hold:

$$\lim_{\vartheta\to 0}\pi_{mn}(\cos\vartheta) = \lim_{\vartheta\to 0}\tau_{mn}(\cos\vartheta) = n(n+1)\delta_{1m}. \quad \textbf{(A15)}$$

3. Properties of the Spherical Bessel Functions

The Wronskian of the spherical Bessel functions of the first and second kind assumes the following expression [36]:

$$j_n(z)\dot{y}_n(z) - \dot{j}_n(z)y_n(z) = \frac{1}{z^2}. \quad \textbf{(A16)}$$

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