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# NON LINEAR HYPERBOLIC–PARABOLIC SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS

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**Abstract.** We prove the well posedness of a class of non linear and non local mixed hyperbolic–parabolic systems in bounded domains, with Dirichlet boundary conditions. In view of control problems, stability estimates on the dependence of solutions on data and parameters are also provided. These equations appear in models devoted to population dynamics or to epidemiology, for instance.

## 1. Introduction

We consider the following non linear system on a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \partial_t u + \nabla\cdot \left(u\, v(t,w)\right) = \alpha(t,x,w)\, u + a(t,x)\,, \\ \partial_t w - \mu\, \Delta w = \beta(t,x,u,w)\, w + b(t,x)\,, \end{cases} \qquad (t,x) \in [0,T] \times \Omega. \tag{1.1}$$

Systems of this form arise, for instance, in predator–prey systems [8] and can be used in the control of parasites, see [10, 19]. A similar mixed hyperbolic–parabolic system is considered, in one space dimension, in [15], where Euler equations substitute the balance law in (1.1).

Motivated by these applications, terms in (1.1) may well contain non local functions of the unknowns. Typically, whenever $u$ is a *predator* and $w$ a *prey*, the velocity $v$ governing the movement of $u$, when computed at a point $x$, i.e., $\left(v\left(t,w\right)\right)(x)$, depends on $w$ through integrals of the form $\int_{\|x-\xi\|\leq\rho} f\left(t,x,\xi,w(t,\xi)\right) \mathrm{d}\xi$ so that $\rho$ is the *horizon* at which the predator feels the prey.

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Under standard assumptions on the functions defining (1.1), we provide the analytical framework where the existence and the uniqueness of solutions to (1.1)–(1.2). Moreover, we obtain a full set of *a priori* and stability estimates on view of the interest about control problems based on these equations, see [2]. To this aim, we equip (1.1) with homogeneous Dirichlet boundary conditions and initial data:

$$\begin{cases} u(t,\xi)=0 \\ w(t,\xi)=0 \end{cases} (t,\xi)\in[0,T]\times\partial\Omega \quad \text{and} \quad \begin{cases} u(0,x)=u_o(x) \\ w(0,x)=w_o(x) \end{cases} x\in\Omega. \qquad (1.2)$$

We stress that the whole construction is settled in $\mathbf{L}^1$, a usual choice for balance laws but less common in the case of the parabolic equation. This choice is motivated by the clear physical meaning of *total population* attached to this norm, whenever solutions are positive – a standard situation in the motivating models. As is well known, in parabolic equations, $\mathbf{L}^2$ or $\mathbf{W}^{k,2}$ are more standard choices, also thanks to the further properties of reflexive spaces, see for instance the recent papers [4, 14].

The introduction of a boundary, with the corresponding boundary conditions, affects the whole analytical structure, differently in the two equations. Indeed, as is well known, the hyperbolic equation for $u$ may well lead to problems that are locally overdetermined, resulting in the boundary condition to be simply neglected, see [1, 9, 18, 22]. On the contrary, the solution to the parabolic equation attains along the boundary the prescribed value, for all positive times, see [12, 17, 20].

We stress that in the hyperbolic case, different definitions of solutions are available, see [1, 18, 21, 22]. Here, we provide Definition 3.10 that unifies different approaches, also allowing to prove an intrinsic uniqueness of solutions, i.e., independent of the way solutions are constructed.

Particularly relevant are the estimates on the dependence of $(u,w)$ on the terms $a,b$ in (1.1), which typically play the role of controls. Indeed, in the applications of (1.1) to biological problems, $a$ and $b$ typically measure the deployment of parasitoids or chemicals that hinder the propagation or reproduction of harmful parasites, see [10, 19]. It is with reference to this context that we care to ensure the positivity of solutions, whenever the data and the controls are positive.

The next section, after the necessary introduction of the notation, presents the result. Proofs and further technical details are deferred to Section 3, where different paragraphs refer to the parabolic problem, to the hyperbolic one and to the coupling.

## 2. Main Results

Throughout, the following notation is used. $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{R}_- = (-\infty, 0]$. If $A \subseteq \mathbb{R}^n$, the characteristic function $\chi_A$ is defined by $\chi_A(x) = 1$ if and only if $x \in A$ and $\chi_A(x) = 0$ if and only if $x \in \mathbb{R}^n \setminus A$. For $x_o \in \mathbb{R}^n$ and $r > 0$, $B(x_o, r)$ is the open sphere centered at $x_o$ with radius $r$. We fix a time $T > 0$ and the following condition on the spatial domain $\Omega$:

($\boldsymbol{\Omega}$) $\Omega$ is a non empty, bounded and connected open subset of $\mathbb{R}^n$, with $\mathbf{C}^{2,\gamma}$ boundary, for a $\gamma \in (0, 1]$.

This condition is mainly motivated by the treatment of the parabolic part. Here, we mostly use the framework in [20, Appendix B, § 48]. Other possible regularity assumptions on $\partial\Omega$ are in [17, Chapter 4, § 4, p. 294].

We pose the following assumptions on the functions appearing in problem (1.1):

($\boldsymbol{v}$) $v : [0,T] \times \mathbf{L}^\infty(\Omega;\mathbb{R}) \to (\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\Omega;\mathbb{R}^n)$ is such that for a constant $K_v > 0$ and for a map $C_v \in \mathbf{L}^\infty_{\mathbf{loc}}([0,T] \times \mathbb{R}_+;\mathbb{R}_+)$ non decreasing in each argument, for all $t, t_1, t_2 \in [0,T]$ and $w, w_1, w_2 \in \mathbf{L}^\infty(\Omega;\mathbb{R})$,

$$\begin{aligned}
&\|v(t,w)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \le K_v \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\|D_x v(t,w)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n})} \le K_v \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\|v(t_1,w_1) - v(t_2,w_2)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \le K_v\big(|t_2 - t_1| + \|w_2 - w_1\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\big)\\
&\left\|D_x^2 v(t,w)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^{n\times n\times n})} \le C_v\big(t, \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\big)\|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\|\nabla\cdot(v(t_1,w_1) - v(t_2,w_2))\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le C_v\big(t, \max_{i=1,2}\|w_i\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\big)\|w_1 - w_2\|_{\mathbf{L}^1(\Omega;\mathbb{R})}.
\end{aligned}$$

($\boldsymbol{\alpha}$) $\alpha : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ admits a constant $K_\alpha > 0$ such that, for a.e. $t \in [0,T]$ and all $w, w_1, w_2 \in \mathbb{R}$

$$\begin{aligned}
\sup_{x\in\Omega} |\alpha(t,x,w_1) - \alpha(t,x,w_2)| &\le K_\alpha\, |w_1 - w_2|\\
\sup_{(x,w)\in\Omega\times\mathbb{R}} \alpha(t,x,w) &\le K_\alpha\,(1 + w)
\end{aligned}$$

and for all $w \in \mathbf{BV}(\Omega;\mathbb{R})$

$$\mathrm{TV}\left(\alpha(t,\cdot,w(t,\cdot))\right) \le K_\alpha\big(1 + \|w\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(w)\big).$$

($\boldsymbol{a}$) $a \in \mathbf{L}^1\left([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R})\right)$ and for all $t \in [0,T]$, $a(t) \in \mathbf{BV}(\Omega;\mathbb{R})$.

($\boldsymbol{\beta}$) $\beta : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ admits a constant $K_\beta > 0$ such that, for a.e. $t \in [0,T]$ and all $u, u_1, u_2, w, w_1, w_2 \in \mathbb{R}$

$$\sup_{x\in\Omega} |\beta(t,x,u_1,w_1) - \beta(t,x,u_2,w_2)| \leq K_\beta \left(|u_1 - u_2| + |w_1 - w_2|\right)$$
$$\sup_{(x,u,w)\in\Omega\times\mathbb{R}\times\mathbb{R}} \beta(t,x,u,w) \leq K_\beta \,.$$

($\boldsymbol{b}$) $b \in \mathbf{L}^1([0,T]; \mathbf{L}^\infty(\Omega;\mathbb{R}))$ and for all $t \in [0,T]$, $b(t) \in \mathbf{BV}(\Omega;\mathbb{R}_+)$.

Note in particular that ($\boldsymbol{v}$) requires to bound $\mathbf{L}^\infty$ norms by means of $\mathbf{L}^1$ norms, a feature typical of non local operators. In fact, referring to predator–prey applications, it is in general reasonable to assume that the $u$ (predator) population moves according to *averages* of the $w$ (prey) population density or of its gradient. This justifies our requiring $v$ in (1.1) to be a *non local* function of $w$.

Since we deal with the bounded domain $\Omega$, these averages need to be computed only inside $\Omega$. To this aim, the modified convolution introduced in [9, § 3], which reads

$$(\rho *_\Omega \eta)(x) = \frac{\int_\Omega \rho(y)\, \eta(x-y)\, \mathrm{d}y}{\int_\Omega \eta(x-y)\, \mathrm{d}y} \tag{2.1}$$

is of help. The quantity $(\rho *_\Omega \eta)(x)$ is an average of the crowd density $\rho$ in $\Omega$ around $x$ as soon as the kernel $\eta$ satisfies

($\boldsymbol{\eta}$) $\eta(x) = \tilde\eta(\|x\|)$, where $\tilde\eta \in \mathbf{C}^2(\mathbb{R}_+;\mathbb{R})$, $\operatorname{spt}\tilde\eta = [0,\ell_\eta]$, $\ell_\eta > 0$, $\tilde\eta' \leq 0$, $\tilde\eta'(0) = \tilde\eta''(0) = 0$ and $\int_{\mathbb{R}^N} \eta(\xi)\, \mathrm{d}\xi = 1$.

In those models where it is reasonable to assume that $u$ moves directed towards the areas with higher/lower density of $w$, i.e., $v$ is parallel to the average gradient of $w$ in $\Omega$, we select:

$$v(t,w) \quad /\!\!/ \quad \frac{\nabla\left(w *_\Omega \eta\right)}{\sqrt{1 + \left\|\nabla\left(w *_\Omega \eta\right)\right\|^2}}, \tag{2.2}$$

where as kernel $\eta$ we choose for instance $\eta(x) = \bar\ell(\ell^4 - \|x\|^4)^4$. Here, $\ell$ has the clear physical meaning of the distance, or *horizon*, at which individuals of the $u$ population *feel* the presence of the $w$ population. The normalization parameter $\bar\ell$ is chosen so that $\int_{\mathbb{R}^2} \eta(x)\, \mathrm{d}x = 1$. A choice like (2.2) is consistent with the requirements ($\boldsymbol{v}$), as proved in [9, Lemma 3.2].

To state what we mean by a *solution* to (1.1), we resort to the standard definitions of solutions, separately, to the hyperbolic and to the parabolic problems constituting (1.1). In the former case, we refer to [18, 21, 22] and in the latter to the classical [20].

**Definition 2.1.** *A pair* $(u, w) \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R}^2)\right)$ *is a solution to* (1.1)–(1.2) *if, setting*

$$c(t,x) = v\left(t, w(x)\right), \quad A(t,x) = \alpha\left(t, x, w(t,x)\right)$$
$$B(t,x) = \beta\left(t, x, u(t,x), w(t,x)\right),$$

*the function* $u$*, according to Definition* 3.10*, solves*

$$\begin{cases} \partial_t u + \nabla\cdot\left(u\, c(t,x)\right) = A(t,x)\, u + a(t,x) & (t,x) \in [0,T] \times \Omega \\ u(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega \end{cases}$$

*and the function* $w$*, according to Definition* 3.1*, solves*

$$\begin{cases} \partial_t w - \mu\, \Delta w = B(t,x)\, w + b(t,x) & (t,x) \in [0,T] \times \Omega \\ w(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ w(0,x) = w_o(x) & x \in \Omega\,. \end{cases}$$

In the present framework, we also verify that, under suitable conditions on the initial data, the solution $(u, w)$ enjoys the following regularity $(u(t), w(t)) \in (\mathbf{BV} \cap \mathbf{L}^\infty)(\Omega;\mathbb{R}^2_+)$ for all $t \in [0,T]$.

We are now ready to state the main result of this work.

**Theorem 2.2.** *Let* $(\boldsymbol{\Omega})$–$(\boldsymbol{v})$–$(\boldsymbol{\alpha})$–$(\boldsymbol{a})$–$(\boldsymbol{\beta})$–$(\boldsymbol{b})$ *hold. For any initial datum* $(u_o, w_o)$ *in* $(\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R}^2)$*, problem* (1.1) *admits a unique solution on* $[0,T]$ *in the sense of Definition* 2.1*. Moreover, the following properties hold:*

*A priori bounds: There exists a constant* $C$ *depending only on* $\Omega$, $K_\alpha$, $K_\beta$, $K_v$ *such that for all* $t \in [0,T]$ *and for all initial data*

$$\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq e^{C\,t}\left(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \tag{2.3}$$
$$\|w(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq e^{C\,t}\left(\|w_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R})}\right). \tag{2.4}$$

*Let* $C_w(t)$ *denote the maximum of the two right hand sides in* (2.3) *and in* (2.4)*; then*

$$\|u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq (\|u_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}) \exp(Ct(1 + C_w(t))) \tag{2.5}$$
$$\|u(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \exp\left(C\,t\,(1 + 2\,C_w(t))\right)\,. \tag{2.6}$$

*Lipschitz continuity in the initial data*: *Let* $(\tilde{u}_o, \tilde{w}_o) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^2)$ *and call* $(\tilde{u}, \tilde{w})$ *the corresponding solution to* (1.1)*. Then, for all* $t \in [0,T]$*,*

$$\begin{aligned} &\|u(t) - \tilde{u}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(t) - \tilde{w}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\leq \mathcal{C}(t)\big(\|u_o - \tilde{u}_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w_o - \tilde{w}_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\big), \end{aligned}$$

*where* $\mathcal{C} \in \mathbf{L}^\infty([0,T]; \mathbb{R}_+)$ *depends on* $\Omega$*,* $K_\alpha$*,* $K_\beta$*,* $K_v$*, on the map* $C_v$*, on norms and total variation of the functions* $a$ *and* $b$ *and of the initial data.*

*Stability with respect to the controls*: *Let* $\tilde{a}$ *satisfy* (***a***)*,* $\tilde{b}$ *satisfy* (***b***) *and call* $(\tilde{u}, \tilde{w})$ *the corresponding solution to* (1.1)*. Then, for all* $t \in [0,T]$*,*

$$\begin{aligned} &\|u(t) - \tilde{u}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(t) - \tilde{w}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\leq \mathcal{C}(t)\big(\|a - \tilde{a}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} + \|b - \tilde{b}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\big), \end{aligned}$$

*where* $\mathcal{C} \in \mathbf{L}^\infty([0,T]; \mathbb{R}_+)$ *depends on* $\Omega$*,* $K_\alpha$*,* $K_\beta$*,* $K_v$*, on the map* $C_v$*, on norms and total variation of the functions* $a, \tilde{a}$ *and* $b, \tilde{b}$ *and of the initial data.*

*Positivity*: *If for all* $(t,x) \in [0,T] \times \Omega$*,* $a(t,x) \geq 0$ *and* $b(t,x) \geq 0$*, then for all initial datum* $(u_o, w_o)$ *with* $u_o(x) \geq 0$ *and* $w_o(x) \geq 0$ *for all* $x \in \Omega$*, the solution* $(u,w)$ *is such that* $u(t,x) \geq 0$ *and* $w(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \Omega$*.*

The proof is deferred to Section 3.

The lower semicontinuity of the total variation with respect to the $\mathbf{L}^1$ distance ensures moreover that bounds on the total variation of the solution can be obtained by means of (3.50) and (3.60).

## 3. Proofs

In the proofs below, we provide all details wherever necessary and precise references for those part that differ only slightly from the cases under consideration.

3.1. **Parabolic Estimates.** Fix $T, \mu > 0$ and let $\Omega$ satisfy ($\mathbf{\Omega}$). This paragraph is devoted to the IBVP

$$\begin{cases} \partial_t w = \mu\,\Delta w + B(t,x)\,w + b(t,x) & (t,x) \in [0,T] \times \Omega \\ w(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ w(0,x) = w_o(x) & x \in \Omega\,. \end{cases} \tag{3.1}$$

The following definition is adapted from [20], see Remark 3.2.

**Definition 3.1.** *A map* $w \in \mathbf{C}^0([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))$ *is a solution to* (3.1) *if* $w(0) = w_o$ *and for all test functions* $\varphi \in \mathbf{C}^2([0,T]\times\overline{\Omega};\mathbb{R})$ *such that* $\varphi(T,x) = 0$ *for all* $x \in \Omega$ *and* $\varphi(t,\xi) = 0$ *for all* $(t,\xi) \in [0,T] \times \partial\Omega$*:*

$$\begin{aligned}\int_0^T \int_\Omega &(w(t,x)\partial_t\varphi(t,x) + \mu w(t,x)\Delta\varphi(t,x) \\ &+ (B(t,x)w(t,x) + b(t,x))\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega w_o(x)\,\varphi(0,x)\,\mathrm{d}x = 0\,.\end{aligned} \tag{3.2}$$

**Remark 3.2.** Let

$$d(x,\partial\Omega) = \inf_{y\in\partial\Omega} \|x-y\|.$$

Recall

$$\|w\|_{\mathbf{L}^1_\delta(\Omega;\mathbb{R})} = \int_\Omega |w(x)|\,d(x,\partial\Omega)\,\mathrm{d}x$$

from [20, Appendix B]. Since $\|w\|_{\mathbf{L}^1_\delta(\Omega;\mathbb{R})} \le \mathcal{O}(1)\|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$, a solution in the sense of Definition 3.1 is also a *weak* $\mathbf{L}^1_\delta$ *solution* in the sense of [20, Definition 48.8, Appendix B].

**Remark 3.3.** In Definition 3.1, it is sufficient to consider test functions $\varphi \in \mathbf{C}^1([0,T]\times\overline{\Omega};\mathbb{R})$ such that for all $t \in [0,T]$, the map $x \mapsto \varphi(t,x)$ is of class $\mathbf{C}^2(\overline{\Omega};\mathbb{R})$ and moreover $\varphi(T,x) = 0$ for all $x \in \Omega$ and $\varphi(t,\xi) = 0$ for all $(t,\xi) \in [0,T]\times\partial\Omega$. This is proved through a standard regularization by means of a convolution with a mollifier supported in $\mathbb{R}_-$.

For $\mu > 0$, the heat kernel is denoted by

$$H_\mu(t,x) = (4\pi\mu t)^{-n/2} \exp\big(-\|x\|^2/(4\mu t)\big),$$

where $t > 0$, $x \in \mathbb{R}^n$. As it is well known, $\|H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} = 1$.

**Proposition 3.4.** *Let* $\Omega$ *satisfy* $(\boldsymbol{\Omega})$ *and fix* $\mu > 0$*. Then, there exists a Green function*

$$G \in \mathbf{C}^\infty((0,+\infty)\times(0,+\infty)\times\Omega\times\Omega;\mathbb{R}_+)\cap\mathbf{C}^0((0,+\infty)\times(0,+\infty)\times\overline{\Omega}\times\overline{\Omega};\mathbb{R}_+)$$

*such that:*

- **(G1)** *For all* $t,\tau \in \mathbb{R}_+$ *and* $x,y \in \Omega$, $G(t,\tau,x,y) = G(t,\tau,y,x)$.
- **(G2)** *For all* $t,\tau \in \mathbb{R}_+$, $\xi \in \partial\Omega$ *and* $y \in \Omega$, $G(t,\tau,\xi,y) = 0$.
- **(G3)** *There exist positive constants* $C, c$ *such that for all* $t,\tau \in \mathbb{R}_+$ *and for all* $x,y \in \Omega$,

$$\begin{aligned}0 \le G(t,\tau,x,y) &\le H_\mu(t-\tau,x-y)\\ |\partial_t G(t,\tau,x,y)| &\le c(t-\tau)^{-(n+2)/2}\exp\big(-C\|x-y\|^2/(t-\tau)\big)\end{aligned}$$

$$\|\nabla_x G(t,\tau,x,y)\| \leq c(t-\tau)^{-(n+1)/2} \exp\big(-C\|x-y\|^2/(t-\tau)\big).$$

**(G4)** *For all* $b \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$ *and all* $w_o \in \mathbf{L}^1(\Omega;\mathbb{R})$*, the IBVP*

$$\begin{cases} \partial_t w = \mu\,\Delta w + b(t,x) & (t,x) \in [0,T]\times\Omega \\ w(t,\xi) = 0 & (t,\xi) \in [0,T]\times\partial\Omega \\ w(0,x) = w_o(x) & x \in \Omega \end{cases} \tag{3.3}$$

*admits a unique solution in the sense of Definition* 3.1*, which is*

$$w(t,x) = \int_\Omega G(t,0,x,y) w_o(y)\,\mathrm{d}y + \int_0^t \int_\Omega G(t,\tau,x,y) b(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\,. \tag{3.4}$$

The Green function depends both on $\mu$ and on $\Omega$ but, for simplicity, we omit this dependence.

**Proof of Proposition 3.4.** Condition **(G1)** follows from [20, Appendix B, § 48.2]. Property **(G2)** comes from [17, Chapter IV, § 16, (16.7)–(16.8) p. 408].

The first bound in **(G3)** follows from [20, Formula (48.4), p.440], the second and the third one from [17, Chapter IV, § 16, Theorem 16.3, p. 413].

To prove **(G4)**, use Remark 3.2 and [20, Proposition 48.9, Appendix B], [20, Corollary 48.10, Appendix B] and the Maximum Principle [20, Proposition 52.7, Appendix B], which ensure the equivalence between (3.2) and (3.4) as soon as either $w_o \geq 0$, $b \geq 0$ or $w_o \leq 0$, $b \leq 0$. The linearity of (3.3) allows to complete the proof. □

**Proposition 3.5.** *Let* $\Omega$ *satisfy* $(\mathbf{\Omega})$*, fix* $\mu > 0$ *and let*

**(P1)** $w_o \in \mathbf{L}^\infty(\Omega;\mathbb{R})$,
**(P2)** $B \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$,
**(P3)** $b \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$.

*Then*

**(1)** *Problem* (3.1) *admits a unique solution in the sense of Definition* 3.1*.*

**(2)** *The solution to* (3.1) *is implicitly given by*

$$\begin{aligned} w(t,x) = {} & \int_\Omega G(t,0,x,y) w_o(y)\,\mathrm{d}y \\ & + \int_0^t \int_\Omega G(t,\tau,x,y)\,\big(B(\tau,y) w(\tau,y) + b(\tau,y)\big)\,\mathrm{d}y\,\mathrm{d}\tau \end{aligned} \tag{3.5}$$

*where* $G$*, independent of* $b$ *and* $B$*, is defined in Proposition* 3.4*.*

**(3)** *The following* a priori *bounds hold for all* $t \in [0,T]$

$$\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq \left(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \exp \int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau \,, \tag{3.6}$$

$$\|w(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \left(\|w_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \exp \int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau \,. \tag{3.7}$$

**(4)** *If* $w_1$*,* $w_2$ *solve* (3.1) *with data* $w_o^1$*,* $w_o^2$ *satisfying* **(P1)***, functions* $B_1$*,* $B_2$ *satisfying* **(P2)** *and functions* $b_1$*,* $b_2$ *satisfying* **(P3)***, then*

$$\begin{aligned} &\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\leq \left(\left\|w_o^1 - w_o^2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b_1 - b_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \exp \int_0^t \|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau \\ &\quad + \|B_1 - B_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \left(\left\|w_o^2\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\ &\qquad \times \exp \int_0^t \left(\|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|B_2(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\right) \mathrm{d}\tau \,. \end{aligned} \tag{3.8}$$

**(5)** *Positivity: if* $b \geq 0$ *and* $w_o \geq 0$*, then* $w \geq 0$*.*

**(6)** *If* $w_o \in \mathbf{BV}(\Omega;\mathbb{R})$ *and* $b(t) \in \mathbf{BV}(\Omega;\mathbb{R})$ *for all* $t \in [0,T]$*, then for all* $t \in [0,T]$ *the following estimate holds :*

$$\begin{aligned} \mathrm{TV}\,(w(t)) \leq{}& \mathrm{TV}\,(w_o) + \int_0^t \mathrm{TV}\,(b(\tau))\,\mathrm{d}\tau + \mathcal{O}(1)\sqrt{t}\|B\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})} \\ &\times \left(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \exp \int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;R)} \, \mathrm{d}\tau \,. \end{aligned} \tag{3.9}$$

We note, for completeness, that in the setting of Proposition 3.5 the following regularity results, not of use in the sequel, can also be obtained:

(7) If $w_o \in \mathbf{C}^1_c(\Omega;\mathbb{R})$, then the solution $w$ is such that $w(t) \in \mathbf{C}^1(\Omega;\mathbb{R})$ for all $t \in [0,T]$.

(8) The solution $w$ is Hölder continuous in time.

**Proof of Proposition 3.5.** We split the proof in a few steps.

**Claim 1:** Problem (3.1) admits at most one solution in the sense of Definition 3.1. Observe that if $w_1$, $w_2$ solve (3.1) in the sense of Definition 3.1, then their difference satisfies

$$\int_0^T \int_\Omega (w_2 - w_1)(\partial_t \varphi + \mu\Delta\varphi + B\varphi) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

for all $\varphi$ as regular as specified in Remark 3.3. By [20, (ii) in Theorem 48.2, Appendix B], we choose as $\varphi$ the strong solution to

$$\begin{cases} \partial_t \varphi + \mu \Delta \varphi + B(t,x) \varphi = f & (t,x) \in [0,T] \times \Omega \\ \varphi(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ \varphi(T,x) = 0 & x \in \Omega, \end{cases}$$

where $f \in \mathbf{C}^0([0,T] \times \overline{\Omega}; \mathbb{R})$. We thus have

$$\int_0^T \int_\Omega (w_2 - w_1)\, f \,\mathrm{d}x\,\mathrm{d}t = 0,$$

so that, by the arbitrariness of $f$, $w_1 = w_2$.

**Claim 2:** If $w \in \mathbf{L}^\infty([0,T]; \mathbf{L}^1(\Omega;\mathbb{R}))$ satisfies (3.5), then (3.6) and (3.7) hold. Consider first (3.6). By **(G3)**, recalling $\|H_\mu(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq 1$, we have

$$\begin{aligned} \|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq{} & \int_\Omega \int_\Omega G(t,0,x,y) |w_o(y)| \,\mathrm{d}y\,\mathrm{d}x \\ & + \int_\Omega \int_0^t \int_\Omega G(t,\tau,x,y) |B(\tau,y)\, w(\tau,y) + b(\tau,y)| \,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}x \\ \leq{} & \int_\Omega \int_\Omega H_\mu(t,x-y) |w_o(y)| \,\mathrm{d}y\,\mathrm{d}x \\ & + \int_\Omega \int_0^t \int_\Omega H_\mu(t-\tau,x-y) |B(\tau,y)\, w(\tau,y) + b(\tau,y)| \,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}x \\ \leq{} & \|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \|w(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \,\mathrm{d}\tau + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}. \end{aligned}$$

An application of Gronwall Lemma [3, Lemma 3.1] yields (3.6). The proof of (3.7) is entirely similar.

**Claim 3:** If $w_1, w_2 \in \mathbf{L}^\infty([0,T]; \mathbf{L}^1(\Omega;\mathbb{R}))$ satisfy (3.5), then (3.8) holds. Note that

$$\begin{aligned} w_1(t,x) - w_2(t,x) ={} & \int_\Omega G(t,0,x,y)\, \left(w_o^1(y) - w_o^2(x)\right) \mathrm{d}y \\ & + \int_0^t \int_\Omega G(t,\tau,x,y) \left(B_1(\tau,y)\, w_1(\tau,y) - B_2(\tau,y)\, w_2(\tau,y)\right) \mathrm{d}y\,\mathrm{d}\tau \\ & + \int_0^t \int_\Omega G(t,\tau,x,y) \left(b_1(t,y) - b_2(t,y)\right) \mathrm{d}y\,\mathrm{d}\tau \\ ={} & \int_\Omega G(t,0,x,y)\, \left(w_o^1(y) - w_o^2(x)\right) \mathrm{d}y \end{aligned}$$

$$+\int_0^t \int_\Omega G(t,\tau,x,y) B_1(\tau,y)\ (w_1(\tau,y) - w_2(\tau,y))\,\mathrm{d}y\,\mathrm{d}\tau$$
$$+\int_0^t \int_\Omega G(t,\tau,x,y)\,\tilde{b}(t,y)\,\mathrm{d}y\,\mathrm{d}\tau\,,$$

where

$$\tilde{b}(t,x) = (B_1(t,x) - B_2(t,x))\ w_2(t,x) + b_1(t,x) - b_2(t,x).$$

Proceeding as in the proof of Claim 2 and exploiting (3.7), we obtain

$$\begin{aligned}
&\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
\le& \Big(\|w_o^1 - w_o^2\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|\tilde{b}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \exp\int_0^t \|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
\le& \Big(\|w_o^1 - w_o^2\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|B_1 - B_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\|w_2\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})} \\
&+ \|b_1 - b_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \exp\int_0^t \|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
\le& \Big(\|w_o^1 - w_o^2\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b_1 - b_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \exp\int_0^t \|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
&+ \|B_1 - B_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \Big(\|w_o^2\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\Big) \\
&\times \exp\int_0^t \Big(\|B_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|B_2(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\Big)\,\mathrm{d}\tau\ .
\end{aligned}$$

**Claim 4:** If $w \in \mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))$ satisfies (3.5), then

$$w \in \mathbf{C}^0([0,T];\mathbf{L}^1(\Omega;\mathbb{R})).$$

Introduce the abbreviation $\tilde{b}(t,x) = B(t,x)\,w(t,x) + b(t,x)$ so that, using (3.6),

$$\begin{aligned}
&\left\|\tilde{b}(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \|B(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le \|B(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\ \Big(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \exp\int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
&\quad + \|b(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}
\end{aligned}$$

and $\|\tilde{b}\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} \le \mathcal{O}(1)$. Compute, using **(G3)**, for $t_2 > t_1 > 0$ and $s,\sigma \in (t_1,t_2)$,

$$\|w(t_2) - w(t_1)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$$

$$\leq \int_\Omega \int_\Omega |G(t_2,0,x,y) - G(t_1,0,x,y)||w_o(y)| \, \mathrm{d}y \, \mathrm{d}x \tag{3.10}$$

$$+ \int_0^{t_1} \int_\Omega \int_\Omega |G(t_2,\tau,x,y) - G(t_1,\tau,x,y)||\tilde{b}(\tau,y)| \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \tag{3.11}$$

$$+ \int_{t_1}^{t_2} \int_\Omega \int_\Omega |G(t_2,\tau,x,y)||\tilde{b}(\tau,y)| \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \,. \tag{3.12}$$

Consider the three terms above separately.

$$\begin{aligned}
(3.10) &\leq \int_\Omega \int_\Omega \int_{t_1}^{t_2} |\partial_t G(s,0,x,y)||w_o(y)| \, \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}x \\
&\leq \int_\Omega \int_\Omega \int_{t_1}^{t_2} \frac{c}{s^{1+n/2}} \exp\left( - \frac{C \left\|x-y\right\|^2}{s}\right) |w_o(y)| \, \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}x \\
&\leq \left\|w_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_{\mathbb{R}^n} \int_{t_1}^{t_2} \frac{c}{s^{1+n/2}} \exp\left( - \frac{C \left\|x\right\|^2}{s}\right) \mathrm{d}s \, \mathrm{d}x \\
&\leq \left\|w_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \frac{c}{C^{n/2}} \int_{\mathbb{R}^n} e^{-\left\|x\right\|^2} \mathrm{d}x \int_{t_1}^{t_2} \frac{1}{s} \, \mathrm{d}s \\
&= \mathcal{O}(1) \ln\left(\frac{t_2}{t_1}\right) \left\|w_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} ,
\end{aligned}$$

which vanishes as $t_2 \to t_1$ since $t_1 > 0$.

$$\begin{aligned}
(3.11) &\leq \int_0^{t_1} \int_\Omega \int_\Omega \int_{t_1}^{t_2} |\partial_t G(s,\tau,x,y)| \left|\tilde{b}(\tau,y)\right| \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leq \int_0^{t_1} \int_\Omega \int_\Omega \int_{t_1}^{t_2} \frac{c}{(s-\tau)^{1+n/2}} \exp\left(\frac{-C\left\|x-y\right\|^2}{s-\tau}\right) \left|\tilde{b}(\tau,y)\right| \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leq \left\|\tilde{b}\right\|_{\mathbf{L}^\infty([0,t_1];\mathbf{L}^1(\Omega;\mathbb{R}))} \int_0^{t_1} \int_{\mathbb{R}^n} \int_{t_1}^{t_2} \frac{c}{(s-\tau)^{1+\frac{n}{2}}} \exp\left(\frac{-C\left\|x\right\|^2}{s-\tau}\right) \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leq \left\|\tilde{b}\right\|_{\mathbf{L}^\infty([0,t_1];\mathbf{L}^1(\Omega;\mathbb{R}))} \frac{c}{C^{n/2}} \int_0^{t_1} \int_{t_1}^{t_2} \frac{1}{(s-\tau)} \, \mathrm{d}s \, \mathrm{d}\tau \int_{\mathbb{R}^n} e^{-\left\|x\right\|^2} \mathrm{d}x \\
&= \mathcal{O}(1) \left\|\tilde{b}\right\|_{\mathbf{L}^\infty([0,t_1];\mathbf{L}^1(\Omega;\mathbb{R}))} \left(t_2 \ln t_2 - t_1 \ln t_1 - (t_2 - t_1) \ln(t_2 - t_1)\right)
\end{aligned}$$

which vanishes as $t_2 \to t_1$ since $t_1 > 0$.

$$\begin{aligned}
(3.12) &\leq c \int_{t_1}^{t_2} (t_2 - \tau)^{-\frac{n}{2}} \int_\Omega \int_\Omega |\tilde{b}(\tau,x)| \exp(-C\left\|x-y\right\|^2/(t_2-\tau)) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leq c \left\|\tilde{b}\right\|_{\mathbf{L}^\infty([0,t_2];\mathbf{L}^1(\Omega;\mathbb{R}))} \int_{t_1}^{t_2} (t_2 - \tau)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-C\left\|x\right\|^2/(t_2-\tau)) \, \mathrm{d}x \, \mathrm{d}\tau
\end{aligned}$$

$$\leq \frac{c|t_2 - t_1|}{C^{n/2}} \|\tilde{b}\|_{\mathbf{L}^\infty([0,t_2];\mathbf{L}^1(\Omega;\mathbb{R}))} \int_{\mathbb{R}^n} \exp(-\|x\|^2)\,\mathrm{d}x$$
$$\leq \mathcal{O}(1)\|\tilde{b}\|_{\mathbf{L}^\infty([0,t_2];\mathbf{L}^1(\Omega;\mathbb{R}))}|t_2 - t_1|,$$

which also vanishes as $t_2 \to t_1$. Adding the three estimates obtained, the $\mathbf{L}^1(\Omega;\mathbb{R})$ continuity of $w$ is proved.

**Claim 5:** There exists a solution to (3.1) in the sense of Definition 3.1 satisfying (3.5). Assume first that $w_o \in \mathbf{C}^0(\overline{\Omega};\mathbb{R})$ with $w_o = 0$ on $\partial\Omega$, $B \in \mathbf{C}^0([0,T] \times \overline{\Omega};\mathbb{R})$ and $b \in \mathbf{C}^0([0,T] \times \overline{\Omega};\mathbb{R})$. From [17, Chapter IV, § 16], we know that (3.1) admits a classical solution, say $w$. Define

$$\tilde{b}(t,x) = B(t,x)\,w(t,x) + b(t,x),$$

so that $w$ satisfies (3.4) by **(G4)** in Proposition 3.4. Hence, $w$ also satisfies (3.5).

Under the weaker regularity **(P1)**, **(P2)**, and **(P3)**, introduce sequences $w_o^\nu \in \mathbf{C}^0(\overline{\Omega};\mathbb{R})$ and $B^\nu, b^\nu \in \mathbf{C}^0([0,T] \times \overline{\Omega};\mathbb{R})$ converging to $w_o$ in $\mathbf{L}^1(\Omega;\mathbb{R})$ and to $B, b$ in $\mathbf{L}^1([0,T] \times \Omega;\mathbb{R})$. Call $w^\nu$ the corresponding classical solution to (3.1) which, by the paragraph above, exists and satisfies

$$\begin{aligned} w^\nu(t,x) = {}& \int_\Omega G(t,0,x,y)w_o^\nu(y)\,\mathrm{d}y \\ & + \int_0^t \int_\Omega G(t,\tau,x,y)(B^\nu(\tau,y)w^\nu(t,y) + b^\nu(t,y))\,\mathrm{d}y\,\mathrm{d}\tau\,. \end{aligned} \tag{3.13}$$

Hence, by Claim 3, $w^\nu$ and $w^{\nu+1}$ satisfy (3.8), so that

$$\begin{aligned} & \left\|w^{\nu+1}(t) - w^\nu(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ & \leq \left(\left\|w_o^{\nu+1} - w_o^\nu\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b^{\nu+1} - b^\nu\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \\ & \times \exp\left(\int_0^t \|B^\nu(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau\right) \\ & + \left\|B^{\nu+1} - B^\nu\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\left(\left\|w_o^{\nu+1}\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \left\|b^{\nu+1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\ & \quad \times \exp\int_0^t \left(\|B^\nu(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \left\|B^{\nu+1}(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\right)\mathrm{d}\tau\,. \end{aligned}$$

By the hypotheses on the sequences $w_o^\nu$, $B^\nu$ and $b^\nu$, $w^\nu$ is a Cauchy sequence in $\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$ converging to a function $w$ in $\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. Moreover, since, by (3.6),

$$\|w^\nu(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq \left(\|w_o^\nu\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b^\nu\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \exp \int_0^t \|B^\nu(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau$$

letting $\nu \to +\infty$, we also have

$$\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq \left(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \exp \int_0^t \|B(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau \tag{3.14}$$

and hence $w \in \mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))$.

Passing to the limit in (3.13), by the Dominated Convergence Theorem, we get that $w$ satisfies (3.5) for a.e. $x \in \Omega$. Moreover, for any $\varphi \in \mathbf{C}^2([0,T]\times \overline{\Omega};\mathbb{R})$, a further application of the Dominated Convergence Theorem allows to pass to the limit $\nu \to +\infty$ in (3.2), proving that $w$ satisfies also (3.2). The $\mathbf{C}^0$ in time – $\mathbf{L}^1$ in space continuity required by Definition 3.1 is proved in Claim 4.

This completes the proof of **(1)** and proves **(2)**. Then **(3)** follows from (3.14) and **(4)** is proved similarly, as in Claim 2.

**Claim 6:** Positivity. As above, consider a more regular and non negative datum $w_o \in \mathbf{C}^0(\overline{\Omega};\mathbb{R}_+)$ with $w_o = 0$ on $\partial\Omega$, $B \in \mathbf{C}^0([0,T] \times \overline{\Omega};\mathbb{R})$ and a non negative $b \in \mathbf{C}^0([0,T] \times \overline{\Omega};\mathbb{R}_+)$. From [17, Chapter IV, § 16], we know that (3.1) admits a classical solution, say $w$. By [17, Chapter I, § 2, Theorem 2.1], we also know that $w \geq 0$. Continue as in the proof of Claim 5 to obtain that in the general case the solution is point-wise almost everywhere limit of non negative classical solutions, completing the proof of **(5)**.

**Claim 7: BV-bound.** We follow the idea of [10, Proposition 2]. First, regularize the initial datum $w_o$ and the function $b$ appearing in the source term as follows: there exist sequences $w_o^h \in \mathbf{C}^\infty(\Omega;\mathbb{R})$ and $b_h(t) \in \mathbf{C}^\infty(\Omega;\mathbb{R})$, for all $t \in [0,T]$, such that

$$\lim_{h\to+\infty} \left\|w_o^h - w_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, \qquad \left\|w_o^h\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \|w_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})},$$
$$\mathrm{TV}\,(w_o^h) \leq \mathrm{TV}\,(w_o),$$

and for all $t \in [0,T]$

$$\lim_{h\to+\infty} \|b_h(t) - b(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, \qquad \|b_h(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \|b(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})},$$
$$\mathrm{TV}\,(b_h(t)) \leq \mathrm{TV}\,(b(t)).$$

According to (3.5), define the sequence $w_h$ corresponding to the sequences $w_o^h$ and $b_h$. By construction and due to the regularity of the Green function

$G$, $w_h(t) \in \mathbf{C}^\infty(\Omega;\mathbb{R})$ for all $t \in [0,T]$. Moreover, exploiting (3.8), it follows immediately that $w_h(t) \to w(t)$ in $\mathbf{L}^1(\Omega;\mathbb{R})$ as $h \to +\infty$ for a.e. $t \in [0,T]$. Compute $\nabla w_h$, using (3.5), the symmetry property of the Green function $G$, see **(G1)** in Proposition 3.4, integration by parts and **(G2)** in Proposition 3.4:

$$\begin{aligned}
\nabla w_h(t,x) &= \int_\Omega \nabla_x G(t,0,x,y) w_o^h(y)\,\mathrm{d}y \\
&+ \int_0^t \int_\Omega \nabla_x G(t,\tau,x,y) B(\tau,y) w_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&+ \int_0^t \int_\Omega \nabla_x G(t,\tau,x,y) b_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&= \int_\Omega \nabla_y G(t,0,y,x) w_o^h(y)\,\mathrm{d}y + \int_0^t \int_\Omega \nabla_x G(t,\tau,x,y) B(\tau,y) w_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&+ \int_0^t \int_\Omega \nabla_y G(t,\tau,y,x) b_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&= -\int_\Omega G(t,0,y,x) \nabla w_o^h(y)\,\mathrm{d}y + \int_0^t \int_\Omega \nabla_x G(t,\tau,x,y) B(\tau,y) w_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&- \int_0^t \int_\Omega G(t,\tau,y,x) \nabla b_h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\,.
\end{aligned}$$

Pass now to the $\mathbf{L}^1$-norm, exploiting **(G3)** in Proposition 3.4 and (3.6):

$$\begin{aligned}
\left\|\nabla w_h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\le \left\|\nabla w_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \int_0^t \left\|\nabla b_h(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
&\quad + \int_0^t \frac{c}{(t-\tau)^{-(n+1)/2}} \left\|w_h(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \left\|B(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \\
&\quad \times \int_\Omega \exp\left(-C\|x-y\|^2/(t-\tau)\right)\mathrm{d}y\,\mathrm{d}\tau \\
&\le \left\|\nabla w_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \int_0^t \left\|\nabla b_h(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\mathrm{d}\tau \\
&\quad + \mathcal{O}(1)\sqrt{t}\left(\left\|w_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b_h\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \left\|B\right\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})} \\
&\quad \times \exp \int_0^t \left\|B(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau\,.
\end{aligned}$$

By the lower semicontinuity of the total variation and the hypotheses on the regularizing sequences $w_o^h$ and $b_h$, passing to the limit $h \to +\infty$ yields (3.9), proving **(6)**.

3.2. **Hyperbolic Estimates.** Fix $T > 0$. This paragraph is devoted to the IBVP

$$\begin{cases} \partial_t u + \nabla\cdot (c(t,x)\, u) = A(t,x)\, u + a(t,x) & (t,x) \in [0,T] \times \Omega \\ u(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega. \end{cases} \tag{3.15}$$

We assume throughout the following conditions:

**(H1)** $u_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R})$
**(H2)** $c \in \left(\mathbf{C}^0 \cap \mathbf{L}^\infty\right)([0,T] \times \Omega;\mathbb{R}^n)$, $c(t) \in \mathbf{C}^1(\Omega;\mathbb{R}^n)$ for all $t \in [0,T]$, $D_x c \in \mathbf{L}^\infty([0,T] \times \Omega;\mathbb{R}^{n\times n})$.
**(H3)** $A \in \mathbf{L}^\infty([0,T] \times \Omega;\mathbb{R})$ and for all $t \in [0,T]$, $A(t) \in \mathbf{BV}(\Omega;\mathbb{R})$.
**(H4)** $a \in \mathbf{L}^1\left([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R})\right)$ and for all $t \in [0,T]$, $a(t) \in \mathbf{BV}(\Omega;\mathbb{R})$.

Note that **(H2)** ensures that $c(t) \in \mathbf{C}^{0,1}(\overline{\Omega};\mathbb{R}^n)$ for any $t \in [0,T]$.

For $(t_o,x_o) \in [0,T] \times \overline{\Omega}$ introduce the characteristic curve [11, § 3.2] exiting $(t_o,x_o)$, i.e., the curve $x = X(t;t_o,x_o)$ where

$$X(\,\cdot\,;t_o,x_o)\colon \quad \begin{array}{ccc} I(t_o,x_o) & \to & \overline{\Omega} \\ t & \mapsto & X(t;t_o,x_o) \end{array} \quad \text{solves} \quad \begin{cases} \dot x = c(t,x), \\ x(t_o) = x_o, \end{cases} \tag{3.16}$$

$I(t_o,x_o)$ being the maximal interval where a solution to the Cauchy problem in (3.16) is defined (with values in $\overline{\Omega}$). For $t \in [0,T]$ and for $x \in \Omega$ define

$$\mathcal{E}(\tau,t,x) = \exp\left(\int_\tau^t \left(A\left(s, X(s;t,x)\right) - \nabla\cdot c\left(s, X(s;t,x)\right)\right) \mathrm{d}s\right) \tag{3.17}$$

and for all $(t,x) \in (0,T] \times \Omega$, if $x \in X(t;[0,t),\partial\Omega) \cap \Omega$, set

$$T(t,x) = \inf\{s \in [0,t]\colon X(s;t,x) \in \Omega\}, \tag{3.18}$$

which is well defined by **(H2)** and Cauchy Theorem. Note that the well posedness of the Cauchy problem (3.16), ensured by **(H2)**, implies that for all $t \in (0,T]$

$$\begin{aligned} &\Omega \subseteq X(t;0,\Omega) \cup X(t;[0,t),\partial\Omega) \subseteq \overline{\Omega} \ \text{ and} \\ &X(t;0,\Omega) \cap X(t;[0,t),\partial\Omega) = \emptyset\,. \end{aligned} \tag{3.19}$$

As is well known, integrating (3.15) along characteristics leads, for $(t,x) \in [0,T] \times \Omega$, to

$$u(t,x) = \begin{cases} u_o\left(X(0;t,x)\right)\mathcal{E}(0,t,x) \\ \quad + \int_0^t a\left(\tau, X(\tau;t,x)\right)\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau & x \in X(t;0,\Omega) \\ \int_{T(t,x)}^t a\left(\tau, X(\tau;t,x)\right)\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau & x \in X(t;[0,t),\partial\Omega). \end{cases} \tag{3.20}$$

The following relation will be of use below, see for instance [3, Chapter 3] for a proof:

$$D_{x_o}X(t;t_o,x_o) = M(t), \text{ the matrix } M \text{ solves} \tag{3.21}$$
$$\begin{cases} \dot M = D_x c\left(t, X(t;t_, x_o)\right) M \\ M(t_o) = \mathrm{Id}\,. \end{cases}$$

We first particularize classical estimates to the present case.

**Lemma 3.6** ([5, Lemma 4.2])**.** *Let* $(\mathbf{\Omega})$ *and* **(H2)** *hold.*

1) *Assume* $u_o \in \mathbf{L}^1(\Omega;\mathbb{R})$, $A \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *and* $a \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. *Then, the map* $u$ *defined in* (3.20) *satisfies for all* $t \in [0,T]$

$$\|u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right)\exp\left(\|A\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})} t\right).$$

2) *Assume*

$$u_o \in \mathbf{L}^\infty(\Omega;\mathbb{R}),\quad A \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R})),\quad a \in \mathbf{L}^1\left([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R})\right).$$

*Then, the map* $u$ *defined in* (3.20) *satisfies for all* $t \in [0,T]$

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq & \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\ & \times \exp\left(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} + \|\nabla\cdot c\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right). \end{aligned}$$

**Lemma 3.7.** *Let* $(\mathbf{\Omega})$ *and* **(H2)** *hold. Assume* $u_o \in \mathbf{L}^\infty(\Omega;\mathbb{R})$ *and* $a \in \mathbf{L}^1\left([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R})\right)$. *Fix* $A_1, A_2 \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$. *Then, the maps* $u_1, u_2$ *defined in* (3.20) *satisfy for all* $t \in [0,T]$

$$\begin{aligned} \|u_2(t) - u_1(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq & \exp\left(t \max\left\{\|A_1\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})}, \|A_2\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})}\right\}\right) \\ & \times \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\|A_2 - A_1\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\,. \end{aligned}$$

The proof is a straightforward adaptation from [5, Lemma 4.3].

The TV bound obtained in the next lemma will be crucial in the sequel.

**Lemma 3.8.** *Let* $(\boldsymbol{\Omega})$–$(\mathbf{H1})$–$(\mathbf{H4})$ *hold. Assume, moreover, that* $A \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$ *and for all* $t \in [0,T]$, $A(t) \in \mathbf{BV}(\Omega;\mathbb{R})$. *Let* $c$ *satisfy* $(\mathbf{H2})$ *and, moreover,* $c(t) \in \mathbf{C}^2(\Omega;\mathbb{R}^n)$ *for all* $t \in [0,T]$ *and* $\nabla\,\nabla\cdot c \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$. *Then, the map* $u$ *defined in* (3.20) *satisfies for all* $t \in [0,T]$.

$$\begin{aligned}
\mathrm{TV}\,(u(t);\Omega) \le{}& \exp\Big(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} + \|D_x c\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))}\Big)\\
&\times\Big(\mathrm{TV}\,(u_o) + \mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^t \mathrm{TV}\,(a(\tau))\,\mathrm{d}\tau\\
&+\Big(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\Big)\\
&\times\int_0^t \Big(\mathrm{TV}\,(A(\tau)) + \|\nabla\,\nabla\cdot c(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\,\mathrm{d}\tau\Big).
\end{aligned}$$

**Proof.** The proof extends that of [9, Lemma 4.4], where a linear conservation law, i.e., with no source term, on a bounded domain is considered.

We first regularize the initial datum $u_o$ and the functions $A$ and $a$ appearing in the source term. In particular, we use the approximation of the initial datum constructed in [9, Lemma 4.3], yielding a sequence $u_o^h \in \mathbf{C}^3(\Omega;\mathbb{R})$ such that

$$\begin{aligned}
&\lim_{h\to+\infty}\left\|u_o^h - u_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, && u_o^h(\xi) = 0 \ \text{ for all } \xi \in \partial\Omega, \\
&\left\|u_o^h\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \left\|u_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, && \mathrm{TV}\,(u_o^h) \le \mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(u_o)\,.
\end{aligned} \tag{3.22}$$

Then, using [13, Formula (1.8) and Theorem 1.17], we regularize the functions $A$ and $a$ as follows. For all $t \in [0,T]$ and $h \in \mathbb{N}\setminus\{0\}$, there exist sequences $A_h(t), a_h(t) \in \mathbf{C}^\infty(\Omega;\mathbb{R})$ such that, for all $t \in [0,T]$,

$$\begin{aligned}
&\lim_{h\to+\infty}\|A_h(t) - A(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, && \lim_{h\to+\infty}\|a_h(t) - a(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, \\
&\|A_h(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \|A(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, && \|a_h(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \|a(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, \\
&\lim_{h\to+\infty}\mathrm{TV}\,(A_h(t)) = \mathrm{TV}\,(A(t))\,, && \lim_{h\to+\infty}\mathrm{TV}\,(a_h(t)) = \mathrm{TV}\,(a(t))\,.
\end{aligned} \tag{3.23}$$

According to (3.20), define the sequence $u_h$ corresponding to the sequences $u_o^h$, $A_h$, $a_h$, where the map $\mathcal{E}$ in (3.17) is substituted by $\mathcal{E}_h$, defined accordingly exploiting $A_h$. By construction, $u_h(t) \in \mathbf{C}^1(\Omega;\mathbb{R})$ for all $t \in [0,T]$, thus we can differentiate it. In particular, we are interested in the $\mathbf{L}^1$–norm of $\nabla u_h(t)$. By (3.19), the following decomposition holds:

$$\|\nabla u_h(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = \|\nabla u_h(t)\|_{\mathbf{L}^1(X(t;0,\Omega);\mathbb{R})} + \|\nabla u_h(t)\|_{\mathbf{L}^1(X(t;[0,t),\partial\Omega);\mathbb{R})}. \tag{3.24}$$

The two terms on the right hand side of (3.24) are treated separately. Focus on the first term: if $x \in X(t;0,\Omega)$, by (3.20)

$$\begin{aligned}\nabla u_h(t,x) = {} & \mathcal{E}_h(0,t,x)\Big(\nabla u_o^h(X(0;t,x))D_xX(0;t,x)\\ & + u_o^h(X(0;t,x))\int_0^t \Big(\nabla A_h(s,X(s;t,x)) - \nabla\nabla\cdot c(s,X(s;t,x))\Big)\\ & \times D_xX(s;t,x)\,\mathrm{d}s\Big) + \int_0^t \mathcal{E}_h(\tau,t,x)\Big(\nabla a_h(\tau,X(\tau;t,x))D_xX(\tau;t,x)\\ & + a_h(\tau,X(\tau;t,x))\int_\tau^t \Big(\nabla A_h(s,X(s;t,x)) - \nabla\nabla\cdot c(s,X(s;t,x))\Big)\\ & \times D_xX(s;t,x)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\,.\end{aligned}$$

Use the change of variables $y = X(0;t,x)$ in the first two lines above involving $u_o^h$, the change of variables $y = X(\tau;t,x)$ in the latter two lines and the bound

$$\|D_xX(\tau;t,x)\| \le \exp\Big(\int_\tau^t \|D_xc(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\,\mathrm{d}s\Big), \tag{3.25}$$

that holds for every $t \in [0,T]$ by (3.21). We thus obtain

$$\begin{aligned}\|\nabla u_h(t)\|_{\mathbf{L}^1(X(t;0,\Omega);\mathbb{R}^n)} = {} & \int_{X(t;0,\Omega)} \|\nabla u_h(t,x)\|\,\mathrm{d}x \qquad (3.26)\\ \le \exp\Big(\int_0^t & \Big(\|A_h(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|D_xc(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\Big)\,\mathrm{d}\tau\Big)\Big(\big\|\nabla u_o^h\big\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\\ + \big\|u_o^h\big\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} & \int_0^t \Big(\|\nabla A_h(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \|\nabla\nabla\cdot c(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\,\mathrm{d}\tau\\ + \int_0^t & \|\nabla a_h(\tau)\|_{\mathbf{L}^1(X(\tau;0,\Omega);\mathbb{R}^n)}\,\mathrm{d}\tau + \int_0^t \|a_h(\tau)\|_{\mathbf{L}^\infty(X(\tau;0,\Omega);\mathbb{R})}\\ \times\Big(\int_\tau^t & \Big(\|\nabla A_h(s)\|_{\mathbf{L}^1(X(s;0,\Omega);\mathbb{R}^n)} + \|\nabla\nabla\cdot c(s)\|_{\mathbf{L}^1(X(s;0,\Omega);\mathbb{R}^n)}\Big)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\Big).\end{aligned}$$

Pass to the second term on the right in (3.24): if $x \in X(t;[0,t),\partial\Omega)$, by (3.20) and (3.25)

$$\begin{aligned}\nabla u_h(t,x) = {} & \int_{T(t,x)}^t \mathcal{E}_h(\tau,t,x)\Big(\nabla a_h\left(\tau,X(\tau;t,x)\right)D_xX(\tau;t,x)\\ & + a_h\left(\tau,X(\tau;t,x)\right)\int_\tau^t \left(\nabla A_h\left(s,X(s;t,x)\right) - \nabla\nabla\cdot c\left(s,X(s;t,x)\right)\right)\\ & \quad \times D_xX(s;t,x)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\,.\end{aligned}$$

For every $t \in [0,T]$, proceed similarly as above using the change of variables $y = X(\tau; t, x)$:

$$\begin{aligned}
&\|\nabla u_h(t)\|_{\mathbf{L}^1(X(t;[0,t),\partial\Omega);\mathbb{R})} = \int_{\Omega\setminus X(t;0,\Omega)} \|\nabla u_h(t,x)\| \,\mathrm{d}x \\
\leq\ & \exp\left( \int_0^t \left( \|A_h(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \|D_x c(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n})} \right) \mathrm{d}\tau \right) \qquad (3.27)\\
&\times \left( \int_0^t \int_{\Omega\setminus X(\tau;0,\Omega)} \|\nabla a_h(\tau,y)\| \,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \|a_h(\tau)\|_{\mathbf{L}^\infty(\Omega\setminus X(\tau;0,\Omega);\mathbb{R})} \,\mathrm{d}\tau \right.\\
&\quad \left. \times \left( \|\nabla A_h\|_{\mathbf{L}^1(\Omega\setminus X([0,t];0,\Omega);\mathbb{R}^n)} + \|\nabla\,\nabla\cdot c\|_{\mathbf{L}^1(\Omega\setminus X([0,t];0,\Omega);\mathbb{R}^n)} \right) \right).
\end{aligned}$$

Inserting the estimates (3.26) and (3.27) into (3.24), we thus obtain

$$\begin{aligned}
&\|\nabla u_h(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq \exp\left( \|A_h\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} + \|D_x c\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))} \right)\\
&\times \left( \|\nabla u_o^h\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \int_0^t \|\nabla a_h(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \,\mathrm{d}\tau \right.\\
&\quad + \left( \|u_o^h\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^t \|a_h(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \,\mathrm{d}\tau \right)\\
&\left. \times \left( \|\nabla A_h\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)} + \|\nabla\,\nabla\cdot c\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)} \right) \right).
\end{aligned}$$

Since $u_h(t) \to u(t)$ in $\mathbf{L}^1(\Omega;\mathbb{R})$, by the lower semicontinuity of the total variation and the hypotheses (3.22)–(3.23) on the regularizing sequences $u_o^h$, $A_h$ and $a_h$, passing to the limit $h \to +\infty$, we complete the proof. □

It is on the basis of next Proposition that we give a definition of solution to (3.15).

**Proposition 3.9.** *Let* $(\mathbf{\Omega})$ *and* **(H2)** *hold. Assume* $u_o \in \mathbf{L}^\infty(\Omega;\mathbb{R})$, $A \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *and* $a \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$. *Then, the following statements are equivalent:*

(1) $u$ *is defined by* (3.20)*, i.e., through integration along characteristics.*
(2) $u \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *is such that for any test function* $\varphi \in \mathbf{C}_c^1((-\infty,T)\times\Omega;\mathbb{R})$,

$$\begin{aligned}
&\int_0^T \int_\Omega \Big( u(t,x)\left(\partial_t\varphi(t,x) + c(t,x)\cdot\nabla\varphi(t,x)\right) \qquad (3.28)\\
&+ \left(A(t,x)u(t,x) + a(t,x)\right)\varphi(t,x) \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_o(x)\varphi(0,x)\,\mathrm{d}x = 0.
\end{aligned}$$

(3) $u \in \mathbf{L}^\infty([0,T] \times \Omega; \mathbb{R})$ *is such that for any test function* $\varphi \in \mathbf{W}^{1,\infty}((-\infty,T) \times \Omega; \mathbb{R})$, *equality* (3.28) *holds.*

**Proof.** (1) $\implies$ (2) The proof exploits arguments similar to [7, Lemma 5.1], see also [8, Lemma 2.7]. Indeed, $u$ defined as in (3.20) is bounded by Item 2) in Lemma 3.6.

Let $\varphi \in \mathbf{C}^1_c((-\infty,T) \times \Omega; \mathbb{R})$. We prove that the equality (3.28) holds with $u$ defined as in (3.20). Notice that, for a fixed time $t \in [0,T]$, by (3.19) the domain $\Omega$ is contained in the disjoint union of $X(t;0,\Omega)$ and $X(t;[0,t),\partial\Omega)$. The first set accounts for the characteristics emanating from the initial datum, the second one for those coming from the boundary. Therefore, to prove that the integral equality (3.28) holds it is sufficient to verify that the following integral equalities hold:

$$\int_0^T \int_{X(t;0,\Omega)} (u\,(\partial_t \varphi + c \cdot \nabla\varphi + A\,\varphi) + a\,\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_o(x)\,\varphi(0,x)\,\mathrm{d}x = 0, \tag{3.29}$$

$$\int_0^T \int_{X(t;[0,t),\partial\Omega)} (u\,(\partial_t \varphi + c \cdot \nabla\varphi + A\,\varphi) + a\,\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.30}$$

In order to prove (3.29), exploiting the change of variables $y = X(0;t,x)$, the first line in (3.20) can be rewritten for $x \in X(t,0,\Omega)$ as

$$u(t,x) = (u_o(y) + \mathcal{A}(t,y))\,\frac{\mathscr{A}(t,y)}{J(t,y)} \qquad \text{where} \qquad y = X(0;t,x)$$

with

$$\begin{aligned}
\mathscr{A}(t,y) &= \exp\Big(\int_0^t A(\tau, X(\tau;0,y))\,\mathrm{d}\tau\Big),\\
J(t,y) &= \exp\Big(\int_0^t \nabla\cdot c(\tau, X(\tau;0,y))\,\mathrm{d}\tau\Big),\\
\mathcal{A}(t,y) &= \int_0^t a\,(\tau, X(\tau;0,y))\,\frac{J(\tau,y)}{\mathscr{A}(\tau,y)}\,\mathrm{d}\tau\,.
\end{aligned}$$

Therefore, the left hand side of (3.29) now reads

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[(u_o(y) + \mathcal{A}(t,y))\,\frac{\mathscr{A}(t,y)}{J(t,y)}\,(\partial_t\varphi\,(t, X(t;0,y))\\
&+c\,(t, X(t;0,y)) \cdot \nabla\varphi\,(t, X(t;0,y)) + A\,(t, X(t;0,y))\,\varphi\,(t, X(t;0,y)))\,J(t,y)\\
&+ a\,(t, X(t;0,y))\,\varphi\,(t, X(t;0,y))\,J(t,y)\Big]\,\mathrm{d}y\,\mathrm{d}t + \int_\Omega u_o(x)\varphi(0,x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} [(u_o(y) + \mathcal{A}(t,y)\mathscr{A}(t,y))\varphi(t, X(t;0,y))] \, \mathrm{d}y \, \mathrm{d}t + \int_\Omega u_o(x)\varphi(0,x) \, \mathrm{d}x \\
&= -\int_\Omega u_o(y)\varphi(0,y) \, \mathrm{d}y + \int_\Omega u_o(x)\varphi(0,x) \, \mathrm{d}x - \int_\Omega \mathcal{A}(0,y)\mathscr{A}(0,y)\varphi(0,y) \, \mathrm{d}y \\
&= 0,
\end{aligned}
$$

since, for all $y \in \Omega$, $\varphi(T,y) = 0$ and, by definition, $\mathcal{A}(0,y) = 0$.

Pass now to (3.30). Here, for all $t \in [0,T]$, we pass from the variables $(\tau, x) \in \Omega^t_{\tau,x}$ to the variables $(\sigma, y) \in \Omega^t_{\sigma,y}$, where

$$
\begin{aligned}
&\Omega^t_{\tau,x} = \{(\tau, x) \colon \tau \in [T(t,x), t] \text{ and } x = X(t, [0,t), \partial\Omega)\}, \\
&\Omega^t_{\sigma,y} = \{(\sigma, y) \colon \sigma \in [0,t] \text{ and } y = X(\sigma, [0,\sigma), \partial\Omega)\}, \quad \sigma = \tau, \ y = X(\tau; t, x).
\end{aligned}
$$

The corresponding Jacobian, which also depends on $t$, is $H(t,\sigma,y)/H(\sigma,\sigma,y)$ where we set

$$
\begin{aligned}
H(t,\sigma,y) &= \exp \int_0^t \nabla \cdot c\,(s, X(s;\sigma,y)) \, \mathrm{d}s \\
\widehat{A}(t,\sigma,y) &= \exp \int_0^t A\,(s, X(s;\sigma,y)) \, \mathrm{d}\sigma \ .
\end{aligned}
$$

Using (3.20), we compute now the right hand side in (3.30) as follows:

$$
\begin{aligned}
&\int_0^T \int_{X(t;[0,t),\partial\Omega)} u(\partial_t \varphi + c \cdot \nabla\varphi + A\varphi)(t,x) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{X(t;[0,t),\partial\Omega)} a\varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_{X(t;[0,t),\partial\Omega)} \int_{T(t,x)}^t a(\tau, X(\tau;t,x)) \mathcal{E}(\tau,t,x) \, \mathrm{d}\tau \, (\partial_t\varphi + c \cdot \nabla\varphi + A\varphi) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^T \int_{X(t;[0,t),\partial\Omega)} a(t,x)\, \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_0^t \int_{X(\sigma,[0,\sigma),\partial\Omega)} a(\sigma,y) \, \frac{\widehat{A}(t,\sigma,y)}{\widehat{A}(\sigma,\sigma,y)} \\
&\qquad \times \Big( \frac{\mathrm{d}\varphi(t, X(t;\sigma,y))}{\mathrm{d}t} + A(t, X(t;\sigma,y))\varphi(t, X(t;\sigma,y)) \Big) \, \mathrm{d}y \, \mathrm{d}\sigma \, \mathrm{d}t \\
&\qquad + \int_0^T \int_{X(t;[0,t),\partial\Omega)} a(t,x)\, \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \Big( \int_0^t \int_{X(\sigma,[0,\sigma),\partial\Omega)} a(\sigma,y) \, \frac{\widehat{A}(t,\sigma,y)}{\widehat{A}(\sigma,\sigma,y)} \, \varphi\,(t, X(t;\sigma,y)) \, \mathrm{d}y \, \mathrm{d}\sigma \Big) \, \mathrm{d}t \\
&= 0,
\end{aligned}
$$

since $\varphi(T,\cdot) \equiv 0$.

(2) $\implies$ (3). Fix $\varphi \in \mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R})$. A standard construction, see [13, § 1.14], ensures the existence of a sequence of functions $\varphi_h \in \mathbf{C}_c^\infty(\mathbb{R}^{n+1};\mathbb{R}_+)$ such that

$$\varphi_h \underset{h\to+\infty}{\longrightarrow} \varphi, \qquad \partial_t\varphi_h \underset{h\to+\infty}{\longrightarrow} \partial_t\varphi, \qquad \nabla_x\varphi_h \underset{h\to+\infty}{\longrightarrow} \nabla_x\varphi$$

in $\mathbf{L}^1_{\mathbf{loc}}((-\infty,T)\times\Omega;\mathbb{R})$ and $\mathbf{L}^1_{\mathbf{loc}}((-\infty,T)\times\Omega;\mathbb{R}^n)$. Call $\chi_h$ a function in $\mathbf{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$ such that $\chi_h(x) = 1$ for all $x\in\Omega$ such that $B(x,1/h)\subseteq\Omega$ and $\|\nabla_x\chi_h\| \leq 2\sqrt{n}\,h$ for all $x\in\mathbb{R}^n$.

Then, we have $\varphi_h\,\chi_h \in \mathbf{C}^1_c((-\infty,T)\times\Omega;\mathbb{R})$. Moreover,

$$\varphi_h\,\chi_h \underset{h\to+\infty}{\longrightarrow} \varphi \quad\text{and}\quad \partial_t(\varphi_h\,\chi_h) \underset{h\to+\infty}{\longrightarrow} \partial_t\varphi \quad \text{in } \mathbf{L}^1_{\mathbf{loc}}((-\infty,T)\times\Omega;\mathbb{R}).$$

Concerning the space gradient, we have

$$\nabla_x(\varphi_h\,\chi_h) = \nabla_x\varphi_h\;\chi_h + \varphi_h\;\nabla_x\chi_h$$

and

$$\begin{aligned} \nabla_x\varphi_h\;\chi_h &\underset{h\to+\infty}{\longrightarrow} \nabla_x\varphi && \text{in } \mathbf{L}^1_{\mathbf{loc}}((-\infty,T)\times\Omega;\mathbb{R})\,; \\ \varphi_h\;\nabla_x\chi_h &\underset{h\to+\infty}{\longrightarrow} \varphi && \text{a.e. in } (-\infty,T)\times\Omega\,. \end{aligned}$$

Therefore, for all $h$ by (2), we have

$$\begin{aligned} 0 = &\int_0^T\!\!\int_\Omega \bigl(u(\partial_t(\varphi_h\chi_h) + c\cdot\nabla(\varphi_h\omega_h)) + (Au(t,x)+a(t,x))(\varphi_h\chi_h)\bigr)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_\Omega u_o(x)\,(\varphi_h\,\chi_h)(0,x)\,\mathrm{d}x \end{aligned}$$

and, by the Dominated Convergence Theorem, (3) follows.

(3) $\implies$ (1). Inspired by [7, Lemma 5.1], we first consider the case $A\in(\mathbf{C}^1\cap\mathbf{W}^{1,\infty})([0,T]\times\Omega;\mathbb{R})$. Assume $u$ satisfies (3) and call $u_*$ the function defined in (3.20). Then, by the above implications (1)⇒(2)⇒(3), the difference $U = u - u_*$ satisfies for all test functions $\tilde\varphi\in\mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R})$ the integral equality

$$\int_0^T\!\!\int_\Omega U\,(\partial_t\tilde\varphi + c\cdot\nabla\tilde\varphi + A\,\tilde\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{3.31}$$

Proceed now exactly as in [7, Lemma 5.1], choosing $\tau\in(0,T]$, a sequence $\chi_h\in\mathbf{C}^1_c(\mathbb{R};\mathbb{R}_+)$ with $\chi_h(t) = 1$ for all $t\in[1/h,\tau-1/h]$ and $|\chi'_h|\leq 2h$. If $\varphi\in\mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R})$, then $(\varphi\,\chi_h)\in\mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R})$. Choosing

$\varphi\,\chi_h$ as $\tilde\varphi$ in (3.31), and passing to the limit $h \to +\infty$ via the Dominated Convergence Theorem, we get

$$\int_0^\tau \int_\Omega U\,(\partial_t\varphi + c\cdot\nabla\varphi + A\,\varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega U(\tau,x)\,\varphi(\tau,x)\,\mathrm{d}x = 0\,. \qquad (3.32)$$

Fix an arbitrary $\eta \in \mathbf{C}^1_c(\Omega;\mathbb{R})$ and let $\varphi$ solve

$$\begin{cases} \partial_t\varphi + c\cdot\nabla\varphi + A\,\varphi = 0 & (t,x)\in\Omega \\ \varphi(t,\xi) = 0 & (t,\xi)\in\partial\Omega \\ \varphi(\tau,x) = \eta(x) & (\tau,x)\in\Omega\,. \end{cases}$$

Note that $\varphi$ can be computed through integration along (backward) characteristics and hence

$$\varphi \in \mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R}).$$

With this choice, (3.32) yields

$$\int_\Omega U(\tau,x)\,\eta(x)\,\mathrm{d}x = 0 \quad \text{for all } \eta \in \mathbf{C}^1_c(\Omega;\mathbb{R}),$$

so that $U(\tau,x) = 0$ for all $x \in \Omega$. By the arbitrariness of $\tau$, we have $U \equiv 0$, hence $u = u_*$.

Let now $A \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$, call $u_*$ the function constructed in (3.20) and assume there is a function $u$ satisfying (3). Construct a sequence $A_h \in (\mathbf{C}^1 \cap \mathbf{W}^{1,\infty})([0,T]\times\Omega;\mathbb{R})$ such that $A_h \underset{h\to+\infty}{\longrightarrow} A$ in $\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. Call $u_h$ the function constructed as in (3.20) with $A_h$ in place of $A$. For any $t \in [0,T]$, we have $u_h(t) \underset{h\to+\infty}{\longrightarrow} u_*(t)$ in $\mathbf{L}^1(\Omega;\mathbb{R})$, by Lemma 3.7. Moreover, for all $\varphi \in \mathbf{W}^{1,\infty}((-\infty,T)\times\Omega;\mathbb{R})$,

$$\begin{aligned} 0 = &\int_0^T \int_\Omega \left(u\,(\partial_t\varphi + c\cdot\nabla\varphi) + (A\,u + a)\,\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_o(x)\,\varphi(0,x)\,\mathrm{d}x \\ &- \int_0^T \int_\Omega \left(u_h\,(\partial_t\varphi + c\cdot\nabla\varphi) + (A_h u_h + a)\,\varphi\right)\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_o(x)\varphi(0,x)\,\mathrm{d}x \\ = &\int_0^T \int_\Omega (u - u_h)\,(\partial_t\varphi + c\cdot\nabla\varphi + A_h\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega (A - A_h)u\varphi\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

The latter summand vanishes since $A_h \to A$ in $\mathbf{L}^1$. The former summand, thanks to the regularity of $A_h$, can be treated by the procedure above, obtaining, for all $\eta \in \mathbf{C}^1_c(\Omega;\mathbb{R})$ and for a sequence of real numbers $\varepsilon_h$ converging to 0,

$$0 = \int_\Omega \left(u(\tau,x) - u_h(\tau,x)\right) \eta(x)\, \mathrm{d}x + \varepsilon_h\,.$$

The above relation ensures that $u_h(\tau,x) \to u(\tau,x)$ for a.e. $x \in \Omega$ as $h \to +\infty$. Therefore, for all $t \in (0,T]$,

$$\begin{aligned}&\|u_*(t) - u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\quad\le \|u_*(t) - u_h(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|u_h(t) - u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \underset{h\to+\infty}{\longrightarrow} 0,\end{aligned}$$

completing the proof. □

**Definition 3.10.** *A map* $u \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *is a solution to* (3.15) *if it satisfies any of the requirements* (1), (2)*, or* (3) *in Proposition 3.9.*

By techniques similar to those in [8], one can verify that a solution to (3.15) in the sense of Definition 3.10 is a *weak entropy solution* also in any of the senses [18, 22], or [1] in the **BV** case, see [21] for a comparison. Here, as is well known, the linearity of the convective part in (3.15) allows to avoid the introduction of any entropy condition, as also remarked in [16].

**Lemma 3.11.** *Let* **(Ω)**–**(H1)**–**(H3)**–**(H4)** *hold. Fix* $c_1, c_2$ *satisfying* **(H2)** *and moreover, for* $i = 1,2$*,* $c_i(t) \in \mathbf{C}^2(\Omega;\mathbb{R}^n)$ *for all* $t \in [0,T]$ *and* $\nabla\,\nabla\cdot c_i \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$*. Then, the maps* $u_1, u_2$ *defined in* (3.20) *satisfy*

$$\begin{aligned}&\|u_2(t) - u_1(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\le \mathcal{O}(1)\left(\|c_1 - c_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} + \|\nabla\cdot(c_1 - c_2)\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\end{aligned}$$

*and a precise expression for the constant* $\mathcal{O}(1)$ *is provided in the proof.*

**Proof.** Following the proof of Lemma 3.8, we first regularize the initial datum $u_o$ as in (3.22) and the functions $A$ and $a$ appearing in the source term as in (3.23), obtaining $u_1^h$ and $u_2^h$ by (3.20). By Proposition 3.9, the difference $u_2^h - u_1^h$ solves

$$\begin{cases} \partial_t(u_2^h - u_1^h) + \nabla\cdot\left(c_2(u_2^h - u_1^h)\right) = A_h(u_2^h - u_1^h) + \alpha_h\\ (u_2^h - u_1^h)(0) = 0,\end{cases}$$

where

$$\alpha_h = -\,\nabla\cdot\Big((c_2 - c_1)u_1^h\Big)$$

in the sense of Definition 3.10. Apply Item 1) in Lemma 3.6 to get

$$\left\|u_2^h(t) - u_1^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \tag{3.33}$$

$$\leq \left\|\nabla\cdot\left((c_2-c_1)u_1^h\right)\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\exp\left(\|A\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})}t\right),$$

where we use the estimate

$$\|A_h(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\leq\|A(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$$

for all $\tau\in[0,T]$. Observe that

$$\begin{aligned}
&\left\|\nabla\cdot\left((c_2(\tau)-c_1(\tau))u_1^h(\tau)\right)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\leq\left\|u_1^h(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\left\|\nabla\cdot\left(c_2(\tau)-c_1(\tau)\right)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\\
&\qquad+\left\|c_2(\tau)-c_1(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}\left\|\nabla u_1^h(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\leq\left(\left\|u_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\left\|a_h\right\|_{\mathbf{L}^1([0,\tau]\times\Omega;\mathbb{R})}\right)\exp\left(\|A\|_{\mathbf{L}^\infty([0,\tau]\times\Omega;\mathbb{R})}t\right)\\
&\qquad\times\left\|\nabla\cdot\left(c_2(\tau)-c_1(\tau)\right)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}+\|c_2(\tau)-c_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}\\
&\qquad\times\exp\left(\|A\|_{\mathbf{L}^1([0,\tau];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))}+\|D_xc_1\|_{\mathbf{L}^1([0,\tau];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))}\right)\\
&\qquad\times\left(\mathrm{TV}\,(u_o)+\mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}+\int_0^\tau\|\nabla\,a_h(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}s\right.\\
&\qquad+\left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}+\int_0^\tau\|a(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}s\right)\\
&\qquad\left.\times\left(\|\nabla\,A_h\|_{\mathbf{L}^1([0,\tau]\times\Omega;\mathbb{R}^n)}+\|\nabla\,\nabla\cdot c_1\|_{\mathbf{L}^1([0,\tau]\times\Omega;\mathbb{R}^n)}\right)\right),
\end{aligned}$$

where we used Item 1) in Lemma 3.6, Lemma 3.8 and the hypotheses (3.22)–(3.23) on the regularizing sequences $u_o^h$, $A_h$ and $a_h$. By the triangular inequality and the above computations,

$$\begin{aligned}
&\|u_2(t)-u_1(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\leq\left\|u_2(t)-u_2^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\left\|u_2^h(t)-u_1^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\left\|u_1(t)-u_1^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\leq\left\|u_2(t)-u_2^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &(3.34)\\
&+\left(\left\|u_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\left\|a_h\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right)\\
&\quad\times\exp\left(\|A\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})}t\right)\int_0^t\|\nabla\cdot\left(c_2(\tau)-c_1(\tau)\right)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau &(3.35)\\
&+\int_0^t\|c_2(\tau)-c_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}\,\mathrm{d}\tau\\
&\quad\times\exp\left(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))}+\|D_xc_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))}\right)
\end{aligned}$$

$$\times \Big( \mathrm{TV}\,(u_o) + \mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^t \|\nabla\, a_h(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}s \tag{3.36}$$

$$+ \Big(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^t \|a(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}s\Big)$$

$$\times \Big(\|\nabla\, A_h\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)} + \|\nabla\,\nabla\cdot c_1\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)}\Big)\Big) \tag{3.37}$$

$$+ \left\|u_1(t) - u_1^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}, \tag{3.38}$$

and in the limit $h \to +\infty$, we treat each term separately. By construction, (3.34) and (3.38) converge to zero as $h \to +\infty$. By the hypotheses (3.22) on $u_o^h$ and (3.23) on $A_h$ and $a_h$, in the limit we thus obtain

$$\begin{aligned}
\|u_2(t) - u_1(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le{}& \Big(\|u_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big)\\
&\times \exp\Big(\|A\|_{\mathbf{L}^\infty([0,t]\times\Omega;\mathbb{R})}t\Big)\int_0^t \|\nabla\cdot(c_2(\tau) - c_1(\tau))\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau\\
&+ \int_0^t \|c_2(\tau) - c_1(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}\,\mathrm{d}\tau\\
&\times \exp\Big(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} + \|D_x c_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))}\Big)\\
&\times \Big(\mathrm{TV}\,(u_o) + \mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^t \mathrm{TV}\,(a(s))\,\mathrm{d}s + \Big(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\\
&+ \int_0^t \|a(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}s\Big)\Big(\int_0^t \mathrm{TV}\,(A(s))\,\mathrm{d}s + \|\nabla\,\nabla\cdot c_1\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)}\Big)\Big),
\end{aligned}$$

concluding the proof. □

**Lemma 3.12.** *Let* $(\mathbf{\Omega})$–**(H2)** *hold. Assume moreover that* $u_o \in \mathbf{L}^\infty(\Omega;\mathbb{R})$, *with* $u_o \ge 0$, $A \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *and* $a \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$, *with* $a \ge 0$. *Then, the solution* $u$ *is positive.*

The proof is an immediate consequence of the representation (3.20).

**Lemma 3.13.** *Let* **(H1)**–**(H2)**–**(H3)**–**(H4)** *hold. Assume, moreover, that* $c(t) \in \mathbf{C}^2(\Omega;\mathbb{R}^n)$ *for all* $t \in [0,T]$ *and* $\nabla\,\nabla\cdot c \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$. *If* $u \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ *is as in* (3.20), *then* $u$ *is* $\mathbf{L}^1$*–Lipschitz continuous in time: for all* $t_1, t_2 \in [0,T]$, *with* $t_1 < t_2$

$$\|u(t_2) - u(t_1)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \mathcal{O}(1)(t_2 - t_1) \tag{3.39}$$

*where* $\mathcal{O}(1)$ *depends on norms of* $c, A, a$ *on the interval* $[0, t_2]$ *and of* $u_o$.

**Proof.** By (3.19), the following decomposition holds

$$\begin{aligned}\|u(t_2) - u(t_1)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = {} & \|u(t_2) - u(t_1)\|_{\mathbf{L}^1(X(t_2;t_1,\Omega);\mathbb{R})} \\ & + \|u(t_2) - u(t_1)\|_{\mathbf{L}^1(X(t_2;[t_1,t_2),\partial\Omega);\mathbb{R})} .\end{aligned} \tag{3.40}$$

Estimate the two latter summands in (3.40) separately. By (3.20)

$$\begin{aligned}
& \|u(t_2) - u(t_1)\|_{\mathbf{L}^1(X(t_2;t_1,\Omega);\mathbb{R})} \\
\le {} & \int_{X(t_2;t_1,\Omega)} |u\left(t_1, X(t_1;t_2,x)\right) \mathcal{E}(t_1,t_2,x) - u(t_1,x)| \, \mathrm{d}x \\
& \qquad + \int_{X(t_2;t_1,\Omega)} \int_{t_1}^{t_2} |a\left(\tau, X(\tau;t_2,x)\right) \mathcal{E}(\tau,t_2,x)| \, \mathrm{d}\tau \, \mathrm{d}x \\
\le {} & \int_{X(t_2;t_1,\Omega)} |u\left(t_1, X(t_1;t_2,x)\right) - u(t_1,x)| \, \mathcal{E}(t_1,t_2,x) \, \mathrm{d}x \qquad (3.41) \\
& + \int_{X(t_2;t_1,\Omega)} |u(t_1,x)| \, |\mathcal{E}(t_1,t_2,x) - 1| \, \mathrm{d}x \qquad (3.42) \\
& + \int_{X(t_2;t_1,\Omega)} \int_{t_1}^{t_2} |a\left(\tau, X(\tau;t_2,x)\right) \mathcal{E}(\tau,t_2,x)| \, \mathrm{d}\tau \, \mathrm{d}x \, . \qquad (3.43)
\end{aligned}$$

To estimate (3.41), we use [6, Lemma 5.1] so that we obtain

$$\begin{aligned}
& \int_{X(t_2;t_1,\Omega)} |u\left(t_1, X(t_1;t_2,x)\right) - u(t_1,x)| \, \mathcal{E}(t_1,t_2,x) \, \mathrm{d}x \\
\le {} & \frac{\|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R}^n)}}{\|D_x c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R}^{n\times n})}} \left(e^{\|D_x c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R}^{n\times n})}(t_2-t_1)} - 1\right) \mathrm{TV}\left(u(t_1)\right) \\
\le {} & \|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R}^n)} e^{\|D_x c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R}^{n\times n})}(t_2-t_1)} \, \mathrm{TV}\left(u(t_1)\right)(t_2-t_1),
\end{aligned}$$

and the total variation of $u$ might be estimated thanks to Lemma 3.8. The bounds for (3.42) and (3.43) follow from the definition (3.17) of $\mathcal{E}$:

$$\begin{aligned}
& \int_{X(t_2;t_1,\Omega)} |u(t_1,x)| |\mathcal{E}(t_1,t_2,x) - 1| \, \mathrm{d}x \\
& \le \|u(t_1)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}(t_2-t_1) \left(\|A\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R})} + \|\nabla\cdot c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R})}\right) \\
& \times \exp\left(\left(\|A\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R})} + \|\nabla\cdot c\|_{\mathbf{L}^\infty([t_1,t_2]\times\Omega;\mathbb{R})}\right)(t_2-t_1)\right) ; \\
& \int_{X(t_2;t_1,\Omega)} \int_{t_1}^{t_2} |a\left(\tau, X(\tau;t_2,x)\right) \mathcal{E}(\tau,t_2,x)| \, \mathrm{d}\tau \, \mathrm{d}x
\end{aligned}$$

$$\leq (t_2 - t_1) \|a\|_{\mathbf{L}^\infty([t_1,t_2];\mathbf{L}^1(\Omega;\mathbb{R}))} \exp\left( \int_{t_1}^{t_2} \|A(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|\nabla \cdot c\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \, \mathrm{d}\tau \right).$$

Consider now the second summand in (3.40). Introduce

$$T_{t_1}(t_2, x) = \inf \left\{ s \in [t_1, t_2] \colon X(s; t_2, x) \in \Omega \right\}$$

and compute

$$\begin{aligned} & \|u(t_2) - u(t_1)\|_{\mathbf{L}^1(X(t_2;[t_1,t_2),\partial\Omega);\mathbb{R})} \\ & \leq \int_{X(t_2;[t_1,t_2),\partial\Omega)} \left| \int_{T_{t_1}(t_2,x)}^{t_2} a\left(\tau, X(\tau; t_2, x)\right) \mathcal{E}(\tau, t_2, x) \, \mathrm{d}\tau \right| \mathrm{d}x \,. \end{aligned}$$

The same procedure used to bound (3.43) applies, completing the proof. □

### 3.3. **Coupling.**

**Proof of Theorem 2.2.** Fix $T > 0$. Define $u_0(t,x) = u_o(x)$ and $w_0(t,x) = w_o(x)$ for all $(t,x) \in [0,T] \times \Omega$. For $i \in \mathbb{N}$, define recursively $u_{i+1}$ and $w_{i+1}$ as solutions to

$$\begin{cases} \partial_t u_{i+1} + \nabla \cdot \left(u_{i+1}\, c_i(t,x)\right) = A_i(t,x)\, u_{i+1} + a(t,x) & (t,x) \in [0,T] \times \Omega \\ u(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega, \end{cases} \tag{3.44}$$

$$\begin{cases} \partial_t w_{i+1} - \mu \Delta w_{i+1} = B_i(t,x) w_{i+1} + b(t,x) & (t,x) \in [0,T] \times \Omega \\ w(t,\xi) = 0 & (t,\xi) \in [0,T] \times \partial\Omega \\ w(0,x) = w_o(x) & x \in \Omega, \end{cases} \tag{3.45}$$

where

$$\begin{aligned} c_i(t,x) &= v\left(t, w_i(x)\right), \quad A_i(t,x) = \alpha\left(t, x, w_i(t,x)\right) \\ B_i(t,x) &= \beta\left(t, x, u_i(t,x), w_i(t,x)\right). \end{aligned} \tag{3.46}$$

We aim to prove that $(u_i, w_i)$ is a Cauchy sequence with respect to the $\mathbf{L}^\infty([0,T]; \mathbf{L}^1(\Omega; \mathbb{R}^2))$ distance as soon as $T$ is sufficiently small.

Observe first that problem (3.45) fits into the framework of Section 3.1, while problem (3.44) fits into the framework of Section 3.2.

Consider the $w$ component. Proposition 3.5 applies, ensuring the existence of a solution to (3.45) for all $i \in \mathbb{N}$. Moreover, if $b \geq 0$ and the initial datum $w_o$ is positive, the solution $w_i$ is positive. By ($\boldsymbol{\beta}$) and (3.46), for all $i \in \mathbb{N}$, $B_i$ satisfies **(P2)** and for all $\tau \in [0,T]$

$$\|B_i(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq K_\beta\,, \tag{3.47}$$

while by ($\boldsymbol{b}$) the function $b$ satisfies **(P3)**. The following uniform bounds on $w_i$ hold for every $i \in \mathbb{N}$: by **(3)** and **(6)** in Proposition 3.5, exploiting also (3.47), for all $\tau \in [0,T]$,

$$\begin{aligned} \|w_i(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\le e^{K_\beta\,\tau}\left(\|w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,\tau]\times\Omega;\mathbb{R})}\right) \qquad (3.48)\\ &=: C_{w,1}(\tau),\\ \|w_i(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\le e^{K_\beta\,\tau}\left(\|w_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,\tau];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \qquad (3.49)\\ &=: C_{w,\infty}(\tau),\\ \mathrm{TV}\left(w_i(\tau,\cdot)\right) &\le \mathrm{TV}\left(w_o\right) + \int_0^\tau \mathrm{TV}\left(b(s)\right)\mathrm{d}s + \mathcal{O}(1)\sqrt{\tau}\,K_\beta\|w_i(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &=: C_w^{\mathrm{TV}}(\tau). \qquad (3.50) \end{aligned}$$

By **(4)** in Proposition 3.5, we get

$$\begin{aligned} &\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \qquad (3.51)\\ &\le \|B_i - B_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\left(\|w_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\\ &\times \exp\int_0^t \left(\|B_i(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|B_{i-1}(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\right)\mathrm{d}\tau\,. \end{aligned}$$

By (3.46), exploiting the hypothesis ($\boldsymbol{\beta}$), we obtain

$$\begin{aligned} &\|B_i - B_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\\ &= \int_0^t\int_\Omega \left|\beta\left(\tau,x,u_i(\tau,x),w_i(\tau,x)\right) - \beta\left(\tau,x,u_{i-1}(\tau,x),w_{i-1}(\tau,x)\right)\right|\mathrm{d}x\,\mathrm{d}\tau\\ &\le K_\beta\left(\|u_i - u_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} + \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right). \end{aligned}$$

Therefore, using also (3.47) and the notation introduced in (3.49), (3.51) becomes

$$\begin{aligned} &\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \qquad (3.52)\\ &\le K_\beta\,e^{t\,K_\beta}C_{w,\infty}(t)\left(\|u_i - u_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} + \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right). \end{aligned}$$

Pass now to the $u$ component. The results of Section 3.2 applies, ensuring the existence of a solution to (3.44) for all $i \in \mathbb{N}$. Moreover, if $a \ge 0$ and the initial datum $u_o$ is positive, the solution $u_i$ is positive, see Lemma 3.12. By ($\boldsymbol{\alpha}$) and (3.46), for every $i \in \mathbb{N}$, we have that $A_i$ satisfies **(H3)** and for all $\tau \in [0,T]$, exploiting (3.49) and (3.50),

$$\begin{aligned}
\|A_i(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\leq K_\alpha\left(1 + C_{w,\infty}(\tau)\right), && (3.53)\\
\mathrm{TV}\left(A_i(\tau,\cdot)\right) &= \mathrm{TV}\,\alpha\left(\tau,\cdot,w_i(\tau,\cdot)\right)\\
&\leq K_\alpha\left(1 + C_{w,\infty}(\tau) + \mathrm{TV}\left(w_i(\tau,\cdot)\right)\right)\\
&\leq K_\alpha\left(1 + C_{w,\infty}(\tau) + C_w^{\mathrm{TV}}(\tau)\right), && (3.54)
\end{aligned}$$

while by ($\boldsymbol{a}$) the function $a$ satisfies **(H4)**. By ($\boldsymbol{v}$), for every $i \in \mathbb{N}$ the function $c_i$ satisfies **(H2)** and, moreover, $c_i(t) \in \mathbf{C}^2(\Omega;\mathbb{R}^n)$ for all $t \in [0,T]$ and $\nabla\,\nabla\cdot c_i \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$. In particular, thanks to ($\boldsymbol{v}$) and (3.48), the following bounds hold for every $i \in \mathbb{N}$ and $t \in [0,T]$:

$$\begin{aligned}
\|\nabla\cdot c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} &\leq K_v \|w_i\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \leq K_v\, t\, C_{w,1}(t), && (3.55)\\
\|D_x c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))} &\leq K_v \|w_i\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \leq K_v\, t\, C_{w,1}(t), && (3.56)\\
\|\nabla\,\nabla\cdot c_i(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} &\leq C_v\left(t, C_{w,1}(t)\right) C_{w,1}(t). && (3.57)
\end{aligned}$$

The following uniform bounds on $u_i$ hold for every $i \in \mathbb{N}$: by Lemma 3.6 and Lemma 3.8, exploiting also (3.48)–(3.50) and (3.53)–(3.57), for all $\tau \in [0,T]$,

$$\begin{aligned}
\|u_i(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\leq \left(\|u_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,\tau]\times\Omega;\mathbb{R})}\right)\exp\left(K_\alpha\,\tau\left(1 + C_{w,\infty}(\tau)\right)\right)\\
&=: C_{u,1}(\tau), && (3.58)
\end{aligned}$$

$$\begin{aligned}
\|u_i(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,\tau];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\\
&\qquad \times \exp\left(K_\alpha\,\tau\left(1 + C_{w,\infty}(\tau)\right) + K_v\,\tau\, C_{w,1}(\tau)\right)\\
&=: C_{u,\infty}(\tau), && (3.59)
\end{aligned}$$

$$\begin{aligned}
\mathrm{TV}\left(u_i(\tau,\cdot)\right) &\leq \exp\left(K_\alpha\,\tau\left(1 + C_{w,\infty}(\tau)\right) + K_v\,\tau\, C_{w,1}(\tau)\right)\\
&\qquad \times \left(\mathrm{TV}\left(u_o\right) + \mathcal{O}(1)\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \int_0^\tau \mathrm{TV}\left(a(s)\right)\mathrm{d}s\right)\\
&\quad + C_{u,\infty}(\tau)\Big(K_\alpha\tau\left(1 + C_{w,\infty}(\tau) + C_w^{\mathrm{TV}}(\tau)\right) + \tau\, C_v\left(\tau, C_{w,1}(\tau)\right) C_{w,1}(\tau)\Big)\\
&=: C_u^{\mathrm{TV}}(\tau). && (3.60)
\end{aligned}$$

By Lemma 3.7 and Lemma 3.11, exploiting (3.53), (3.58) and (3.60), we get

$$\|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq C_{u,1}(t)\int_0^t \|\nabla\cdot\left(c_i(\tau) - c_{i-1}(\tau)\right)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}\tau$$

$$+ C_u^{\mathrm{TV}}(t) \int_0^t \|c_i(\tau) - c_{i-1}(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \,\mathrm{d}\tau + \exp\left(t\, K_\alpha \left(1 + C_{w,\infty}(t)\right)\right)$$
$$\times \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \|A_i - A_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}. \tag{3.61}$$

By (3.46), exploiting the hypothesis ($\boldsymbol{\alpha}$), we obtain

$$\begin{aligned} &\|A_i - A_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \\ &= \int_0^t \int_\Omega \left|\alpha\left(\tau, x, w_i(\tau,x)\right) - \alpha\left(\tau, x, w_{i-1}(\tau,x)\right)\right| \mathrm{d}x \,\mathrm{d}\tau \\ &\leq K_\alpha \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}. \end{aligned} \tag{3.62}$$

By (3.46), exploiting the hypothesis ($\boldsymbol{v}$) and (3.48), we obtain

$$\begin{aligned} &\|\nabla\cdot\left(c_i(\tau) - c_{i-1}(\tau)\right)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \\ &\quad \leq C_v\left(t, C_{w,1}(t)\right) \|w_i(\tau) - w_{i-1}(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}, \end{aligned} \tag{3.63}$$

$$\|c_i(\tau) - c_{i-1}(\tau)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \leq K_v \|w_i(\tau) - w_{i-1}(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}. \tag{3.64}$$

Hence, inserting (3.62), (3.63), and (3.64) into (3.61) yields

$$\begin{aligned} \|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\leq \Big(C_{u,1}(t) C_v(t, C_{w,1}(t)) + K_v C_u^{\mathrm{TV}}(t) \\ &+ K_\alpha \exp\left(t\, K_\alpha \left(1 + C_{w,\infty}(t)\right)\right) \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\Big) \\ &\times \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}. \end{aligned} \tag{3.65}$$

Collecting together (3.52) and (3.65), we obtain

$$\begin{aligned} &\|w_{i+1} - w_i\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} + \|u_{i+1} - u_i\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ &\leq C_{u,w}(t)\, t \left(\|w_i - w_{i-i}\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} + \|u_i - u_{i-1}\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))}\right), \end{aligned}$$

where

$$\begin{aligned} C_{u,w}(t) = K_\beta e^{t\, K_\beta} C_{w,\infty}(t) + \Big(&C_{u,1}(t) C_v(t, C_{w,1}(t)) + K_v C_u^{\mathrm{TV}}(t) \\ &+ K_\alpha \, \exp\left(t K_\alpha \left(1 + C_{w,\infty}(t)\right)\right) \left(\|u_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\Big). \end{aligned}$$

Choosing a sufficiently small $t_* > 0$, we ensure that $(u_i, w_i)$ is a Cauchy sequence in the complete metric space $\mathbf{L}^\infty\left([0,t_*]; \mathbf{L}^1(\Omega;\mathbb{R}^2)\right)$. Call $(u_*, w_*)$ its limit.

Then, the bounds (2.3) and (2.4) directly follow from (3.48) and (3.49) by the lower semicontinuity of the $\mathbf{L}^\infty$ norm with respect to the $\mathbf{L}^1$ distance. The same procedure applies to get (2.5) and (2.6) from (3.58) and (3.59). If

$a \geq 0$, $b \geq 0$ and both components of the initial datum $(u_o, w_o)$ are positive, then also the components of $(u_*, w_*)$ are positive.

We now prove that $(u_*, w_*)$ solves (1.1) in the sense of Definition 2.1. Note that by $(\boldsymbol{v})$, the sequence $v(\cdot, w_i)$ converges to $v(\cdot, w_*)$ in $\mathbf{L}^\infty([0, t_*];\mathbf{L}^1(\Omega; \mathbb{R}))$. Similarly, by $(\boldsymbol{\alpha})$ and $(\boldsymbol{\beta})$, $\alpha(\cdot, \cdot, w_i)$ and $\beta(\cdot, \cdot, u_i, w_i)$ converge to $\alpha(\cdot, \cdot, w_*)$ and $\beta(\cdot, \cdot, u_*, w_*)$. Two applications of the Dominated Convergence Theorem ensure that the integral equality (3.28) for the hyperbolic problems and (3.2) for the parabolic problem do hold.

By (3.59), we also have $u_* \in \mathbf{L}^\infty([0, t_*] \times \Omega; \mathbb{R})$. Moreover, Lemma 3.13 ensures that

$$u_* \in \mathbf{C}^0\left([0, t_*]; \mathbf{L}^1(\Omega; \mathbb{R})\right),$$

using also (3.53)–(3.57).

By construction, we have $w_* \in \mathbf{C}^0\left([0, t_*]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$. Indeed, the uniform bound (3.49) shows that

$$w_* \in \mathbf{L}^\infty([0, t_*] \times \Omega; \mathbb{R}) \subseteq \mathbf{L}^\infty\left([0, t_*]; \mathbf{L}^1(\Omega; \mathbb{R})\right).$$

Moreover, a further application of the Dominated Convergence Theorem shows that $w_*$ satisfies (3.5). Hence, proceeding as in Claim 4 in the proof of Proposition 3.5, we have that $w_* \in \mathbf{C}^0\left([0, t_*]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$.

Thus, $(u_*, w_*)$ satisfies the requirements in Definition 2.1. Moreover, this solution $(u_*, w_*)$ can be uniquely extended to all $[0, T]$. The proof is identical to [10, Theorem 2.2, Step 6].

Following the same techniques used in [10, Theorem 2.2, Step 7], we can prove also the Lipschitz continuous dependence of the solution to (1.1) on the initial data. Let $(u_o, w_o)$ and $(\tilde{u}_o, \tilde{w}_o)$ be two sets of initial data. Call $(u, w)$ and $(\tilde{u}, \tilde{w})$ the corresponding solutions to (1.1) in the sense of Definition 2.1. The proof is based on (3.6), (3.47), (3.53), Item 1) in Lemma 3.6 and computations analogous to those leading to (3.52) and (3.65) now yield

$$\begin{aligned}
&\|u(t) - \tilde{u}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(t) - \tilde{w}(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\leq \|u_o - \tilde{u}_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \exp\left(K_\alpha\, t\left(1 + K_{w,\infty}(t)\right)\right) + \|w_o - \tilde{w}_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} e^{K_\beta\, t} \\
&+ K_\beta\, e^{t\, K_\beta} K_{w,\infty}(t) \Big( \int_0^t \|u(\tau) - \tilde{u}(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(\tau) - \tilde{w}(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \, \mathrm{d}\tau \Big) \\
&+ K_1(t) \int_0^t \|w(\tau) - \tilde{w}(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \, \mathrm{d}\tau,
\end{aligned}$$

where

$$K_{w,\infty}(t) = \min\left\{C_{w,\infty}(t), C_{\tilde{w},\infty}(t)\right\},$$

$$K_1(t) = \min\left\{C_{u,1}(t)\, C_v(t, C_{w,1}(t)) + K_v\, C_u^{\mathrm{TV}}(t) + K_\alpha\, C_{u,\infty}(t), \right. \qquad (3.66)$$
$$\left. C_{\tilde u,1}(t)\, C_v(t, C_{\tilde w,1}(t)) + K_v\, C_{\tilde u}^{\mathrm{TV}}(t) + K_\alpha\, C_{\tilde u,\infty}(t)\right\},$$

and $C_{\tilde w,1}$, $C_{\tilde w,\infty}$, $C_{\tilde w}^{\mathrm{TV}}$, $C_{\tilde u,1}$, $C_{\tilde u,\infty}$, $C_{\tilde u}^{\mathrm{TV}}$ are defined accordingly to (3.48), (3.49), (3.50), (3.58), (3.59), (3.60), corresponding to the initial datum $(\tilde u_o, \tilde w_o)$. Then, Gronwall Lemma [3, Lemma 3.1] yields

$$\begin{aligned}
&\|u(t) - \tilde u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(t) - \tilde w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le \left(\|u_o - \tilde u_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w_o - \tilde w_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right) \int_0^t \mathcal{K}_o(\tau) \exp\left(\int_\tau^t \mathcal{K}(s)\,\mathrm{d}s\right)\mathrm{d}\tau\,,
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{K}_o(\tau) &= \exp\left(\max\left\{\left(K_\alpha\, \tau\, (1 + K_{w,\infty}(\tau))\right), K_\beta\, \tau\right\}\right), \\
\mathcal{K}(\tau) &= K_\beta\, e^{\tau\, K_\beta} K_{w,\infty}(\tau) + K_1(\tau).
\end{aligned}$$

Uniqueness of solution readily follows.

We focus now on the stability of (1.1) with respect to the controls $a$ and $b$. Let $a, \tilde a$ satisfy ($\boldsymbol{a}$), $b, \tilde b$ satisfy ($\boldsymbol{b}$). Call $(u,w)$ and $(\tilde u, \tilde w)$ the solutions to (1.1) corresponding to the functions $a, b$ and $\tilde a, \tilde b$ respectively. Similarly to the previous step, by (3.6), (3.47), (3.53), Item 1) in Lemma 3.6 and computations analogous to those leading to (3.52) and (3.65), we obtain

$$\begin{aligned}
&\|u(t) - \tilde u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(t) - \tilde w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le \|a - \tilde a\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \exp\left(K_\alpha\, t\, (1 + K_{w,\infty}(t))\right) + \left\|b - \tilde b\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} e^{K_\beta\, t} \\
&+ K_\beta e^{tK_\beta} K_{w,\infty}(t) \left(\int_0^t \|u(\tau) - \tilde u(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|w(\tau) - \tilde w(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\mathrm{d}\tau\right) \\
&+ K_1(t) \int_0^t \|w(\tau) - \tilde w(\tau)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\mathrm{d}\tau\,,
\end{aligned}$$

where $K_{w,\infty}(t)$ and $K_1(t)$ are defined as in (3.66), with the main difference that the *tilde*-versions of $C_{*,*}$ now corresponds to the functions $\tilde a$ and $\tilde b$. An application of Gronwall Lemma [3, Lemma 3.1] yields the desired estimate. □

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