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On the existence and the uniqueness of the solution to a fluid-structure interaction problem

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Abstract

In this paper we consider the linearized version of a system of partial differential equations arising from a fluid-structure interaction model. We prove the existence and the uniqueness of the solution under natural regularity assumptions.

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1. Introduction

The mathematical analysis and the numerical approximation of problems involving the interaction of fluids and solids are essential for the modeling and simulation of a variety of applications related to engineering, physics, and biology.

We consider a model presented in [4,6] based on a fictitious domain approach and the use of a distributed Lagrange multiplier. The considered formulation is the evolution of a model originated from a finite element approach of the immersed boundary method [5,7,3]. The immersed boundary method has been introduced by Peskin and his collaborators in several seminal

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papers [33,35,26,34] where the interaction between the fluid and the solid was modeled by a suitably defined Dirac delta function and the numerical approximation was performed by finite differences. One of the differences of the immersed boundary method with respect to other possible approaches is that the discretization is performed by using two fixed meshes: one for the fluid domain (artificially extended to include the immersed solid) and one for the reference configuration of the solid. This choice has the advantage that the computational meshes need not be updated at each time step. On other hand, the intersection between the fluid mesh and the image of the solid mesh into the actual solid configuration needs to be evaluated; in our approach the Lagrange multiplier is responsible for such coupling. The numerical analysis of the problem shows appealing properties related to its stability [32,4] and the numerical investigations demonstrate the superiority of the finite element approach with respect to the original finite difference scheme in terms of mass conservation. Higher order time discretization has been investigated in [8].

In this paper we address the study of the existence and uniqueness of the continuous solution. The solution has four components: fluid velocity $\mathbf{u}$ and pressure p (extended into the solid region in the spirit of the fictitious domain), the position of the solid domain inside the fluid, seen as a mapping $\mathbf{X}$ from a reference configuration, and the Lagrange multiplier $\boldsymbol{\lambda}$ supported in the solid reference domain that is used to enforce the coupling between the solid and the fluid. The problem is highly non linear; in particular the unknown $\mathbf{X}$ defines mathematically the region occupied by the solid at a given time. We consider a linearization of the problem with respect to the variable $\mathbf{X}$ and, for simplicity, we neglect the convective term of the Navier–Stokes equations.

Our existence and uniqueness proof is based on a Faedo–Galerkin approximation as done in [41] for the study of the Navier–Stokes equations. We extend the results of [18] where the coupling of the incompressible Navier–Stokes equations with a linear elasticity model in a fixed domain is considered.

Existence and uniqueness results for models related to fluid-structure interactions have a limited but not empty occurrence in the literature. In particular, some authors discussed the existence of weak solutions in the case of a fluid containing rigid solids or elastic bodies whose behavior is described by a finite number of modes [12,15–17,19,21–23,38,40,39]. Other results are available for the existence of weak solutions in the case of a fluid enclosed in a solid membrane [42,11,28,27,29–31] or interacting with a plate [20]; the typical example of application is the blood flowing in a vessel [36]. The existence of the solution in the case of viscoelastic particles immersed in a Newtonian fluid is discussed in [25] using the Eulerian description for both fluid and solid. Local-in-time existence and uniqueness of strong solutions for a model involving an elastic structure immersed in a fluid is analyzed in [13,14,37,9,10].

In the next section we recall the strong formulation of our model. Section 3 presents the fictitious domain approach together with its variational formulation. The linearized problem is described in Section 4 and the main existence and uniqueness result for the velocity $\mathbf{u}$ and the position of the solid $\mathbf{X}$ is stated and proved in Section 5. Finally, Section 6 is devoted to the existence and uniqueness of the pressure p and the multiplier $\boldsymbol{\lambda}$.

2. Setting of the problem

In this section we recall the formulation of the fluid-structure interaction problem presented in [6]. We assume that we are given a Lipschitz and convex domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is occupied by a fluid and a solid. We denote by Ω_t^f and Ω_t^s the regions where the fluid and the solid are respectively located at time t, so that Ω is the interior of $\overline{\Omega}_t^f \cup \overline{\Omega}_t^s$. The regularity of the two subdomains will be made more precise later on as a consequence of Assumption 1. For simplicity we assume that $\partial\Omega_t^s \cup \partial\Omega$ is empty, that is the solid is immersed in the fluid, and the moving interface $\partial\Omega_t^f \cap \partial\Omega_t^s$ is denoted by Γ_t.

We denote by $\mathbf{u}$, $\boldsymbol{\sigma}$, and ρ the velocity, stress tensor, and mass density, respectively, and we use subscripts f or s to refer to the fluid or to the solid. We assume that the densities ρ_f and ρ_s are positive constants.

The following equations represent the strong form of the problem we are interested in, corresponding to the interaction of an incompressible fluid and an incompressible immersed elastic structure.

$$\begin{aligned} &\rho_f \dot{\mathbf{u}}_f = \operatorname{div} \boldsymbol{\sigma}_f && \text{in } \Omega_t^f \\ &\operatorname{div} \mathbf{u}_f = 0 && \text{in } \Omega_t^f \\ &\rho_s \dot{\mathbf{u}}_s = \operatorname{div} \boldsymbol{\sigma}_s && \text{in } \Omega_t^s \\ &\operatorname{div} \mathbf{u}_s = 0 && \text{in } \Omega_t^s \\ &\mathbf{u}_f = \mathbf{u}_s && \text{on } \Gamma_t \\ &\boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_s \mathbf{n}_s && \text{on } \Gamma_t. \end{aligned} \tag{1}$$

The following initial and boundary conditions are imposed on $\partial\Omega$.

$$\begin{aligned} &\mathbf{u}_f(0) = \mathbf{u}_{f0} && \text{on } \Omega_0^f, \\ &\mathbf{u}_s(0) = \mathbf{u}_{s0} && \text{on } \Omega_0^s, \\ &\mathbf{u}_f(t) = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The fluid stress tensor is defined by the Navier–Stokes law as it is common for Newtonian fluids

$$\boldsymbol{\sigma}_f = -p_f \mathbb{I} + \nu_f \varepsilon(\mathbf{u}_f), \tag{3}$$

where $\varepsilon(\mathbf{u}) = (1/2)\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right)$ is the symmetric gradient and ν_f represents the viscosity of the fluid.

The solid domain Ω_t^s is the image of a reference domain $\mathcal{B} = \Omega_0^s$, which we assume to have a Lipschitz continuous boundary. Let $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$ be the mapping that associates to each point $\mathbf{s} \in \mathcal{B}$ a point $\mathbf{x} \in \Omega_t^s$. When it is needed in order to avoid confusion we will use the notation $\mathbf{X}(\mathbf{s}, t)$ to denote the dependence on both space and time. We assume that $\mathbf{X}$ is one to one and that, for all $t \in [0, T]$, $\|\mathbf{X}(\mathbf{s}_1, t) - \mathbf{X}(\mathbf{s}_2, t)\| \ge \gamma \|\mathbf{s}_1 - \mathbf{s}_2\|$ for all $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{B}$ for a positive constant γ. In particular, $\mathbf{X}(t)$ is invertible with Lipschitz inverse.

We denote by $\mathbb{F} = \nabla_s \mathbf{X}$ the deformation gradient; its determinant is denoted by $|\mathbb{F}|$. We have that $|\mathbb{F}|$ is constant in time since the fluid and the solid are incompressible; it is not restrictive to assume that $\mathbf{X}(\mathbf{s}, 0) = \mathbf{X}_0(\mathbf{s}) = \mathbf{s}$ for $\mathbf{s} \in \mathcal{B}$, so that $|\mathbb{F}| = 1$ for all t.

When dealing with moving domains it is essential to be precise with respect to the Eulerian and Lagrangian descriptions of the involved quantities. In (1), we used the dot over the velocity in order to denote the material time derivative. The Eulerian description of the fluid gives $\dot{\mathbf{u}}_f = \partial \mathbf{u}_f / \partial t + \mathbf{u}_f \cdot \nabla \mathbf{u}_f$. The spatial description of the material velocity in the solid, where the Lagrangian representation is used, reads

$$\mathbf{u}_s(\mathbf{x},t)=\frac{\partial \mathbf{X}(\mathbf{s},t)}{\partial t}\Big|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)} \tag{4}$$

so that $\dot{\mathbf{u}}_s(\mathbf{x},t)=\partial^2\mathbf{X}(\mathbf{s},t)/\partial t^2|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)}$.

Following what we did in [7], we consider a viscous-hyperelastic solid structure and we define the Cauchy stress tensor as the sum of two contributions $\boldsymbol{\sigma}_s=\boldsymbol{\sigma}_s^f+\boldsymbol{\sigma}_s^s$. There is a fluid-like part of the stress

$$\boldsymbol{\sigma}_s^f=-p_s\mathbb{I}+\nu_s\varepsilon(\mathbf{u})_s \tag{5}$$

for a positive constant viscosity ν_s and an elastic part $\boldsymbol{\sigma}_s^s$. The elastic part of the stress $\boldsymbol{\sigma}_s^s$ can be written in terms of the first Piola–Kirchhoff stress tensor $\mathbb{P}$ with a change of variables from the Eulerian to the Lagrangian framework:

$$\mathbb{P}(\mathbb{F}(\mathbf{s},t))=|\mathbb{F}(\mathbf{s},t)|\boldsymbol{\sigma}_s^s(\mathbf{x},t)\mathbb{F}^{-\top}(\mathbf{s},t) \qquad \text{for } \mathbf{x}=\mathbf{X}(\mathbf{s},t). \tag{6}$$

Following [6] we are going to consider a linear dependence of $\mathbb{P}$ on $\mathbb{F}$, namely

$$\mathbb{P}(\mathbb{F})=\kappa\mathbb{F}=\kappa\,\nabla_s\mathbf{X}. \tag{7}$$

The following notation will be used throughout the paper.

If D is a domain in $\mathbb{R}^d$, we denote by $W^{s,p}(D)$ the Sobolev space on D ($s\in\mathbb{R}$, $1\le p\le\infty$), and by $\|\cdot\|_{s,p,D}$ its norm (see, for example, [1]). As usual we write $H^s(D)=W^{s,2}(D)$ and omit p in the norm and seminorm when it is equal to 2. Moreover, bold characters denote vector valued functions and the corresponding functional spaces. The dual space of a Hilbert space X will be denoted with X'. The notation $(\cdot,\cdot)_D$ stands for the scalar product in $L^2(D)$ and the duality pairing is denoted by brackets $\langle\cdot,\cdot\rangle$. The subscript indicating the domain is omitted if the domain is Ω, while we shall always use it for the reference domain $\mathcal{B}$. We will make use of the space $H_0^1(D)$ of functions in $H^1(D)$ with zero trace on the boundary of D and of its dual $H^{-1}(D)$. Moreover, $L_0^2(D)$ denotes the subspace of $L^2(D)$ of functions with zero mean value on D.

We denote by $\mathscr{D}(D)$ the space of C^∞ functions with compact support in D. When X is a Banach space, we denote by $L^p(0,T;X)$ ($1\le p\le\infty$) the space of L^p-integrable functions from $(0,T)$ into X, which is a Banach space with the norm

$$\|v\|_{L^p(X)}=\left(\int_0^T\|v(t)\|_X^p\,dt\right)^{1/p}.$$

Analogously, the space $C^m([0,T];X)$ denotes the space of functions from $[0,T]$ to X which are continuous up to the m-th derivative in t.

Finally, we are going to use the following spaces:

$$\begin{aligned}
&\mathscr{V}_0=\{\mathbf{v}\in\mathscr{D}(\Omega)^d:\operatorname{div}\mathbf{v}=0\}\\
&\mathbf{H}_0=\text{ the closure of }\mathscr{V}_0\text{ in }\mathbf{L}^2(\Omega)\\
&\mathbf{V}_0=\text{ the closure of }\mathscr{V}_0\text{ in }\mathbf{H}_0^1(\Omega).
\end{aligned} \tag{8}$$

3. Fictitious domain approach and Lagrange multiplier

We extend the fluid velocity and the pressure into the solid domain by introducing new unknowns with the following meaning:

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } \Omega_t^f \\ \mathbf{u}_s & \text{in } \Omega_t^s \end{cases} \qquad p = \begin{cases} p_f & \text{in } \Omega_t^f \\ p_s & \text{in } \Omega_t^s \end{cases}. \tag{9}$$

The condition that the material velocity of the solid is equal to the velocity of the fictitious fluid is expressed by

$$\frac{\partial \mathbf{X}(\mathbf{s},t)}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{s},t),t) \quad \text{for } \mathbf{s} \in \mathcal{B}. \tag{10}$$

We introduce the following bilinear form:

$$\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) = (\nabla_s \boldsymbol{\mu}, \nabla_s \mathbf{z})_{\mathcal{B}} + (\boldsymbol{\mu},\mathbf{z})_{\mathcal{B}} \quad \forall \boldsymbol{\mu},\ \mathbf{z} \in \mathbf{H}^1(\mathcal{B}). \tag{11}$$

It is obvious that for all $\boldsymbol{\mu}, \mathbf{z} \in \mathbf{H}^1(\mathcal{B})$

$$\begin{aligned}
&\mathbf{c}(\mathbf{z},\mathbf{z}) = \|\mathbf{z}\|^2_{1,B} = \|\mathbf{z}\|^2_{0,\mathcal{B}} + \|\nabla_s \mathbf{z}\|^2_{0,B} \\
&\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) \le \|\boldsymbol{\mu}\|_{1,B}\|\mathbf{z}\|_{1,B} \\
&\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) = 0 \text{ for all } \boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B}) \text{ implies } \mathbf{z} = 0.
\end{aligned}$$

System (1) can be formulated as follows.

Problem 1. Given $\mathbf{u}_0 \in \mathbf{H}^1_0(\Omega)$, $\mathbf{u}_{s0} \in \mathbf{H}^1(\Omega^s_0)$, and $\mathbf{X}_0(\mathbf{s}) = \mathbf{s}$ for $\mathbf{s} \in \mathcal{B}$, for almost every $t \in]0,T]$, find $(\mathbf{u}(t), p(t)) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$, $\mathbf{X}(t) \in \mathbf{H}^1(\mathcal{B})$, and $\boldsymbol{\lambda}(t) \in \mathbf{H}^1(\mathcal{B})$ such that it holds

$$\begin{aligned}
&\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) \\
&\qquad -(\operatorname{div}\mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot,t))) = 0 \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) && \text{(12a)} \\
&(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega) && \text{(12b)} \\
&\delta_\rho \left(\frac{\partial^2 \mathbf{X}}{\partial t^2}(t), \mathbf{z}\right)_{\mathcal{B}} + \kappa(\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t),\mathbf{z}) = 0 \quad \forall \mathbf{z} \in \mathbf{H}^1(\mathcal{B}) && \text{(12c)} \\
&\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}}{\partial t}(t)\right) = 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B}) && \text{(12d)} \\
&\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, && \text{(12e)} \\
&\mathbf{X}(0) = \mathbf{X}_0 \quad \text{in } \mathcal{B}, \qquad \frac{\partial \mathbf{X}}{\partial t}(0) = \mathbf{u}_{s0} \quad \text{in } \mathcal{B}. && \text{(12f)}
\end{aligned}$$

Here $\delta_\rho = \rho_s - \rho_f$ and

$$a(\mathbf{u}, \mathbf{v}) = (\nu\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) \quad \text{with } \nu = \begin{cases} \nu_f & \text{in } \Omega_t^f \\ \nu_s & \text{in } \Omega_t^s \end{cases}$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{\rho_f}{2}\left((\mathbf{u}\cdot\nabla\,\mathbf{v}, \mathbf{w}) - (\mathbf{u}\cdot\nabla\,\mathbf{w}, \mathbf{v})\right).$$

We assume that $\nu \in L^\infty(\Omega)$ and that there exists a positive constant $\nu_0 > 0$ such that $\nu \geq \nu_0 > 0$ in Ω, hence the following Korn's inequality holds true for all $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$

$$a(\mathbf{u}, \mathbf{u}) \geq \mathbf{k}\|\nabla\,\mathbf{u}\|_{0,\Omega}^2. \tag{13}$$

We add the following compatibility conditions for the initial velocity

$$\operatorname{div}\mathbf{u}_0 = 0, \quad \text{and} \quad \mathbf{u}_0|_{\Omega_0^s} = \mathbf{u}_{s0}. \tag{14}$$

The second condition is related to the fact that we are assuming $\mathcal{B} = \Omega_0^s$.

4. Linearized problem

We fix a function $\overline{\mathbf{X}}$ which satisfies the following assumption.

Assumption 1. Let $\overline{\mathbf{X}} \in C^1([0, T]; \mathbf{W}^{1,\infty}(\mathcal{B}))$ be invertible with Lipschitz inverse for all $t \in [0, T]$, with $\overline{\mathbf{X}}(\mathbf{s}, 0) = \mathbf{s}$ for $\mathbf{s} \in \mathcal{B}$. In addition, we assume that

$$\overline{J}(t) = \det(\nabla_s \overline{\mathbf{X}}(t)) = 1 \quad \text{for all } t. \tag{15}$$

This assumption and the fact that $\mathcal{B}$ has a Lipschitz continuous boundary imply that also $\overline{\mathbf{X}}(\mathcal{B}, t)$ has a Lipschitz continuous boundary.

From now one we are going to neglect the convective term so that our problem will read as follows.

Problem 2. Given $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u}_{s0} \in \mathbf{H}^1(\mathcal{B})$, and $\mathbf{X}_0(\mathbf{s}) = \mathbf{s}$ for $\mathbf{s} \in \mathcal{B}$, for almost every $t \in]0, T]$ find $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, $\mathbf{X}(t) \in \mathbf{H}^1(\mathcal{B})$, and $\boldsymbol{\lambda}(t) \in \mathbf{H}^1(\mathcal{B})$ such that it holds

$$\begin{aligned}
&\rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t)) \\
&\qquad + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}\circ\overline{\mathbf{X}}) = 0 && \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) && \text{(16a)}\\
&(\operatorname{div}\mathbf{u}(t), q) = 0 && \forall q \in L_0^2(\Omega) && \text{(16b)}\\
&\delta_\rho\left(\frac{\partial^2\mathbf{X}}{\partial t^2}(t), \mathbf{z}\right)_{\mathcal{B}} + \kappa(\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 && \forall \mathbf{z} \in \mathbf{H}^1(\mathcal{B}) && \text{(16c)}\\
&\mathbf{c}\left(\boldsymbol{\mu}, (\mathbf{u}\circ\overline{\mathbf{X}})(t) - \frac{\partial\mathbf{X}}{\partial t}(t)\right) = 0 && \forall \boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B}) && \text{(16d)}\\
&\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, && && \text{(16e)}
\end{aligned}$$

$$\mathbf{X}(0)=\mathbf{X}_0 \quad \text{in } \mathcal{B}, \qquad \frac{\partial \mathbf{X}}{\partial t}(0)=\mathbf{u}_{s0} \quad \text{in } \mathcal{B}. \tag{16f}$$

In the previous equations we used the notation $\mathbf{v}\circ\overline{\mathbf{X}}=\mathbf{v}(\overline{\mathbf{X}}(\cdot,t))$ and $(\mathbf{u}\circ\overline{\mathbf{X}})(t)=\mathbf{u}(\overline{\mathbf{X}}(\cdot,t),t)$.

Let us split the second order in time Equation (16c) into a system of two differential equations of first order in time by introducing a new unknown $\mathbf{w}=\frac{\partial \mathbf{X}}{\partial t}$. Then Problem 2 becomes:

Problem 3. Given $\mathbf{u}_0\in\mathbf{H}^1_0(\Omega)$, $\mathbf{u}_{s0}\in\mathbf{H}^1(\mathcal{B})$, and $\mathbf{X}_0(\mathbf{s})=\mathbf{s}$ for $\mathbf{s}\in\mathcal{B}$, for almost every $t\in]0,T]$ find $(\mathbf{u}(t),p(t))\in\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$, $(\mathbf{X}(t),\mathbf{w}(t))\in\mathbf{H}^1(\mathcal{B})\times\mathbf{H}^1(\mathcal{B})$, and $\boldsymbol{\lambda}(t)\in\mathbf{H}^1(\mathcal{B})$ such that it holds

$$\rho_f\frac{d}{dt}(\mathbf{u}(t),\mathbf{v})+a(\mathbf{u}(t),\mathbf{v})-(\operatorname{div}\mathbf{v},p(t))$$
$$+\mathbf{c}(\boldsymbol{\lambda}(t),\mathbf{v}\circ\overline{\mathbf{X}})=0 \qquad \forall\mathbf{v}\in\mathbf{H}^1_0(\Omega) \tag{17a}$$

$$(\operatorname{div}\mathbf{u}(t),q)=0 \qquad \forall q\in L^2_0(\Omega) \tag{17b}$$

$$\delta_\rho\left(\frac{\partial\mathbf{w}}{\partial t}(t),\mathbf{z}\right)_{\mathcal{B}}+\kappa(\nabla_s\mathbf{X}(t),\nabla_s\mathbf{z})_{\mathcal{B}}-\mathbf{c}(\boldsymbol{\lambda}(t),\mathbf{z})=0 \qquad \forall\mathbf{z}\in\mathbf{H}^1(\mathcal{B}) \tag{17c}$$

$$\left(\frac{\partial\mathbf{X}}{\partial t}(t),\mathbf{y}\right)_{\mathcal{B}}=(\mathbf{w}(t),\mathbf{y})_{\mathcal{B}} \qquad \forall\mathbf{y}\in\mathbf{L}^2(\mathcal{B}) \tag{17d}$$

$$\mathbf{c}\left(\boldsymbol{\mu},(\mathbf{u}\circ\overline{\mathbf{X}})(t)-\mathbf{w}(t)\right)=0 \qquad \forall\boldsymbol{\mu}\in\mathbf{H}^1(\mathcal{B}) \tag{17e}$$

$$\mathbf{u}(0)=\mathbf{u}_0 \quad \text{in } \Omega, \tag{17f}$$

$$\mathbf{X}(0)=\mathbf{X}_0 \quad \text{in } \mathcal{B}, \qquad \mathbf{w}(0)=\mathbf{u}_{s0} \quad \text{in } \mathcal{B}. \tag{17g}$$

We set

$$\mathbb{K}_t=\{(\mathbf{v},\mathbf{z}(t))\in\mathbf{V}_0\times\mathbf{H}^1(\mathcal{B}):\mathbf{c}(\boldsymbol{\mu},\mathbf{v}\circ\overline{\mathbf{X}}(t)-\mathbf{z}(t))=0\ \forall\boldsymbol{\mu}\in\mathbf{H}^1(\mathcal{B})\}. \tag{18}$$

We observe that (14) implies that $(\mathbf{u}_0,\mathbf{u}_{s0})\in\mathbb{K}_0$.

Problem 3 is equivalent to the following one.

Problem 4. Given $(\mathbf{u}_0,\mathbf{u}_{s0})\in\mathbb{K}_0$ and $\mathbf{X}_0(\mathbf{s})=\mathbf{s}$ for $\mathbf{s}\in\mathcal{B}$, for almost every $t\in]0,T]$, find $(\mathbf{u}(t),\mathbf{w}(t))\in\mathbb{K}_t$ and $\mathbf{X}(t)\in\mathbf{H}^1(\mathcal{B})$ such that

$$\begin{aligned}
&\rho_f\frac{d}{dt}(\mathbf{u}(t),\mathbf{v})+a(\mathbf{u}(t),\mathbf{v})+\delta_\rho\left(\frac{\partial\mathbf{w}}{\partial t}(t),\mathbf{z}(t)\right)_{\mathcal{B}}\\
&\qquad+\kappa(\nabla_s\mathbf{X}(t),\nabla_s\mathbf{z}(t))_{\mathcal{B}}=0 && \forall(\mathbf{v},\mathbf{z}(t))\in\mathbb{K}_t\\
&\left(\frac{\partial\mathbf{X}}{\partial t}(t),\mathbf{y}\right)_{\mathcal{B}}=(\mathbf{w}(t),\mathbf{y})_{\mathcal{B}} && \forall\mathbf{y}\in\mathbf{L}^2(\mathcal{B})\\
&\mathbf{u}(0)=\mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{w}(0)=\mathbf{u}_{s0} \quad \text{in } \mathcal{B},\\
&\mathbf{X}(0)=\mathbf{X}_0 \quad \text{in } \mathcal{B},
\end{aligned} \tag{19}$$

In the following section we are going to prove existence and uniqueness of the solution to Problem 4. In the next section we will show existence and existence for the pressure p and the multiplier λ as well.

5. Existence and uniqueness

We start this section by showing existence and uniqueness of the solution to Problem 4 by following the Galerkin approximation technique used in [41, Chapt. III.1]. The proof of the next theorem will be obtained in several steps.

Theorem 1. *We set* $\mathbf{X}_0(\mathbf{s}) = \mathbf{s}$ *for* $\mathbf{s} \in \mathcal{B}$. *Let* $\overline{\mathbf{X}} \in C^1([0, T]; W^{1,\infty}(\mathcal{B}))$ *be such that Assumption 1 is satisfied. Then, given* $\mathbf{u}_0 \in \mathbf{V}_0$ *and* $\mathbf{u}_{s0} \in \mathbf{H}^1(\mathcal{B})$ *satisfying the compatibility condition* (14)*, for a.e.* $t \in (0, T)$ *there exist* $(\mathbf{u}(t), \mathbf{w}(t)) \in \mathbb{K}_t$ *and* $\mathbf{X}(t) \in \mathbf{H}^1(\mathcal{B})$ *satisfying Problem 4 and*

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0, T; \mathbf{H}_0) \cap L^2(0, T; \mathbf{V}_0)\\
&\mathbf{w} \in L^\infty(0, T; \mathbf{L}^2(\mathcal{B})) \cap L^2(0, T; \mathbf{H}^1(\mathcal{B}))\\
&\mathbf{X} \in L^\infty(0, T; \mathbf{H}^1(\mathcal{B})) \quad \text{with } \frac{\partial \mathbf{X}}{\partial t} \in L^\infty(0, T; \mathbf{L}^2(\mathcal{B})) \cap L^2(0, T; \mathbf{H}^1(\mathcal{B})).
\end{aligned}$$

5.1. Basis in $\mathbb{K}_t$

We introduce a basis in $\mathbb{K}_t$ that will be used for the Galerkin approximation of our problem. Let $\boldsymbol{\psi}_j$ $(j \in \mathbb{N})$ be the complete set of eigenfunctions for the eigenvalue problem: find $\lambda_f \in \mathbb{R}$ and $\boldsymbol{\psi} \in \mathbf{V}_0$ with $\boldsymbol{\psi} \neq 0$ such that

$$a(\boldsymbol{\psi}, \mathbf{v}) = \lambda_f (\boldsymbol{\psi}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_0. \tag{20}$$

It is well known that the eigenvalues are positive and can be enumerated in an increasing sequence going to $+\infty$. The associated eigenfunctions $\{\boldsymbol{\psi}_j\}_{j=1}^\infty$ are orthogonal with respect to the scalar product in $\mathbf{L}^2(\Omega)$ and to the bilinear form $a(\cdot, \cdot)$. We normalize them with respect to the $\mathbf{L}^2(\Omega)$ norm, so that $\|\boldsymbol{\psi}_j\|_{0,\Omega} = 1$ for all $j \in \mathbb{N}$.

Moreover, let $\boldsymbol{\chi}_j$ $(j \in \mathbb{N})$ be the complete set of eigenfunctions for the eigenvalue problem: find $\lambda_s \in \mathbb{R}$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\mathcal{B})$ with $\boldsymbol{\chi} \neq 0$ such that

$$\mathbf{c}(\boldsymbol{\mu}, \boldsymbol{\chi}) = \lambda_s (\boldsymbol{\chi}, \boldsymbol{\mu})_{\mathcal{B}} \qquad \forall \boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B}). \tag{21}$$

Also the eigenvalues of (21) are positive and can be enumerated in increasing sequence going to $+\infty$. We have that $\{\boldsymbol{\chi}_j\}_{j=1}^\infty$ are orthogonal we respect to the scalar product in $\mathbf{L}^2(\mathcal{B})$ and to the bilinear form $\mathbf{c}(\cdot, \cdot)$. We normalize them with respect to $\mathbf{c}$ so that $\mathbf{c}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j) = 1$ for all $j \in \mathbb{N}$.

Proposition 2. *For* $j \in \mathbb{N}$ *and* $t \in [0, T]$*, let us set* $\boldsymbol{\varphi}_j(t) = \boldsymbol{\psi}_j \circ \overline{\mathbf{X}}(t) \in \mathbf{H}^1(\mathcal{B})$. *Then, for each* $t \in [0, T]$, $\{\boldsymbol{\varphi}_j(t)\}_{j=1}^\infty$ *is a basis of* $\mathbf{H}^1(\mathcal{B})$.

Proof. Given $\mathbf{z} \in \mathbf{H}^1(\mathcal{B})$ we will show that it can be written as a combination of the $\{\boldsymbol{\varphi}_j(t)\}$'s. Thanks to the assumptions on $\overline{\mathbf{X}}$, we have that $\mathbf{v}_z(t) = \mathbf{z} \circ \overline{\mathbf{X}}(t)^{-1} \in \mathbf{H}^1(\overline{\Omega_t^s})$ where $\overline{\Omega_t^s} = \overline{\mathbf{X}}(\mathcal{B}, t)$.

Let $\tilde{\mathbf{v}}_z(t) \in \mathbf{H}^1_0(\Omega)$ be an extension of $\mathbf{v}_z(t)$ to Ω, so that $\tilde{\mathbf{v}}_z(t)|_{\overline{\Omega^s_t}} = \mathbf{v}_z(t)$. Then we can write $\tilde{\mathbf{v}}_z(t)$ in terms of the basis functions $\boldsymbol{\psi}_j$, that is

$$\tilde{\mathbf{v}}_z(t) = \sum_{j=1}^{\infty} \alpha_j(t)\boldsymbol{\psi}_j.$$

By construction we have that $\tilde{\mathbf{v}}_z(\overline{\mathbf{X}}(\cdot,t),t) = \mathbf{z} \in \mathbf{H}^1(\mathcal{B})$, hence we obtain

$$\mathbf{z} = \tilde{\mathbf{v}}_z(\overline{\mathbf{X}}(\cdot,t),t) = \sum_{j=1}^{\infty} \alpha_j(t)\boldsymbol{\psi}_j \circ \overline{\mathbf{X}} = \sum_{j=1}^{\infty} \alpha_j(t)\boldsymbol{\varphi}_j(t). \quad \square$$

As a consequence of the previous proposition, a basis in $\mathbb{K}_t$ is given by $\{(\boldsymbol{\psi}_j, \boldsymbol{\varphi}_j(t))\}_{j=1}^{\infty}\}$.

5.2. Galerkin approximation

We introduce a Galerkin approximation of the solution of Problem 4. Let us consider $\mathbf{V}^m_0 = \mathrm{span}(\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_m)$, $\mathbf{W}^m(t) = \mathrm{span}(\boldsymbol{\varphi}_1(t), \dots, \boldsymbol{\varphi}_m(t))$, and $\mathbf{H}^m = \mathrm{span}(\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_m)$. We define a subspace $\mathbb{K}^m_t$ of $\mathbb{K}_t$ generated by the first m basis functions in $\mathbb{K}_t$ as follows

$$\mathbb{K}^m_t = \{(\mathbf{v}, \mathbf{z}(t)) \in \mathbf{V}^m_0 \times \mathbf{W}^m : \mathbf{c}(\boldsymbol{\varphi}_i(t), \mathbf{v} \circ \overline{\mathbf{X}}(t) - \mathbf{z}(t)) = 0 \text{ for } i = 1, \dots, m\}. \tag{22}$$

It is clear that if $(\mathbf{v}, \mathbf{z}(t)) \in \mathbb{K}^m_t$ then

$$\mathbf{v} = \sum_{j=1}^{m} \alpha_j \boldsymbol{\psi}_j \qquad \mathbf{z}(t) = \sum_{j=1}^{m} \alpha_j \boldsymbol{\varphi}_j(t)$$

for the same coefficients $\{\alpha_j\}$. The Galerkin approximation of the solution of Problem 4 is given by

$$\begin{aligned} &\mathbf{u}^m(t) = \sum_{j=1}^{m} \alpha_j^{(m)}(t)\boldsymbol{\psi}_j, \quad \mathbf{w}^m(t) = \sum_{j=1}^{m} \alpha_j^{(m)}(t)\boldsymbol{\varphi}_j(t), \\ &\mathbf{X}^m(t) = \sum_{j=1}^{m} \beta_j^{(m)}(t)\boldsymbol{\chi}_j \end{aligned} \tag{23}$$

such that

$$\begin{aligned} &\rho_f \frac{d}{dt}(\mathbf{u}^m(t), \mathbf{v}) + a(\mathbf{u}^m(t), \mathbf{v}) + \delta_\rho \left(\frac{\partial \mathbf{w}^m}{\partial t}(t), \mathbf{z}(t) \right)_{\mathcal{B}} \\ &\quad + \kappa(\nabla_s \mathbf{X}^m(t), \nabla_s \mathbf{z}(t))_{\mathcal{B}} = 0 && \forall (\mathbf{v}, \mathbf{z}(t)) \in \mathbb{K}^m_t \\ &\left(\frac{\partial \mathbf{X}^m}{\partial t}(t), \mathbf{y} \right)_{\mathcal{B}} = (\mathbf{w}^m(t), \mathbf{y})_{\mathcal{B}} && \forall \mathbf{y} \in \mathbf{H}^m \end{aligned} \tag{24}$$

$$\mathbf{u}^m(0) = \mathbf{u}_0^m \text{ in } \Omega, \qquad \mathbf{w}^m(0) = \mathbf{u}_{s0}^m \text{ in } \mathcal{B}$$
$$\mathbf{X}^m(\mathbf{s}, 0) = \mathbf{X}_0^m \text{ for } \mathbf{s} \in \mathcal{B}.$$

The initial conditions in (24) are obtained by projecting the initial data, that is

$$\begin{aligned}
\mathbf{u}_0^m &= \sum_{j=1}^m \alpha_{0j}^{(m)} \boldsymbol{\psi}_j \quad \text{with } \alpha_{0j}^{(m)} = \frac{(\mathbf{u}_0, \boldsymbol{\psi}_j)}{(\boldsymbol{\psi}_j, \boldsymbol{\psi}_j)} \\
\mathbf{u}_{s0}^m &= \sum_{j=1}^m \alpha_{0j}^{(m)} \boldsymbol{\varphi}_j(0) \\
\mathbf{X}_0^m &= \sum_{j=1}^m \beta_{0j}^{(m)} \boldsymbol{\chi}_j \quad \text{with } \beta_{0j}^{(m)} = \frac{(\mathbf{s}, \boldsymbol{\chi}_j)}{(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j)},
\end{aligned} \tag{25}$$

where we have taken into account the compatibility assumption (14).

Using (23) in (24) we obtain for $(\mathbf{v}, \mathbf{z}) = (\boldsymbol{\psi}_i, \boldsymbol{\varphi}_i(t))$ and $\mathbf{y} = \boldsymbol{\chi}_i$ the following system:

$$\begin{aligned}
&\rho_f \sum_{j=1}^m \alpha_j'(t)(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) + \sum_{j=1}^m \alpha_j(t) a(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) \\
&\quad + \delta_\rho \left(\sum_{j=1}^m \left(\alpha_j'(t)\boldsymbol{\varphi}_j(t) + \alpha_j(t)\boldsymbol{\varphi}_j'(t) \right), \boldsymbol{\varphi}_i(t) \right)_{\mathcal{B}} \\
&\quad + \kappa \sum_{j=1}^m \beta_j(t)(\nabla_s \boldsymbol{\chi}_j, \nabla_s \boldsymbol{\varphi}_i(t))_{\mathcal{B}} = 0 \\
&\sum_{j=1}^m \beta_j'(t)(\boldsymbol{\chi}_j, \boldsymbol{\chi}_i)_{\mathcal{B}} = \sum_{j=1}^m \alpha_j(t)(\boldsymbol{\varphi}_j(t), \boldsymbol{\chi}_i)_{\mathcal{B}},
\end{aligned} \tag{26}$$

where we omitted the superscript m in order to simplify the notation.

Thanks Proposition 2, $\boldsymbol{\varphi}_j(t) \in \mathbf{H}^1(\mathcal{B})$ can be written in terms of the basis $\{\boldsymbol{\chi}_i\}_{i=1}^\infty$ as follows

$$\boldsymbol{\varphi}_j(t) = \sum_{r=1}^\infty \delta_{jr}(t)\boldsymbol{\chi}_r \qquad \text{with } \delta_{jr}(t) = \mathbf{c}(\boldsymbol{\varphi}_j(t), \boldsymbol{\chi}_r), \tag{27}$$

therefore $\delta_{jr}(t)$ inherits the regularity in time of $\boldsymbol{\varphi}_j(t)$.

Lemma 3. *Under Assumption 1, we have for $j \in \mathbb{N}$*

$$\left\| \sum_{r=1}^\infty \delta_{jr}^2(t) c_r \right\|_{L^\infty(0,T)} \le C \|\overline{\mathbf{X}}\|^2_{L^\infty(L^\infty(\mathcal{B}))} \|\boldsymbol{\psi}_j\|^2_{0,\Omega}$$

$$\left\| \sum_{r=1}^{\infty} \delta_{jr}^2(t) \right\|_{L^\infty(0,T)} \le C \|\overline{\mathbf{X}}\|^2_{L^\infty(\mathbf{W}^{1,\infty}(\mathcal{B}))} \|\boldsymbol{\psi}_j\|^2_{1,\Omega} \tag{28}$$

$$\left\| \sum_{r=1}^{\infty} (\delta'_{jr}(t))^2 c_r \right\|_{L^\infty(0,T)} \le C \|\overline{\mathbf{X}}\|^2_{W^{1,\infty}(L^\infty(\mathcal{B}))} \|\boldsymbol{\psi}_j\|^2_{1,\Omega},$$

where $c_r = \|\boldsymbol{\chi}_r\|^2_{0,\mathcal{B}} = \frac{1}{\lambda_{sr}}$ *(see (21)).*

Proof. For each $j \in \mathbb{N}$, $\boldsymbol{\psi}_j \in \mathbf{V}_0$ is an eigenfunction of (20) with $\|\boldsymbol{\psi}_j\|^2_{0,\Omega}=1$ and $\|\varepsilon(\boldsymbol{\psi}_j)\|^2_{0,\Omega} = \lambda_{fj}$. Hence $\boldsymbol{\varphi}_j$ is continuous from $[0,T]$ into $\mathbf{H}^1(\mathcal{B})$ with the time derivative in $L^\infty(0,T;\mathbf{L}^2(\mathcal{B}))$. Taking into account the properties of the eigensolutions of (21) we have

$$\|\boldsymbol{\varphi}_j(t)\|^2_{0,\mathcal{B}} = \sum_{r=1}^{\infty} \delta_{jr}^2(t) \|\boldsymbol{\chi}_r\|^2_{0,\mathcal{B}} = \sum_{r=1}^{\infty} \delta_{jr}^2(t) c_r.$$

Hence we have:

$$\left\| \sum_{r=1}^{\infty} \delta_{jr}^2(t) c_r \right\|_{L^\infty(0,T)} = \|\boldsymbol{\varphi}_j(t)\|^2_{L^\infty(\mathbf{L}^2(\mathcal{B}))} \le C \|\overline{\mathbf{X}}\|^2_{L^\infty(\mathbf{L}^\infty(\mathcal{B}))} \|\boldsymbol{\psi}_j\|^2_{0,\Omega}.$$

Similarly, we set $d_r = \|\nabla_s \boldsymbol{\chi}_r\|^2_{0,\mathcal{B}} = \frac{\lambda_{sr}-1}{\lambda_{sr}}$ where λ_{sr} are the eigenvalues of (21). It is easy to see that $0 < \frac{\lambda_{s1}-1}{\lambda_{s1}} \le d_r < 1$. Then we have

$$\|\nabla_s \boldsymbol{\varphi}_j(t)\|^2_{0,\mathcal{B}} = \sum_{r=1}^{\infty} \delta_{jr}^2(t) \|\nabla_s \boldsymbol{\chi}_r\|^2_{0,\mathcal{B}} = \sum_{r=1}^{\infty} \delta_{jr}^2(t) d_r.$$

The above equations imply that for each $t \in [0,T]$ the series on the right hand side is convergent and that the first two inequalities in (28) hold true. Let us now show the third one. We have that the time derivative of $\boldsymbol{\varphi}_j(t)$ is given by $\frac{\partial \boldsymbol{\varphi}_j}{\partial t}(t) = \nabla \boldsymbol{\psi}_j \frac{\partial \overline{\mathbf{X}}}{\partial t}(t)$, hence it belongs to $L^\infty(0,T;\mathbf{L}^2(\mathcal{B}))$; moreover we have

$$\frac{\partial \boldsymbol{\varphi}_j}{\partial t}(t) = \sum_{r=1}^{\infty} \delta'_{jr}(t) \boldsymbol{\chi}_r,$$

from which we obtain

$$\|\boldsymbol{\varphi}'_j(t)\|^2_{0,\mathcal{B}} = \sum_{r=1}^{\infty} (\delta'_{jr})^2 c_r$$

and we conclude again that the series on the right hand side is convergent and that the estimate in the second inequality of (28) is verified. □

Using the expression (27) into (26), we arrive at

$$
\begin{aligned}
&\rho_f \alpha_i'(t) + a(\boldsymbol{\psi}_i, \boldsymbol{\psi}_i)\alpha_i(t) \\
&\qquad + \delta_\rho \sum_{j=1}^{m} \Big(C_{ij}(t)\alpha_j'(t) + D_{ij}(t)\alpha_j(t) \Big) + \kappa \sum_{j=1}^{m} \delta_{ij}(t) d_j \beta_j(t) = 0 \\
&\beta_i'(t) = \sum_{j=1}^{m} \alpha_j(t) B_{ji}(t),
\end{aligned}
$$

where $B(t)$, $C(t)$, $D(t)$, and $E(t)$ are real matrices in $\mathbb{R}^{m\times m}$ with elements

$$
\begin{aligned}
&B_{ji}(t) = \delta_{ji}(t) && C_{ij}(t) = \sum_{r=1}^{\infty} \delta_{jr}(t)\delta_{ir}(t)c_r \\
&D_{ij}(t) = \sum_{r=1}^{\infty} \delta'_{jr}(t)\delta_{ir}(t)c_r && E_{ij}(t) = \delta_{ij}(t)d_j.
\end{aligned}
\tag{29}
$$

Let $\boldsymbol{\alpha}^{(m)}(t)$ and $\boldsymbol{\beta}^{(m)}(t)$ be the vector valued functions with components $\alpha_j^{(m)}(t)$ and $\beta_j^{(m)}(t)$, respectively. We have obtained the following system of linear ordinary differential equations

$$
\begin{aligned}
&\big(\rho_f \mathbb{I}_m + \delta_\rho C(t)\big)(\boldsymbol{\alpha}^{(m)}(t))' + \big(a(\boldsymbol{\psi}_i, \boldsymbol{\psi}_i)\mathbb{I}_m + D(t)\big)\boldsymbol{\alpha}^{(m)}(t) \\
&\qquad + E(t)\boldsymbol{\beta}^{(m)}(t) = 0 \\
&(\boldsymbol{\beta}^{(m)}(t))' = B^T \boldsymbol{\alpha}^{(m)}(t) \\
&\boldsymbol{\alpha}^{(m)}(0) = \boldsymbol{\alpha}_0^{(m)} \\
&\boldsymbol{\beta}^{(m)}(0) = \boldsymbol{\beta}_0^{(m)},
\end{aligned}
\tag{30}
$$

where $\boldsymbol{\alpha}_0^{(m)}$ and $\boldsymbol{\beta}_0^{(m)}$ are the vectors with components $\alpha_{0j}^{(m)}$ and $\beta_{0j}^{(m)}$ (see (25)).

Lemma 4. *Under Assumption 1, the matrices $B(t)$, $C(t)$, $D(t)$, and $E(t)$ given by (29) are well defined and continuous in $[0,T]$. Moreover, the matrix $\rho_f \mathbb{I}_m + \delta_\rho C(t)$ is invertible with continuous inverse.*

Proof. By definition (27), it is clear that the elements of $B(t)$ and $E(t)$ are continuous in $[0,T]$. Let us consider the elements of $C(t)$. Thanks to the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
\|C_{ij}\|_{L^\infty(0,T)} &= \left| \sum_{r=1}^{\infty} \delta_{jr}(t)\delta_{ir}(t)c_r \right| \\
&\le \left(\sum_{r=1}^{\infty} \delta_{jr}^2(t)c_r \right)^{1/2} \left(\sum_{r=1}^{\infty} \delta_{ir}^2(t)c_r \right)^{1/2} \\
&\le \|\boldsymbol{\varphi}_j\|_{L^\infty(\mathbf{L}^2(\mathcal{B}))} \|\boldsymbol{\varphi}_i\|_{L^\infty(\mathbf{L}^2(\mathcal{B}))}.
\end{aligned}
$$

Since $\boldsymbol{\varphi}_j(t)$ for $j \in \mathbb{N}$ is continuous in $[0, T]$ with values in $\mathbf{H}^1(\mathcal{B})$, the series $\sum_{r=1}^{\infty} \delta_{jr}^2(t) c_r$ is continuous in $[0, T]$. This implies that the elements of $C(t)$ are also continuous in $[0, T]$.

A similar argument shows that $\|D_{ij}\|_{L^\infty(0,T)}$ is bounded.

Now we show that $\rho_f \mathbb{I}_m + \delta_\rho C(t)$ is invertible. Since $C(t)$ is symmetric, it is enough to show that C is also positive semidefinite that is $x^T C x \geq 0$ for all $x \in \mathbb{R}^m$. This can be obtained by direct computation as follows

$$
\begin{aligned}
x^T C x &= \sum_{i,j=1}^{m} x_i \left(\sum_{r=1}^{\infty} \delta_{ir}(t) c_r \delta_{jr}(t) \right) x_j \\
&= \sum_{r=1}^{\infty} c_r \left(\sum_{i=1}^{m} x_i \delta_{ir}(t) \right) \left(\sum_{j=1}^{m} x_j \delta_{jr}(t) \right) \\
&= \sum_{r=1}^{\infty} c_r \left(\sum_{i=1}^{m} x_i \delta_{ir}(t) \right)^2 \geq 0. \quad \square
\end{aligned}
$$

Proposition 5. *The system of ordinary differential equations* (30) *has a unique solution* $\boldsymbol{\alpha}^{(m)} \in C^1([0, T])$ *and* $\boldsymbol{\beta}^{(m)} \in C^1([0, T])$.

Proof. As a consequence of Lemma 4, the matrix $\rho_f \mathbb{I}_m + \delta_\rho C(t)$ is invertible with continuous inverse, hence the standard theory for systems of linear first order ordinary differential equations gives that (30) has a unique solution in $C^1([0, T])$. $\square$

The above proposition yields the existence of the solution of (24), stated in the following theorem

Theorem 6. *There exists a unique solution* $(\mathbf{u}^m(t), \mathbf{w}^m(t)) \in \mathbb{K}_t^m$ *and* $\mathbf{X}^m(t) \in \mathbf{H}^m$ *of* (24) *and* (25) *with*

$$
(\mathbf{u}^m, \mathbf{w}^m) \in C^1([0, T]; \mathbb{K}_t^m), \quad \mathbf{X}^m \in C^1([0, T]; \mathbf{H}^m). \tag{31}
$$

5.3. A priori estimates

We have the following a priori estimates for the solution of (24) and (25).

Proposition 7. *The following bounds hold true with* $C > 0$ *independent of* m*:*

$$
\|\mathbf{u}^m\|_{L^\infty(\mathbf{L}^2(\Omega))} + \|\mathbf{u}^m\|_{L^2(\mathbf{H}_0^1(\Omega))} \leq C \left(\|\mathbf{u}_0^m\|_{0,\Omega} + \|\mathbf{u}_{s0}^m\|_{0,\mathcal{B}} + |\mathcal{B}|^{1/2} \right) \tag{32a}
$$

$$
\|\mathbf{w}^m\|_{L^\infty(\mathbf{L}^2(\mathcal{B}))} + \|\mathbf{w}^m\|_{L^2(\mathbf{H}^1(\mathcal{B}))} \leq C \left(\|\mathbf{u}_0^m\|_{0,\Omega} + \|\mathbf{u}_{s0}^m\|_{0,\mathcal{B}} + |\mathcal{B}|^{1/2} \right) \tag{32b}
$$

$$
\|\mathbf{X}^m\|_{L^\infty(\mathbf{H}^1(\mathcal{B}))} \leq C \left(\|\mathbf{u}_0^m\|_{0,\Omega} + \|\mathbf{u}_{s0}^m\|_{0,\mathcal{B}} + |\mathcal{B}|^{1/2} \right) \tag{32c}
$$

$$\left\|\frac{\partial \mathbf{X}^m}{\partial t}\right\|_{L^\infty(\mathbf{L}^2(\mathcal{B}))}+\left\|\frac{\partial \mathbf{X}^m}{\partial t}\right\|_{L^\infty(\mathbf{H}^1(\mathcal{B}))}\le C\big(\|\mathbf{u}_0^m\|_{0,\Omega}+\|\mathbf{u}_{s0}^m\|_{0,\mathcal{B}}+|\mathcal{B}|^{1/2}\big),\tag{32d}$$

where $|\mathcal{B}|$ stands for the measure of $\mathcal{B}$.

Proof. By definition (23) we have that

$$\frac{\partial \mathbf{u}^m}{\partial t}\in L^2(0,T;\mathbf{V}_0)\qquad \frac{\partial \mathbf{w}^m}{\partial t}\in L^2(0,T;\mathbf{H}^1(\mathcal{B}))\qquad \frac{\partial \mathbf{X}^m}{\partial t}\in L^2(0,T;\mathbf{H}^1(\mathcal{B}))$$

implying that

$$\begin{aligned}
2\left(\frac{\partial \mathbf{u}^m}{\partial t}(t),\mathbf{u}^m(t)\right)&=\frac{d}{dt}\|\mathbf{u}^m(t)\|^2_{0,\Omega}\\
2\left(\frac{\partial \mathbf{w}^m}{\partial t}(t),\mathbf{w}^m(t)\right)_{\mathcal{B}}&=\frac{d}{dt}\|\mathbf{w}^m(t)\|^2_{0,\mathcal{B}}\\
2\left(\nabla_s\frac{\partial \mathbf{X}^m}{\partial t}(t),\nabla_s\mathbf{X}^m(t)\right)_{\mathcal{B}}&=\frac{d}{dt}\|\nabla_s\mathbf{X}^m(t)\|^2_{0,\mathcal{B}}.
\end{aligned}$$

Let us take $(\mathbf{v},\mathbf{z}(t))=(\mathbf{u}^m(t),\mathbf{w}^m(t))$ in the first equation in (24), then

$$\begin{aligned}
&\frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{u}^m(t)\|^2_{0,\Omega}+\nu_0\|\varepsilon(\mathbf{u}^m(t))\|^2_{0,\Omega}\\
&\qquad+\frac{\delta_\rho}{2}\frac{d}{dt}\|\mathbf{w}^m(t)\|^2_{0,\mathcal{B}}+\kappa(\nabla_s\mathbf{X}^m(t),\nabla_s\mathbf{w}^m(t))_{\mathcal{B}}\le 0.
\end{aligned}$$

Thanks to the fact that $(\mathbf{u}^m(t),\mathbf{w}^m(t))\in\mathbb{K}_t^m$, we have that $\mathbf{w}^m(t)$ belongs to $\mathbf{H}^1(\mathcal{B})$ and the second equation in (24) implies that it is equal to the time derivative of $\mathbf{X}^m(t)$, so that the last inequality can be rewritten as

$$\begin{aligned}
&\frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{u}^m(t)\|^2_{0,\Omega}+\nu_0\|\varepsilon(\mathbf{u}^m(t))\|^2_{0,\Omega}\\
&\qquad+\frac{\delta_\rho}{2}\frac{d}{dt}\|\mathbf{w}^m(t)\|^2_{0,\mathcal{B}}+\frac{\kappa}{2}\frac{d}{dt}\|\nabla_s\mathbf{X}^m(t)\|^2_{0,\mathcal{B}}\le 0.
\end{aligned}$$

Integrating on $(0,t)$ with $t\in(0,T]$ and taking into account (13), we arrive at

$$\begin{aligned}
&\rho_f\|\mathbf{u}^m(t)\|^2_{0,\Omega}+2\mathbf{k}\int_0^t\|\nabla\mathbf{u}^m(\tau)\|^2_{0,\Omega}\,d\tau+\delta_\rho\|\mathbf{w}^m(t)\|^2_{0,\mathcal{B}}+\kappa\|\nabla_s\mathbf{X}^m(t)\|^2_{0,\mathcal{B}}\\
&\qquad\le\rho_f\|\mathbf{u}_0^m\|^2_{0,\Omega}+\delta_\rho\|\mathbf{u}_{s0}^m\|^2_{0,\mathcal{B}}+\kappa\|\mathbf{X}_0^m\|^2_{0,\mathcal{B}}.
\end{aligned}\tag{33}$$

Thanks to (25), the last inequality implies (32a). In order to obtain (32b), we observe that a.e. in t $\mathbf{w}^m(t)=(\mathbf{u}^m\circ\overline{\mathbf{X}})(t)$ in $\mathbf{H}^1(\mathcal{B})$. Therefore

$$\nabla_s\mathbf{w}^m(t)=(\nabla\mathbf{u}^m\circ\overline{\mathbf{X}})(t)\,\nabla_s\overline{\mathbf{X}}(t)$$

and

$$\begin{aligned}
\| \nabla_s \mathbf{w}^m \|^2_{L^2(\mathbf{L}^2(\mathcal{B}))} &= \int_0^T \| \nabla_s \mathbf{w}^m(t) \|^2_{0,\mathcal{B}}\, dt \\
&= \int_0^T \| (\nabla \mathbf{u}^m \circ \overline{\mathbf{X}})(t)\, \nabla_s \overline{\mathbf{X}}(t) \|^2_{0,\mathcal{B}}\, dt \\
&\le \| \nabla_s \overline{\mathbf{X}}(t) \|^2_{L^\infty(\mathbf{L}^\infty(\mathcal{B}))} \int_0^T \| (\nabla \mathbf{u}^m \circ \overline{\mathbf{X}})(t) \|^2_{0,\mathcal{B}}\, dt \\
&\le C \int_0^T \| \nabla \mathbf{u}^m \|^2_{0,\Omega^s_t}\, dt \le C \| \nabla \mathbf{u}^m \|^2_{L^2(\mathbf{L}^2(\Omega))}
\end{aligned}$$

which together with (33) gives (32b). It remains to bound $\mathbf{X}^m$. Since

$$\frac{\partial \mathbf{X}^m(t)}{\partial t} = \mathbf{w}^m(t),$$

we obtain (32d) directly. Moreover, the inequality (33) gives the estimate for $\| \nabla_s \mathbf{X}^m \|_{L^\infty(\mathbf{L}^2(\mathcal{B}))}$. Let us now estimate $\mathbf{X}^m(t)$ in the $\mathbf{L}^2(\Omega)$-norm. Integrating in time the last equation, we obtain

$$\mathbf{X}^m(t) = \mathbf{X}^m_0 + \int_0^t \mathbf{w}^m(\tau) d\tau. \tag{34}$$

After some computations, we get

$$\begin{aligned}
\| \mathbf{X}^m(t) \|^2_{0,\mathcal{B}} &\le \| \mathbf{X}^m_0 \|^2_{0,\mathcal{B}} + \left\| \int_0^t \mathbf{w}^m(\tau) d\tau \right\|^2_{0,\mathcal{B}} \\
&\le \| \mathbf{X}^m_0 \|^2_{0,\mathcal{B}} + \left\| t \int_0^t |\mathbf{w}^m(\tau)|^2 d\tau \right\|_{0,\mathcal{B}}^2 \\
&\le \| \mathbf{X}^m_0 \|^2_{0,\mathcal{B}} + t \int_0^t \| \mathbf{w}^m(\tau) \|^2_{0,\mathcal{B}} d\tau.
\end{aligned}$$

It follows

$$\begin{aligned}
\| \mathbf{X}^m \|^2_{L^2(\mathbf{L}^2(\mathcal{B}))} &\le \| \mathbf{X}^m_0 \|^2_{0,\mathcal{B}} + T \| \mathbf{w}^m \|^2_{L^2(\mathbf{L}^2(\mathcal{B}))} \\
\| \mathbf{X}^m \|^2_{L^\infty(\mathbf{L}^2(\mathcal{B}))} &\le \| \mathbf{X}^m_0 \|^2_{0,\mathcal{B}} + T^2 \| \mathbf{w}^m \|^2_{L^\infty(\mathbf{L}^2(\mathcal{B}))}. \quad \square
\end{aligned}$$

5.4. Passing to the limit

Step 1: $\mathbf{u}^m$ converges to $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_0) \cap L^2(0,T;\mathbf{V}_0)$.
The a priori estimate (32a) shows the existence of an element $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_0)$ and of a subsequence $m' \to \infty$ such that

$$\mathbf{u}^{m'} \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;\mathbf{H}_0).$$

This means that for each $\mathbf{v} \in L^1(0,T;\mathbf{H}_0)$

$$\int_0^T (\mathbf{u}^{m'}(t) - \mathbf{u}(t), \mathbf{v}(t))\,dt \to 0 \qquad \text{as } m' \to \infty. \tag{35}$$

Since $\mathbf{u}^{m'}$ is also bounded in $L^2(0,T;\mathbf{V}_0)$, we can extract another subsequence (still denoted $\mathbf{u}^{m'}$) that converges weakly to $\mathbf{u}^* \in L^2(0,T;\mathbf{V}_0)$, that is

$$\mathbf{u}^{m'} \rightharpoonup \mathbf{u}^* \quad \text{in } L^2(0,T;\mathbf{V}_0).$$

The above convergence means that

$$\int_0^T \langle \mathbf{u}^{m'}(t) - \mathbf{u}^*(t), \mathbf{v}(t)\rangle\,dt \to 0 \qquad \text{as } m' \to \infty \quad \forall \mathbf{v} \in L^2(0,T;\mathbf{V}_0'). \tag{36}$$

By the Riesz representation theorem, we can identify $\mathbf{H}_0$ with $\mathbf{H}_0'$, so that

$$\mathbf{V}_0 \subset \mathbf{H}_0 = \mathbf{H}_0' \subset \mathbf{V}_0'.$$

Moreover the duality pairing between $\mathbf{V}_0'$ and $\mathbf{V}_0$ can be identified to the scalar product in $\mathbf{H}_0$ for $\mathbf{u} \in \mathbf{V}_0$ and $\mathbf{v} \in \mathbf{H}_0$, that is

$$\langle \mathbf{v}, \mathbf{u}\rangle = (\mathbf{v}, \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{H}_0,\ \forall \mathbf{u} \in \mathbf{V}_0.$$

Comparing (35) and (36) with $\mathbf{v} \in L^2(0,T;\mathbf{H}_0)$, we obtain that

$$\mathbf{u} = \mathbf{u}^* \in L^\infty(0,T;\mathbf{H}_0) \cap L^2(0,T;\mathbf{V}_0). \tag{37}$$

Step 2: $\mathbf{w}^m$ converges to $\mathbf{w}$ in $L^\infty(0,T;\mathbf{L}^2(\mathcal{B})) \cap L^2(0,T;\mathbf{H}^1(\mathcal{B}))$.
With arguments similar to those used above, we obtain from (32b) the following convergence for some subsequences of $\mathbf{w}^{m'}$:

$$\begin{aligned} &\mathbf{w}^{m'} \overset{*}{\rightharpoonup} \mathbf{w} && \text{in } L^\infty(0,T;\mathbf{L}^2(\mathcal{B})) \\ &\mathbf{w}^{m'} \rightharpoonup \mathbf{w}^* && \text{in } L^2(0,T;\mathbf{H}^1(\mathcal{B})). \end{aligned} \tag{38}$$

Using again the Riesz representation theorem, we have that $\mathbf{H}^1(\mathcal{B}) \subset \mathbf{L}^2(\mathcal{B}) \subset \mathbf{H}^1(\mathcal{B})'$ and we can conclude that

$$\mathbf{w} = \mathbf{w}^* \in L^\infty(0,T;\mathbf{L}^2(\mathcal{B})) \cap L^2(0,T;\mathbf{H}^1(\mathcal{B})). \tag{39}$$

Step 3: The limit $(\mathbf{u}(t),\mathbf{w}(t))$ is contained in $\mathbb{K}_t$.
By construction $(\mathbf{u}^m(t),\mathbf{w}^m(t)) \in \mathbb{K}_t^m$, that is

$$\mathbf{c}(\boldsymbol{\varphi}_i(t), (\mathbf{u}^m \circ \overline{\mathbf{X}})(t) - \mathbf{w}^m(t)) = 0 \quad \text{for } i = 1,\dots,m.$$

Since $\mathbf{u}^m \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathbf{V}_0)$, we have that $\mathbf{u}^m \circ \overline{\mathbf{X}} \rightharpoonup \mathbf{u} \circ \overline{\mathbf{X}}$ in $L^2(0,T;\mathbf{H}^1(\mathcal{B}))$.

Let us consider a scalar function $\phi \in C^\infty(0,T)$. Then we have that $\phi\boldsymbol{\varphi}_i$ belongs to $L^2(0,T;\mathbf{H}^1(\mathcal{B}))$ and that

$$\begin{aligned} 0 &= \int_0^T \phi(t)\mathbf{c}(\boldsymbol{\varphi}_i(t), (\mathbf{u}^m \circ \overline{\mathbf{X}})(t) - \mathbf{w}^m(t))\,dt \\ &= \int_0^T \mathbf{c}(\phi(t)\boldsymbol{\varphi}_i(t), (\mathbf{u}^m \circ \overline{\mathbf{X}})(t) - \mathbf{w}^m(t))\,dt. \end{aligned}$$

Recalling that the bilinear form $\mathbf{c}(\cdot,\cdot)$ is the scalar product in $\mathbf{H}^1(\mathcal{B})$, we can pass to the limit as $m \to \infty$. The weak convergence of $\mathbf{u}^m \circ \overline{\mathbf{X}}$ and of $\mathbf{w}^m$ in $L^2(0,T;\mathbf{H}^1(\mathcal{B}))$ implies

$$\int_0^T \phi(t)\mathbf{c}(\boldsymbol{\varphi}_i, (\mathbf{u} \circ \overline{\mathbf{X}})(t) - \mathbf{w}(t))\,dt = 0 \quad \text{for } i = 1,\dots,m. \tag{40}$$

The last equality is valid for each i, and by linearity for all finite linear combinations of $\boldsymbol{\varphi}_i(t)$. Using Proposition 2, by continuity, Equation (40) is still valid for all $\boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B})$ and implies that

$$\mathbf{c}(\boldsymbol{\mu}, (\mathbf{u} \circ \overline{\mathbf{X}})(t) - \mathbf{w}(t)) = 0 \quad \forall \boldsymbol{\mu} \in \mathbf{H}^1(\mathcal{B})$$

holds true in the sense of distributions on $(0,T)$, so that we conclude that $(\mathbf{u}(t),\mathbf{w}(t))$ belongs to $\mathbb{K}_t$.

Step 4: Limit of $\mathbf{X}^m$ and $\partial\mathbf{X}^m/\partial t$.
Since $\mathbf{X}^m$ and $\|\partial\mathbf{X}^m/\partial t\|$ are bounded in $L^\infty(0,T;\mathbf{H}^1(\mathcal{B}))$ and $L^\infty(0,T;\mathbf{L}^2(\mathcal{B}))$, respectively, there exist $\mathbf{X} \in L^\infty(0,T;\mathbf{H}^1(\mathcal{B}))$, $\mathbf{Y} \in L^\infty(0,T;\mathbf{L}^2(\mathcal{B}))$, and subsequences m' such that

$$\begin{aligned} &\mathbf{X}^{m'} \overset{*}{\rightharpoonup} \mathbf{X} && \text{in } L^\infty(0,T;\mathbf{H}^1(\mathcal{B})) \\ &\frac{\partial \mathbf{X}^{m'}}{\partial t} \overset{*}{\rightharpoonup} \mathbf{Y} && \text{in } L^\infty(0,T;\mathbf{L}^2(\mathcal{B})) \end{aligned} \tag{41}$$

in the sense that

$$\int_0^T \langle \mathbf{X}^{m'}(t) - \mathbf{X}(t), \mathbf{y}(t)\rangle_{\mathcal{B}}\, dt \to 0 \qquad \text{as } m \to \infty \quad \forall \mathbf{y} \in L^1(0,T;\mathbf{H}^1(\mathcal{B})')$$

$$\int_0^T \left(\frac{\partial \mathbf{X}^{m'}}{\partial t}(t) - \mathbf{Y}(t), \mathbf{y}(t)\right)_{\mathcal{B}} \to 0 \qquad \text{as } m \to \infty \quad \forall \mathbf{y} \in L^1(0,T;\mathbf{L}^2(\mathcal{B})).$$

Let us consider a scalar function $\phi(t)$ which is continuously differentiable in $[0,T]$ and $\phi(T) = 0$, and let us denote by $\phi'(t)$ its derivative. Then for $j = 1, \dots, m$

$$\int_0^T \left(\frac{\partial \mathbf{X}^m}{\partial t}(t), \boldsymbol{\chi}_j\right)_{\mathcal{B}} \phi(t)\, dt = -\int_0^T \left(\mathbf{X}^m(t), \phi'(t)\boldsymbol{\chi}_j\right)_{\mathcal{B}} dt - (\mathbf{X}_0^m, \boldsymbol{\chi}_j)_{\mathcal{B}}\phi(0).$$

From (25) we have that $\mathbf{X}_0^m \to \mathbf{X}_0$ strongly in $\mathbf{H}^1(\mathcal{B})$, hence we can pass to the limit and obtain

$$\int_0^T \left(\mathbf{Y}(t), \boldsymbol{\chi}_j\right)_{\mathcal{B}} \phi(t)\, dt = -\int_0^T \left(\mathbf{X}(t), \phi'(t)\boldsymbol{\chi}_j\right)_{\mathcal{B}} dt - (\mathbf{X}_0, \boldsymbol{\chi}_j)_{\mathcal{B}}\phi(0).$$

The above relation is valid for all finite linear combinations $\mathbf{y}$ of $\boldsymbol{\chi}_j$ with $j = 1, \dots, m$. Moreover, it depends linearly and continuously on $\mathbf{y} \in \mathbf{L}^2(\mathcal{B})$; hence, it is valid for all $\mathbf{y} \in \mathbf{L}^2(\mathcal{B})$. Taking $\phi \in \mathscr{D}(0,T)$ and integrating by parts, we get the following equation in the sense of distributions:

$$\left(\frac{\partial \mathbf{X}}{\partial t}(t), \mathbf{y}\right)_{\mathcal{B}} = (\mathbf{Y}(t), \mathbf{y})_{\mathcal{B}} \quad \forall \mathbf{y} \in \mathbf{L}^2(\mathcal{B}).$$

Step 5: $\partial \mathbf{X}/\partial t(t) = \mathbf{w}(t)$.
We have (see (24))

$$\left(\frac{\partial \mathbf{X}^m}{\partial t}(t) - \mathbf{w}^m(t), \boldsymbol{\chi}_i\right)_{\mathcal{B}} = 0 \qquad \forall \boldsymbol{\chi}_i \; i = 1, \dots, m.$$

The convergence of $\mathbf{w}^m$ and of $\mathbf{X}^m$ obtained in (38) and (41) implies that

$$\mathbf{Y} = \mathbf{w} \in L^\infty(0,T;\mathbf{L}^2(\mathcal{B})),$$

therefore the limits $\mathbf{X}$ and $\mathbf{w}$ satisfy the second equation in (19).

Step 6: Passing to the limit in Equation (24).
Let $\phi(t)$ be defined as before. We have:

$$\int_0^T \left(\frac{\partial \mathbf{u}^m}{\partial t}, \boldsymbol{\psi}_j\right) \phi(t)\, dt = -\int_0^T (\mathbf{u}^m(t), \boldsymbol{\psi}_j\phi'(t))\, dt - (\mathbf{u}^m(0), \boldsymbol{\psi}_j)\phi(0)$$

and

$$\int_0^T \left(\frac{\partial \mathbf{w}^m}{\partial t}, \boldsymbol{\varphi}_j(t)\right)_{\mathcal{B}} \phi(t)\,dt = -\int_0^T (\mathbf{w}^m(t), \boldsymbol{\varphi}_j(t)\phi'(t))_{\mathcal{B}}\,dt$$
$$-\int_0^T (\mathbf{w}^m(t), \boldsymbol{\varphi}'_j(t)\phi(t))_{\mathcal{B}}\,dt - (\mathbf{w}^m(0), \boldsymbol{\varphi}_j(0))_{\mathcal{B}}\phi(0).$$

Using these relations in (24) we obtain

$$\begin{aligned}
&-\rho_f \int_0^T (\mathbf{u}^m(t), \boldsymbol{\psi}_j\phi'(t))\,dt + \int_0^T a(\mathbf{u}^m(t), \boldsymbol{\psi}_j\phi(t))\,dt \\
&\quad -\delta_\rho \int_0^T (\mathbf{w}^m(t), \boldsymbol{\varphi}_j(t)\phi'(t))_{\mathcal{B}}\,dt - \delta_\rho \int_0^T (\mathbf{w}^m(t), \frac{\partial \boldsymbol{\varphi}_j}{\partial t}(t)\phi(t))_{\mathcal{B}}\,dt \\
&\quad +\kappa \int_0^T (\nabla_s \mathbf{X}^m(t), \nabla_s \boldsymbol{\varphi}_j(t)\phi(t))_{\mathcal{B}}\,dt \\
&= \rho_f(\mathbf{u}^m(0), \boldsymbol{\psi}_j)\phi(0) + \delta_\rho(\mathbf{w}^m(0), \boldsymbol{\varphi}_j(0))_{\mathcal{B}}\phi(0).
\end{aligned}$$

For j fixed, passing to the limit yields

$$\begin{aligned}
&-\rho_f \int_0^T (\mathbf{u}(t), \boldsymbol{\psi}_j\phi'(t))\,dt + \int_0^T a(\mathbf{u}(t), \boldsymbol{\psi}_j\phi(t))\,dt \\
&\quad -\delta_\rho \int_0^T (\mathbf{w}(t), \boldsymbol{\varphi}_j(t)\phi'(t))_{\mathcal{B}}\,dt - \delta_\rho \int_0^T (\mathbf{w}(t), \frac{\partial \boldsymbol{\varphi}_j}{\partial t}(t)\phi(t))_{\mathcal{B}}\,dt \\
&\quad +\kappa \int_0^T (\nabla_s \mathbf{X}(t), \nabla_s \boldsymbol{\varphi}_j(t)\phi(t))_{\mathcal{B}}\,dt \\
&= \rho_f(\mathbf{u}_0, \boldsymbol{\psi}_j)\phi(0) + \delta_\rho(\mathbf{u}_{s0}, \boldsymbol{\varphi}_j(0))_{\mathcal{B}}\phi(0).
\end{aligned} \tag{42}$$

Each element $(\mathbf{v}, \mathbf{z}(t)) \in \mathbb{K}_t^m$ can be written as

$$\mathbf{v} = \sum_{j=1}^m a_j \boldsymbol{\psi}_j$$
$$\mathbf{z}(t) = \sum_{j=1}^m a_j \boldsymbol{\varphi}_j(t) = \sum_{j=1}^m a_j \boldsymbol{\psi}_j \circ \overline{\mathbf{X}}(t).$$

Let us denote by $\mathbf{z}'(t)$ the time derivative of $\mathbf{z}(t)$. We have

$$\mathbf{z}'(t) = \sum_{j=1}^{m} a_j \nabla \boldsymbol{\psi}_j \circ \overline{\mathbf{X}}(t) \frac{\partial \overline{\mathbf{X}}}{\partial t}(t)$$

which, due to the regularity of $\overline{\mathbf{X}}$, is continuous from $[0, T]$ into $\mathbf{L}^2(\mathcal{B})$.

We write (42) as follows

$$\begin{aligned} &-\rho_f \int_0^T (\mathbf{u}(t), \mathbf{v}\phi'(t))\,dt + \int_0^T a(\mathbf{u}(t), \mathbf{v}\phi(t))\,dt \\ &\quad - \delta_\rho \int_0^T (\mathbf{w}(t), \mathbf{z}(t)\phi'(t))_{\mathcal{B}}\,dt - \delta_\rho \int_0^T (\mathbf{w}(t), \mathbf{z}'(t)\phi(t))_{\mathcal{B}}\,dt \\ &\quad + \kappa \int_0^T (\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z}(t)\phi(t))_{\mathcal{B}}\,dt \\ &= \rho_f(\mathbf{u}_0, \mathbf{v})\phi(0) + \delta_\rho(\mathbf{u}_{s0}, \mathbf{z}(0))_{\mathcal{B}}\phi(0). \end{aligned} \tag{43}$$

All the terms depend linearly and continuously on $(\mathbf{v}, \mathbf{z}) \in C^1([0, T]; \mathbb{K}_t^m)$, hence for each $t \in [0, T]$ Equation (43) holds true for all $(\mathbf{v}, \mathbf{z}) \in C^1([0, T]; \mathbb{K}_t)$.

Taking $\phi \in \mathscr{D}(0, T)$ and integrating by parts with respect to t, we arrive at

$$\begin{aligned} &\rho_f \int_0^T \left(\frac{\partial \mathbf{u}}{\partial t}(t), \mathbf{v}\right)\phi(t)\,dt + \int_0^T a(\mathbf{u}(t), \mathbf{v})\phi(t)\,dt \\ &\quad + \delta_\rho \int_0^T \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}(t)\right)_{\mathcal{B}} \phi(t)\,dt + \kappa \int_0^T (\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z}(t))_{\mathcal{B}}\phi(t)\,dt = 0, \end{aligned}$$

which implies that the first equation in (19) holds true in the sense of distributions on $(0, T)$.

Step 7: Initial conditions.
It remains to check that $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{w}(0) = \mathbf{u}_{s,0}$ and $\mathbf{X}(0) = \mathbf{X}_0$.

Since $\frac{\partial \mathbf{X}}{\partial t} = \mathbf{w} \in L^2(0, T; \mathbf{H}^1(\mathcal{B}))$, then $\mathbf{X}$ is continuous from $[0, T]$ to $\mathbf{H}^1(\mathcal{B})$, and we can pass to the limit in (34) for $t = 0$ arriving at $\mathbf{X}(0) = \mathbf{X}_0$.

We recall that

$$(\mathbf{u}, \mathbf{w}) \in L^2(0, T; \mathbf{V}_0 \times \mathbf{H}^1(\mathcal{B})).$$

Moreover, since $(\mathbf{u}(t), \mathbf{w}(t)) \in \mathbb{K}_t$ and $\mathbb{K}_t \subset \mathbf{V}_0 \times \mathbf{H}^1(\mathcal{B})$, we have that

$$(\mathbf{u}, \mathbf{w}) \in L^2(0, T; \mathbb{K}_t).$$

In order to prove the continuity of **u** and **w** in the correct spaces for the initial conditions, we can use a general interpolation theorem. If we can show

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{w}}{\partial t}\right) \in L^2(0, T; \mathbf{V}_0' \times \mathbf{H}^1(\mathcal{B})') \tag{44}$$

then it follows

$$(\mathbf{u}, \mathbf{w}) \in C^0([0, T]; \mathbf{H}_0 \times \mathbf{L}^2(\mathcal{B})).$$

Given $(\mathbf{v}, \mathbf{z}(t)) \in \mathbb{K}_t$ the first equation in (19) can be written as

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t}(t), \mathbf{v}\right) + \delta_\rho \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}(t)\right)_{\mathcal{B}} = -\langle A\mathbf{u}(t), \mathbf{v}\rangle - \kappa(\nabla_s \mathbf{X}(t), \nabla_s(\mathbf{z}(t)))_{\mathcal{B}},$$

where $A : \mathbf{V}_0 \to \mathbf{V}_0'$ is the linear and continuous operator associated with the bilinear form a. Since $\mathbf{u} \in L^2(0, T; \mathbf{V}_0)$, the function $A\mathbf{u}$ belongs to $L^2(0, T; \mathbf{V}_0')$. Taking into account that $\mathbf{X} \in L^2(0, T; \mathbf{H}^1(\mathcal{B}))$ we obtain also

$$\int_0^T (\nabla_s \mathbf{X}(t), \nabla_s(\mathbf{v} \circ \overline{\mathbf{X}}(t)))_{\mathcal{B}}\, dt \le C \| \nabla_s \mathbf{X}(t)\|_{L^2(\mathbf{L}^2(\mathcal{B}))} \|\mathbf{v}\|_{\mathbf{V}_0}. \tag{45}$$

It follows that

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{w}}{\partial t}\right) \in L^2(0, T; \mathbb{K}_t'),$$

which, from Hahn–Banach theorem implies (44).

The general interpolation theory of Lions–Magenes [24] and [41, Lemma 1.2] implies that **u** is continuous form $[0, T]$ to $\mathbf{H}_0$ and **w** is continuous from $[0, T]$ to $\mathbf{L}^2(\mathcal{B})$.

It remains to check that $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{w}(0) = \mathbf{u}_{s0}$. We multiply the first equation in (19) by a scalar function $\phi(t)$, continuously differentiable on $[0, T]$ with $\phi(T) = 0$, and integrate with respect to t

$$\int_0^T \rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v})\phi(t)\, dt + \int_0^T \delta_\rho \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}(t)\right)_{\mathcal{B}} \phi(t)\, dt$$
$$+ \int_0^T a(\mathbf{u}(t), \mathbf{v})\phi(t)\, dt + \int_0^T \kappa(\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z}(t))_{\mathcal{B}}\phi(t)\, dt = 0.$$

Integration by parts in the first two integrals gives for $(\mathbf{v}, \mathbf{z}(t)) \in \mathbb{K}_t$

$$\rho_f \int_0^T \frac{d}{dt}(\mathbf{u}(t),\mathbf{v})\phi(t)\,dt + \delta_\rho \int_0^T \left(\frac{\partial \mathbf{w}}{\partial t}(t),\mathbf{z}(t)\right)_{\mathcal{B}} \phi(t)\,dt$$
$$= -\rho_f \int_0^T (\mathbf{u}(t),\mathbf{v})\phi'(t)\,dt - \delta_\rho \int_0^T \left(\mathbf{w}(t),\mathbf{z}(t)\phi'(t) + \frac{\partial \mathbf{z}}{\partial t}(t)\phi(t)\right)_{\mathcal{B}} dt$$
$$- \rho_f(\mathbf{u}(0),\mathbf{v})\phi(0) - \delta_\rho(\mathbf{w}(0),\mathbf{z}(0))_{\mathcal{B}}\phi(0),$$

which inserted in the previous equation gives

$$- \rho_f \int_0^T (\mathbf{u}(t),\mathbf{v})\phi'(t)\,dt - \delta_\rho \int_0^T \left(\mathbf{w}(t),\mathbf{z}(t)\phi'(t) + \frac{\partial \mathbf{z}}{\partial t}(t)\phi(t)\right)_{\mathcal{B}} dt$$
$$+ \int_0^T a(\mathbf{u}(t),\mathbf{v})\phi(t)\,dt + \int_0^T \kappa(\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z}(t))_{\mathcal{B}}\phi(t)\,dt$$
$$= \rho_f(\mathbf{u}(0),\mathbf{v})\phi(0) + \delta_\rho(\mathbf{w}(0),\mathbf{z}(0))_{\mathcal{B}}\phi(0).$$

Comparing the last equality with (43) gives for $\phi(0) \neq 0$

$$\rho_f(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) + \delta_\rho(\mathbf{w}(0) - \mathbf{u}_{s0}, \mathbf{z}(0))_{\mathcal{B}} = 0 \tag{46}$$

for all $(\mathbf{v},\mathbf{z}(0)) \in \mathbb{K}_0$. Recalling that $\mathcal{B} = \Omega_0^s$, $\mathbf{w}(0) = (\mathbf{u} \circ \overline{\mathbf{X}})(t)(0)$, and the compatibility assumption (14), we can write the last equation as

$$\rho_f(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v})_{\Omega_0^f} + \rho_s(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v})_{\Omega_0^s} = 0,$$

that is, with obvious notation, $(\rho(\mathbf{u}(0) - \mathbf{u}_0), \mathbf{v})_\Omega = 0$ which gives that $\mathbf{u}(0) = \mathbf{u}_0$ in Ω. Substituting in (46) we obtain also that $\mathbf{w}(0) = \mathbf{u}_{s0}$ in $\mathcal{B}$.

5.5. Uniqueness

Let us assume that $(\mathbf{u}_1, \mathbf{w}_1, \mathbf{X}_1)$ and $(\mathbf{u}_2, \mathbf{w}_2, \mathbf{X}_2)$ are two solutions of (19). Since the problem is linear the differences $\hat{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\hat{\mathbf{w}} = \mathbf{w}_1 - \mathbf{w}_2$ and $\hat{\mathbf{X}} = \mathbf{X}^1 - \mathbf{X}^2$ satisfy the same equations as $\mathbf{u}, \mathbf{w}, \mathbf{X}$ with vanishing initial conditions, that is

$$\rho_f\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}(t),\mathbf{v}\right) + a(\hat{\mathbf{u}}(t),\mathbf{v}) + \delta_\rho\left(\frac{\partial \hat{\mathbf{w}}}{\partial t}(t),\mathbf{z}(t)\right)_{\mathcal{B}}$$
$$+ \kappa(\nabla_s \hat{\mathbf{X}}(t), \nabla_s \mathbf{z}(t))_{\mathcal{B}} = 0 \qquad \forall(\mathbf{v},\mathbf{z}(t)) \in \mathbb{K}_t$$
$$\left(\frac{\partial \hat{\mathbf{X}}}{\partial t}(t),\mathbf{y}\right)_{\mathcal{B}} = (\hat{\mathbf{w}}(t),\mathbf{y})_{\mathcal{B}} \qquad \forall \mathbf{y} \in \mathbf{L}^2(\mathcal{B})$$
$$\hat{\mathbf{u}}(0) = 0 \quad \text{in } \Omega, \quad \hat{\mathbf{w}}(0) = 0 \quad \text{in } \mathcal{B}, \quad \hat{\mathbf{X}}(0) = 0 \quad \text{in } \mathcal{B}.$$

We take $(\mathbf{v}, \mathbf{z}(t)) = (\hat{\mathbf{u}}(t), \hat{\mathbf{w}}(t))$ in the first equation and use the fact that $\hat{\mathbf{w}}(t) = \frac{\partial \hat{\mathbf{X}}}{\partial t}(t)$, so that we get

$$\rho_f \left(\frac{\partial \hat{\mathbf{u}}}{\partial t}(t), \hat{\mathbf{u}}(t)\right) + a(\hat{\mathbf{u}}(t), \hat{\mathbf{u}}(t)) + \delta_\rho \left(\frac{\partial \hat{\mathbf{w}}}{\partial t}(t), \hat{\mathbf{w}}(t)\right)_{\mathcal{B}} + \kappa \left(\nabla_s \hat{\mathbf{X}}(t), \nabla_s \frac{\partial \hat{\mathbf{X}}}{\partial t}(t)\right)_{\mathcal{B}} = 0. \tag{47}$$

Thanks to (37), (39), (41), (44), Step 5, and [41, Lemma III.1.2], we can write

$$\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}(t), \hat{\mathbf{u}}(t)\right) = \frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}}(t)\|^2_{0,\Omega}$$
$$\left(\frac{\partial \hat{\mathbf{w}}}{\partial t}(t), \hat{\mathbf{w}}(t)\right)_{\mathcal{B}} = \frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{w}}(t)\|^2_{0,\mathcal{B}}$$
$$\left(\nabla_s \hat{\mathbf{X}}(t), \nabla_s \frac{\partial \hat{\mathbf{X}}}{\partial t}(t)\right)_{\mathcal{B}} = \frac{1}{2}\frac{d}{dt}\| \nabla_s \hat{\mathbf{X}}(t)\|^2_{0,\mathcal{B}}.$$

Inserting these equalities into (47) gives

$$\frac{\rho_f}{2}\frac{d}{dt}\|\hat{\mathbf{u}}(t)\|^2_{0,\Omega} + \mathbf{k}\| \nabla \hat{\mathbf{u}}(t)\|^2_{0,\Omega} + \frac{\delta_\rho}{2}\frac{d}{dt}\|\hat{\mathbf{w}}(t)\|^2_{0,\mathcal{B}} + \frac{\kappa}{2}\frac{d}{dt}\| \nabla_s \hat{\mathbf{X}}(t)\|^2_{0,\mathcal{B}} \le 0,$$

which integrated from 0 to t implies

$$\frac{\rho_f}{2}\|\hat{\mathbf{u}}(t)\|^2_{0,\Omega} + \frac{\delta_\rho}{2}\|\hat{\mathbf{w}}(t)\|^2_{0,\mathcal{B}} + \frac{\kappa}{2}\| \nabla_s \hat{\mathbf{X}}(t)\|^2_{0,\mathcal{B}} \le 0.$$

Therefore $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, $\mathbf{w}_1(t) = \mathbf{w}_2(t)$, and $\mathbf{X}_1(t) = \mathbf{X}_2(t)$ for all t.

6. Recovery of the pressure and of the Lagrange multiplier

In order to obtain existence and uniqueness of the solution of Problem 3, we need to show that starting from the solution $(\mathbf{u}, \mathbf{w}, \mathbf{X})$ of Problem 4, we can define a Lagrange multiplier $\boldsymbol{\lambda}$ and a pressure p so that $(\mathbf{u}, p, \mathbf{X}, \mathbf{w}, \boldsymbol{\lambda})$ satisfies (17a)-(17g).

Proposition 8. *Let $(\mathbf{u}, \mathbf{w}, \mathbf{X})$ be the solution of Problem 4, then there exists $\boldsymbol{\lambda} \in L^2(0, T; \mathbf{H}^1(\mathcal{B}))$ such that for all $t \in (0, T)$*

$$\mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = \delta_\rho \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}\right)_{\mathcal{B}} + \kappa(\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z})_{\mathcal{B}} \qquad \forall \mathbf{z} \in \mathbf{H}^1(\mathcal{B}). \tag{48}$$

Proof. Since $\mathbf{c}$ is equal to the scalar product in $\mathbf{H}^1(\mathcal{B})$, it is enough to show that the right hand side is a linear functional on $\mathbf{H}^1(\mathcal{B})$. The linearity is obvious. We check now that both terms are continuous. Since $\frac{\partial \mathbf{w}}{\partial t} \in L^2(0, T; \mathbf{H}^1(\mathcal{B})')$ and $\mathbf{X} \in L^2(0, T; \mathbf{H}^1(\mathcal{B}))$ we have

$$\int_0^T \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}\right)_{\mathcal{B}} dt \le \left\|\frac{\partial \mathbf{w}}{\partial t}\right\|_{L^2(\mathbf{H}^1(\mathcal{B})')} \|\mathbf{z}\|_{\mathbf{H}^1(\mathcal{B})}$$

$$\int_0^T (\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z})_{\mathcal{B}}\, dt \le \|\mathbf{X}\|_{L^2(\mathbf{H}^1(\mathcal{B}))} \|\mathbf{z}\|_{\mathbf{H}^1(\mathcal{B})}.$$

These inequalities imply that the right hand side of (48) is a continuous functional on $L^2(0, T; \mathbf{H}^1(\mathcal{B}))$. Therefore, from the Lax–Milgram lemma, we obtain existence and uniqueness of the solution $\boldsymbol{\lambda} \in L^2(0, T; \mathbf{H}^1(\mathcal{B}))$. □

The above proposition allows us to split the first equation in (19) into two equations as follows:

$$\begin{aligned}
&\rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v} \circ \overline{\mathbf{X}}(t)) = 0 && \forall \mathbf{v} \in \mathbf{V}_0\\
&\delta_\rho \left(\frac{\partial \mathbf{w}}{\partial t}(t), \mathbf{z}\right)_{\mathcal{B}} + \kappa (\nabla_s \mathbf{X}(t), \nabla_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 && \forall \mathbf{z} \in \mathbf{H}^1(\mathcal{B}).
\end{aligned} \tag{49}$$

In order to obtain the solution of Problem 3, it remains to show the existence of p.

Proposition 9. *Let* $(\mathbf{u}, \mathbf{w}, \mathbf{X})$ *and* $\boldsymbol{\lambda}$ *be the solutions of Problem 4 and of* (48). *Then there exists a unique* $p \in L^2(0, T; L^2_0(\Omega))$ *such that* $(\mathbf{u}, p, \mathbf{X}, \mathbf{w}, \boldsymbol{\lambda})$ *is the solution of Problem 3.*

Proof. The existence and uniqueness of $(\mathbf{u}, \mathbf{w}, \mathbf{X})$ and $\boldsymbol{\lambda}$ are stated in Theorem 1 and in Proposition 8, respectively. The pressure p can be obtained as the solution of the following equation

$$(p(t), \operatorname{div} \mathbf{v}) = \rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v} \circ \overline{\mathbf{X}}(t)) \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega). \tag{50}$$

In order to see that this problem defines a function $p(t)$ satisfying the required regularity, we can use standard arguments originating from the Banach closed range theorem (see, for instance [2, Theorem 4.1.4]). We need to show that the right-hand side of (50) is a linear and continuous functional on $\mathbf{H}^1_0(\Omega)$ belonging to the polar set of the kernel of the divergence operator in $\mathbf{H}^1_0(\Omega)$. Let us denote the right-hand side of (50) by $\ell(\mathbf{v})$; the continuity of ℓ can be shown as follows

$$\begin{aligned}
\left|\int_0^T \ell(\mathbf{v})\, dt\right| &\le \int_0^T |\ell(\mathbf{v})|\, dt\\
&\le C\left(\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(H^{-1}(\Omega))} + \|\mathbf{u}\|_{L^2(\mathbf{H}^1_0(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(\mathbf{H}^1(\mathcal{B}))}\right) \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}.
\end{aligned}$$

Moreover, it is clear that ℓ belongs to the polar set of the kernel of the divergence operator in $\mathbf{H}^1_0(\Omega)$: this is exactly what is stated in the first equation of (49).

From the closed range theorem, it follows that there exists $p(t)$ satisfying (50) such that

$$\|p\|_{L^2(L^2_0(\Omega))} \le (1/\beta) \|\ell\|_{L^2(H^{-1}(\Omega))},$$

where β is the inf-sup constant associated with the divergence operator in $\mathbf{H}_0^1(\Omega)$ (see [41, (I.1.51) and Prop. I.1.2]). $\square$

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