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ON THE EFFECTIVENESS OF CONVOLUTIVE TYPE VARIATIONAL PRINCIPLES IN THE NUMERICAL SOLUTION OF VISCOELASTIC PROBLEMS

ANGELO CARINI, FRANCESCA LEVI AND FRANCESCO GENNA

Dedicated to the memory of Professor Morton Gurtin

With reference to the nonaging linear viscoelastic problem, three convolutive type variational formulations existing in the literature are critically reviewed: the Gurtin formulation, the split Gurtin formulation and the Huet formulation. The formulations are used for the numerical solution of the hereditary viscoelastic problem through spatial and temporal discretizations considering both a finite time range and using a step-by-step method of time marching. Several numerical examples are included and numerical results are compared with the aim of investigating the effectiveness of the variational formulations in the numerical solution of the viscoelastic problem.

1. Introduction

This paper discusses the numerical use of variational principles for linear viscoelasticity. As observed by Tonti [1973], variational formulations for initial value problems are not possible in the classical context of the Calculus of Variations. This is because the linear operators relevant to linear initial value problems are not self-adjoint with respect to the standard bilinear form. A variational formulation, valid simultaneously for the given problem and the adjoint problem, was provided by Morse and Feshbach [1953]. By using this method it is always possible to rephrase any linear problem in a variational way. This method considers, alongside with the original problem, another artificial one governed by the adjoint operator. The use of the adjoint operator, whether explicitly or implicitly, inspired a significant number of papers, especially for diffusion problems.

The earliest variational formulations for the viscoelastic problem, although under very restrictive assumptions, date back to Biot [1956], Olszak and Perzyna [1959], Breuer and Onat [1964], Hlaváček [1966], and Christensen [1968; 1971].

The first true variational formulation for initial value problems dates back to Gurtin's work [1963; 1964a; 1964b]. It is based on the use of convolution integrals and it is valid for a large class of linear time-dependent problems, including viscoelasticity, elastodynamics, and the heat conduction problem.

Gurtin's method was applied to the linear theory of viscoelasticity by many authors (Schapery [1964], Leitman [1966], Taylor et al. [1970], Brilla [1972], Reddy [1976], just to name a few).

It should be emphasized that Gurtin's variational principles and relevant generalizations are not extremum principles. Nevertheless, Benthien and Gurtin [1970] showed that the Laplace transform of the Gurtin-type stationarity formulations are in fact minimum formulations in the transformed domain.

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Rafalski [1969] has the merit of having obtained extremal formulations for linear problems with initial values. The limit of his results is that it is necessary to study the whole history, from the time of the initial assignment of data to infinity. His work was later perfected by Reiss [1978] and by Reiss and Haug [1978].

Huet [1992], through the use of pseudoconvolutive and pseudobiconvolutive bilinear forms, although under restrictive assumptions, obtained two principles, extensions of the minimum principles of the total potential energy and the complementary energy related to linear elasticity.

Tonti [1984] generalized the method to any nonlinear problem. The latter approach was applied also to the linear viscoelasticity case in [Carini et al. 1995; Carini and De Donato 2004].

In [Carini and Mattei 2015], minimum-stationary type formulations are obtained for the linear viscoelastic problem. In the following these formulations are called "split Gurtin's formulations".

To the authors' knowledge, there does not appear to be any significant work in the literature concerning the use of variational formulations for the numerical solution of the linear viscoelastic problem. In particular, variational principles based on convolutive bilinear forms with respect to time do not seem to have had any place so far in computational procedures for the numerical determination of the viscoelastic response of solids or structures subjected to external actions. This paper tries to start filling this gap with reference to Gurtin's formulation and those related to it.

The paper is organized as follows. In Section 2, the linear viscoelastic problem is introduced. In Section 3, Gurtin's variational formulation and those derived from it are presented, i.e., the split Gurtin formulation and the Huet formulation. In Section 4, by using the finite element technique in space and the Ritz method in time, a discretized formulation is applied and several numerical examples are shown with the aim of testing the effectiveness of the presented Gurtin type variational formulations in the numerical solution of the linear viscoelastic problem.

2. The linear viscoelastic problem

Consider a solid body $\Omega \subset \mathbb{R}^3$ made of a linear viscoelastic material, possibly anisotropic. An orthogonal Cartesian reference system is used, with coordinates x_r, $r = 1, 2, 3$. The components of vectors, second order, and fourth order tensors are indicated with the usual indicial notation. Einstein's convention over repeated indices is adopted.

Denote by V the volume of the region Ω and by Γ its external surface, with unit outward normal $n_i(x_r)$. Let $u_i(x_r, t)$, $\epsilon_{ij}(x_r, t)$ and $\sigma_{ij}(x_r, t)$ be, respectively, the displacement, strain, and stress fields at the point $x_r \in \Omega$, at the time $t \in [0, 2T]$. The solid may be subjected to a history of body forces $b_i(x_r, t)$, to a history of surface tractions $p_i(x_r, t)$ acting on the loaded region Γ_p of the boundary, and to a history of prescribed displacements $\bar{u}_i(x_r, t)$ acting on the constrained boundary Γ_u, with $\Gamma_u \cup \Gamma_p = \Gamma$.

The body is undisturbed for $t < 0$ and the whole loading history is supposed to be defined in the given time range $t \in [0, 2T]$, $2T$ being the end time of the loading process.

Here, we deal only with nonaging materials (i.e., we consider only the hereditary viscoelasticity case), for which the direct constitutive law, that relates a known strain field $\epsilon_{ij}(x_r, t)$ to the corresponding stress field $\sigma_{ij}(x_r, t)$, can be written in the Boltzmann form as

$$\sigma_{ij}(x_r, t) = \int_{0^-}^{t} R_{ijhk}(x_r, t - \tau)\, \mathrm{d}\epsilon_{hk}(x_r, \tau), \tag{2-1}$$

where the integral has to be meant in the Stieltjes sense, and $R_{ijhk}(x_r, t)$, with $t > 0$ (we assume that $R_{ijhk}(x_r, t) = 0$ for $t < 0$), is the relaxation tensor, or kernel.

We assume that the relaxation tensor satisfies the same symmetry properties enjoyed also by the linear elasticity constitutive tensors:

$$R_{ijhk}(x_r,t)=R_{jihk}(x_r,t)=R_{ijkh}(x_r,t)=R_{hkij}(x_r,t) \quad \forall x_r\in\Omega,\ \forall t\in[0,2T] \tag{2-2}$$

and that the inequalities

$$R^0_{ijhk}(x_r)\,\lambda_{ij}\,\lambda_{hk}>0, \qquad R^\infty_{ijhk}(x_r)\,\lambda_{ij}\,\lambda_{hk}>0 \tag{2-3}$$

hold for every $x_r\in\Omega$ and for every nonvanishing symmetric second order tensor λ_{ij}, with $R^0_{ijhk}(x_r)$ and $R^\infty_{ijhk}(x_r)$ being defined, respectively, as

$$R^0_{ijhk}(x_r):=\lim_{t\to 0}R_{ijhk}(x_r,t), \qquad R^\infty_{ijhk}(x_r):=\lim_{t\to+\infty}R_{ijhk}(x_r,t). \tag{2-4}$$

We further assume that the constitutive law (2-1) is invertible, with the inverse form written as

$$\epsilon_{ij}(x_r,t)=\int_{0^-}^{t}C_{ijhk}(x_r,t-\tau)\,\mathrm{d}\sigma_{hk}(x_r,\tau) \tag{2-5}$$

in which the symbol $C_{ijhk}(x_r,t)$ denotes the creep tensor, or kernel, which enjoys the same properties as $R_{ijhk}(x_r,t)$.

For the conditions of invertibility of (2-1), see for instance [Fabrizio 1992; Bozza and Gentili 1995].

The equilibrium equations in terms of displacements (Navier's equations) are

$$-\frac{\partial}{\partial x_i}\int_{0^-}^{t}R_{ijhk}(x_r,t-\tau)\,\tfrac{1}{2}\,\mathrm{d}\big(u_{h/k}+u_{k/h}\big)(\tau)=b_j \quad \text{in } \Omega\times[0,2T], \tag{2-6a}$$

$$n_i\int_{0^-}^{t}R_{ijhk}(x_r,t-\tau)\,\tfrac{1}{2}\,\mathrm{d}\big(u_{h/k}+u_{k/h}\big)(\tau)=p_j \quad \text{on } \Gamma_p\times[0,2T] \tag{2-6b}$$

with u_i satisfying the kinematic boundary conditions $u_i=\bar{u}_i$ on Γ_u. Equation (2-6a) can be written in the operatorial form

$$\mathcal{L}\boldsymbol{u}=\boldsymbol{b} \tag{2-7}$$

with $\boldsymbol{u}$ the displacement vector with components u_i, $\boldsymbol{b}$ the vector of the known terms with components b_i and $\mathcal{L}$ the Navier operator.

3. Variational formulations in viscoelasticity

In viscoelasticity, the lack of symmetry of the constitutive operators of (2-1) and (2-5), with respect to standard bilinear forms (see, for example, [Tonti 1973; 1984]), has severely hindered the development of variational formulations. Some results have been obtained in the past making reference to *convolutive* bilinear forms [Gurtin 1963; Tonti 1973]. Given any two functions $f(x_r,t)$ and $g(x_r,t)$, we introduce the following notation for the basic convolution integral, such as the one adopted in (2-1) and (2-5):

$$f(x_r,t)\circ g(x_r,t):=\int_{0^-}^{t}f(x_r,t-\tau)\mathrm{d}g(x_r,\tau)=\int_{0^-}^{t}g(x_r,t-\tau)\mathrm{d}f(x_r,\tau)=g(x_r,t)\circ f(x_r,t). \tag{3-1}$$

The symbol $\circ$ introduced in the above formula and used by several authors (see, for instance, [Huet 1992; Charpin and Sanahuja 2017]), is a simplification of the symbol $\otimes$ adopted, for instance, by Mandel [1966] and Leitman and Fisher [1973]. The symbol $\circ$ is not to be confused with the standard convolution symbol $*$, see, e.g., [Gurtin and Sternberg 1962].

In this section we present and discuss the variational formulations adopted in the next section for the numerical determination of the viscoelastic response of solids and structures subjected to external actions. In particular, Gurtin's formulation and the formulations derived from it, i.e., the split Gurtin formulation and the Huet formulation, will be taken into consideration. Tonti's "extended" formulation will be used as a comparison. The formulation based on adding the adjoint equation (see [Morse and Feshbach 1953]) is somehow related to Gurtin's formulation.

3.1. ***The method of the adjoint equation and Gurtin's formulation.*** Gurtin's formulation can be derived directly from the method of adding the adjoint equation (see [Morse and Feshbach 1953]). In fact, consider the Navier equation (2-6a) (here we suppose, for simplicity, homogeneous boundary conditions) and write the adjoint equation with respect to the standard bilinear form (see the definition (3-8)) as

$$-\frac{\partial}{\partial x_i}\int_{2T}^{t} R_{ijhk}(x_r,\tau-t)\tfrac{1}{2}\,\mathrm{d}\big(v_{h/k}+v_{k/h}\big)(\tau)=a_j \quad \text{in } \Omega\times[0,2T] \tag{3-2}$$

or, in operatorial form

$$\widetilde{\mathcal{L}}\boldsymbol{v}=\boldsymbol{a} \tag{3-3}$$

with $\boldsymbol{v}$ the unknown vector of the adjoint problem with components v_i and $\boldsymbol{a}$ an arbitrary known term with components a_i. The two equations, the original one and the adjoint one, can be represented in symmetrical matrix form as

$$\boldsymbol{S}\boldsymbol{z}=\boldsymbol{h} \tag{3-4}$$

with

$$\boldsymbol{S}:=\begin{bmatrix}0 & \widetilde{\mathcal{L}}\\ \mathcal{L} & 0\end{bmatrix},\qquad \boldsymbol{z}:=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix},\qquad \boldsymbol{h}:=\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}. \tag{3-5}$$

By the symmetry of the operator $\boldsymbol{S}$, with respect to the standard bilinear form, the solution $\boldsymbol{z}$ of (3-4) can be obtained by the solution of the stationarity problem

$$F^{MF}(\boldsymbol{z})=\operatorname*{stat}_{\boldsymbol{z}'}F^{MF}(\boldsymbol{z}'), \tag{3-6}$$

where $\boldsymbol{z}'$ is any vector belonging to the domain of $\boldsymbol{S}$ and

$$F^{MF}(\boldsymbol{z}')=F^{MF}(\boldsymbol{u}',\boldsymbol{v}')=\tfrac{1}{2}\langle\boldsymbol{S}\boldsymbol{z}',\boldsymbol{z}'\rangle-\langle\boldsymbol{h},\boldsymbol{z}'\rangle=\langle\mathcal{L}\boldsymbol{u}',\boldsymbol{v}'\rangle-\langle\boldsymbol{b},\boldsymbol{v}'\rangle-\langle\boldsymbol{a},\boldsymbol{u}'\rangle, \tag{3-7}$$

where $\langle\boldsymbol{b},\boldsymbol{v}'\rangle$ is the standard bilinear form with time integral in the sense of Stieltjes

$$\langle\boldsymbol{b},\boldsymbol{v}'\rangle:=\int_\Omega\int_{0^-}^{2T} b_i(x,t)\,\mathrm{d}v_i'(x,t)\,\mathrm{d}\Omega. \tag{3-8}$$

If we take

$$\boldsymbol{a}(t)=\boldsymbol{b}(2T-t) \tag{3-9}$$

and by making the change of variables $2T-\tau=\bar{\tau}$ e $2T-t=\bar{t}$, the adjoint equation (3-2) becomes

$$-\frac{\partial}{\partial x_i}\int_{0}^{\bar{t}} R_{ijhk}(x_r,\bar{t}-\bar{\tau})\,\tfrac{1}{2}\,\mathrm{d}\big(v_{h/k}+v_{k/h}\big)(2T-\bar{\tau})=b_j(\bar{t}) \quad \text{in } \Omega\times[0,2T], \tag{3-10}$$

which, compared with the starting equation (2-6a) allows us to easily deduce that

$$\boldsymbol{v}(t) = \boldsymbol{u}(2T - t) \tag{3-11}$$

and, therefore, F^{MF} is a function only of $\boldsymbol{u}'$:

$$F^{MF}(\boldsymbol{u}') = \langle \mathcal{L}\boldsymbol{u}', \boldsymbol{u}'(2T-t)\rangle - \langle \boldsymbol{b}, \boldsymbol{u}'(2T-t)\rangle - \langle \boldsymbol{b}(2T-t), \boldsymbol{u}'\rangle = \langle \mathcal{L}\boldsymbol{u}', \boldsymbol{u}'\rangle_c - 2\langle \boldsymbol{b}, \boldsymbol{u}'\rangle_c, \tag{3-12}$$

where

$$\langle \boldsymbol{b}, \boldsymbol{u}'\rangle_c := \int_\Omega \int_{0^-}^{2T} b_i(x, 2T-t)\, \mathrm{d}u_i'(x,t)\, \mathrm{d}\Omega \tag{3-13}$$

is a bilinear form convolutive (in the sense of Stieltjes) with respect to time but standard with respect to space.

In the case of homogeneous boundary conditions, the functional of the total potential energy type obtained by Gurtin [1963] coincides with (3-12), the one obtained by the method of the adjoint equation. Gurtin derived his functional starting from the bilinear form (3-13). In the case of inhomogeneous boundary conditions, Gurtin's functional takes the more general form

$$\begin{aligned} F^G(u_i') = \frac{1}{2}\int_\Omega \int_{0^-}^{2T} \int_{0^-}^{2T-t} R_{ijhk}(x, 2T-t-\tau)\, \mathrm{d}\epsilon_{ij}'(x,t)\ \mathrm{d}\epsilon_{hk}'(x,\tau)\, \mathrm{d}\Omega \\ - \int_\Omega \int_{0^-}^{2T} b_j(x, 2T-t)\, \mathrm{d}u_j'(x,t)\, \mathrm{d}\Omega - \int_{\Gamma_p} \int_{0^-}^{2T} p_j(x, 2T-t)\, \mathrm{d}u_j'(x,t)\, \mathrm{d}\Gamma_p \end{aligned} \tag{3-14}$$

under the constraints $\epsilon_{ij}' = \frac{1}{2}(u_{i/j}' + u_{j/i}')$ and $u_i' = \bar{u}_i$ on Γ_u

3.2. ***Split Gurtin's formulation.*** In [Carini and Mattei 2015] five new variational formulations for the viscoelastic problem are derived, one of which is of the minimum type. Here, these new formulations are conveniently interpreted as obtained from the decomposition of the time interval $[0, 2T]$ into two subintervals ($[0, T]$ and $[T, 2T]$) of equal length (see, also, [Huet 1999; 2001a; 2001b]). The unknowns are thus formally doubled: the variables defined in the first subinterval and those defined in the second. Accordingly, the strain and stress fields can be written, respectively, as

$$\epsilon_{ij}(t) = \begin{cases} \epsilon_{1_{ij}}(t) & \text{for } t \in [0, T], \\ \epsilon_{2_{ij}}(t) & \text{for } t \in [T, 2T], \end{cases} \tag{3-15}$$

$$\sigma_{ij}(t) = \begin{cases} \sigma_{1_{ij}}(t) & \text{for } t \in [0, T], \\ \sigma_{2_{ij}}(t) & \text{for } t \in [T, 2T], \end{cases} \tag{3-16}$$

(from here on, unless strictly necessary, we will omit indicating the dependence on the space coordinate x_r) where subscript 1 refers to the quantities defined over the time interval $[0, T]$, and subscript 2 indicates quantities defined over $[T, 2T]$. As a consequence, the constitutive law operator is split into suboperators, that can be arranged into a two-by-two matrix symmetric with respect to Gurtin's convolutive bilinear form (see Carini and Mattei [2015]).

The direct constitutive law (2-1), by virtue of (3-15) and (3-16) and thanks to Boltzmann's superposition principle, can be split as follows:

$$\sigma_{1_{ij}}(t) = \int_{0^-}^{t} R_{ijhk}(t-\tau)\,\mathrm{d}\epsilon_{1_{hk}}(\tau) \quad \text{for } t \in [0, T] \tag{3-17}$$

$$\sigma_{2_{ij}}(t) = \int_{0^-}^{T} R_{ijhk}(t-\tau)\,\mathrm{d}\epsilon_{1_{hk}}(\tau) + \int_{T}^{t} R_{ijhk}(t-\tau)\,\mathrm{d}\epsilon_{2_{hk}}(\tau) \quad \text{for } t \in [T, 2T]. \tag{3-18}$$

Equations (3-17) and (3-18) can be rewritten in the compact, and symmetric, matricial form

$$\boldsymbol{\sigma} = \mathbf{L}\,\boldsymbol{\epsilon} \tag{3-19}$$

with

$$\mathbf{L} := \begin{bmatrix} A & B \\ \widetilde{B} & 0 \end{bmatrix} := \begin{bmatrix} \int_{0^-}^{T} R_{ijhk}(t-\tau)\,\mathrm{d}(.) & \int_{T}^{t} R_{ijhk}(t-\tau)\,\mathrm{d}(.) \\ \int_{0^-}^{t} R_{ijhk}(t-\tau)\,\mathrm{d}(.) & 0 \end{bmatrix} \begin{array}{l} \text{for } t \in [T, 2T] \\ \text{for } t \in [0, T] \end{array} \tag{3-20}$$

$$\boldsymbol{\epsilon} := \begin{bmatrix} \epsilon_{1_{ij}}(t) \\ \epsilon_{2_{ij}}(t) \end{bmatrix}; \qquad \boldsymbol{\sigma} := \begin{bmatrix} \sigma_{2_{ij}}(t) \\ \sigma_{1_{ij}}(t) \end{bmatrix} \tag{3-21}$$

in which the superscript $\widetilde{\ }$ denotes the adjoint operator with respect to the adopted convolutive bilinear form.

The convolutive bilinear form was introduced by Gurtin [1963] (see also [Tonti 1973]) and, for our purposes, fixed a material point x_r of the viscoelastic solid, and a time $2T > 0$, from here on is defined (in the Stieltjes sense) as

$$(\sigma'_{ij}, \epsilon''_{ij})_c := \sigma'_{ij}(2T) \circ \epsilon''_{ij}(2T) := \int_{0^-}^{2T} \sigma'_{ij}(2T-t)\,\mathrm{d}\epsilon''_{ij}(t), \tag{3-22}$$

where quantities with one or two apexes denote generic symmetric stress and strain tensors, and $2T$ denotes the end time of the considered loading history.

Operator A is self-adjoint with respect to (3-22). The operatorial formulation (3-19) is therefore symmetric, and it is equivalent to the constitutive law (3-17)–(3-18). Operator A is also, in general, positive semidefinite. In fact, the quadratic form

$$\phi(T) = (A\,\epsilon'_{1ij}, \epsilon'_{1ij})_c = \int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon'_{1hk}(\tau)\,\mathrm{d}\epsilon'_{1ij}(t) \tag{3-23}$$

has the physical meaning of the free energy per unit volume of the material (see [Mandel 1966]), a nonnegative quantity, which explains the positive semidefinition of operator A of (3-20). Note that this operator has not the form of (2-1), owing to its upper integration limit. Therefore, even though we have assumed the constitutive operator (2-1) to be invertible, nothing can be said, in general, about the invertibility of operator A.

A completely analogous path allows one to introduce a symmetric split formulation for the inverse constitutive law, expressed in terms of the creep kernel, which we skip here for the sake of brevity (see [Carini and Mattei 2015]).

From this new split approach, exploiting symmetry only, Carini and Mattei [2015] were able to derive several saddle-point formulations, of which the first one (of the total potential energy type) is especially interesting for the purposes of the present work and reads

$$F^{SG}(u_{1i}, u_{2i}) = \min_{u'_{1i}} \operatorname*{stat}_{u'_{2i}} F^{SG}(u'_{1i}, u'_{2i}), \tag{3-24}$$

where

$$\begin{aligned} F^{SG}[u'_{1i}, u'_{2i}] = \frac{1}{2}\int_\Omega \big(A\epsilon'_{1ij}(2T)\circ\epsilon'_{1ij}(2T) + 2\widetilde{B}\epsilon'_{1ij}(2T)\circ\epsilon'_{2ij}(2T)\big)\,\mathrm{d}\Omega \\ -\int_\Omega b_{2i}(2T)\circ u'_{1i}(2T)\,\mathrm{d}\Omega - \int_\Omega b_{1i}(2T)\circ u'_{2i}(2T)\,\mathrm{d}\Omega \\ -\int_{\Gamma_p} p_{2i}(2T)\circ u'_{1i}(2T)\,\mathrm{d}\Gamma - \int_{\Gamma_p} p_{1i}(2T)\circ u'_{2i}(2T)\,\mathrm{d}\Gamma. \end{aligned} \tag{3-25}$$

Here $u_{1i}(t)$ and $u_{2i}(t)$ are the exact solution of the problem, and $u'_{1i}(t)$, $u'_{2i}(t)$, $\epsilon'_{1ij}(t)$, and $\epsilon'_{2ij}(t)$ are arbitrary but compatible displacement and strain fields.

A mixed minimum formulation was also derived in [Carini and Mattei 2015], by transforming the problem operator $\boldsymbol{L}$ of (3-20) into a positive definite one. Despite looking very promising, precisely because it is a true minimum principle, this formulation is quite hard to be adopted as a basis for further calculations, since it involves operator A^{-1}, which, even assuming it to exist, is extremely difficult to be suitably characterized. Note that the inversion of operator A on the main diagonal of $\boldsymbol{L}$ in (3-20) requires, as a necessary condition, that it be positive definite, not just positive semidefinite as it was shown to be in general. This happens, for example, when the relaxation tensor $R_{ijhk}(x_r, t)$ is completely monotonic (see [Del Piero and Deseri 1996]).

3.3. ***Huet's formulation.*** The extremum theorems derived by Huet [1992] are valid only under severe restrictions and were proposed for linear quasistatic viscoelasticity without aging, with material, isotropic or not, for which a free energy density per unit volume $\phi(T)$ and a dissipation power density $D(T)$ can be expressed from the relaxation function or the creep function. As demonstrated by Mandel [1966] and Brun [1965; 1969], by their extensions of the results obtained by Staverman and Schwarzl [1952a; 1952b], this occurs at least for the behaviors which can be dealt with through an internal variable approach. An essential ingredient for Huet's approach are Brun's [1965; 1969] formal identities giving viscoelastic analogues to the Clapeyron equation of elasticity. Huet [1992] introduced a pseudoconvolutive formalism based on results obtained by Brun [1965; 1969]. In particular Brun gave a generalization of the Clapeyron equation for the nonaging viscoelastic case, which expresses ϕ in terms of ϵ_{ij} and σ_{ij} only:

$$\phi(T) = \frac{1}{2}\Big(\int_{0^-}^{T} - \int_{T}^{2T}\Big)\,\sigma_{ij}(2T-t)\,\mathrm{d}\epsilon_{ij}(t). \tag{3-26}$$

This is a bilinear form in σ_{ij} and ϵ_{ij}, which restitutes the Staverman and Schwarzl formula when replacing σ_{ij} by its value given by the constitutive equation:

$$\begin{aligned} \phi(T) &= \frac{1}{2}\Big(\int_{0^-}^{T} - \int_{T}^{2T}\Big)\,\sigma_{ij}(2T-t)\,\mathrm{d}\epsilon_{ij}(t) \\ &= \frac{1}{2}\Big(\int_{0^-}^{T} - \int_{T}^{2T}\Big)\int_{0^-}^{2T-t} R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon_{ij}(\tau)\,\mathrm{d}\epsilon_{hk}(t) \\ &= \int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon_{ij}(\tau)\,\mathrm{d}\epsilon_{hk}(t). \end{aligned}$$

Huet introduced a pseudoconvolution $a\Box b$ of two functions $a(t)$ and $b(t)$ from the forms exhibited by the free energy:

$$a\Box b=\Big(\int_{0^-}^{T}-\int_{T}^{2T}\Big)\,a(2T-t)\,\mathrm{d}b(t). \tag{3-27}$$

It is important to note that the pseudoconvolution $a\Box b$ is not commutative in general:

$$a\Box b\neq b\Box a. \tag{3-28}$$

This means that, in contrast with the bilinear form associated to the Staverman and Schwarzl quadratic one, the pseudoconvolution is not symmetric even for kernels R_{ijhk} having the universal symmetries of linear elasticity. This is the basic reason for which the minimum theorems derived by Huet [1992] are subjected to restrictions.

To overcome the problem, Huet introduced the so-called "t-symmetrizing" virtual field. A t-symmetrizing virtual field $\tilde{\epsilon}_{ij}$ satisfies

$$\int_{\Omega}R_{ijhk}\circ\tilde{\epsilon}_{ij}\Box\epsilon_{hk}\,\mathrm{d}\Omega=\int_{\Omega}R_{ijhk}\circ\epsilon_{ij}\Box\tilde{\epsilon}_{hk}\,\mathrm{d}\Omega. \tag{3-29}$$

The total potential energy type formulation introduced by Huet, a minimum theorem for the displacements, can be expressed as follows:

> *among all the kinematically admissible (that is $\tilde{\epsilon}_{ij}=\frac{1}{2}(\tilde{u}_{i/j}+\tilde{u}_{j/i})$ and $\tilde{u}_i=\bar{u}_i$ on Γ_u) and "t-symmetrizing" virtual displacement and strain field histories, the actual solution field histories is the solution of the minimum problem*
>
> $$F^H(\hat{u})=\min_{\tilde{u}_i}F^H(\tilde{u}_i), \tag{3-30}$$
>
> *where*

$$\begin{aligned}F^H(\tilde{u}_i)&=\frac{1}{2}\int_{\Omega}R_{ijhk}\circ\tilde{\epsilon}_{ij}\Box\tilde{\epsilon}_{hk}\,\mathrm{d}\Omega-\int_{\Omega}F_i\Box\tilde{u}_i\,\mathrm{d}\Omega-\int_{\Gamma_p}p_i\Box\tilde{u}_i\,\mathrm{d}\Gamma\\&=\frac{1}{2}\int_{\Omega}\int_{0^-}^{T}\int_{0^-}^{T}R_{ijhk}(2T-t-\tau)\,\mathrm{d}\tilde{\epsilon}_{ij}(\tau)\,\mathrm{d}\tilde{\epsilon}_{hk}(t)\,\mathrm{d}\Omega\\&\quad-\int_{\Omega}\Big(\int_{0^-}^{T}-\int_{T}^{2T}\Big)F_i(2T-t)\,\mathrm{d}\tilde{u}_i(t)\,\mathrm{d}\Omega-\int_{\Gamma_p}\Big(\int_{0^-}^{T}-\int_{T}^{2T}\Big)p_i(2T-t)\,\mathrm{d}\tilde{u}_i(t)\,\mathrm{d}\Gamma.\end{aligned} \tag{3-31}$$

In general, the solution $\hat{u}_i$ of the problem (3-30) sought without the constraint (3-29) on $\tilde{u}_i$ and $\tilde{\epsilon}_{ij}$ differs from the true solution. In the following we will call $\hat{u}_i$ a "fictitious" solution. In the case $u(t)$ remains constant from T onwards, it is possible to show, however, that the "fictitious" solution "contains" in itself the real one if one sets, in it, $t=T$.

In fact, in this case, imposing the stationarity of Huet's functional (3-31) one gets the following equilibrium conditions (using the notation previously introduced, i.e., subscript 1 and 2 refer to the quantities defined over [0, T] and [T, $2T$], respectively):

$$-\frac{\partial}{\partial x_i}\Big[\int_{0^-}^{T}R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon_{1hk}(\tau)\Big]=b_{2j}(2T-t)\quad\text{in }\Omega \tag{3-32}$$

$$n_i\Big[\int_{0^-}^{T}R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon_{1hk}(\tau)\Big]=p_{2j}(2T-t)\quad\text{on }\Gamma_p \tag{3-33}$$

with $0\le t\le T$. As noted, in this equilibrium condition one has explicit functions of both t and T.

It can now be seen that the solution $\hat{\epsilon}_{1ij}(t, T)$ to problem (3-32)–(3-33) coincides, at time $t = T$, with the solution of the starting problem. In fact, setting $t = T$ in (3-32)–(3-33), one obtains

$$-\frac{\partial}{\partial x_i}\Big[\int_{0^-}^{T} R_{ijhk}(T-\tau)\,\mathrm{d}\epsilon_{1hk}(\tau)\Big] = b_{2j}(T) \quad \text{in } \Omega \tag{3-34}$$

$$n_i\Big[\int_{0^-}^{T} R_{ijhk}(T-\tau)\,\mathrm{d}\epsilon_{1hk}(\tau)\Big] = p_{2j}(T) \quad \text{on } \Gamma_p \tag{3-35}$$

which coincide with the true equilibrium equations for $\epsilon_{1ij}(t)$ at time $t = T$ if the loading terms fulfill the following continuity condition at time $t = T$:

$$b_{2i}(T) = b_{1i}(T); \qquad p_{2i}(T) = p_{1i}(T). \tag{3-36}$$

In a step-by-step time marching procedure, for each step, a displacement field constant over time and satisfying the kinematic boundary conditions can be considered a “t-symmetrizing” virtual field.

3.4. ***Tonti’s extended formulation.*** Consider the problem (2-7). Since $\mathcal{L}$ is not a symmetric operator with respect to the standard bilinear form, it is not possible to reformulate the problem as a classical variational statement. Nevertheless, Tonti [1984] proved that it is possible to transform (2-7) into the “extended” linear symmetric form

$$\widetilde{\mathcal{L}}\mathcal{K}\mathcal{L}\boldsymbol{u} = \widetilde{\mathcal{L}}\mathcal{K}\boldsymbol{b}, \tag{3-37}$$

where $\widetilde{\mathcal{L}}$ is the adjoint of $\mathcal{L}$ and $\mathcal{K}$ is a linear positive definite, invertible operator, symmetric with respect to the standard bilinear form, the so-called “integrating” or “kernel” operator, with $D(\mathcal{K}) \supseteq R(\mathcal{L})$ (where $D(\mathcal{K})$ is the domain of the operator $\mathcal{K}$ and $R(\mathcal{L})$ is the range of operator $\mathcal{L}$) and $R(\mathcal{K}) \subseteq D(\widetilde{\mathcal{L}})$. The transformation of (2-7) into the equivalent symmetric form (3-37) leads to what is called an “extended variational formulation”. In fact, it is easy to prove that the solution of problem (3-37) minimizes the following “extended” functional [Tonti 1984]:

$$F^{ext}(\boldsymbol{u}') = \tfrac{1}{2}\langle \mathcal{L}\boldsymbol{u}' - 2\boldsymbol{b}, \mathcal{K}\mathcal{L}\boldsymbol{u}'\rangle \tag{3-38}$$

where $\langle \cdot, \cdot \rangle$ is the standard bilinear form. Tonti’s statement is invertible: a function $\boldsymbol{u}'$ which minimizes the functional (3-38) is also the solution $\boldsymbol{u}$ of the problem (3-37), if this problem admits a solution. The choice of the kernel operator $\mathcal{K}$ is crucial in determining the conditioning of the problem which arises from imposing the stationarity of the functional (3-38). For instance, if the kernel is chosen as simply the identity operator, than the extremal property of the extended functional (3-38) reduces to a least square formulation, well known for its bad conditioning. Here, we consider, for the linear viscoelastic problem, the formulation proposed by Carini, Gelfi, Marchina [1995] in which $\mathcal{K} = \mathcal{S}^{-1}$ where $\mathcal{S}$ is the linear elastic part of $\mathcal{L}$. In the next section the formulation proposed by Carini, Gelfi and Marchina [1995] will be used as a comparison.

3.5. ***Particularization of the previous functionals to plane frame structures.*** In the case of plane frame structures, the classical hypothesis of Bernoulli–Navier and a first-order beam theory are assumed. Let the kinematic of a straight beam be defined by the vector $\boldsymbol{u} = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ where u and v are the x and y displacements of the centroid of the cross section at x (x is the longitudinal axis of the beam and y the transversal one).

Denoting by $N(x,t)$ and $M(x,t)$ the axial force and the bending moment, the constitutive viscoelastic law may be written as

$$N(x,t)=\int_{t_0^-}^{t} R(x,t-\tau)A(x)\,\mathrm{d}\epsilon(x,\tau), \tag{3-39}$$

$$M(x,t)=\int_{t_0^-}^{t} R(x,t-\tau)J(x)\,\mathrm{d}\chi(x,\tau), \tag{3-40}$$

where $\epsilon=\partial u/\partial x$ and $\chi=-\partial^2 v/\partial x^2$ are the axial strain and curvature, respectively, while $A(x)$, $J(x)$ and $R(x,t)$ are the cross-section area, its moment of inertia and the relaxation function, respectively. Then, denoting with $p(x,t)$ and $q(x,t)$ the longitudinal and transversal components of the distributed load, the general beam equilibrium equations become

$$-\frac{\partial}{\partial x}\int_{t_0^-}^{t} R(x,t-\tau)A(x)\frac{\partial\,\mathrm{d}u(x,\tau)}{\partial x}=p(x,t) \tag{3-41}$$

$$\frac{\partial^2}{\partial x^2}\int_{t_0^-}^{t} R(x,t-\tau)J(x)\frac{\partial^2\,\mathrm{d}v(x,\tau)}{\partial x^2}=q(x,t). \tag{3-42}$$

Using the previous notation, in the case of structures composed by a number m of straight beams (the index e is added to any quantity relevant to the generic beam e, $e=1,\dots,m$), Gurtin's functional, split Gurtin's functional and Huet's functional are, respectively,

$$\begin{aligned}F^G(u,v)=\sum_{e=1}^{m}\bigg\{\frac{1}{2}\int_0^{l^e}\int_{0^-}^{2T}\int_{0^-}^{2T-t}&\big[R^e(x,2T-t-\tau)A^e\,\mathrm{d}\epsilon^e(x,t)\,\mathrm{d}\epsilon^e(x,\tau)\\&+R^e(x,2T-t-\tau)J^e\,\mathrm{d}\chi^e(x,t)\,\mathrm{d}\chi^e(x,\tau)\big]\,\mathrm{d}x\\&-\int_0^{l^e}\int_{0^-}^{2T}p^e(x,2T-t)\,\mathrm{d}u^e(x,t)\,\mathrm{d}x-\int_0^{l^e}\int_{0^-}^{2T}q^e(x,2T-t)\,\mathrm{d}v^e(x,t)\,\mathrm{d}x\bigg\},\end{aligned}$$

$$\begin{aligned}&F^{SG}(u_1,u_2,v_1,v_2)\\&=\sum_{e=1}^{m}\bigg\{\frac{1}{2}\int_0^{l^e}\int_{0^-}^{T}\int_{0^-}^{T}\big[R^e(x,2T-t-\tau)A^e\,\mathrm{d}\epsilon_1^e(x,t)\,\mathrm{d}\epsilon_1^e(x,\tau)+R^e(x,2T-t-\tau)J^e\,\mathrm{d}\chi_1^e(x,t)\,\mathrm{d}\chi_1^e(x,\tau)\big]\,\mathrm{d}x\\&\quad+\int_0^{l^e}\int_{T}^{2T}\int_{0^-}^{2T-t}\big[R^e(x,2T-t-\tau)A^e\,\mathrm{d}\epsilon_1^e(x,\tau)\,\mathrm{d}\epsilon_2^e(x,t)+R^e(x,2T-t-\tau)J^e\,\mathrm{d}\chi_1^e(x,\tau)\,\mathrm{d}\chi_2^e(x,t)\big]\,\mathrm{d}x\\&\quad-\int_0^{l^e}\Big[\int_{0^-}^{T}p_2^e(x,2T-t)\,\mathrm{d}u_1^e(x,t)\,\mathrm{d}x+\int_T^{2T}p_1^e(x,2T-t)\,\mathrm{d}u_2^e(x,t)\,\mathrm{d}x\Big]\\&\quad-\int_0^{l^e}\Big[\int_{0^-}^{T}q_2^e(x,2T-t)\,\mathrm{d}v_1^e(x,t)\,\mathrm{d}x+\int_T^{2T}q_1^e(x,2T-t)\,\mathrm{d}v_2^e(x,t)\,\mathrm{d}x\Big]\bigg\}\end{aligned}$$

$$\begin{aligned}F^H(u,v)=\sum_{e=1}^{m}\bigg\{&\frac{1}{2}\int_0^{l^e}\int_{0^-}^{T}\int_{0^-}^{T}[R^e(x,2T-t-\tau)A^e\,\mathrm{d}\epsilon^e(x,t)\,\mathrm{d}\epsilon^e(x,\tau)+R^e(x,2T-t-\tau)J^e\,\mathrm{d}\chi^e(x,t)\,\mathrm{d}\chi^e(x,\tau)]\,\mathrm{d}x\\&-\int_0^{l^e}\Big(\int_{0^-}^{T}-\int_T^{2T}\Big)p^e(x,2T-t)\,\mathrm{d}u^e(x,t)\,\mathrm{d}x-\int_0^{l^e}\Big(\int_{0^-}^{T}-\int_T^{2T}\Big)q^e(x,2T-t)\,\mathrm{d}v^e(x,t)\,\mathrm{d}x\bigg\}\end{aligned}$$

4. Numerical simulations

In the case of *planar frame structures*, standard finite elements with two nodes will be considered, with three degrees of freedom in each node, with linear interpolation functions for the "extensional" degrees of freedom r_1 and r_2 and with cubic interpolation functions for the "bending" degrees of freedom r_3, r_4, r_5, r_6 (see Fig. 12c). Each beam can be subdivided into several straight finite elements. In the following we will suppose, for simplicity, that each element has constant cross-section and homogeneous material.

In the case of a *three-dimensional body*, let's subdivide the solid into N finite elements. In the following we will suppose, for the sake of simplicity, that each element is constituted by a homogeneous material (i.e., $R^e_{ijhk} = R^e_{ijhk}(t)$) and we will assume zero given displacements on Γ_u. Let $\boldsymbol{u}^e(\boldsymbol{x}, t)$ be the displacement vector of an internal point $\boldsymbol{x}$ of the finite element e ($e = 1, \dots, N$), u^e, v^e, w^e being the displacement components along x, y, z, respectively.

In the following we will denote with $\boldsymbol{\delta}(t)$ (with components $\delta_1, \dots, \delta_{n_s}$) the vector of the n_s degrees of freedom of the assembled solid, with respect to the global reference system both in the case of three-dimensional bodies and in the case of plane frame structures.

Time discretization. Each component $\delta_i(t)$ of the spatial degrees of freedom vector of the assembled solid is now written as a function of n_t time degrees of freedom $\boldsymbol{\beta}_i$ (with components $\beta_{i1}, \beta_{i2}, \dots, \beta_{in_t}$) through time interpolation functions collected into the matrix $\boldsymbol{M}(t)$, with $t \in 2T$:

$$\boldsymbol{\delta}(t) = \begin{bmatrix} \delta_1(t) \\ \vdots \\ \delta_{n_s}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{m}_1(t) & & \\ & \ddots & \\ & & \boldsymbol{m}_{n_s}(t) \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_{n_s} \end{bmatrix} = \boldsymbol{M}(t)\boldsymbol{B}. \tag{4-1}$$

In every example shown hereafter, all the spatial degrees of freedom $\delta_i(t)$ are discretized with respect to time using the same interpolation functions, i.e.,

$$\boldsymbol{m}_1 = \dots = \boldsymbol{m}_{n_s} = \boldsymbol{m} = [m_1 \ \cdots \ m_{n_t}]. \tag{4-2}$$

In this way, $n_s \times n_t$ is the total number of degrees of freedom.

In the examples discussed in the next section two different procedures will be adopted, specified case by case. In a first one the solution is sought in a single step over the entire time interval $2T$. In this case the numerical response is not obtained by means of a step-by-step method, but is given immediately on the whole temporal range without the need of a time integration procedure. In a second procedure the variational technique is applied on subintervals of the original integration time range $2T$, therefore becoming a step-by-step procedure.

Here we introduce three examples of calculation of the structural effects of creep. For the first two examples we use the hereditary Kelvin–Voigt model (see Figure 12, bottom). For the Kelvin–Voigt model, the relaxation function R and the creep function C for the uniaxial case are

$$R(t-\tau) = E_0 - \frac{E_0^2}{E_0+E_1}\left(1 - e^{-(E_0+E_1)(t-\tau)/\eta}\right) = E_\infty + (E_0 - E_\infty)e^{-(t-\tau)/T^*}, \tag{4-3}$$

$$C(t-\tau) = \frac{1}{E_0} + \frac{1}{E_1}\left(1 - e^{-E_1(t-\tau)/\eta}\right) = \frac{1}{E_\infty} + \left(\frac{1}{E_0} + \frac{1}{E_1}\right)e^{-(t-\tau)/\tau^*}, \tag{4-4}$$

where

$$E_\infty = \frac{E_0 E_1}{E_0+E_1}; \quad \tau^* = \frac{\eta}{E_1}; \quad T^* = \frac{\eta}{E_0+E_1}. \tag{4-5}$$

The extension to the multiaxial case is straightforward.

In all the examples, negative exponential time interpolation functions are used:

$$\boldsymbol{m} = \left[1, e^{-t/t^*}, e^{-t/10t^*}, e^{-t/100t^*}, \dots\right], \tag{4-6}$$

where t^* is a parameter to be defined and has the meaning of a relaxation time. For the examples considered herein, these interpolation functions were empirically found to be better than polynomial or damped polynomial functions.

The asymptotic character of the viscoelastic responses over large time intervals justifies the above choice of the negative exponential interpolation functions which appear more suitable for a good approximation of the structural behavior even in the presence of a small number of temporal degrees of freedom, especially when solving the problem over the entire time interval. Polynomial time interpolation functions could also be chosen, but only for short enough time intervals.

The sequence of relaxation times t^*, $10t^*$, $100t^*$, ..., adopted for the time interpolation functions in (4-6), has provided good numerical results. Both this type of sequence and the numerical values of t^*, in the absence of theoretical indications, have been found by trial and error.

For each example, significant displacements and/or stresses are plotted versus time, for different time intervals and, in the case of a single step solution (first procedure, see above) for 1, 2, 3, 4, and 5 time degrees of freedom. In each diagram both the exact solution and the percentage error are also plotted.

Example A (viscoelastic beam on elastic support). The structure is a Bernoulli–Navier fully fixed beam of length $2l$, with an elastic support (a spring of stiffness k_m) in the middle; the beam is homogeneous and has a constant cross-section. A uniformly distributed transversal load q is applied at time $t_0 = 0$ and remains constant. The beam is subdivided into two finite elements of the type described previously.

First, we consider *Gurtin's formulation*. After the spatial discretization, Gurtin's functional takes the form

$$F^G = \frac{1}{2}\,k_m \int_{0^-}^{2T} v'(l, 2T-t)\,\mathrm{d}v'(l,t) + \frac{1}{2}\left[\frac{24J}{l^3}\right]\int_{0^-}^{2T} \mathrm{d}v'(l,t)\int_{0^-}^{2T-t} R(2T-t-\tau)\,\mathrm{d}v'(l,\tau) - l\int_{0^-}^{2T} q(2T-t)\,\mathrm{d}v'(l,t), \tag{4-7}$$

where $v(l,t)$, the only nonzero δ degree of freedom, is the vertical displacement of the middle section of the beam.

We start by discussing the results obtained finding the entire solution in a single step. In Figure 1, the ratio between the displacement $v(l,t)$ of the middle section at time t and the initial elastic displacement $v(l,0)$ is plotted. The time shape functions are of the negative exponential type (4.6) with $t^* = 70$, 130 and 180 days, the time $2T$ ranges from 0 to 1500 days and the number of time degrees of freedom from 1 to 5. After 750 days, $v(l)$ remains practically constant over time. The diagrams show that the presented method allows a good interpretation of the viscoelastic behavior of the structure. With three time degrees of freedom the solution is satisfactory within the entire time range.

The accuracy of the numerical results depends on the choices of t^*. In this example, for $t^* = 130$ days very precise results are obtained with just two temporal degrees of freedom. The optimal determination of t^* remains an open problem.

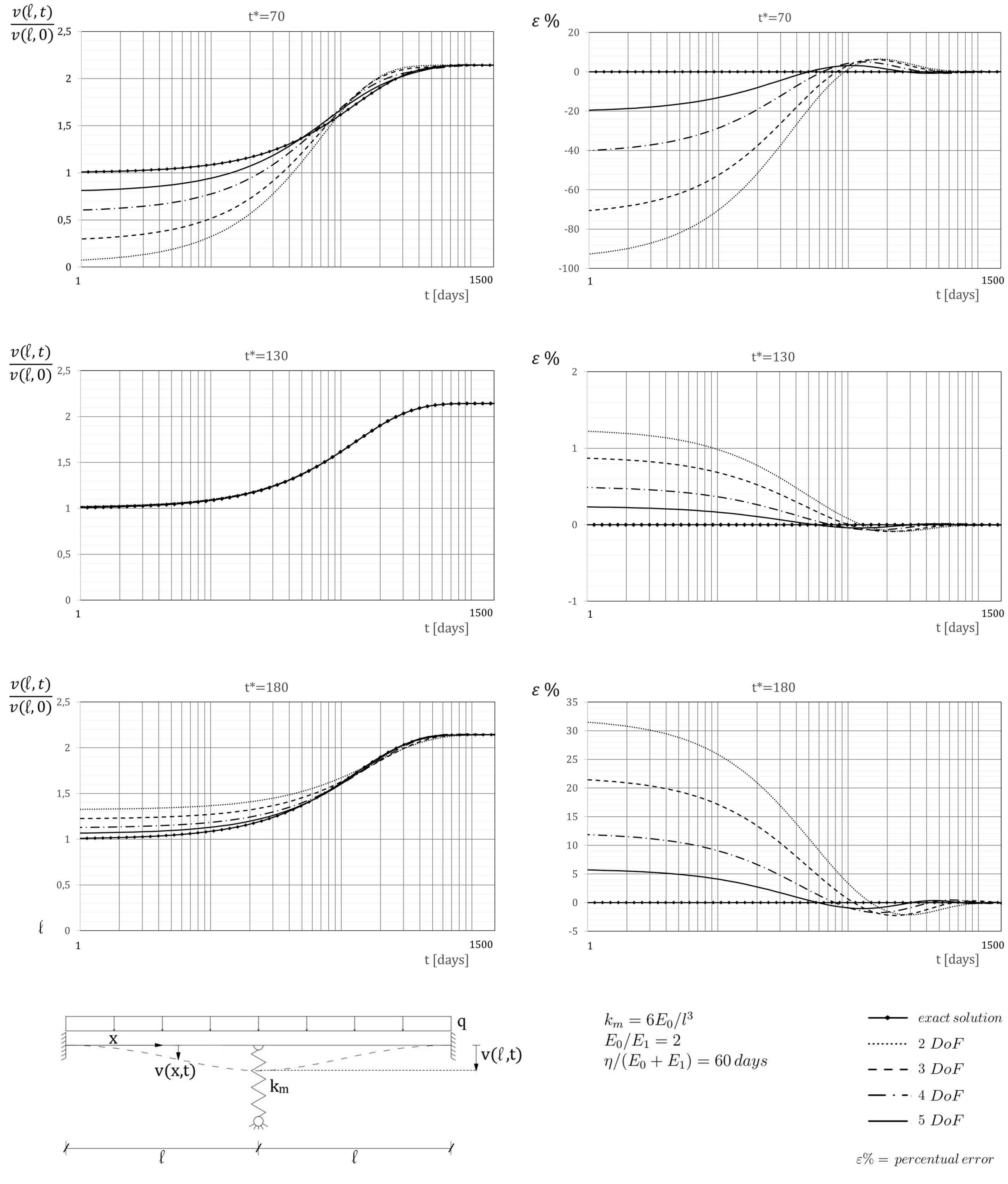


Figure 1. Example A: Beam on elastic support. Numerical results using Gurtin's formulation.

Figure 2 shows the numerical results obtained by splitting the loading time interval $2T = 1500$ days into two equal parts and using the *split Gurtin's formulation* (3-24). After the spatial discretization, functional (3-25) takes the form

$$\begin{aligned} F^{SG} = \frac{1}{2} k_m \int_{0^-}^{T} v_2'(l, 2T-t)\, \mathrm{d}v_1'(l,t) + \frac{1}{2} k_m \int_{T}^{2T} v_1'(l, 2T-t)\, \mathrm{d}v_2'(l,t) \\ + \frac{1}{2}\left[\frac{24J}{l^3}\right] \int_{0^-}^{T} \mathrm{d}v_1'(l,t) \int_{0^-}^{T} R(2T-t-\tau)\, \mathrm{d}v_1'(l,\tau) \\ + \frac{1}{2}\left[\frac{24J}{l^3}\right] \int_{0^-}^{T} \mathrm{d}v_1'(l,t) \int_{T}^{2T-t} R(2T-t-\tau)\, \mathrm{d}v_2'(l,\tau) \\ + \frac{1}{2}\left[\frac{24J}{l^3}\right] \int_{T}^{2T} \mathrm{d}v_2'(l,t) \int_{0^-}^{2T-t} R(2T-t-\tau)\, \mathrm{d}v_1'(l,\tau) \\ - l \int_{0^-}^{T} q_2(2T-t)\, \mathrm{d}v_1'(l,t) - l \int_{T}^{2T} q_1(2T-t)\, \mathrm{d}v_2'(l,t) \end{aligned} \tag{4-8}$$

Here we have considered from 2 to 5 time degrees of freedom both in the first and in the second subinterval. Of course, the accuracy increases significantly in the case of splitting, due to a doubling of the degrees of freedom. As it is possible to see in Figure 2, a constant time function is sufficient to approximate well the solution in the second subinterval. Figure 3 shows the numerical results for the same example, obtained using n_t degrees of freedom on the first subinterval and $(n_t - 1)$ time degrees of freedom on the second one. Note that the solution in the first subinterval significantly worsens compared to the case of an equal number of degrees of freedom in both subintervals.

Figure 4 shows the case of $(n_t - 2)$ degrees of freedom in the second subinterval: the solution in the first subinterval further deteriorates respect to the case of $(n_t - 1)$ number of degrees of freedom in the second subinterval. To understand the reason for this strange behavior, considering that just one degree of freedom is enough to obtain a good approximation in the second subinterval, the exact solution in the second subinterval was replaced in the functional, thus reducing the variational problem to a minimum problem only in the first subinterval. The obtained numerical result implies $v_1(l)$ becoming unbounded.

This strong instability can be explained by analyzing the equation obtained by imposing the stationarity of the functional, that is,

$$\int_{0^-}^{T} R(2T-t-\tau)\, \mathrm{d}v(l,\tau) = f(t) \tag{4-9}$$

or, in operatorial form,

$$Av = f \tag{4-10}$$

with $f(t)$ the known term obtained by the sum of the original known term and the one resulting from the integral on the second subinterval.

The equation $Av = f$ is called well-posed or properly posed if A is bijective and the inverse operator A^{-1} is continuous. Otherwise the equation is called ill-posed or improperly posed. Our operator A is not surjective, then equation $Av = f$ is not solvable for all f, and A is also not injective, so that the equation may have more than one solution. In general, linear integral equations of the first kind with continuous (as in our case) or weakly singular kernels provide typical examples of ill-posed problems.

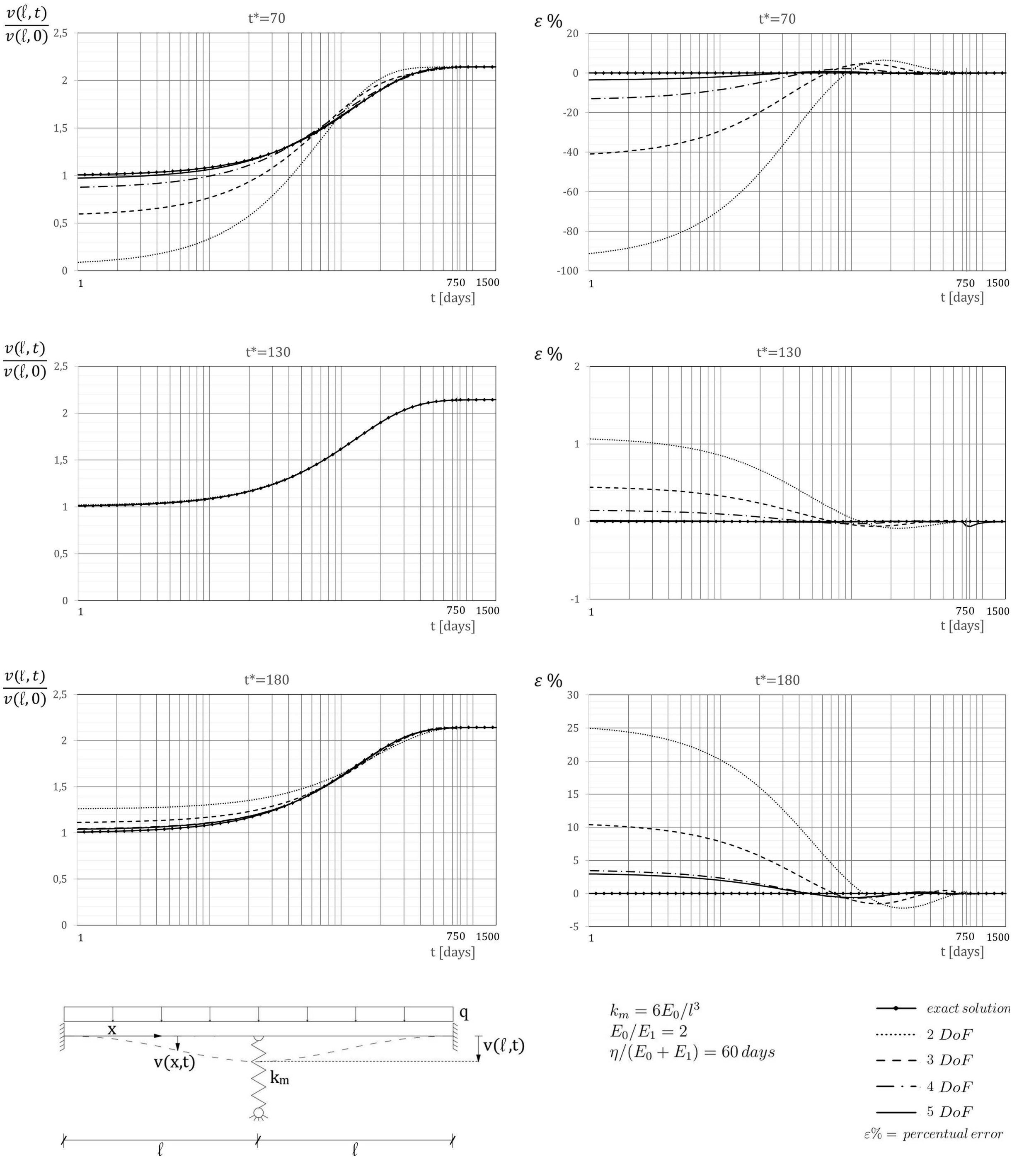


Figure 2. Example A: Beam on elastic support. Numerical results using split Gurtin's formulation with same number of time degrees of freedom in the first and in the second time subinterval.

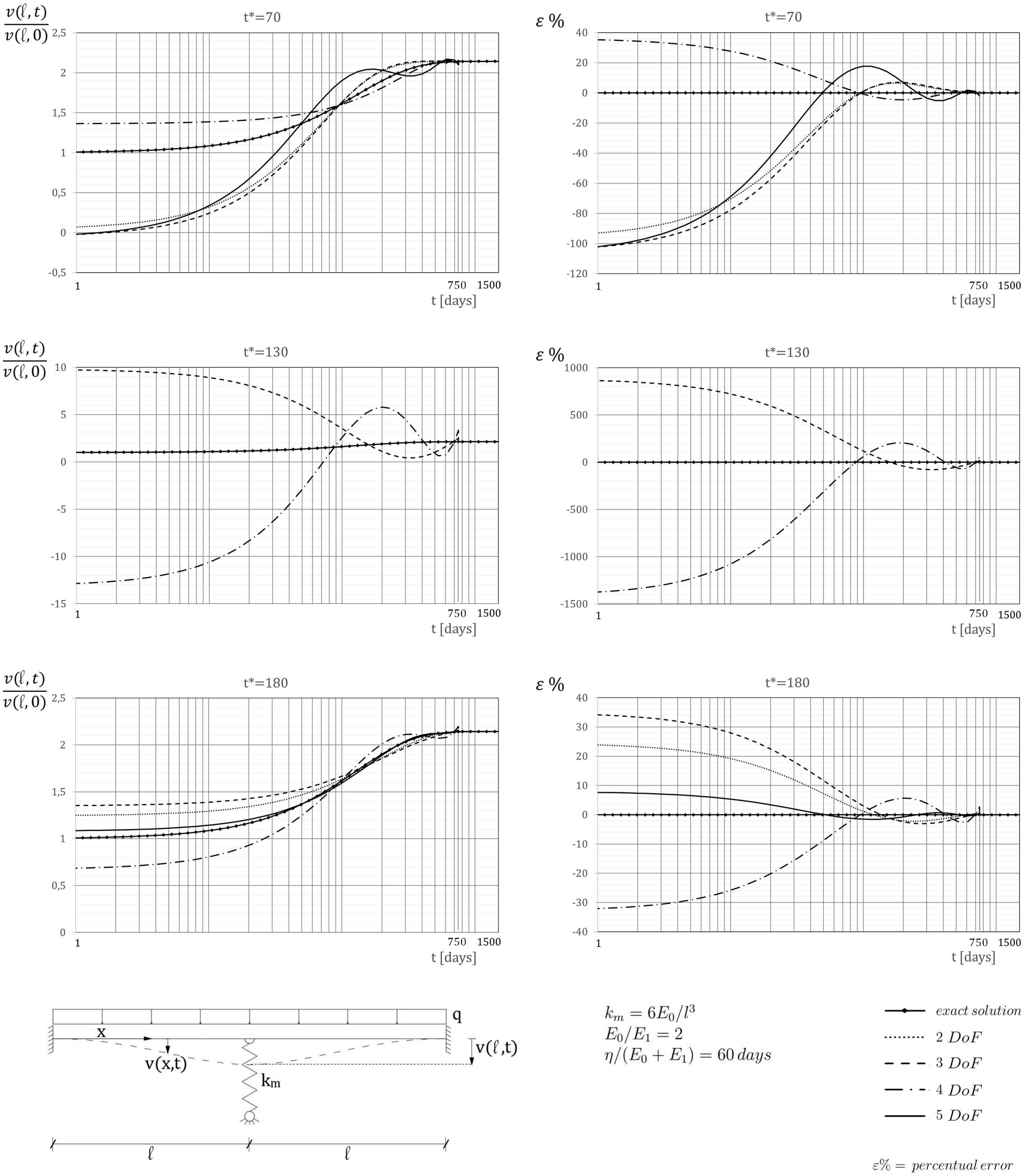


Figure 3. Example A: Beam on elastic support. Numerical results using split Gurtin's formulation with n degrees of freedom in the first subinterval and $(n-1)$ DoF in the second subinterval.

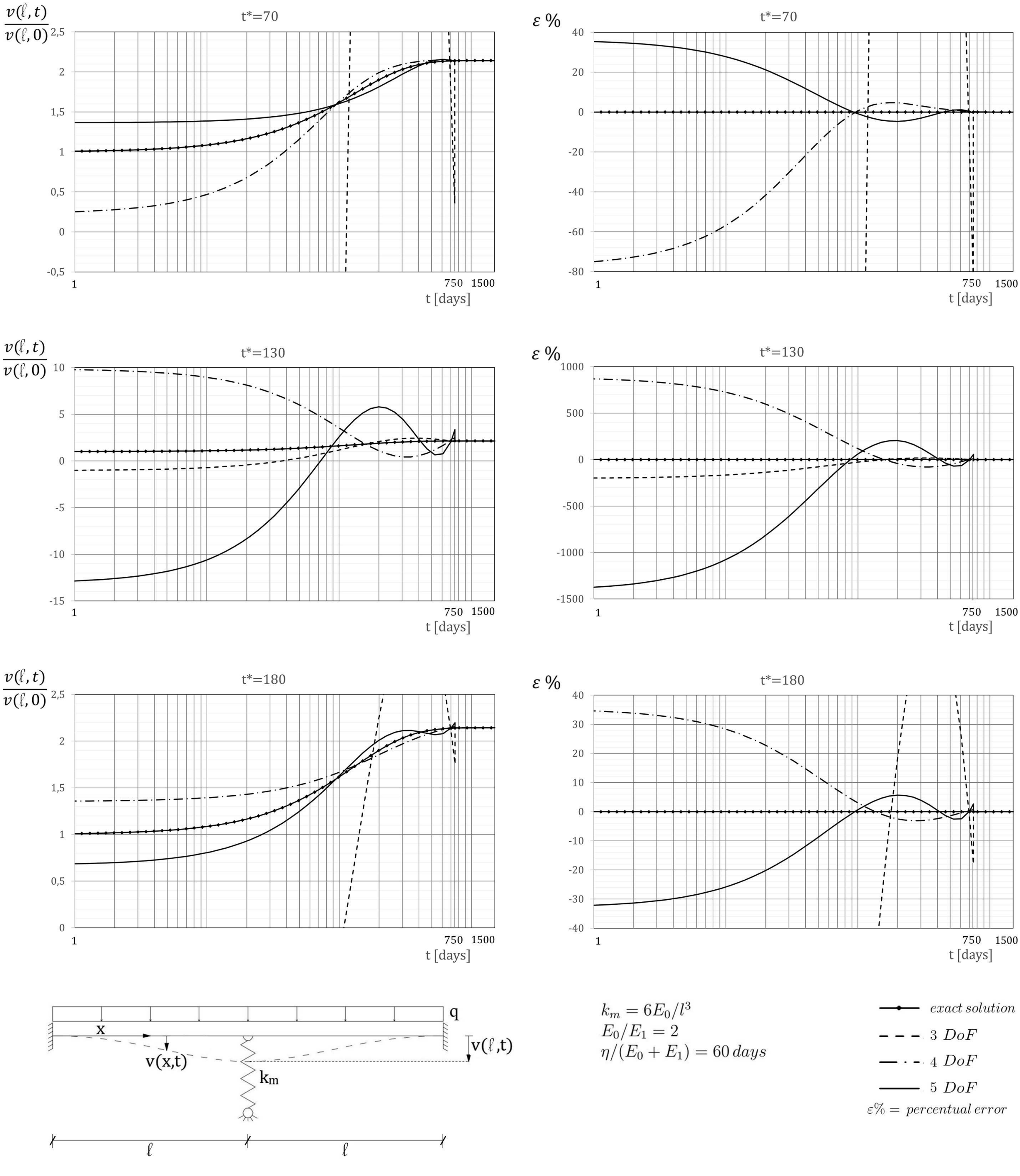


Figure 4. Example A: Beam on elastic support. Numerical results using split Gurtin's formulation with n degrees of freedom in the first subinterval and $(n-2)$ DoF in the second subinterval.

One of the fundamental facts in the theory of operatorial equations of the first kind is that the inverse of a completely continuous operator is not bounded. So, even if f_a and f_b are two mutually close elements in a Hilbert space and both equations $Av = f_a$ and $Av = f_b$ are solvable, the respective solutions $v_a = A^{-1} f_a$ and $v_b = A^{-1} f_b$ can differ strongly from each other. Therefore, a small difference in the known term of the starting equation can lead to a large error in the solution (see, for instance, [Kress 1989]).

The ill-posed nature of an equation, of course, has consequences on its numerical treatment. The fact that an operator A does not have a bounded inverse means that the condition numbers of its finite-dimensional approximations grow with the quality of the approximation. This means that increasing the accuracy of the approximation for the operator A will cause the approximate solution to the equation to become less and less reliable.

To at least partially mitigate these instabilities, a Tikhonov "regularization" was considered (see, for example, [Kress 1989]). A regularizing "penalty" term $\frac{1}{2}\alpha \int_{0^-}^{T} v'^2(l,t)\,\mathrm{d}t$, dependent on a parameter α, has therefore been added to the functional (4-8). This means replacing the given equation of the first kind with the following equation of the second kind:

$$\alpha v(l,t) + \int_{0^-}^{T} R(2T-t-\tau)\,\mathrm{d}v(l,\tau) = f(t). \tag{4-11}$$

The choice of α is very important but linked in general to great difficulties, and we made it by trial and error. The results of this regularization, shown in Figure 5, is not very satisfactory. Figure 6 shows the regularized result obtained by adding the modified quadratic term $\frac{1}{2}\alpha \int_{0^-}^{T} v'^2(l, 2T-t)\,\mathrm{d}t$ to the functional (4-8). In this case a good result is obtained by using two temporal degrees of freedom and $\alpha = 0.007925$.

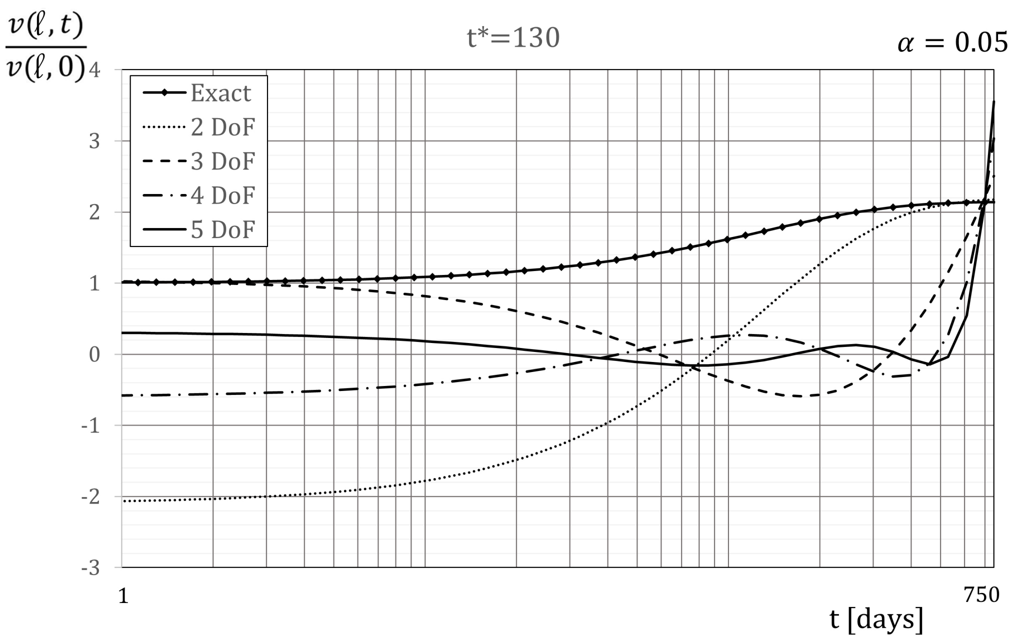


Figure 5. Example A: Beam on elastic support. Numerical results using a Tikhonov regularization with the addition of $\frac{1}{2}\alpha \int_{0^-}^{T} v'^2(l,t)\,\mathrm{d}t$ to functional (4-8) ($t^* = 130$ days, $\alpha = 0.05$).

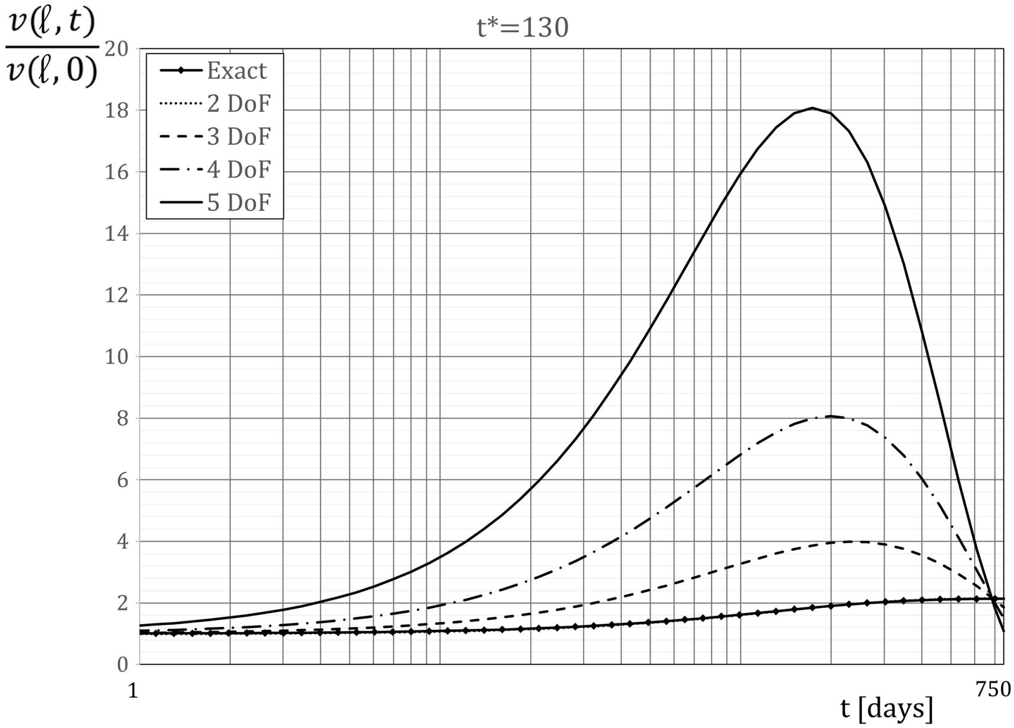


Figure 6. Example A: Beam on elastic support. Numerical results using a Tikhonov regularization with the addition of $\frac{1}{2}\alpha \int_{0^-}^{T} v'^2(l, 2T - t)\, \mathrm{d}t$ to functional (4-8). ($t^* = 130$ days, $\alpha = 0.007925$ for 2 DoF; $\alpha = 0.06594$ for 3 DoF; $\alpha = 0.29431$ for 4 DoF; $\alpha = 1.40348$ for 5 DoF).

Let us now analyze the behavior of *Huet's formulation*.

Because of the theoretical restrictions summarized in Section 3, Huet's formulation does not work numerically if applied in a single loading step, unless during the step the solution tends to becomes constant. For this reason, in the next examples we have considered step-by-step procedures only, with a reasonably large number of time steps.

Once again because of the same theoretical restrictions, standard step-by-step techniques, in the case of Huet's formulation, become rapidly unstable with an increasing number of time degrees of freedom in a single time step: actually, they seem to work only by adopting a single time degree of freedom in each step. By standard we mean a procedure in which (i) the time integrals are computed only on the current time step and (ii) the initial conditions at each time step are the end conditions at the previous time step.

Nevertheless, considering the convolutive nature of the involved functionals, one is lead to adopt nonstandard step-by-step procedure in which (i) all the time integrals are computed starting from the global loading initial time $t = 0$, and (ii) the time interpolation functions are defined as the sum of Heaviside step functions, each starting at the beginning of a new time step. In this way, the degrees of freedom can be obtained in a simple recursive form and the solution seems to be more accurate with respect to a standard one with the same numbers of unknowns. This happens because in this way the past history is "remembered". In the sequel, we will apply this idea also to Gurtin's approach.

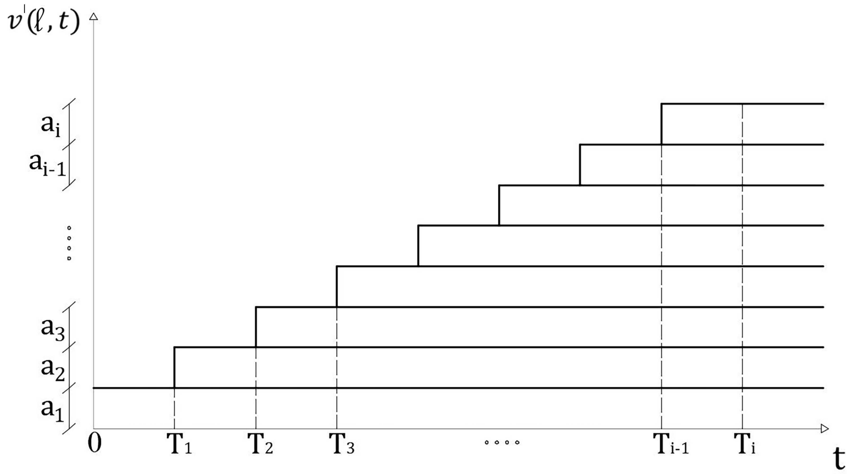


Figure 7. Example A: Beam on elastic support, nonstandard step-by-step procedure. Step time interpolation functions.

Since the integrals in Huet's functional are defined between 0 and T, in this case the solution must be found in this interval only. The time interval $[0, T]$ is subdivided into n temporal subintervals defined by the sequence of instants $T_0, T_1, \ldots, T_i, \ldots, T_n = T$. The analysis is carried out in sequence in the intervals $T_0 \leq t \leq T_1$, $T_0 \leq t \leq T_2$, etc., always assuming, as initial conditions, the conditions at the initial time $t = 0$.

Following this idea, all the time interpolation functions used in the previous steps are prolonged to the current temporal step by means of the Heaviside function. This means that the approximating function at the step $T_{i-1} - T_i$ is represented using a sum of step time interpolation functions (see Figure 7)

$$v'(l,t) = \sum_{n=1}^{i} a_n H(t - T_{n-1}), \qquad T_{i-1} \leq t \leq T_i \tag{4-12}$$

with $a_1, \ldots, a_{i-1}$ already known from the previous steps. The time marching procedure starts from the initial time, adding a new interpolation function to each step. By imposing the stationarity of the Huet functional, rewritten for the example here studied and for time interval $T_0 \leq t \leq T_i$

$$\begin{aligned} &F^H(v'(l,t,T_i)) \\ &= \frac{1}{2} k_m v'^2(l,T_i) + \frac{1}{2}\left[\frac{24J}{l^3}\right] \int_{0^-}^{T_i}\int_{0^-}^{T_i} R(2T_i - t - \tau)\,\mathrm{d}v'(l,\tau)\,\mathrm{d}v'(l,t) - l\int_{0^-}^{T_i} q(2T_i - t)\,\mathrm{d}v'(l,t) \end{aligned} \tag{4-13}$$

and making $t \to T_i$, we go back to the starting equation (see Section 3.3). Using (4-12) as an admissible function, we obtain

$$a_n = \frac{q(2T_n - T_{n-1})l - \sum_{i=1}^{n-1} a_i\big[(24J/l^3)\,R(2T_n - T_{i-1} - T_{n-1}) + k_m\big]}{(24J/l^3)\,R(2T_n - 2T_{n-1}) + k_m}, \qquad n \geq 2. \tag{4-14}$$

Figure 8 shows the results from 0 to 750 days for $t \to T_i$, and, as a test, $t \to T_{i-1} + \Delta T_i/2$, $t \to T_{i-1} + \Delta T_i/4$, $t \to T_{i-1} + \Delta T_i/8$, $t \to T_{i-1}$ (with $\Delta T_i = T_i - T_{i-1}$). Surprisingly, the best results are obtained for $t \to T_{i-1} + \Delta T_i/4$ and not for $t \to T_i$, as instead explained by the theory summarized in Section 3.3. In Figures 9, 10, and 11 the results obtained using both the Huet formulation and the Gurtin formulation (now used with the same nonstandard step-by-step procedure) are reported for a constant load, for a load that varies linearly over time and for a load that varies sinusoidally over time, respectively. It should be noted that with Gurtin's formulation one has not the problem of finding an optimal t. Gurtin's results are slightly better than Huet's results for the first two cases while they are slightly worse for the last case.

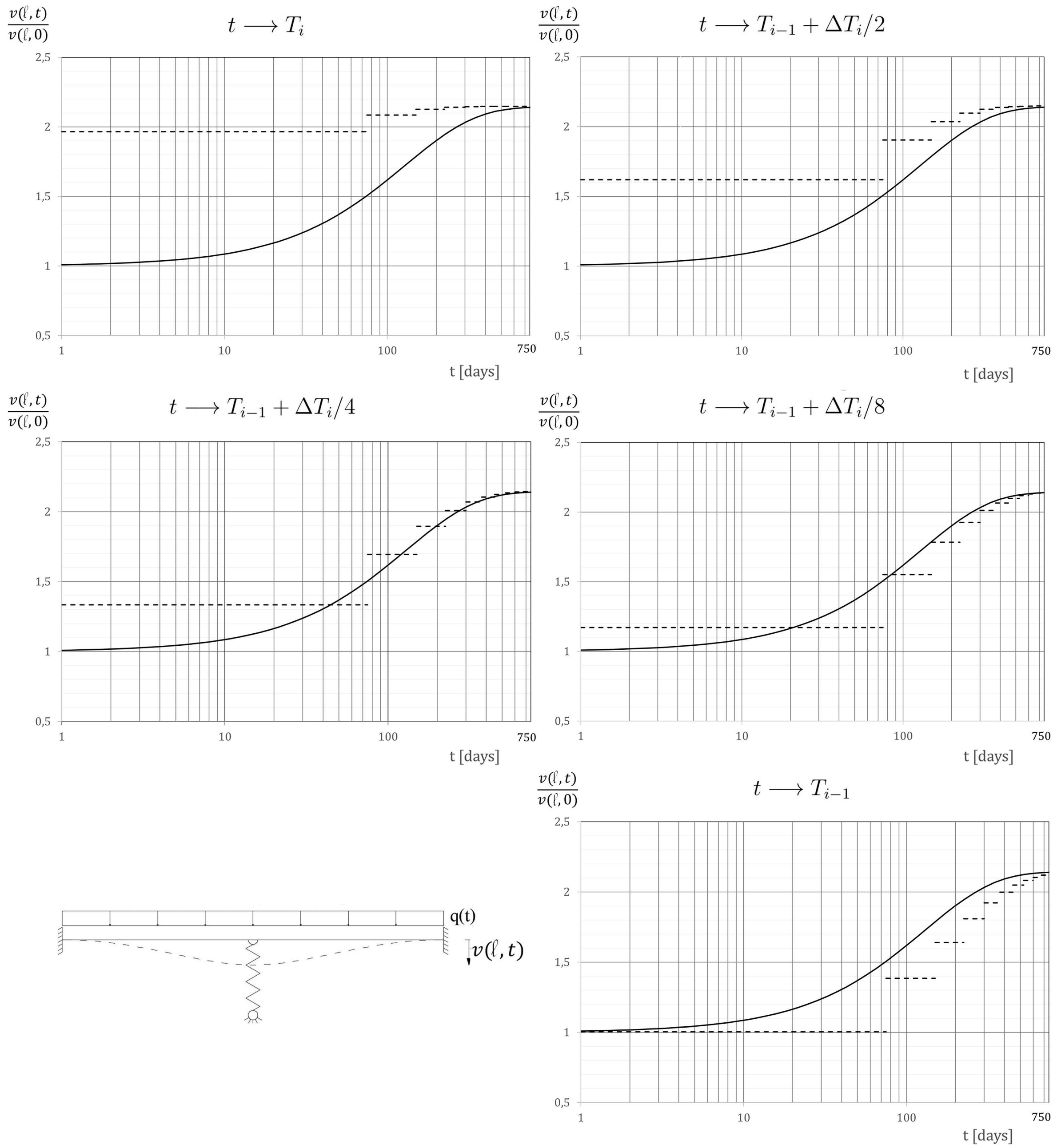


Figure 8. Example A: Beam on elastic support. Numerical results using step time interpolation functions in the case of constant load.

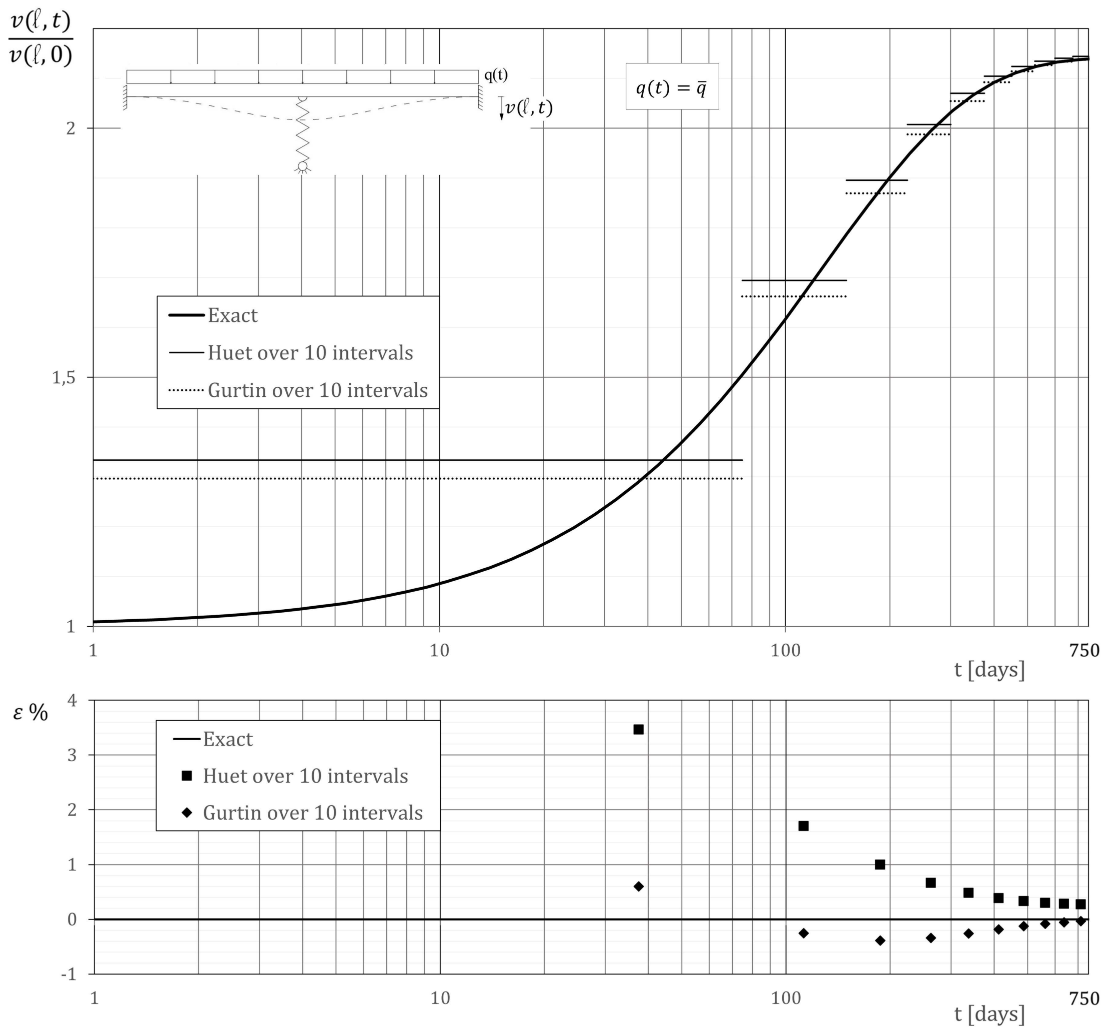


Figure 9. Example A: Beam on elastic support. Numerical results using step time interpolation functions in the case of constant load, using Huet's formulation and Gurtin's formulation.

Example B (cable-stayed bridge). The structure (Figure 12) represents a simple cable stayed bridge. The girder and the piers are of concrete with geometrical and rheological properties constant along the axis; the stays are made of steel. The uniform load q is applied on the girder at time t_0 = 28 days and remains constant in time. The structure is discretized with seven beam elements having the characteristics shown in Figure 12. The reference solution is obtained with the program ABAQUS [ABAQUS 2018].

Once again, the solution is here found in a single time step. Negative exponential time interpolation functions are used, as in the first solutions of the previous example. The results obtained in [Carini et al. 1995], using the same interpolation functions, are compared with the present ones and, therefore, as in that work, $t^* = 120$ days is taken. Physical phenomena start at 28 days because the structure is designed in reinforced concrete. Two and three temporal degrees of freedom are considered. The numerical results,

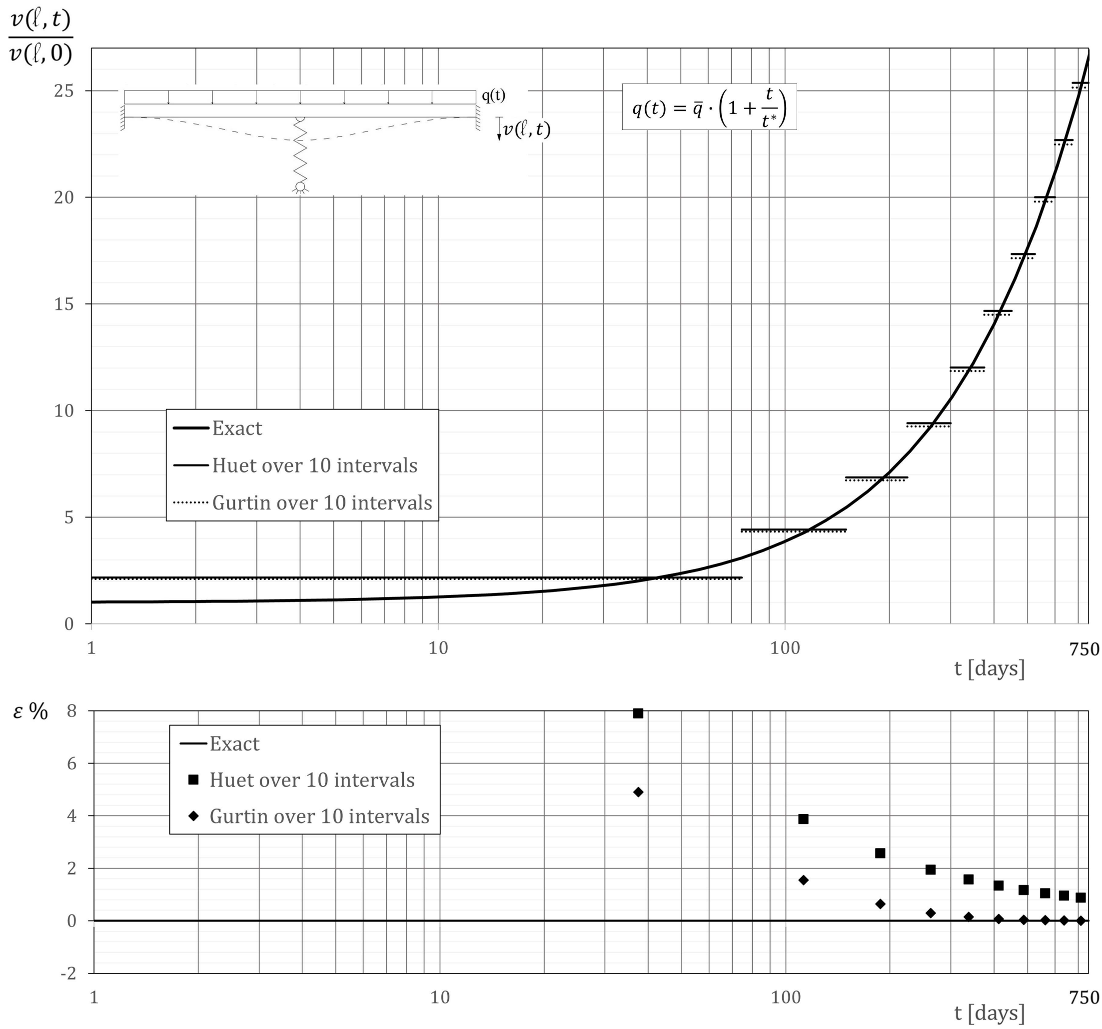


Figure 10. Example A: Beam on elastic support. Numerical results using step time interpolation functions in the case of load q linearly varying over time, using Huet's formulation and Gurtin's formulation.

using Gurtin's approach, are reported in Figure 13 for three distinct time intervals: 28–100, 28–1000 and 28–10000 days. Figure 13 also shows the reference solution obtained with the ABAQUS program with a considerable degree of precision and, therefore, to be considered "exact" from an engineering point of view. As can be seen, the approximate solution is already good with only two temporal degrees of freedom over the entire time interval analyzed, showing errors lower than 5% almost everywhere.

Figure 14 shows the results obtained with the split Gurtin's formulation and the same interpolation functions. The results are not very different from those obtained with the unsplit Gurtin's formulation, in spite of a doubling of the unknowns. Table 1 compares the conditioning indexes of the matrices of the coefficients of the resolving systems in the case of use of the least squares method, of the method based on Tonti's extended functionals, and of Gurtin's method. The results related to the least square

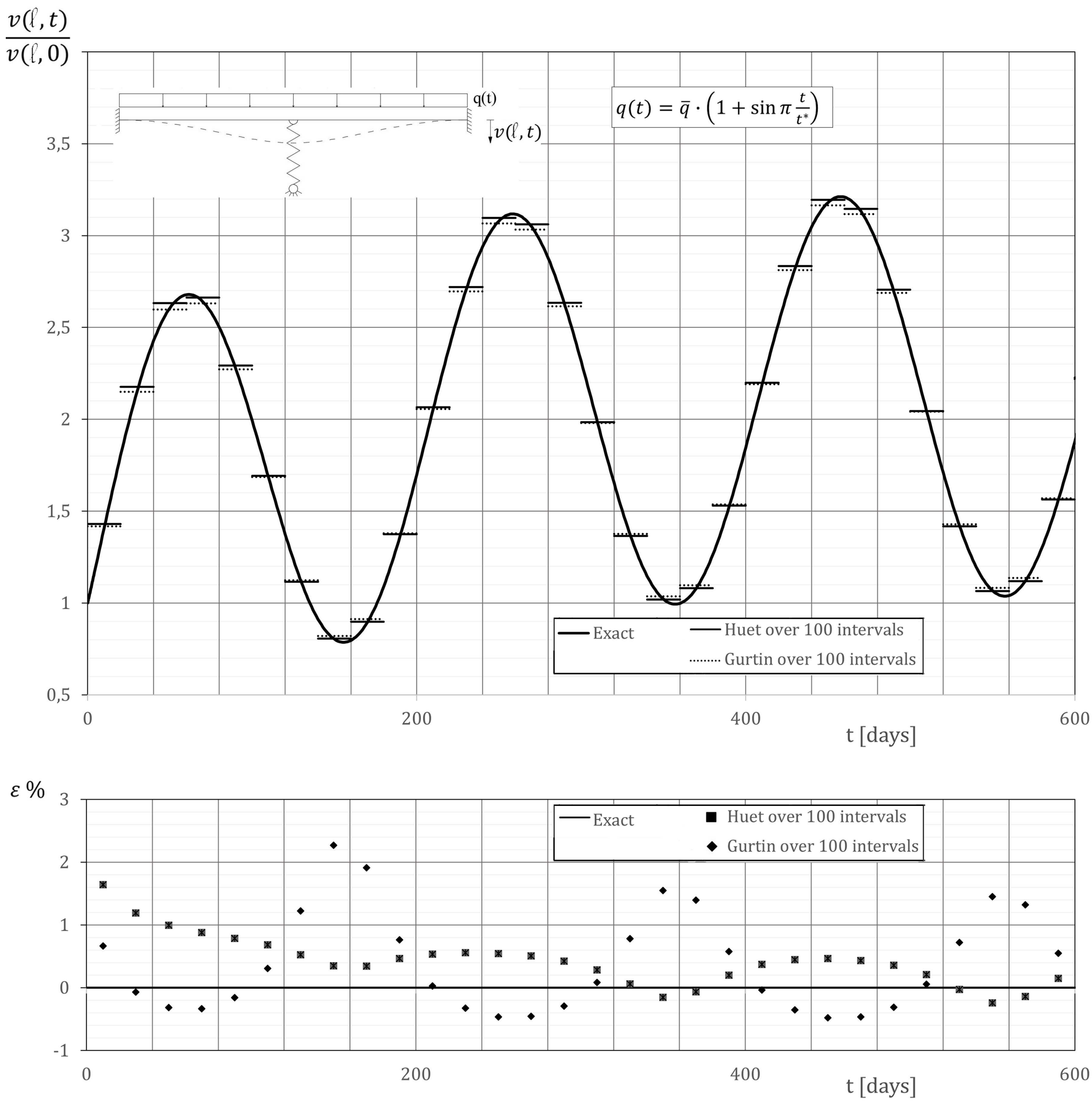
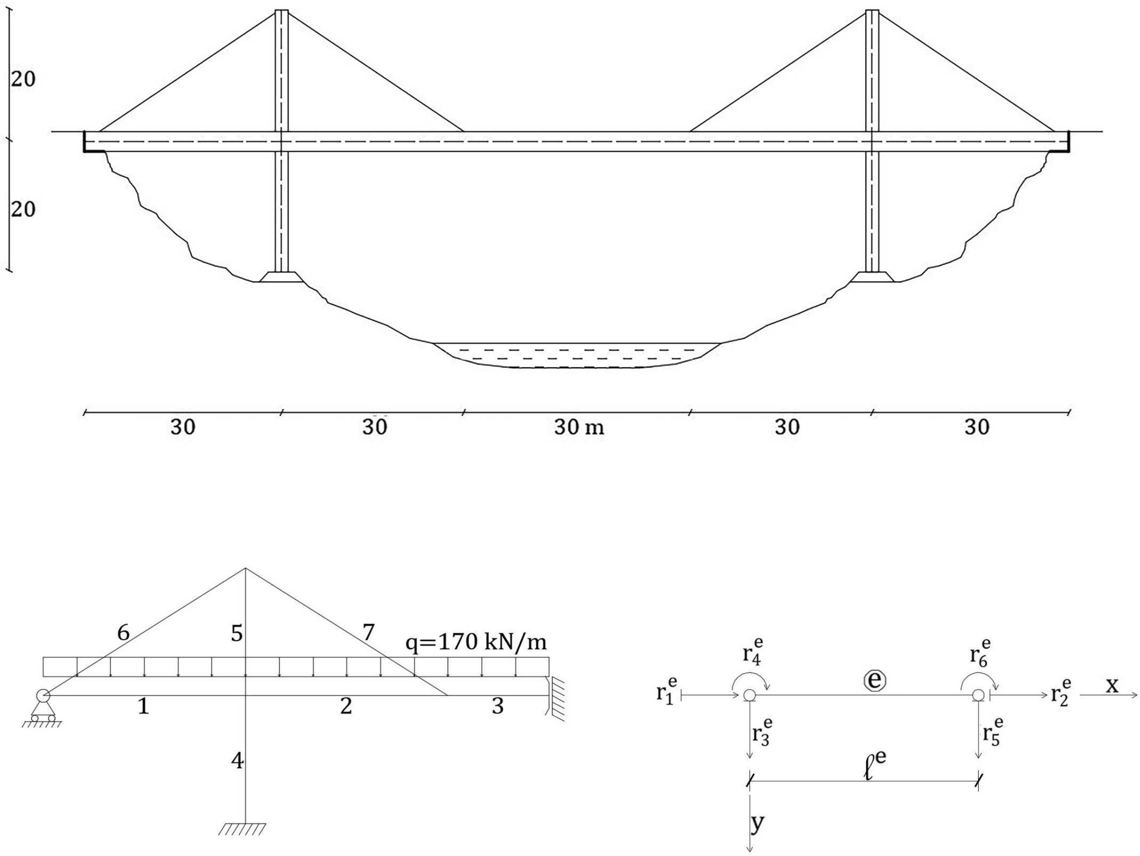
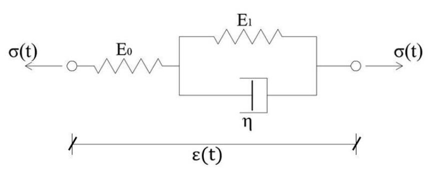


Figure 11. Example A: Beam on elastic support. Numerical results using step time interpolation functions in the case of load q sinusoidally varying over time, using Huet's formulation and Gurtin's formulation.

method and to the Tonti method are taken from Carini, Gelfi, Marchina [1995]. The results with 1 or 2 temporal degrees of freedom, obtained by the different methods, are comparable to each other. It is noted that the conditioning index of Gurtin's method is of the same order of magnitude as that associated to Tonti's extended functional, orders of magnitude lower than the method of least squares. Furthermore, the Gurtin method is much less expensive because it does not require the inversion of the elastic operator, as necessary in the case of Tonti's extended functional where the inverse of the elastic operator acts as a preconditioning operator.



RHEOLOGICAL PROPERTIES

KELVIN-VOIGT	ELEM.	E_0 kN/m^2	E_1 kN/m^2	η $gg \cdot kN/m^2$
	1	3.80e7	1.90e7	2.70e9
	2	3.80e7	1.90e7	2.70e9
	3	3.80e7	1.90e7	2.70e9
	4	3.00e7	1.50e7	2.70e9
	5	3.00e7	1.50e7	2.70e9
	6	steel $E = 2.10e8$		
	7	steel $E = 2.10e8$		

GEOMETRICAL PROPERTIES

ELEM.	AREA m^2	J m^4
1	5.50	8.17
2	5.50	8.17
3	5.50	8.17
4	5.00	2.86
5	2.00	0.67
6	0.04	-
7	0.04	-

Figure 12. Example B. Top: cable-stayed bridge. Center left: finite element mesh. Center right: beam finite element. Bottom: geometrical and rheological properties.

The conditioning indexes related to the split Gurtin method are similar to those of the unsplit Gurtin method, except for the case related to the interval 28–10000 days and for 2 and 3 degrees of freedom where the conditioning index jumps to very high levels. Using time interpolation functions that are still negative exponential but with different relaxation times (10 t^* and 100 t^* for the second and third degrees of freedom, respectively) yielded conditioning indices comparable to those for the Gurtin approach (see the last column of Table 1).

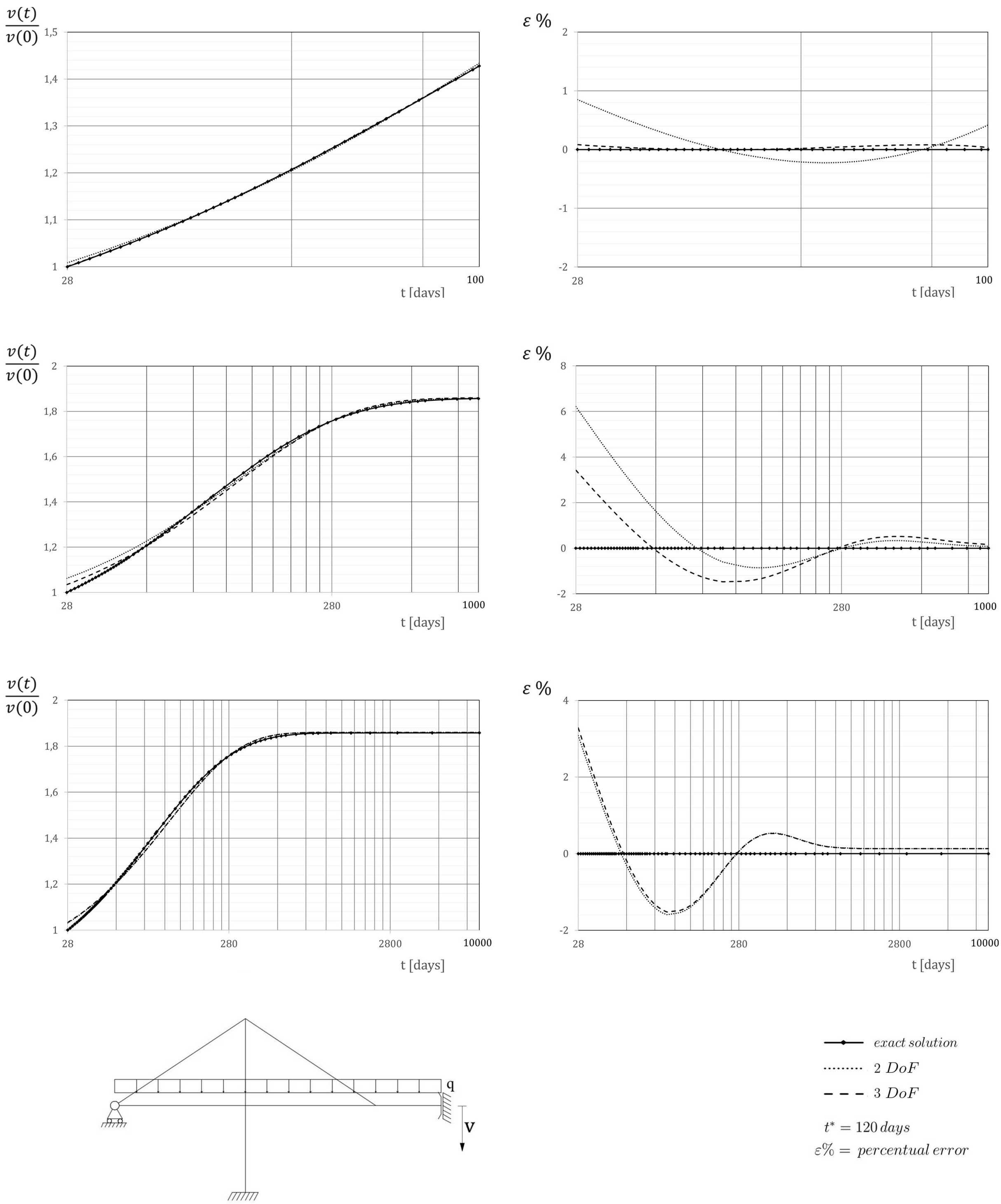


Figure 13. Example B: Cable-stayed bridge. Numerical results using Gurtin's approach.

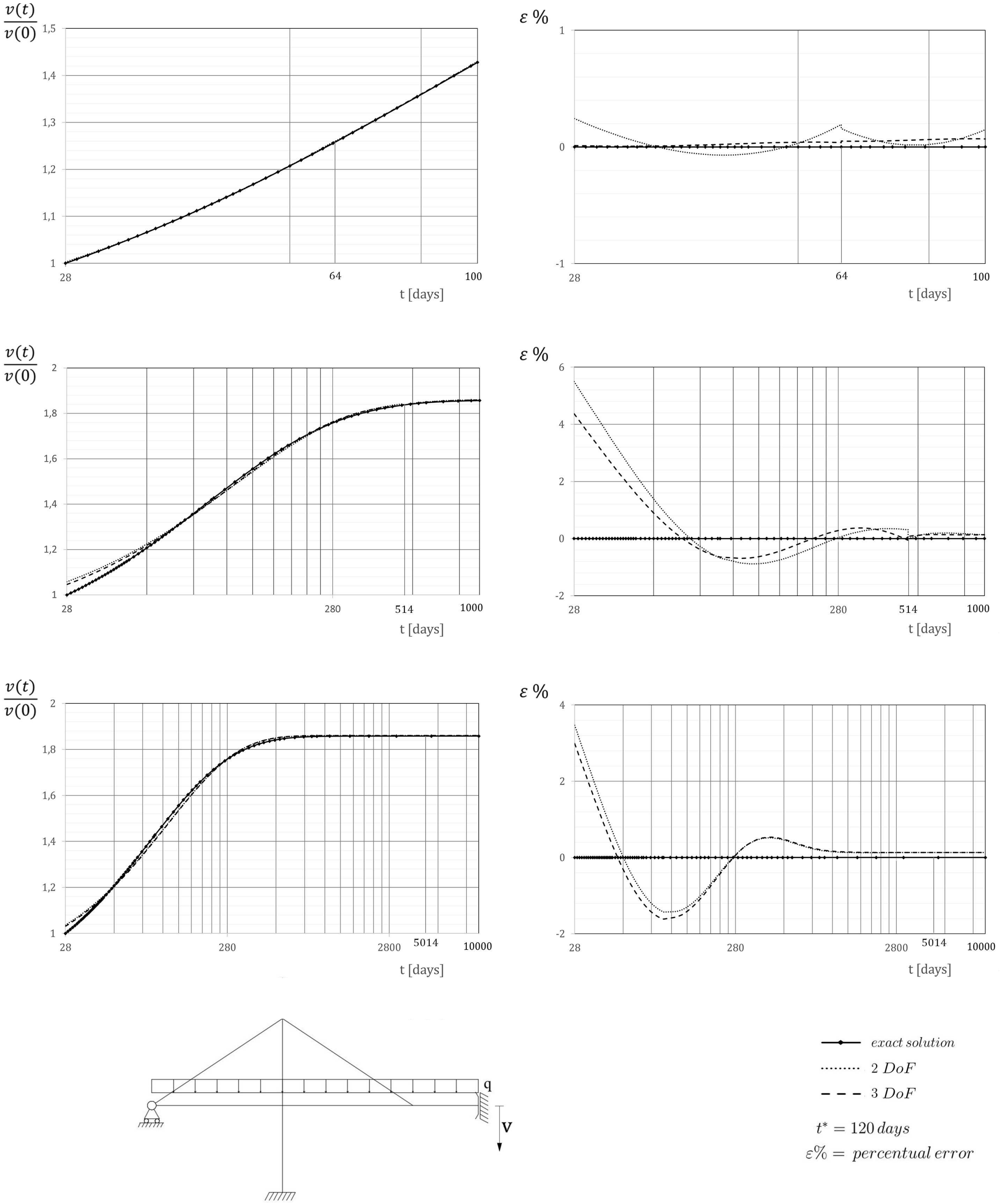


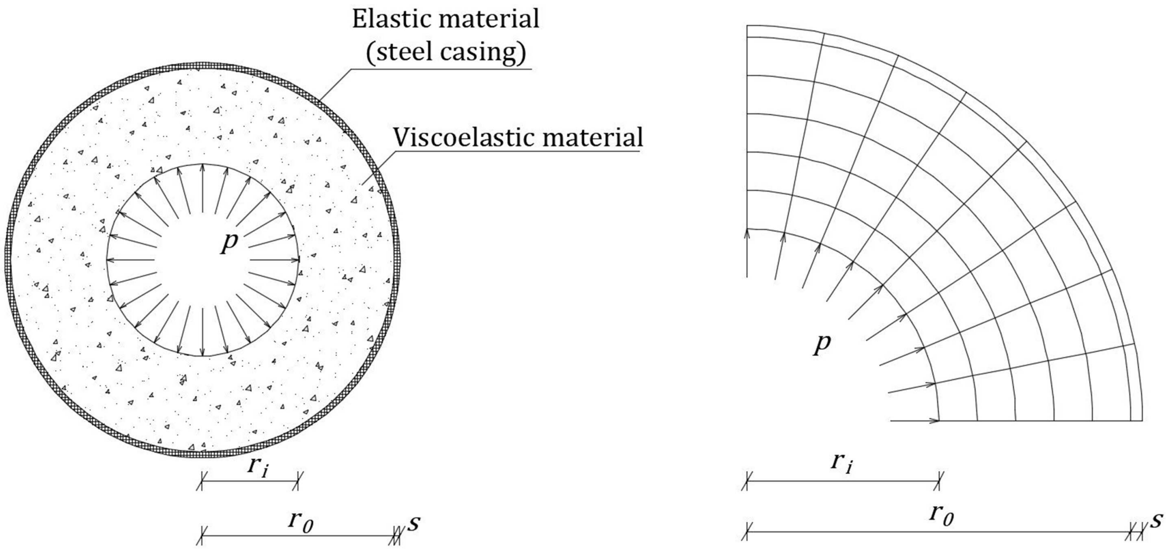
Figure 14. Example B: Cable-stayed bridge. Numerical results using the split Gurtin's approach.

$2T$ (days)	Time d.o.f.	Conditioning index				
		Least square	Extended functional	Gurtin	Split Gurtin	Split Gurtin*
28–100	1	1.555×10^7	3.635×10^3	4.032×10^3	5.638×10^3	
	2	7.389×10^8	2.118×10^5	4.655×10^5	2.735×10^6	2.459×10^7
	3	2.705×10^9	2.328×10^7	1.049×10^{10}	2.367×10^{11}	2.542×10^{13}
28–1000	1	5.944×10^6	2.248×10^3	3.117×10^3	3.681×10^3	
	2	2.195×10^8	1.051×10^5	2.350×10^5	5.640×10^6	3.361×10^5
	3	1.715×10^{10}	4.932×10^6	1.112×10^6	1.158×10^8	9.123×10^8
28–10000	1	8.891×10^6	2.064×10^3	3.001×10^3	3.058×10^3	
	2	2.695×10^9	1.001×10^6	2.095×10^7	5.969×10^{24}	3.878×10^7
	3	5.796×10^{12}	8.667×10^7	1.754×10^9	6.036×10^{36}	1.537×10^8
		$\widetilde{\mathcal{L}}\mathcal{L}\boldsymbol{u} = \widetilde{\mathcal{L}}\boldsymbol{b}$ $\langle\cdot,\cdot\rangle$ standard	$\widetilde{\mathcal{L}}\mathcal{K}\mathcal{L}\boldsymbol{u} = \widetilde{\mathcal{L}}\mathcal{K}\boldsymbol{b}$ $\langle\cdot,\cdot\rangle$ standard	$\mathcal{L}\boldsymbol{u} = \boldsymbol{b}$ $\langle\cdot,\cdot\rangle_c$		
Shape functions		$[1, e^{-t/t^*}, e^{-2t/t^*}]$		$[1, e^{-t/t^*}, e^{-0.1t/t^*}]$		$[1,\ e^{-t/t^*}, e^{-0.1t/t^*}]$ (first subinterval), $[1,\ e^{-0.1t/t^*}, e^{-0.01t/t^*}]$ (second subinterval)

Table 1. Cable-stayed bridge. Conditioning indexes of the coefficient matrices.

Example C (reinforced cylinder under internal pressure). With this example we want to test the effectiveness of Gurtin's approach in the case of more complex geometry and load history. In this example a plane strain cylinder of viscoelastic material is surrounded by a steel casing and subjected to an internal pressure history. Shown in Figure 15 are the geometric dimensions of the solid (top left), the mechanical properties of the materials (center; the material properties are assumed as in [Zienkiewicz et al. 1968]), the finite element mesh used (top right). In particular, a quarter of cylinder is subdivided in 48 four nodes isoparametric finite elements, and two load histories; the first (bottom left) is a pressure suddenly applied at time $t = 28$ days and then held constant over time, the second (bottom right) is a pressure suddenly applied at time $t = 28$ days, held constant for 100 days, suddenly doubled for the next 100 days and then suddenly removed. The reference solution has been obtained by means of ABAQUS.

Using Gurtin's approach, and with reference to the pressure history (Figure 15, bottom left), in Figure 16 the radial displacements with the relevant percentage errors are plotted while the variation of radial stress and variation of the tangential stress are plotted in Figure 17. Table 2 reports the conditioning indexes of the coefficient matrices. Figure 18 shows the results obtained with Gurtin's formulation in the case of the more complex loading history (Figure 15, bottom right). The results were obtained by solving the problem in three time steps, adopting a variation of the nonstandard step-by-step procedure described earlier. In the case of the three loading steps, in the first step the solution for the first 100 days was calculated. Then the solution for 100 more days was calculated using the functional for the first 200 days and using the solution already found in the first interval. Finally, the solution was found in the third interval, between 228 and 328 days by considering the functional over the entire time interval, starting at 28 days and using the solution already found in the first two intervals. Basically, a step-by-step procedure was used with large time steps and using two temporal degrees of freedom for each step, only. As shown in Figure 18, Gurtin's approach leads to very good results with low computational cost.



PROPERTIES

Geometrical	$r_0 = 4\,in$	$r_i = 2\,in$	$s = 4/33\,in$
Rheological	Steel casing	$E = 3 \times 10^7\,lb/in^2$	$\nu = 0.3015$
	Viscoelastic material	Volumetric part (elastic)	$K = 1 \times 10^5\,lb/in^2$
		Deviatoric part σ(t) σ(t) ε(t)	$G = 3.75 \times 10^4\,lb/in^2$, Relaxation time= $10\,days$

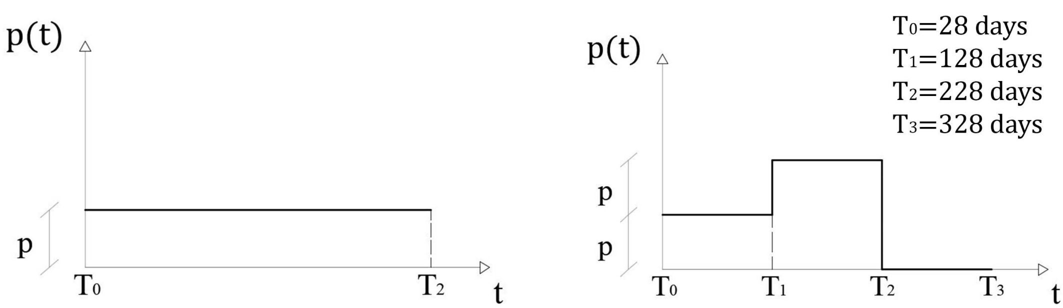


Figure 15. Example C. Top left: reinforced cylinder under internal pressure. Top right: finite element mesh. Center: geometrical properties and rheological properties. Bottom left: first load history. Bottom right: second load history.

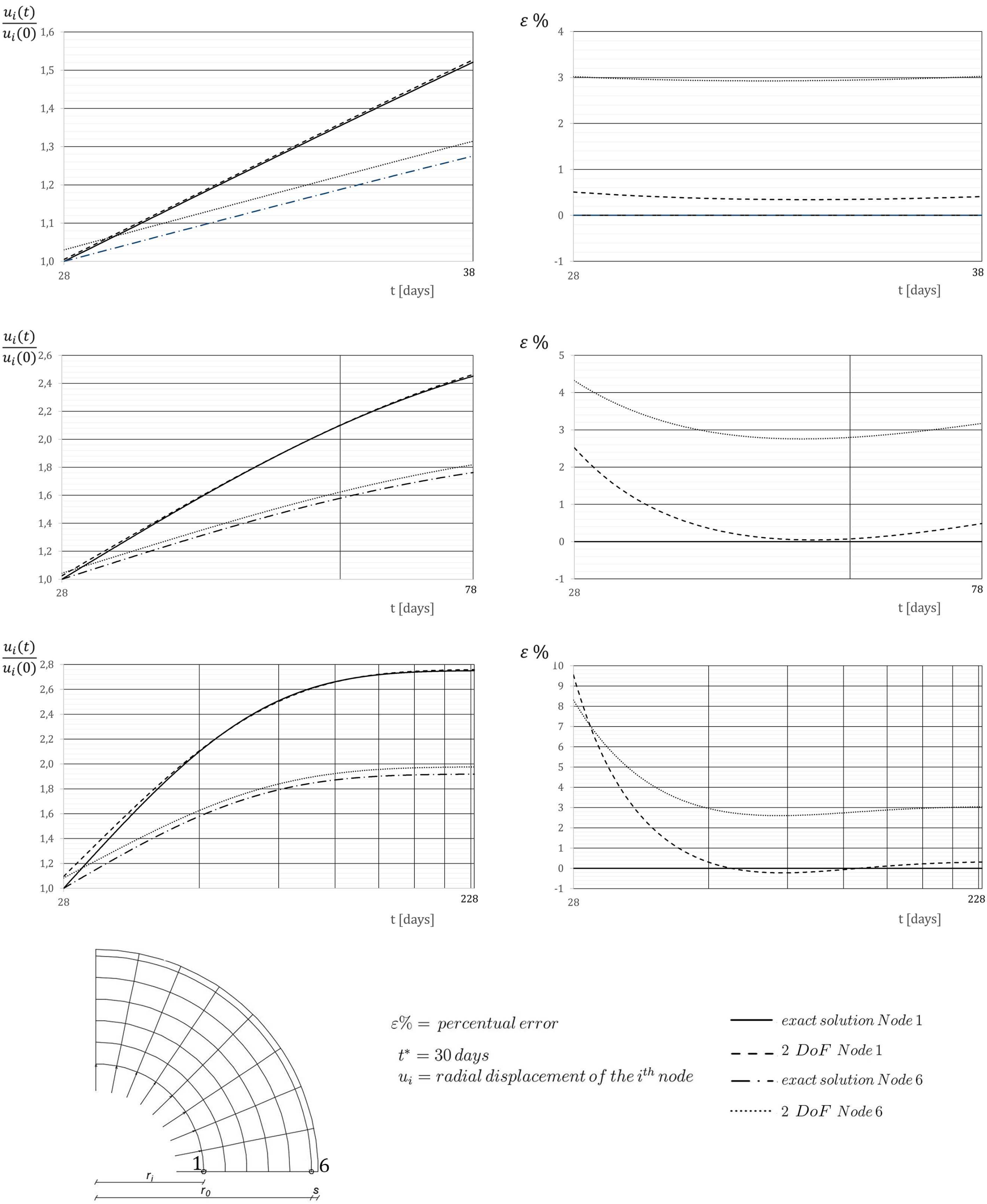


Figure 16. Example C: Reinforced cylinder under suddenly applied constant internal pressure. Radial displacements plots.

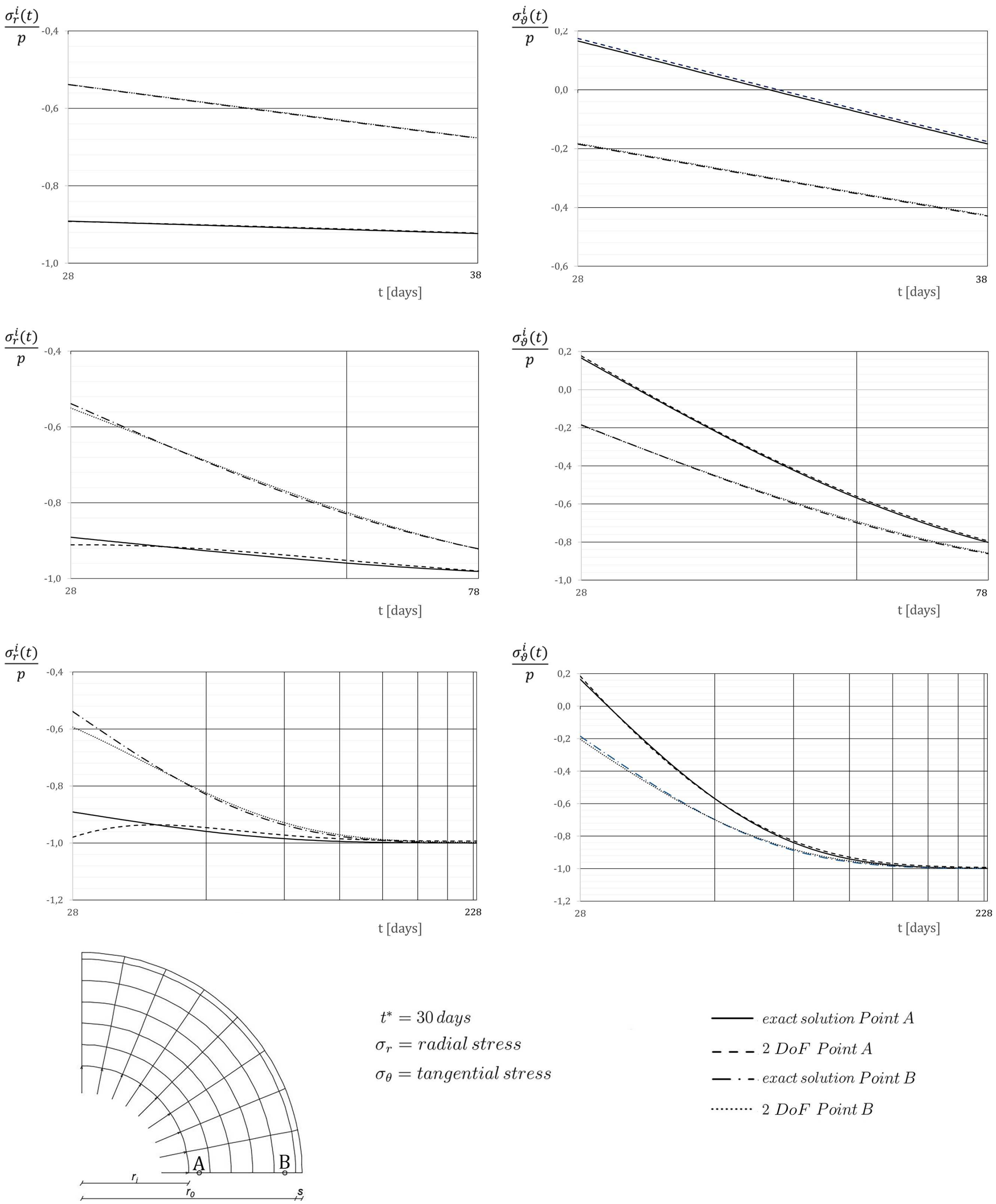


Figure 17. Example C: Reinforced cylinder under suddenly applied constant internal pressure. Radial and tangential stress plots.

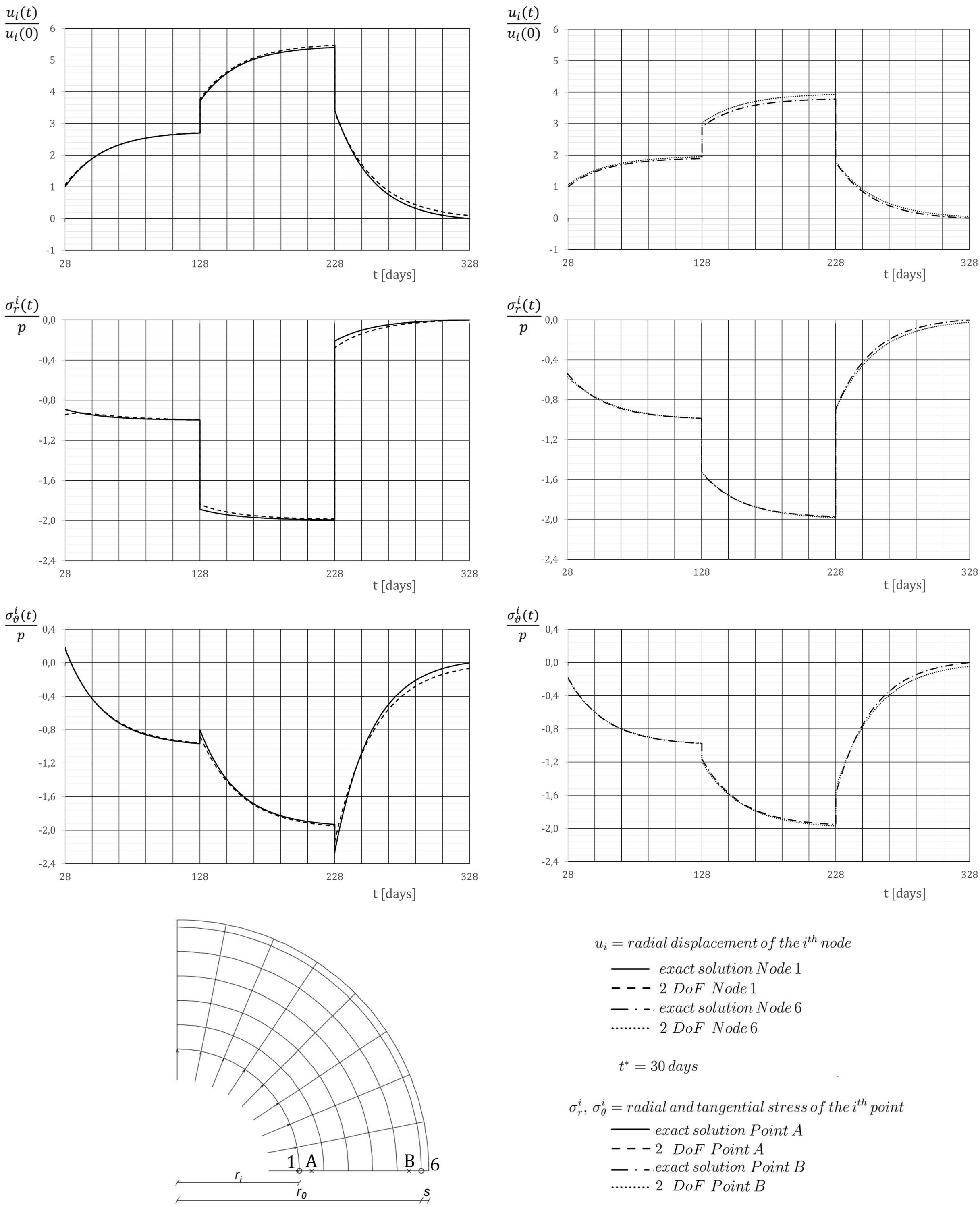


Figure 18. Example C: Reinforced cylinder under internal pressure depicted in Figure 15, bottom right.

$2T$ (days)	degrees of freedom	Conditioning index Gurtin's functional
28-38	1	9.4777×10^5
	2	3.2489×10^8
	3	2.6005×10^{13}
28-78	1	1.6799×10^6
	2	3.3209×10^7
	3	3.8077×10^{10}
28-228	1	2.4283×10^6
	2	1.5416×10^8
	3	8.1938×10^8
		$\mathcal{L}\boldsymbol{u} = \boldsymbol{b}$ $\langle\cdot,\cdot\rangle_c$

Table 2. Reinforced cylinder under suddenly applied constant internal pressure: conditioning indexes of the coefficient matrices.

5. Conclusions

To the authors' knowledge, no significant work seems to exist in the literature concerning the use of variational formulations for the numerical solution of the linear viscoelastic problem. In particular, variational principles based on convolutional bilinear forms with respect to time do not seem to have had any place so far in computational procedures for the numerical determination of the viscoelastic response of solids or structures subject to external actions. This paper aims at investigating the effectiveness of numerical procedures based on these variational formulations.

Three variational formulations for the linear viscoelastic problem are critically reviewed and numerically tested with the purpose of investigating their effectiveness in the numerical solution of viscoelastic problem. Numerical examples with reference to hereditary Kelvin–Voigt material model are presented.

In the case of finding the viscoelastic solution over a finite time interval in a single step, Gurtin's formulation works well in the sense that the numerical solutions are obtained with errors lower than 5%, using only two or three temporal degrees of freedom; by carefully choosing the temporal interpolation functions, Gurtin's formulation leads to numerical results comparable to those obtained with Tonti's "extended" variational formulation in terms of accuracy, but with lower computational costs.

With Gurtin's formulation, the results are more sensitive to the choice made of interpolation functions over time. Polynomial interpolation functions are appropriate only for short time intervals. The interpolation functions that have given the best results are the negative exponential functions because they capture well the typically damped trend of the viscoelastic solution. The conditioning indexes are comparable to those obtained with the "extended" Tonti's formulation. In the case of using the Gurtin formulation in a step-by-step procedure, the results are comparable to those obtained with a traditional time integration technique.

The split Gurtin formulation gives good results only for an equal number of degrees of freedom in the first and second time subinterval. For a lower number in the former there is no solution while for a lower number in the latter there arise numerical instabilities that, in the worst cases, are difficult to overcome even with the help of a Tikhonov regularization. From the numerical point of view it does not seem to provide any improvement over the unsplit formulation. In any case, the doubling of unknowns makes the method less competitive than the unsplit Gurtin's method. The split Gurtin functional, however, could have advantages from the theoretical point of view, as shown in Carini and Mattei [2015]. In fact, the formulation is of the min-max type and, by the adopted time splitting, one is able to isolate a part of the functional that represents the free energy of the system and is, therefore, in general, semidefinite positive.

In the case of the Gurtin and split Gurtin functionals, the problem of the optimal choice of the relaxation time to be used in the negative exponential interpolation functions remains unsolved.

Huet's formulation, in general, can provide a wrong solution. It provides the true solution only under very restrictive assumptions, as in the case of integration times long enough for the solution to have become constant, but even in these cases the formulation has numerical instabilities similar to those presented by the split Gurtin formulation. Huet's formulation works well only for short time intervals embedded in a standard step-by-step procedure. From a theoretical point of view it is interesting because the quadratic part of the functional has the physical meaning of a free energy.

In conclusion, of the three formulations presented here and numerically tested, Gurtin's original formulation seems to be the most effective in numerically solving the linear hereditary viscoelastic problem.

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