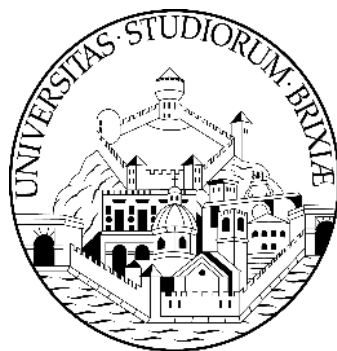




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**ESSAYS ON THE STRATEGIC FOUNDATIONS OF ASSET PRICING MODELS**

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*To Elena and Leonardo*



## Abstract

Modern asset pricing is based on General Equilibrium Theory (GET), which makes use of Walrasian Equilibrium (WE) to characterize the outcome of a market economy. However, GET and WE remain tacit about how trades take place in the economy and the way individuals reason and take actions. The purpose of this thesis is to revisit static asset pricing models from an epistemic, game-theoretical perspective. The difference between traditional and epistemic game theory is that while the former assumes *Nash equilibrium* behavior to abstract from the potential uncertainty that a player faces about the actions played by her opponents, the latter considers the reasoning process through which each player attempts to deal with such uncertainty. The thesis is composed of a preface and three chapters. **Chapter 1** is dedicated to reviewing the most relevant concepts of epistemic game theory, including the necessary mathematical prerequisites and the construction of epistemic type structures. In **Chapter 2** we study rationalizability and bounded reasoning about rationality in a strategic version of the static Investment CAPM, which is a particular case of a *network game with linear best-replies*. When the number of firms (i.e., the players of the game) is large, if there is a unique rationalizable outcome, the Investment CAPM is a reasonable approximation. However, rationalizability in large games requires very sophisticated reasoning skills. For this reason, we simulate outcomes consistent with rationality and bounded reasoning about rationality. We reproduce many documented puzzles in the empirical estimation of the Investment CAPM and provide a rational explanation. However, from a strategic and epistemic perspective, one limitation of the model in **Chapter 2** is the assumption of a Stochastic Discount Factor (SDF). For this reason, in **Chapter 3** we consider a market economy with state-contingent claims and a class of trading games describing *auctions with limit orders and price priority* for the same assets. We show that every profile of actions (i.e., limit orders) is rationalizable. We introduce the notion of mutual  $\pi$ -belief in Walrasian markets which characterizes an epistemic state in which every investor (i.e., a player of the game) believes to be a price-taker as in GET. If investors are rational and mutually  $\pi$ -believe in Walrasian markets, in the case of pay-as-bid auctions (i.e., buyers pay the price they quote), we find that asset prices are consistent with WE of the economy, although asset allocations may be not. In the case of pay-as-clear auctions (i.e., buyers pay the market price), we find that asset prices are either consistent with a WE of the economy, or, as a consequence of weakly dominated

actions such as "market orders, inflated. We conclude that the existence of an SDF should be taken far from granted, even when individuals are rational and believe in Walrasian markets.

## Sommario (Abstract)

La moderna teoria dei prezzi degli attivi (*asset pricing*) si fonda sulla Teoria dell'Equilibrio Generale (GET), che utilizza l'Equilibrio Walrasiano (WE) per caratterizzare prezzi e allocazioni in un'economia di mercato. Tuttavia, la GET e il WE restano taciti riguardo alle modalità con cui gli scambi avvengono nell'economia e al modo in cui gli individui ragionano e prendono decisioni. Lo scopo di questa tesi è di riesaminare i modelli statici di asset pricing adottando l'approccio epistemico alla teoria dei giochi. La differenza tra la teoria dei giochi tradizionale e quella epistemica risiede nel fatto che, mentre la prima assume un comportamento di *Equilibrio di Nash* per astrarre dall'incertezza che un giocatore può affrontare riguardo alle azioni intraprese dai suoi avversari, la seconda considera invece il processo di ragionamento attraverso il quale ogni giocatore cerca di gestire tale incertezza. La tesi è composta da una prefazione e tre capitoli. Il **Capitolo 1** è dedicato alla revisione dei concetti più rilevanti della teoria dei giochi epistemica, includendo i prerequisiti matematici necessari e la costruzione delle strutture di "tipi" epistemici (*epistemic type structure*). Nel **Capitolo 2** si studiano la razionalizzabilità (*rationalizability*) e il ragionamento limitato sulla razionalità (*bounded reasoning about rationality*) in una versione strategica dell'Investment CAPM statico, che rappresenta un caso particolare di *gioco su rete con risposte ottime lineari* (*network game with linear best replies*). Quando il numero di aziende (giocatori) è elevato, se esiste un unico esito razionalizzabile, l'Investment CAPM ne costituisce un'approssimazione ragionevole. Tuttavia, la razionalizzabilità nei giochi di grandi dimensioni richiede capacità di ragionamento molto sofisticate da parte dei giocatori. Per questa ragione, vengono simulate situazioni coerenti con la razionalità e con un ragionamento limitato sulla razionalità. Tali simulazioni riproducono molti dei paradossi documentati nelle stime empiriche dell'Investment CAPM e ne forniscono una spiegazione razionale. Da una prospettiva strategica ed epistemica, tuttavia, una limitazione del modello del **Capitolo 2** è l'assunzione di un Fattore di Sconto Stocastico (SDF). Per questo motivo, nel **Capitolo 3** si considera un'economia di mercato con titoli dipendenti dallo stato del mondo (*state-contingent claims*) e una classe di giochi di negoziazione (*trading games*) che descrivono *aste con ordini limite e priorità di prezzo* sugli stessi attivi. Si dimostra che ogni profilo di azioni (ossia ordini limite) è razionalizzabile. Si introduce quindi la nozione di  $\pi$ -credenza reciproca in mercati Walrasiani, che caratterizza uno stato epistemico in cui ogni investitore (giocatore) crede di essere *price-taker*, come nella GET.

In questa situazione, assumendo che gli investitori siano razionali, nel caso di aste *pay-as-bid* (i.e., gli acquirenti pagano il prezzo da loro offerto), si riscontra che i prezzi degli attivi sono coerenti con l'Equilibrio Walrasiano dell'economia, sebbene le allocazioni degli attivi possano non esserlo. Nel caso di aste *pay-as-clear* (i.e., gli acquirenti pagano il prezzo di mercato), si osservano invece prezzi degli attivi coerenti con un Equilibrio Walrasiano oppure, come conseguenza diretta di azioni debolmente dominate quali gli “ordini a mercato”, gonfiati. Si conclude che l'esistenza di un Fattore di Sconto Stocastico non può essere considerata come un fatto ovvio, nemmeno quando gli individui sono razionali e credono, in senso lato, nell'efficienza dei mercati.

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In *Théodicée*, Gottfried Leibniz attempted to prove that *we live in the best of all possible worlds*. Leibniz was a mathematician and philosopher who embraced a metaphysical journey in his work. I was never very good at philosophy, but now I realize that he was right. The ideas presented in this thesis have been long in my mind, probably since I graduated from Bocconi University in 2013. If it were not for a matter of circumstances, they would never have been put on paper. A huge thanks goes to Roberto Savona, without his invaluable advice and support, I could not conclude my PhD program. I am very grateful to all the faculty members of the PhD program in Analytics for Economics and Management of Università degli Studi di Brescia for their professionalism, kindness, and support provided in all these years. A special thanks goes to Renato Camodeca and Alex Almici, with whom I co-authored several articles in the past and provided me with the right motivation to go through this PhD program. A huge thanks also goes to Quaestio Capital Management SGR S.p.A., including its shareholders and directors, and in particular to my colleagues Alessandro Potestà and Christian Prinoth: this journey started together and continues together. I also thank Giordano Beani for the insightful discussions on markets and investors' psychology during my time at Fideuram Asset Management Ireland dac. I am also seriously indebted to many other people who have contributed to my professional and intellectual journey and I hope they do not get offended if I do not mention them explicitly. They are simply too many, and I am grateful to everyone. However, the greatest thanks goes to my family, Elena and Leonardo, and to my parents, Ilia and Donato. They constantly remind me that the most important thing in life is to be happy. And happy I was when I wrote the contents of the next chapters while deciding what to do next in my life.

## Preface

Since my time at university, I always had mixed feelings about the unconditional adoption of the General Equilibrium Theory (GET) in finance. On the one hand, the parsimonious description of a market economy and the elegance of Walrasian equilibrium (WE) had the same impressing effect they had on many others. On the other hand, I always felt a bit uncomfortable with the two crucial aspects of GET. First, GET does not provide a description of how individuals interact with each other, that is, what actions they take to trade in goods or financial assets and how these actions translate into prices and allocations. Second, even if we solved the previous issue, it would still remain unclear how every agent in the economy *reasons* about the behavior of the others and how *rational behavior* may result in prices and allocations consistent with a WE. As Gerard Debreu (1959) elegantly remarked,

*“Allegiance to rigor dictates the axiomatic form of the analysis where the theory, in the strict sense, is logically entirely disconnected from its interpretations.”*

(Debreu, 1959, *Theory of Value*, Preface, p. X)

Although these issues may appear to some extent as obvious, it took some time before they were acknowledged as serious limitations. In fact, it was the seminal paper of Shapley & Shubik (1977) that stimulated the subsequent vast scientific production in the field of *trading games*.

*“The Walrasian model of a competitive economy describes equilibrium without reference to the mechanics of trading. Our purpose is to present a class of games in which equilibrium prices emerge as the result of strategic interaction.”* (Shapley and Shubik, 1977, JPE 85(5), p. 937)

However, almost all relevant contributions in this field impose Nash equilibrium as an additional axiom to obtain prices and allocations, without offering any epistemic motivation for its adoption. In this regard, the main issue with Nash equilibrium comes with its implicit epistemic assumption that players know, or hold correct belief about, the actions being simultaneously chosen by their respective opponents, which is clearly a very strong requirement.

*“The use of mutually correct beliefs in the epistemic characterization of the Nash equilibrium has led to criticisms of the Nash equilibrium as a solution concept. The question is how do players ever come to have such correct beliefs or common knowledge about what the other players are choosing (cf. Skyrms, 1990). This seems to go against the very idea of a game in strategic form, where the players choose simultaneously, without knowing the choices of the other players. Tan & Werlang (1988) were among the first to express this concern, which has become increasingly prevalent in epistemic game theory.”* (Pacuit and Roy, 2025, Stanford Encyclopedia of Philosophy, Epistemic Foundations of Game Theory, 2025)

Tan and Werlang (1988), Aumann and Brandenburger (1995) and Polak (1999) provided sufficient epistemic conditions for Nash equilibrium, which, however, appears very demanding, that is, unrealistic, within the context of trading games.

Using the vast literature on trading games and the language of epistemic game theory, the purpose of this Ph.D. thesis is to address the *strategic* and *epistemic* foundations of static asset pricing models. The first chapter (**Chapter 1**) of this thesis is dedicated to reviewing the most relevant concepts of epistemic game theory, including the necessary mathematical prerequisites and the construction of epistemic type structures. Game-theoretic models start from a clear description of the actions and information available to the players, and how these actions translate into players’ outcomes. In addition, epistemic models are explicit about how players make use of their available information to *reason* about their respective elements of uncertainty, such as the actions taken by the other players, in order to pick their best available actions. This latter aspect marks the fundamental distinction with traditional, i.e. non-epistemic, game theory, which, assuming Nash equilibrium, or variant of, remains silent on *strategic thinking*, therefore falling again in the spectrum of axiomatic theories.

In **Chapter 1** we also review *rationalizability*, a solution concept motivated by the joint behavioral-epistemic hypothesis of *rationality* and *common belief in rationality* RCBR, and in **Chapter 2** we study rationalizability in a strategic version of the static Investment CAPM (cf. Zhang, 2017), which is a particular case of a *network game with linear best-replies*. The Investment CAPM is an example of a supply-side asset pricing model which predicts that, in a Walrasian equilibrium, stock returns are equal to investment returns, the latter being defined as the marginal return of a firm’s productive assets. Considering the more general case of network games with linear best-replies, we provide necessary and sufficient conditions for the existence of a unique rationalizable outcome, which is a

Nash equilibrium. When a unique rationalizable outcome exists in the strategic Investment CAPM, *stock returns* quickly converge to *investment returns* as the number of firms increases. However, since common belief in rationality may be a too demanding epistemic requirement in games with a large number of players, we study the implications for the relation between stock and investment returns, as well as *valuations*, under the alternative assumption of rationality and *mutual belief in rationality*. We develop a specific simulation methodology reminiscent of *level-k thinking* and replicate the behavior of an econometrician estimating the static investment CAPM on simulated data with the *generalized method of moments* (GMM). Calibrating the model on the top 1000 listed US companies by market capitalization, with the exclusion of a restricted number of sectors (e.g., financial firms), we find that rationality and mutual belief in rationality is able to explain several empirical puzzles in the empirical asset pricing literature while generating average stock returns, as well as *size* and *value* risk-premia, consistent with the values observed in the calibration sample.

One limitation of the model presented in **Chapter 2** is that it takes for granted the existence of a *stochastic discount factor* (SDF) to price stocks once companies have paid dividends to their existing shareholders. The existence of an SDF is often motivated by the absence of arbitrage opportunities; yet, this is another example of the axiomatic characterization of asset prices, just as WE. If we look at what happens in the real world, a person in charge of rebalancing a portfolio of stocks does not know the execution prices of the orders that she sends to the market, and may not even be sure that the orders will be executed in full, that is, there are no truly price-taking agents. In this regard, the vast literature of trading games that emerged between the end of the 1970s and the beginning of the 1990s provided a link between the set of Nash equilibria of a trading game and that of the Walrasian equilibria of the corresponding market economy, but it remained silent on the epistemic conditions supporting Nash Equilibrium outcomes in trading games. For this reason, in **Chapter 3** we consider a market economy with state-contingent claims and, following Dubey (1982), a class of trading games describing *auctions with limit orders and price priority* for those assets, in which the market price of each asset is determined as the bid price of the lowest filled buyer. In this class of games, which describes a trading protocol similar to that adopted by many stock markets to organize *closing auctions*, we show that every profile of actions is rationalizable. Hence, (Bayesian) Nash equilibria cannot be motivated by identifying sufficient conditions for the existence of a unique rationalizable outcome, imposing the need of a different approach from that followed in **Chapter 2**. For this reason, we introduce the notion of *mutual  $\pi$ -belief in Walrasian*

*markets*, which, roughly speaking, identifies an epistemic state in which every player believes that, in her absence, financial assets will be priced competitively; with some degree of approximation, it can also be interpreted as a belief in the lack of arbitrage opportunities. Assuming mutual  $\pi$ -belief in Walrasian in markets, we show under which conditions the players' rational behavior results in competitive asset prices, that is, financial assets being priced as in a WE, providing an epistemic characterization of SDFs. In particular, we will show that the result depends on the payment rule characterizing the game and on the initial distribution of assets (i.e., the players' endowments). While in *pay-as-bid* auctions rationality and mutual  $\pi$ -belief in Walrasian markets always result in competitive prices if the initial allocations are sufficiently different from those of a WE, the result breaks down in *pay-as-clear* auctions. In particular, we find that, in pay-as-clear auctions, the joint hypothesis of rationality and mutual  $\pi$ -belief in Walrasian markets is also consistent with incautious actions inflating asset prices. In this regard, a notable example of incautious actions is the use of *market orders*, particularly on the side of buyers, from which we can also draw interesting *normative* conclusions for the microstructure of financial markets.

To the best of my knowledge, the results contained in chapters 2 and 3 are novel in the literature and find a *locus* in the intersection between game theory and finance. We also believe that our results may be helpful in *microfounding* agent-based models in finance and linking financial economics to artificial intelligence.

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## CHAPTER 1

# Epistemic Game Theory

### 1. Introduction

Modern asset pricing, both *demand side* (consumption based) and *supply side* (investment-based), is based on the general equilibrium theory of Arrow and Debreu (1954). General equilibrium is an axiomatic approach to characterize the outcome of a price-mediated economy, and it is tacit about how trades take place and the way individuals reason and take actions. Based on this observation, why should we believe in general equilibrium models? Several answers have been provided for the specific case of trading in consumption goods, although with a traditional game-theoretical approach, of which notable examples are Shapley and Shubik (1977), Dubey and Shubik (1978, 1980), Dubey (1982) and Sahi and Yao (1989), among others. The purpose of this thesis is to revisit static asset pricing models from an epistemic, game-theoretical perspective. The difference between traditional and epistemic game theory is that while the former assumes *Nash equilibrium*<sup>1</sup> behavior to abstract from the potential uncertainty that a player faces about the actions played by her opponents, the latter considers the reasoning process through which each player attempts to deal with such uncertainty. This is a crucial aspect, especially in trading games and, consequently, in asset pricing. As Brandenburger (2014) observes, the epistemic research program puts back (Bayesian) decision theory at the core of economic models, as originally advocated by von Neumann and Morgenstern (1944) in their pioneering work in the theory of games. In this regard, the scope of this chapter is to provide a synthetic but comprehensive overview of epistemic game theory. The remainder of this section offers a mathematical preamble which summarizes some essential definitions and results, from the theory of sets to general topology and measure theory. **Section 2** introduces the definition of static economic models and games, while **Section 3** presents the construction of hierarchies of beliefs and epistemic type structures, the canonical tools to express, in a formal way, statements such as “player  $f$  believes that her opponents do something” and “player  $f$  believes that her opponents believe something else”. In this way, it is possible to characterize the outcome of a

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<sup>1</sup>See **Section 4.1** for a precise definition of equilibrium as a solution concept for a static game.

game starting from the preferences and the (assumed) reasoning process of the players. Different assumptions lead to different solution concepts, and the most relevant are examined in **Section 4**, namely, Nash equilibrium and rationalizability. **Section 5** is dedicated to establishing a first link between the theory of games and that of general equilibrium, a crucial point given our interest in asset pricing models. Finally, **Section 6** briefly describes the content of the next two chapters. Both focus on static games, leaving the study of dynamic models for future research. There are in fact technical complications when dealing with dynamic models from an epistemic perspective that could be addressed only after a thorough analysis of static games.

**1.1. Mathematical preamble.** This subsection is dedicated to refresh or clarifying several mathematical aspects that are particularly relevant for game-theoretic models. The relevant references for this section are Halmos (1958), Bourbaki (1968, 1987), MacLane (1969), Bernardo and Smith (1994), Aliprandis and Border (1996), Kelley (2005), Munkres (2014), Rao (2016) and Battigalli et al. (2024a, 2024b). We review some elementary theory of sets in **Section 1.1.1**, while in **Section 1.1.2** we recall the notion of a product of a family of sets. In **Section 1.1.3** we review some essential elements of general topology, while in **Section 1.1.4** we define vectors and matrices as real-valued selections on arbitrary finite sets and discuss how to generalize standard linear algebraic operations. In **Sections 1.1.5-6** we review some essential aspects of measure and probability theory that are fundamental to properly formulate concepts such as Bayesian decision makers and hierarchies of beliefs. Finally, **Section 1.1.6** provides a brief overview of the general structure of the *decision* (i.e. *optimization*) problems that we shall discuss in the remainder of the thesis. The preamble has the ambition to be a synthetic and comprehensive summary of the most "delicate" mathematical aspects of game-theoretic models. In addition, the preamble serves as a general occasion to clarify the notation and conventions that will be used later. In this regard, as in most of the literature and mathematical textbooks, the meaning of *if* in a *definition* should be intended, with abuse of language, as *if and only if* (i.e., *iff*); unless the context may be subject to interpretation, we will stick to this abuse of language for ease of exposure. In addition, when a definition is evident from the context, we simply use  $=$  instead of  $:=$ .

The reader may wonder why this preamble is needed, since this is after all a doctoral thesis. We realized that after some years, many definitions and results are taken a bit too much for granted, with their nuisance getting blurred as the mind becomes inevitably more devoted to the applications of higher personal interest. Also, notation and conventions may vary across fields, even when

contiguous or falling under the same discipline. Finally, we are of the opinion that a bit of math is never bad. *One theorem per day keeps boredom away.*

1.1.1. *Set theory.* Game-theoretic models are an application of the theory of sets of Zermelo and Fraenkel, of which axioms, as well as the axiom of choice, are assumed to be valid throughout the chapters; put together, these axioms provide the basis for the ZFC theory of sets, which is the most commonly adopted in applied mathematics. A central role in ZFC is played by its language, which consists of logical symbols (e.g.,  $\wedge$  for the logic AND), symbols for logic relations (e.g.,  $=$  to denote equality), punctuation (curved parentheses and commas) and variables, denoted by characters, possibly marked with subscripts and superscripts. Together, they are known as syntactical elements. A *formula* is a *string* composed of a finite number of syntactical elements. A *semantic* specifies how formulas are given meaning through interpretations, and we generally use formulas to represent concepts such as properties of *objects* or relations among them. In this regard, a formula is well-formed if it is grammatically correct, that is, it makes correct use of the syntactical elements. A formula that involves an input variable and an output variable obtained by applying a certain syntax is *well-defined* if it returns a unique output for a given input, regardless of how the latter is represented syntactically. Informally, we view a *set* as a container of distinct objects, and we take for granted basic operations such as unions and intersections. We generally use a capital letter to denote a set, and the corresponding small letter to denote a typical element. To avoid confusion, we also say that  $A$  is a set with a typical element  $a \in A$ . Two sets are equal if they contain the same elements; therefore, the order in which elements may be listed is irrelevant. We denote by  $|X|$  the *cardinality* of a given set  $X$ ; if  $X$  has a finite number of elements, say  $n \in \mathbb{N}$ ,  $|X| = n$  and we say that  $X$  is finite or a finite set. Given two sets  $A$  and  $B$ , the (Kuratowski) *ordered pair*  $\langle a, b \rangle$ , where  $a \in A$  and  $b \in B$ , is defined as  $\langle a, b \rangle := \{a, \{a, b\}\}$ ;  $\langle a, b \rangle$  carries the canonical interpretation of a pair of elements appearing in the specific order in which  $b$  is preceded by  $a$ . Given two sets  $A$  and  $B$ , the *ordered binary Cartesian product*, denoted with  $A \times B$ , is the set of all ordered pairs  $\langle a, b \rangle$ , where  $a \in A$  and  $b \in B$ , that is,  $A \times B := \{\langle a, b \rangle : a \in A \wedge b \in B\}$ . Given two sets  $A$  and  $B$ , a *binary relation*  $R$  between  $A$  and  $B$  is a subset of  $A \times B$ ; we also say that  $R$  is a relation for short. For notational convenience, we also write  $aRb$  to express the predicate  $\langle a, b \rangle \in R \subseteq A \times B$ . Given a set  $X$ , a binary relation on  $X$  is a subset of  $X \times X$ . A binary relation on  $X$  is said to be:

- *complete* or *total* if, for every  $x, x' \in X$ ,  $xRx'$  or  $x'Rx$ ;
- *reflexive*, if for every  $x \in X$  we have  $xRx$ ;

- *transitive*, if for every  $x, x', x'' \in X$  such that  $xRx'$  and  $x'R x''$ , then  $xRx''$ ;
- *euclidean*, if for every  $x, x', x'' \in X$  such that  $xRx'$  and  $xRx''$  then  $x'R x''$ ;
- *symmetric*, if for every  $x, x' \in X$ ,  $xRx' \implies x'R x$ .

An *equivalence* relation is a reflexive and euclidean relation, and a relation is reflexive and euclidean if and only if it is reflexive, symmetric, and transitive. A *weak order* on  $X$  is a complete and transitive binary relation on  $X$ ; a complete relation is reflexive, which implies that a weak order is reflexive. As the name suggests, a weak order on  $X$  can be used to establish an order of the elements of the same set. To give an example, utility functions in neoclassical consumer theory are obtained from assuming preferences consistent with a weak order over different bundles of consumption goods. More generally, a directed set is a pair  $(X, \preceq)$  where  $X$  is a set and  $\preceq$  is a binary, reflexive, transitive and directed relation on  $X$ ; a directed relation  $\preceq$  on  $X$  is a weak order if and only if it is total, that is,  $x \preceq y$  or  $y \preceq x$  for every  $x, y \in X$ . In this regard, for  $n \in \mathbb{N}$ , we denote by  $[n]$  the set of the first  $n$  natural numbers directed by their usual order. We also say that a binary relation  $\preceq$  is symmetric if  $x \preceq y \implies y \preceq x$ , and antisymmetric if  $x \preceq y \wedge y \preceq x \implies x = y$ . A binary relation is a total order if it is complete, reflexive, transitive, and antisymmetric.

Given a set  $X$ , we denote by  $2^X$  the set of all subsets of  $X$ ; the notation is motivated by the following result: every finite set  $X$  has  $2^{|X|}$  distinct subsets. A *correspondence* is a triplet  $\phi = (X, Y, R)$  where  $X$  and  $Y$  are two sets and  $R \subseteq X \times Y$  a binary relation, which is said to be the *graph* of  $\phi$  and often denoted as  $\text{Gr } \phi$ . A correspondence  $\phi = (X, Y, R)$  is often represented as

$$\phi : X \rightrightarrows Y$$

$$x \mapsto \phi(x),$$

intending  $\phi$  as a formula mapping  $x$  to an element of  $2^Y$ . A *function*, or *mapping*,  $f$  from the set  $X$  (*domain of  $f$* ) to the set  $Y$  (*codomain of  $f$* ) is indicated as  $f : X \rightarrow Y$ , and is a relation between  $X$  and  $Y$  such that  $xfy \wedge x fz \implies y = z$ . Any relation satisfying the previous property is said to be injective; hence, a function is an injective relation. Notice that a necessary condition for a formula to define a function is that of being well-formed and well-defined. An equivalent notation for a function  $f$  is also

$$f : X \rightarrow Y$$

$$x \mapsto f(x),$$

intending  $f(\cdot)$  as a formula that associates one and only one element  $y \in Y$  with an element  $x \in X$ ; with abuse of language, it is customary to use the term function to denote the corresponding formula and, abusing notation, write  $y = f(x)$ . Given the sets  $X, Y$  and  $Z$ , and two functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , we denote by  $g \circ f$  the *composite function*

$$h : X \rightarrow Z$$

$$x \mapsto g(f(x)).$$

Given a function  $f : X \rightarrow Y$ , we denote by  $f(X)$  the image set of  $f$ , defined as

$$f(X) = \{y \in Y : \exists x \in X, y = f(x)\}.$$

The set of all functions from  $X$  to  $Y$  is denoted by  $Y^X$ . A function  $f : X \rightarrow Y$  is *injective* or *one-to-one* if  $xfy \wedge zfy \implies x = z$ , and *surjective* or *onto* if  $\forall y \in Y, \exists x \in X, xRy$ ; if  $f$  is injective (resp. surjective), it is also said that  $f$  is an injection (resp. surjection). A function  $f : X \rightarrow Y$  is *bijective* if it is one-to-one and onto, and it is also called a bijection. Given two sets  $X$  and  $Y$  and a relation  $R \subseteq X \times Y$ , the image set of  $X$  under  $R$  is  $R(X) := \{y \in Y : \exists x \in X, \langle x, y \rangle \in R\}$ ; we denote by

$$R^{-1}(Y) := \{x \in X : \exists y \in Y, \langle x, y \rangle \in R\}$$

the inverse image set of  $Y$  under  $R$ . Similarly, for  $A \subseteq X$ ,  $R(A) := \{y \in Y : \exists x \in A, \langle x, y \rangle \in R\}$ , and for  $B \subseteq Y$ ,

$$R^{-1}(B) := \{x \in X : \exists y \in B, \langle x, y \rangle \in R\}.$$

We say that  $f$  is invertible if there exists a function  $g : R(X) \rightarrow X$ , known as the inverse of  $R$ , such that  $x = g(y) \iff y = R(x)$ . It is immediate to verify that a function admits on its whole co-domain an inverse if and only if it is a bijection. Given a set  $X$  and a subset  $A$  of  $X$ , we denote by  $\mathbb{I}_A : X \rightarrow \{0, 1\}$  an indicator function, that is, a function that maps  $x$  to 1 if  $x \in A$  and 0 otherwise. Two sets  $X$  and  $Y$  are said to be isomorphic if there exists a bijection  $f$  between  $X$  and  $Y$  that preserves one or more properties of interest; we write  $X \cong Y$  or  $Y \cong X$  and say that  $f$  and its inverse  $f^{-1}$  are isomorphisms between  $X$  and  $Y$ . A function  $f : X \rightarrow Y$  is called an *embedding* of  $X$  in  $Y$  if the function  $g : X \rightarrow f(X)$ ,  $x \mapsto g(x) = f(x)$  is an isomorphism between  $X$  and  $f(X)$ ; we also say that  $f$  embeds the set  $X$  in  $Y$ . An embedding is said to be canonical if it is generated by a canonical isomorphism. An embedding allows one to establish an equivalence relation between a set  $X$  and subset of another set  $Y$ ; for this reason,  $X$  is often *identified* as a subset of  $Y$  if there

is a canonical embedding between the same two sets.

It is immediate to show that the ordered Cartesian product is neither exactly commutative nor exactly associative, that is,  $A \times B \neq B \times A$  and  $(A \times B) \times C \neq A \times (C \times B)$  respectively. However, it is immediate to show the existence of a bijection preserving the "unpacked" or "flattened" content; hence  $A \times B \cong B \times A$  and  $(A \times B) \times C \cong A \times (C \times B)$  to the extent that the order and structure of the previous sets are not considered relevant properties. In other words, the ordered Cartesian product is commutative and associative up to an isomorphism. In general, we write  $S_1 \times S_2 \times \dots \times S_n$ ,  $n \in \mathbb{B}$  meaning that  $S_1 \times (S_2 \times \dots \times S_n)$ , and, proceeding by induction, we obtain a set of nested ordered pairs; an element of  $S_1 \times S_2 \times \dots \times S_n$  is denoted as  $\langle s_1, s_2, \dots, s_n \rangle$  and is said to be a  $n$ -ordered tuple, or just ordered tuple. Given a set  $A \subseteq \mathbb{R}$ , we say that  $a^*$  is the supremum (resp. infimum) of  $A$  if  $a^*$  is the smallest (resp. largest) real number such that for every  $a \in A$ ,  $a \leq a^*$  (resp.  $a^* \leq a$ ), where  $\leq$  denotes the usual direction (i.e. directed relation) on real numbers. If  $a^* \in A$  we say that  $A$  has a maximum (resp. minimum) and that  $a^*$  is the maximum (resp. minimum) of  $A$ . Given a function  $f : X \rightarrow \mathbb{R}$ , we say that  $f$  has a maximum in  $B \subseteq X$  if the image set  $f(B)$  has a maximum in  $y^*$ , and denote by  $\operatorname{argmax}_{x \in B} f(X)$  the set  $f^{-1}(\{y^*\})$ , which corresponds to the subset of  $X$  such that each of its elements is assigned to  $y^*$ . We also write  $y^* = \max_{x \in B} f(B)$  and say that  $y^*$  is the maximum of  $f$ . If  $B \subset X$  we speak of a constrained maximum, and a non-constrained, or global, maximum otherwise. The definitions for the minimum, supremum, and infimum of a function are analogous. As a matter of notation, we adopt the usual convention to denote with  $[a, b]$  where  $a, b \in \mathbb{R}$ , the set  $\{x \in \mathbb{R} : x \leq b \wedge a \leq x\}$ . We also write  $x \geq a$  meaning that  $a \leq x$ , and  $a \leq x \leq b$  for  $x \leq b \wedge a \leq x$ , as usual. Needless to say, we follow all the other usual conventions to denote algebraic operations between real numbers. We take for granted the notion of induction and transfinite induction.

**1.1.2. Product of a family of sets.** We let  $I$  be a countable set,  $\{X_i\}_{i \in I} := \{X_i : i \in I\}$  a family of sets indexed by  $I$  and  $\bar{X} := \cup_{i \in I} X_i$ . Given a family  $\{X_i\}_{i \in I}$ , a selection or *profile* is a function  $x : I \rightarrow \bar{X}$  such that  $i \mapsto x_i \in X_i$  for every  $i \in I$ , and we write  $x = (x_i)_{i \in I}$ . Notice that a selection always exists by the *axiom of choice* that we assumed to be valid in precedence. Given a family  $\{X_i\}_{i \in I}$ , we denote by  $X$  or  $\prod_{i \in I} X_i$  the set of selections defined on  $I$ , that is,

$$X = \prod_{i \in I} X_i := \{x \in \bar{X}^I : \forall i \in I, x_i \in X_i\}.$$

We also refer to  $X$  as the product of the family of sets  $\{X_i\}_{i \in I}$  or unordered Cartesian product of  $\{X_i\}_{i \in I}$ . With an abuse of notation, we also write  $\times_{i \in I} X_i = \prod_i X_i$ . It is immediate to show that the

product of a family of set is commutative up to isomorphism, that is, for every bijection  $\sigma : I \rightarrow I$ ,  $\prod_{i \in I} X_i$  is isomorphic to  $\prod_{i \in I} X_{\sigma(i)}$ . Given a subset  $J$  of  $I$ , and a family  $\{X_i : i \in I\}$ ,  $J \subseteq I$ , we let  $X_J = \prod_{j \in J} X_j$  the set of selections *restricted* to  $J$ , that is,

$$X_J := \prod_{j \in J} X_j := \{x_J \in (\cup_{i \in J} X_i)^J : \forall i \in J, x_J(i) \in X_i\};$$

notice that, if  $I = J$ , we obtain  $X_I = \prod_{i \in I} X_i = X$ . For every  $i \in I$ , we define the function

$$\text{proj}_i : \prod_{i \in I} X_i \rightarrow \bar{X}$$

$$(x_i)_{i \in I} \rightarrow x_i,$$

which is said to be a canonical projection mapping. Given a subset  $J$  of  $I$ ,  $J \subseteq I$ , the projection function of  $X$  *onto*  $X_J$  is the mapping

$$\text{proj}_{X_J} : \prod_{i \in I} X_i \rightarrow \prod_{i \in J} X_i$$

$$(x)_{i \in I} \rightarrow (x)_{i \in J}.$$

Using projections, it is easy to verify that, for every  $J \in 2^I$ , there exists an isomorphism between  $\prod_{i \in I} X_i$  and  $\prod_{i \in J} X_i \times \prod_{i \in I \setminus J} X_i$ , as well as between  $\prod_{i \in I} X_i$  and  $\prod_{i \in I \setminus \{j\}} X_i \times X_j$  for  $j \in I$ , since for every  $i \in I$ ,  $X_i$  is also isomorphic to  $X_{\{i\}} = \prod_{i \in \{i\}} X_i$ . Thus, the product of a family of sets is commutative and associative up to isomorphism.

Consistent with the game theoretical literature, we follow the convention stipulated in Section 4.1 of Chapter 1 in Bourbaki (1987) to identify products of sets up to canonical bijections. In particular, we identify the product of two families of sets by the product of a family indexed by the product of the respective indices (i.e., by means of a unique index for all the sets involved in the product). For this reason, several abuses of notation are accepted and will be recurring in this thesis, consistent with the literature; in particular, abusing notation, we write  $\prod_{i \in I} X_i = \prod_{i \in J} X_i \times \prod_{i \in I \setminus J} X_i$  and  $\prod_{i \in I} X_i = \prod_{i \in I \setminus \{j\}} X_i \times X_j$ , where  $J \in 2^I$  and  $i \in I$  respectively. In this regard, we also let  $-i := I \setminus \{i\}$ ; thus  $X = X_i \times X_{-i} = X_{-i} \times X_i$ . In addition, given two disjoint subsets  $I'$  and  $I''$  of  $I$ , we denote by  $(x_{I'}, x_{I''})$  a selection  $x_K \in X_K$ , where  $K := I' \cup I''$ , such that  $x_K(i) = x_{I'}(i)$  if  $i \in I'$  or  $x_K(i) = x_{I''}(i)$  if  $i \in I''$ ; notice that  $(x_{I'}, x_{I''}) = (x_{I''}, x_{I'})$  as the order in which the two restricted selections appear is irrelevant to identify the corresponding selection on  $I' \cup I''$ . When  $I' = \{i\}$  for some  $i \in I$ , with abuse of notation, we also write  $x = (x_i, x_{-i})$  instead of  $(x_{\{i\}}, x_{I \setminus \{i\}})$ .

Consequently, given a function  $f : X \rightarrow Y$ , where  $X = \prod_{i \in I} X_i$ , we write equivalently

$$\begin{aligned} f &: X_i \times X_{-i} \rightarrow Y \\ (x_i, x_{-i}) &\mapsto f(x_i, x_{-i}). \end{aligned}$$

That is, the functions of more variables are viewed as functions of selections. In this regard, the section of  $f : X \rightarrow Y$  at  $x_i \in X_i$  is defined as a function  $f_{x_i} : X_{-i} \rightarrow Y$  such that  $x_{-i} \mapsto f(x_i, x_{-i})$ . The definition also extends to the subsets of  $I$ ; given a set  $J \subseteq I$ ,

$$\begin{aligned} f_{x_J} &: X_{-J} \rightarrow Y \\ x_{-J} &\mapsto f(x_J, x_{-J}), \end{aligned}$$

where  $-J$  is a short-hand notation for  $I \setminus J$ . This notation is very convenient as it allows us to fix one or more components of a profile and evaluate the range of values that can be achieved for a certain function given the values taken by the others.

EXAMPLE. Given two families  $\{X_i\}_{i \in I}$  and  $\{Y_i\}_{i \in I}$ , there is a canonical bijection between the nested or iterated product  $\prod_{j \in \{1,2\}} Z_j$ , where  $Z_1 = \prod_{i \in I} X_i$  and  $Z_2 = \prod_{i \in I} Y_i$ , and the flattened product  $\prod_{s \in \{1,2\} \times I} Z'_s$ , where, for every  $i \in I$ ,  $Z'_{\langle 1,i \rangle} = X_i$  and  $Z'_{\langle 2,i \rangle} = Y_i$ . In this regard, we often abuse the notation and write  $Z_1 \times Z_2 = \prod_{i \in I} (X_i \times Y_i)$  instead of  $Z_1 \times Z_2 \cong \prod_{i \in I} (X_i \times Y_i)$ . Such an abuse is frequent in the construction and application of epistemic type structures.

1.1.3. *General topology.* Literally speaking, the word topology means the study of places. In mathematics, topology generalizes the familiar notions of convergence and approximation in metric spaces, and it plays a major role in establishing many important results in game theory. To grasp the idea of why topology is so relevant, let us consider a collection of distinct objects, such as cars. When we deal with a set of numbers, we find it intuitive to define the notion of distance of two points, and, accordingly, that of open and closed sets, from which follow those of neighborhood of a point, limit of a sequence, and so on. However, in the case of sets such as one composed of cars or apples, there might not be a unique and natural way, or a way at all, to introduce a notion of distance. Besides, as any (valid) notion distance remains somewhat arbitrary, so is the corresponding definition of an open set that eventually commands results about approximation and convergence. For this reason, in topology we start from defining open sets.

Given a generic set  $X$ , a topology of  $X$  is a collection  $\tau := \{S : S \subseteq X\}$  of subsets of  $X$  such that:

- $X, \emptyset$  are both in  $\tau$ ;

- for every finite family of subsets of  $\tau$  their intersection is in  $\tau$ ;
- for every arbitrary family of subsets of  $\tau$  their union is in  $\tau$ .

We say that a topology of  $X$  is closed under arbitrary unions and finite intersections. A topological space is a pair  $(X, \tau)$  where  $\tau$  is a topology of  $X$ . Given a topological space  $(X, \tau)$ , a subset of  $C$  of  $X$  is open in the  $\tau$ -topology if  $C \in \tau$ . Abusing language and notation, we also say that  $X$  is a topological space, when there is no need to be explicit about  $\tau$ . A subset  $C$  of  $X$  is closed in the  $\tau$ -topology if its complement with respect to  $X$  is an element of  $\tau$  (i.e. a closed set is the complement of an open set). A set can be open and closed in the same topology, and we say that it is clopen; it is immediate to check that a clopen set always exists, as  $X$  and  $\emptyset$  are both open and closed, one being the complement of the other and both included in every topology of  $X$ .

As said, a topology is introduced to generalize the definition of a neighborhood of a point for arbitrary sets. Recall that, given a set  $X$ , a semi-metric is a function  $d : X \times X \rightarrow \mathbb{R}$  such that  $d(x, y) \geq 0$ ,  $d(x, x) = 0$  and  $d(x, y) = d(y, x)$ , and that a semi-metric  $d$  is said to be a metric if  $d(x, y) = 0 \implies x = y$ . Every semi-metric induces a topology. Let  $I$  be a finite set of  $n \in \mathbb{N}$  elements; given a set  $X \subseteq \mathbb{R}^I$ , the Euclidean distance of two points  $x, y \in X$ , is  $d_e(x, y) = \sqrt{\sum_{i \in I} (x_i - y_i)^2}$  and the topology generated by  $d_e$  is the collection of open sets  $B(x, \varepsilon) = \{x' \in X : \sum_{i \in I} (x_i - x'_i)^2 < \varepsilon^2\}$  for  $\varepsilon \in \mathbb{R}$ , which corresponds to the usual definition of family of open balls in real analysis. Given a semi-metric  $d$  defined on a set  $X$ , we denote by  $\tau_d$  the topology generated by  $d$  on  $X$ ; in particular, given a finite set  $I$ , the topological space  $(\mathbb{R}^I, \tau_{d_e})$  is said to be Euclidean, and, abusing notation, we also say that  $\mathbb{R}^I$  is an Euclidean space to refer to  $(\mathbb{R}^I, \tau_{d_e})$ .

When we cannot or do not want to specify a metric for a given set  $X$ , we can define, more generally, the neighborhood of a point  $x \in X$  as a set  $U \subseteq X$  such that  $x \in U$  and  $U \in \tau$ ; hence, every open set is a neighborhood of its elements. A topological space  $(X, \tau)$  is said to be metrizable if there exists a metric  $d : X \times X \rightarrow \mathbb{R}$  that generates the topology  $\tau$  on  $X$ . A topological space  $(X, \tau)$  is said to be Hausdorff, or  $T_2$ , if for every two points  $x$  and  $y$  in  $X$ , there exists a neighborhood  $U$  of  $x$  and a neighborhood  $V$  of  $y$  with respect to the topology  $\tau$  such that  $U \cap V = \emptyset$ . We also say that  $X$  is Hausdorff, with the usual abuse of notation. A metric space is a pair  $(X, d)$  where  $d : X \times X \rightarrow \mathbb{R}$  is a metric on  $X$ ; with abuse of notation, we also say that  $X$  is a metric space when we do not need to be explicit about  $d$ . It is immediate to show that every metric space is a topological Hausdorff space with respect to the topology generated by the corresponding metric. Every metric space  $(\mathbb{R}^I, d_e)$ , where  $I$  is a finite set, is said to be Euclidean.

Given a metric space  $(X, d)$ , a sequence is a function  $\chi : D \rightarrow X$ , where  $D \subseteq \mathbb{N}$ . A sequence  $\chi$  is Cauchy in  $(X, d)$  if  $D = \mathbb{N}$  and  $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, d(\chi(m), \chi(n)) \leq \varepsilon$ . A sequence  $x$  converges to a point  $x^*$  if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(\chi(n), x^*) \leq \varepsilon$ ; we also write  $\lim_{n \rightarrow \infty} x = x^*$  or  $x \rightarrow x^*$  for  $n \rightarrow \infty$ , following the usual convention. Let  $(I, \preceq)$  be a directed set; given a topological space  $(X, \tau)$ , a net is a function  $\chi : I \rightarrow X$  and converges to a point  $x^*$  in  $X$  if for every open neighborhood  $U$  of  $x^*$  there exists  $i \in I$  such that  $\chi(i) \in U$  and  $i' \preceq i \implies \chi(i') \in U$ ; we also write  $\chi \xrightarrow{\tau_x} x^*$  to stress that convergence depends on the specific topology considered. In other words, a net generalizes the notion of a sequence in metric spaces. A metric space is said to be complete if every Cauchy sequence is convergent, while a topological space  $(X, \tau)$  is said to be completely metrizable if  $\tau$  is generated by a metric  $d$  and the corresponding metric space  $(X, d)$  is complete. A topological space  $(X, \tau)$  is said to be separable if there exists a net  $a : \mathbb{N} \rightarrow X$ , where  $\mathbb{N}$  is directed by its usual order, such that for every  $b \in X$  and every neighborhood  $U$  of  $b$ , there exists  $i \in \mathbb{N}$  such that  $a(i) \in U$ . A topological space  $(X, \tau)$  is said to be Polish if it is separable and completely metrizable; with the usual abuse of language notation, we also say that  $X$  is Polish when we do not need to be specific on the topology  $\tau$  that makes  $(X, \tau)$  Polish, and that  $\tau$  is a Polish topology of  $X$ .

Given a topological space  $(X, \tau_X)$ , an open cover of  $X' \subseteq X$ , with respect to the topology  $\tau_X$ , is a collection of open sets  $\{U_i : i \in I\}$ , where  $I$  is a generic index set such that  $X' \subseteq \bigcup_{i \in I} U_i$ . We say that  $X'$  is compact in  $\tau_X$  if every open cover has a finite subcover, that is, if every open cover of  $X'$  is included in a finite cover of  $X'$ . A topological space  $(X, \tau_X)$  is compact if  $X$  is compact in  $\tau_X$ , and it is easy to verify that every closed subset of a  $X$  is compact in  $\tau_X$  if  $X$  is compact in  $\tau_X$ . In this regard, we also recall two notable results. The first, known as the *finite intersection property* of compact sets, is that every countable (finite or infinite) family of closed subsets with pairwise non-empty intersection has non-empty intersection. The second is that, by the Heine-Borel theorem, a subset of  $\mathbb{R}^n$  is compact with respect to the topology generated by the Euclidean metric if and only if it is closed and bounded.

Given two topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ , and a function  $f : X \rightarrow Y$ , we say that  $f$  is continuous at a point  $x \in X$  if for every neighborhood  $V$  of  $f(x)$ ,  $f^{-1}(V) \subseteq U$  for  $U$  such that  $x \in U$  and  $\exists T \in \tau_X, U \subseteq T$ , that is,  $f$  is said to be continuous at  $x$  if the inverse image of every neighborhood of  $f(x)$  is a neighborhood of  $x$ ;  $f$  is said to be continuous if for every open set  $V \subseteq Y, f^{-1}(V) \in \tau_X$ . Observe that a function  $f : X \rightarrow Y$  may be continuous or not

depending on the topologies chosen for  $X$  and  $Y$ . A homeomorphism<sup>2</sup> between  $(X, \tau_X)$  and  $(Y, \tau_Y)$  is a bijection  $f : X \rightarrow Y$  such that  $f$  and its inverse  $f^{-1}$  are continuous; a continuous bijective function with continuous inverse is also said to be *bicontinuous*. In other words, a homeomorphism is an isomorphism that preserves topological properties; in particular, the image of an open set  $U \in \tau_x$  is an open set in  $Y$ , that is,  $f(U) \in \tau_Y$ , and the inverse image of an open set  $V$  in  $U$  is open in  $\tau_X$ , that is,  $f^{-1}(V) \in \tau_X$ . It follows that a homeomorphism preserves the compactness and convergence of nets, that is,  $X' \subseteq X$  is compact in  $\tau_X$  if and only if  $f(X)$  in  $\tau_Y$ , and a net  $x$  converges in  $\tau_X$  if and only if the net generated by the image of each element of  $x$  under  $f$  converges in  $\tau_Y$ . A topological embedding is an embedding that is bicontinuous.

Given a set  $X$ , the topology  $\tau$  of  $X$  is said to be coarser than the topology  $\tau'$  of  $X$  if  $\tau \subseteq \tau'$ ; since every topology is closed under arbitrary union,  $\tau$  is coarser than  $\tau'$  if every open set in  $\tau$  is open in  $\tau'$ . Given two topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ . We also say that  $\tau'$  is finer than  $\tau$ . Given two topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ , and a family of functions  $\{f_i : X \rightarrow Y\}_{i \in I}$ , where  $I$  is a generic set, the weak topology of  $X$  generated by  $\{f_i : X \rightarrow Y\}_{i \in I}$  is the coarsest topology of  $X$  such that, for every  $i \in I$ ,  $f_i$  is continuous. If  $X$  is equipped with the weak topology generated by  $\mathcal{F} = \{f_i : X \rightarrow Y\}_{i \in I}$ , where  $(Y, \tau_Y)$  is a given topological space, a net  $\chi$  on  $X$  converges to a point  $x^* \in X$  if and only if, for every  $f \in \mathcal{F}$ , the net  $f \circ \chi$  converges to  $f(x^*)$  in  $\tau_Y$ . Given a family of sets  $\{X_i\}_{i \in I}$ , the product topology of the set of selections  $\times_{i \in I} X_i = \Pi_{i \in I} X_i$  on  $I$  is the weak topology generated by the projection mapping family  $\{\text{proj}_{X_i}\}_i$ . Under the product topology, the product of a family of sets is associative up to homeomorphism, hence the identification of Cartesian products up to bijections adopted in the previous section. By Tychonoff's product theorem, given a family of compact spaces  $\{(X_i, \tau_{X_i})\}_{i \in I}$ ,  $\Pi_{i \in I} X_i$  is compact in the product topology. The product of sets, each endowed with a Polish topology, is also Polish with respect to the corresponding product topology. As remarked in Aliprantis and Border (2006), weak topologies and topologies generated by a metric are generally sufficient in applications. In other words, most of the sets in which we may be interested are usually endowed with a weak or norm topology; in this sense, we speak of "usual" or "natural" topologies. For a finite set, the natural or usual topology is the one including each element of the same set, which is known as discrete topology.

1.1.4. *Vector spaces and linear algebra.* Let  $n$  be a natural number and consider the sets  $\mathbb{R}^{[n]}$  and  $\mathbb{R}^n$ . The former is the set of functions  $f : [n] \rightarrow \mathbb{R}$ , that is,  $\Pi_{i \in [n]} \mathbb{R}$ , whereas the latter is that

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<sup>2</sup>Not to be confused with an homomorphism.

of the ordered  $n$ -tuples of real numbers, that is, nested ordered pairs of real numbers. Traditionally, linear algebra is built in  $\mathbb{R}^n$ , however, since there is a canonical bijection between  $\mathbb{R}^n$  and  $\prod_{i \in [n]} \mathbb{R}$ , we can easily generalize the concept of vector space and related operations to  $\mathbb{R}^I$ , where  $I$  is an arbitrary finite set. To clarify the distinction, consider the case of  $n = 2$ . Rather than defining a vector as an ordered pair  $\langle x_1, x_2 \rangle = \{x_1, \{x_1, x_2\}\}$  in  $\mathbb{R}^2$ , we can define it, for example, as a function  $x : \{1, 2\} \rightarrow \mathbb{R}$  such that  $i \mapsto x(i) = x_i \in \mathbb{R}$ . Recalling that a function is a relation, a vector is viewed as a set of ordered pairs, such as  $x = \{\langle 1, x_1 \rangle, \langle 2, x_2 \rangle\}$ , instead of as an ordered tuple.<sup>3</sup>

In the remainder of this subsection, we fix two finite sets, denoted by  $I$  and  $J$ , with typical elements  $i$  and  $j$ , respectively; we also define  $m := |I|$  and  $n := |J|$ .<sup>4</sup> If we interpret  $I$  as a *row* index set and  $J$  as a *column* index set, we define:

- a row vector as a function  $x : J \rightarrow \mathbb{R}$  such that  $j \mapsto x(j) = x_j$ ,  $x_j \in \mathbb{R}$ ;
- a column vector as a function  $x : I \rightarrow \mathbb{R}$  such that  $i \mapsto x(i) = x_i$ ,  $x_i \in \mathbb{R}$ ;
- a matrix is as a function  $A : I \times J \rightarrow \mathbb{R}$  such that  $(i, j) \mapsto A(i, j) = a_{i,j}$ ,  $a_{i,j} \in \mathbb{R}$ .

Row and column vectors are real-valued selections, that is, elements of  $\mathbb{R}^I$  and  $\mathbb{R}^J$  respectively. We also write  $A = (a_{ij})_{I \times J}$  instead of  $A = (a_{i,j})_{I \times J}$  for ease of notation; notice the presence or absence of the comma to separate the two indices  $i$  and  $j$ , which we may omit when there is no risk of confusion. We also let  $\mathcal{M}(I, J) := \mathbb{R}^{I \times J}$  be the set of  $I \times J$  matrices, intending that the first argument of  $\mathcal{M}(\cdot, \cdot)$  is the set of row indices. Notice that, since  $\mathbb{R}^n$  is isomorphic to  $\mathbb{R}^J$ , it is customary to abuse notation and write  $x \in \mathbb{R}^n$  even when  $x$  is a selection from  $J$  to  $\mathbb{R}$ ; we shall refrain from this abuse of notation as much as possible, as it can be a source of confusion. Where it is clear from the context, we write  $x \neq y$  as a shorthand notation to denote a pair of vectors  $x, y \in \mathbb{R}^I : \exists i \in I, x(i) \neq y(i)$ . We also write  $x > y$  to denote vectors  $x, y \in \mathbb{R}^I : \forall i \in I, x(i) > y(i)$ ,  $x \geq y$  to denote a vector  $x \in \mathbb{R}^I : x \neq y \wedge \forall i \in I, x(i) \geq y(i)$ . If  $x, y \in \mathbb{R}^{[n]}$ ,  $n \in \mathbb{N}$ , we write  $x \geq^L y$  if  $x_i = y_i$  for every  $i \in [n]$  or if there exists  $m \leq n$  such that  $x_l = y_l$  for every  $l \in [m-1]$  and  $x_m > y_m$ ; the latter case is also specifically identified by writing  $(x_i)_{i \in [n]} >_L (y_i)_{i \in [n]}$ . This is an example of total order on  $\mathbb{R}^{[n]}$ , which is known as lexicographic order.

The sum of two vectors  $x, y \in \mathbb{R}^I$  is defined as  $x + y := (x_i + y_i)_{i \in I} \in \mathbb{R}^I$ , while, for every  $\alpha \in \mathbb{R}$ ,  $\alpha x := (\alpha x_i)_{i \in I} \in \mathbb{R}^I$ . The analog definitions follow if  $x, y \in \mathbb{R}^J$ . Similarly, given two matrices  $A, B \in$

<sup>3</sup>For those familiar with modern programming languages, such as Python, the idea is the same as performing matrix and vector multiplications using `Series` and `DataFrame` in *Pandas*, a Python package to work with tabular data, where the elements to be multiplied are identified by their labels (indices) rather than the order in which they appear. We rely on Python and the class `Pandas DataFrame` to run simulations in **Section 5 of Chapter 2**.

<sup>4</sup>It is customary, although not necessary, to fix  $I = [m]$  and  $J = [n]$  for  $m, n \in \mathbb{N}$ .

$\mathcal{M}(I, J)$ ,  $A + B := (a_{i,j} + b_{i,j})_{I \times J} \in \mathcal{M}(I, J)$  and, for every  $\alpha \in \mathbb{R}$ ,  $\alpha A = (\alpha a_{i,j})_{I \times J} \in \mathcal{M}(I, J)$ . Equipped with these operations, we say that  $\mathbb{R}^I, \mathbb{R}^J$  and  $\mathcal{M}(I, J)$  are vector spaces. To keep the discussion self-contained, we limit our attention to a few additional notions and results. A matrix  $A = (a_{i,j})_{I \times J}$  is square if  $m = n$  and symmetric if  $A = A^T$ . Given a row vector  $x = (x_i)_{i \in I}$ , we denote by  $x \cdot y$  the scalar multiplication of two vectors in  $x, y \in \mathbb{R}^I$ , that is,  $x \cdot y := \sum_{j \in I} x_j y_j$ , which is clearly a commutative operation. The product of a matrix  $A = (a_{i,j})_{I \times J}$  and a vector  $y \in \mathbb{R}^J$  is defined as the column vector  $Ay := \left( \sum_{j \in J} a_{i,j} y_j \right)_{i \in I}$ . Given a vector  $x \in \mathbb{R}^I$  and a matrix  $B = (b_{i,j})_{I \times J}$ , the product of  $x$  times  $B$  is defined as the row vector  $xB := \left( \sum_{i \in I} b_{i,j} x_i \right)_{j \in J}$ ; it follows that  $x \cdot (Ay) = (xA) \cdot y := xAy$ . Given two matrices  $A \in \mathcal{M}(I, J)$  and  $B \in \mathcal{M}(n, m)$ , the product of  $A$  and  $B$  is defined as the matrix  $\left( \sum_{j \in I} a_{i,j} b_{j,i} \right)_{I \times I}$ , and it follows that  $AB = B^T A^T = (BA)^T$ , that is, the product of two matrices is not commutative unless both are symmetric. It is easy to verify that the product of two matrices is associative and distributive. Given a square matrix  $A$ , we write  $A^k$  for  $k \in \mathbb{N}$  to denote the product of  $A$  iterated  $k$  times, e.g.  $A^1 = A$ ,  $A^2 = AA$  and  $A^3 = (AA)A$ . A square matrix  $A : I \times I \rightarrow \mathbb{R}$  is said to be invertible if there exists a matrix  $A^{-1} \in \mathcal{M}(I, I)$  such that  $AA^{-1} = A^{-1}A = \mathcal{I}_I$ , where  $\mathcal{I}_I$  is a matrix, known as the identity matrix (of size  $m$ ), such that  $\mathcal{I}(i, i) = 1$  for every  $i \in I$  and  $\mathcal{I}(i, j) = 0$  for  $i \neq j$ ; evidently,  $\mathcal{I}_m A = A \mathcal{I}_m = A$  for every  $A \in \mathcal{M}(\mathcal{I}, \mathcal{I})$ . A pair  $(A, b)$ , where  $A \in \mathcal{M}(I, J)$ ,  $m \leq n$ , and  $b \in \mathbb{R}^I$  such that  $b \neq 0$ , is a solvable system of linear equations in  $x \in X \subseteq \mathbb{R}^I$  if there exists  $x^* \in X$  such that  $Ax^* = b$ ; if  $X = \mathbb{R}^I$ ,  $(A, b)$  denotes a solvable system if and only there exists a unique  $x^*$  such that  $Ax^* = b$  if and only if  $m = n$  and the matrix  $A$  is invertible.

Given a square matrix  $A \in \mathcal{M}(J, J)$ , we define the determinant of  $A$  as its image through the function  $\det : \mathcal{M}(I, I) \rightarrow \mathbb{R}$  such that  $A \mapsto \det(A) = \sum_{\sigma \in I^I} \text{sgn}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}$  where:

- $\sigma : I \rightarrow I$  is a bijection between  $I$  and itself, known as *permutation* on  $I$ ;
- $\text{sgn}$  (reads: sign) is a function defined on the set of permutations of  $I$  such that  $\text{sgn}(\sigma) = 1$  if there is none or an even number of pairs  $\langle i, j \rangle \in I \times I$  such that  $i < j \implies \sigma(i) > \sigma(j)$  (inversions), and  $\text{sgn}(\sigma) = -1$  otherwise.

A matrix  $A \in \mathcal{M}(I, I)$  is invertible if and only if  $\det(A) \neq 0$ . Given a square matrix  $A \in \mathcal{M}(I, I)$ , we say that  $\lambda \in \mathbb{C}$ , where  $\mathbb{C}$  is the set of complex numbers, is an eigenvalue of  $A$  and  $x \in \mathbb{R}^I$  is the corresponding eigenvector if  $Ax = \lambda x$  and  $x \neq 0$ . An eigenvalue of  $A$  exists if and only if  $(A - \lambda \mathcal{I}_I)x = 0_I$  where  $0_I$  is a vector (that is, a selection) equal to zero for every  $i \in I$ . Given a  $B \in \mathcal{M}(I, I)$  and a vector  $x \in \mathbb{R}^I$ ,  $Bx = 0_I$  if and only if  $\det(B) = 0$ ; therefore, an eigenvalue

of  $\lambda$  is a solution of the equation  $\det(A - \lambda \mathcal{I}_I) = 0$ , which always exists in  $\mathbb{C}$  as  $\det(A - \lambda \mathcal{I}_I)$  is a polynomial of degree  $n$  in  $\lambda$  with coefficients in  $\mathbb{R}$ . Denoting by  $|c| = \sqrt{a^2 + b^2}$  the modulus of a complex number  $c = a + ib \in \mathbb{C}$ , the spectral radius of a matrix  $A \in \mathcal{M}(I, I)$  is the image of  $A$  through  $\varrho : \mathcal{M}(I, I) \rightarrow \mathbb{R}$  such that  $A \mapsto \sup\{\lambda \in \mathbb{C} : \exists x \in \mathbb{R}^I, Ax = \lambda x\}$ , that is, the spectral radius of a square matrix is the least upper bound of the modulus of its complex eigenvalues. For every matrix  $A \in \mathcal{M}(I, I)$ , the spectral radius is bounded above by the maximum of the sum of the absolute values of its row entries, that is,  $\varrho(A) \leq \max\{\sum_{j \in I} |a_{i,j}| : i \in I\}$ . Additional properties of the spectral radius will be discussed in **Chapter 2**.

Let  $A \in \mathcal{M}(J, J)$  be a matrix and  $c \in \mathbb{R}^J$  and denote by  $\mathbb{N}_0$  the set  $\mathbb{N} \cup \{0\}$ . We say that a sequence  $x : \mathbb{N}_0 \rightarrow \mathbb{R}^J$  is a solution of the linear system of difference equations with transition matrix, or homogeneous term,  $A$  and forcing, or inhomogeneous, term  $c$  if, for every  $k \in \mathbb{N}$ ,  $x_{k+1} = Ax_k + c$ . We say that a vector  $x^* \in \mathbb{R}^J$  is a fixed point, or a steady state, of the same system of difference equations if  $x^* = Ax^* + c$ . Observe that, if  $c \neq (0)_{j \in J}$ , and a fixed point  $x^*$  exists, it must be the case that  $x^* = (\mathcal{I}_n - A)^{-1}c$ . Hence, if  $c \neq (0)_{j \in J}$ , a fixed point exists if and only if  $(\mathcal{I}_n - A)$  is invertible, that is, if and only if the eigenvalues of  $A$  have a modulus different from one (i.e. *they do not lie on the unit circle*). In addition, it can be shown that every solution  $x$  of the linear system of difference equations defined by  $A$  and  $c$  converges, with respect to any metric  $d$  defined over  $\mathbb{R}^J$ , to the unique fixed point  $x^*$  of the same system if and only if  $\varrho(A) < 1$ . The usual topology of  $\mathbb{R}^J$  is that induced by the Euclidean metric  $d_e(x, y) := \left(\sum_{i \in I} x(i)^2\right)^{\frac{1}{2}}$ . Every metric space  $(\mathbb{R}^I, d_e)$  is said to be a Euclidean space. It is common to abuse the notation and write  $(\mathbb{R}^n, d)$  or just  $\mathbb{R}^n$  for  $(\mathbb{R}^I, d_e)$ , but we shall refrain as much as possible from this abuse since it can often be misleading in game-theoretic models.

We can use the properties of the generalized Cartesian product, and the notation for selection based on its restrictions, to provide many different representations of the same system of equations. In particular, given a solvable system of equation  $(A, b)$  in  $x \in \mathbb{R}^I$ , for every  $S \subseteq I$ , we can write  $x = (x_S, x_{-S})$  and  $b = (b_S, b_{-S})$ , and  $A = (A_{S \times S}, A_{S \times -S}, A_{-S \times S}, A_{-S \times -S})$ . It follows that  $b_J = A_{J \times J}x_J + A_{J \times -J}x_{-J}$  and  $b_{-J} = A_{-J \times J}x_J + A_{-J \times -J}x_{-J}$ . In this regard, it is also customary to represent the system  $(A, b)$  as

$$\begin{bmatrix} b_J \\ b_{-J} \end{bmatrix} = \begin{bmatrix} A_{J \times J} & A_{J \times -J} \\ A_{-J \times J} & A_{-J \times -J} \end{bmatrix} \begin{bmatrix} x_J \\ x_{-J} \end{bmatrix}.$$

We can also obtain a system of equations  $b = Ay$ , where  $A \in \mathcal{M}(\{1, 2\} \times I, \{1, 2\} \times I)$  and  $x, b \in \mathbb{R}^{\{1, 2\} \times I}$ , combining two subsystems  $\bar{b} = \bar{A} \cdot \bar{x} + \underline{A} \cdot \underline{x}$  and  $\underline{b} = \bar{A}' \cdot \bar{x} + \underline{A}'$ , such that

$$y : \{1, 2\} \times I \rightarrow \mathbb{R},$$

$$(s, i) \mapsto \begin{cases} \bar{x}(i) & s = 1 \\ \underline{x}(i) & s = 2, \end{cases}$$

$$b : \{1, 2\} \times I \rightarrow \mathbb{R},$$

$$(s, i) \mapsto \begin{cases} \bar{x}(i) & s = 1 \\ \underline{x}(i) & s = 2, \end{cases}$$

and

$$A : (\{1, 2\} \times I) \times (\{1, 2\} \times J) \rightarrow \mathbb{R},$$

$$((s, i), (t, j)) \mapsto \begin{cases} \bar{A}_{i,j} & s = 1, t = 1 \\ \underline{A}_{ij} & s = 1, t = 2 \\ \bar{A}'_{i,j} & s = 2, t = 1 \\ \underline{A}'_{ij} & s = 2, t = 2, \end{cases}$$

and, consequently, represent the same system as

$$\begin{bmatrix} \bar{b} \\ \underline{b} \end{bmatrix} = \begin{bmatrix} \bar{A} & \underline{A} \\ \bar{A}' & \underline{A}' \end{bmatrix} \begin{bmatrix} \bar{y} \\ \underline{y} \end{bmatrix}.$$

We shall rely on this result when dealing with the system of linear difference equations involving two selections, particularly in **Chapter 2**. In this regard, we also write

$$A = \begin{bmatrix} \bar{A} & \underline{A} \\ \bar{A}' & \underline{A}' \end{bmatrix},$$

and refer to the right-hand side of the previous equation as block decomposition of  $A$ . A matrix  $A$  is said to be a block diagonal if it admits a block decomposition and if  $\bar{A}, \underline{A}'$  or  $\bar{A}', \underline{A}$  have all elements equal to zero. Additional results on block-diagonal matrices will be discussed in **Chapter 2**.

1.1.5. *Measure theory and integration.* Let  $X$  be a set. An *algebra* of  $X$  is a collection  $\mathcal{A}$  of subsets of  $X$  such that:

- for every finite family  $\{A_i\}_{i \in I}$  of subsets of  $\mathcal{A}$ ,  $\cup_{i \in I} A_i \in \mathcal{A}$ ;
- if  $A_i \in \mathcal{A}$ , then  $X \setminus A_i \in \mathcal{A}$ .

In other words, an algebra of  $X$  is a collection of subsets that is closed under finite unions and complementation. A  $\sigma$ -algebra of  $X$  is a collection  $\mathcal{S}$  of subsets of  $X$  such that:

- for every countable family  $\{A_i\}_{i \in I}$  of subsets of  $\mathcal{S}$ ,  $\cup_{i \in I} A_i \in \mathcal{S}$ ;
- if  $A_i \in \mathcal{S}$ , then  $X \setminus A_i \in \mathcal{S}$ .

Hence, a  $\sigma$ -algebra of  $X$  is an algebra of the same set that is closed under countable unions. Given a pair  $(X, \mathcal{A})$ , where  $X$  is a set and  $\mathcal{A}$  is an algebra of  $X$ , a *finite measure*, or *charge*, is a function  $\mu : \mathcal{A} \rightarrow \mathbb{R}$  such that:

- $0 \leq \mu(A) < \infty$  for every  $A \in \mathcal{A}$ ;
- $\mu(\emptyset) = 0$ ;
- for every finite family  $\{A_i\}_{i \in I}$  of disjoint subsets of  $\mathcal{A}$ ,  $\mu(\cup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$  (finite additivity).

Given a pair  $(X, \mathcal{S})$ , where  $X$  is a set and  $\mathcal{S}$  is a  $\sigma$ -algebra of  $X$ , a *measure* is a function  $\mu : \mathcal{S} \rightarrow \mathbb{R}$  such that:

- $0 \leq \mu(A) < \infty$  for every  $A \in \mathcal{S}$ ;
- $\mu(\emptyset) = 0$ ;
- for every countable family  $\{A_i\}_{i \in I}$  of disjoint subsets of  $\mathcal{S}$ ,  $\mu(\cup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$  ( $\sigma$ -additivity).

The definitions of *coarser* and *finer* for a pair of algebras or  $\sigma$ -algebras defined on the same set are identical to those given for topologies. Every measure is also a charge, and a sub- $\sigma$ -algebra of  $\mathcal{S}$  is a set  $\mathcal{U} \subset \mathcal{S}$  which is a  $\sigma$ -algebra of  $X$ . A *probability measure* is a measure such that  $\mu(X) = 1$ , and a probability space is a triplet  $(X, \mathcal{S}, \mu)$ , where  $X$  is a set,  $\mathcal{S}$  is a  $\sigma$ -algebra of  $X$  and  $\mu$  a probability measure  $\mu : \mathcal{S} \rightarrow \mathbb{R}$ . Given a probability space  $(X, \mathcal{S}, \mu)$  and a set  $A \subseteq \mathcal{S}$ , let  $\mu_A : \mathcal{S} \rightarrow \mathbb{R}$ ,  $S \mapsto \mu(S \cap A)$  be the restriction of  $\mu$  to  $A$ . Given a topological space  $(X, \tau_X)$ , the Borel  $\sigma$ -algebra of  $X$ , denoted by  $\mathcal{B}_{\tau_X}(X)$ , is the smallest, i.e., the coarsest  $\sigma$ -algebra that contains all the open sets in  $\tau_X$ . We also refer to *the Borel  $\sigma$ -algebra* of a topological space  $X$  as its *Borel  $\sigma$ -field*, and given a probability space  $(X, \mathcal{B}_{\tau_X}(X), \mu)$ , we say that  $\mu$  is a Borel probability measure. Given a topological space  $(X, \tau_X)$ , we denote by  $\Delta_{\tau_X}(X)$  the set of all Borel probability measures on  $X$ . With an abuse of notation, when there is no need to specify the topology of  $X$

adopted, for example when  $X$  is endowed with its natural topology, it is customary to write  $\mathcal{B}(X)$  and  $\Delta(X)$  instead of  $\mathcal{B}_{\tau_X}(X)$  and  $\Delta_{\tau_X}(X)$ , respectively.

A fundamental topic in measure theory is that of *integration*. Given a set  $X$  and a  $\sigma$ -algebra  $\mathcal{S}$  of  $X$ , the pair  $(X, \mathcal{S})$  is said to be a measurable space. Given two measurable spaces  $(X, \mathcal{S}_X)$  and  $(Y, \mathcal{S}_Y)$ , a function  $f : X \rightarrow Y$  is said to be measurable for every set  $V \in \mathcal{S}_Y$ ,  $f^{-1}(V) \in \mathcal{S}_X$ . In the remainder of this paragraph, we fix a generic measurable space  $(X, \mathcal{S})$  and endow the set of real numbers with the Borel  $\sigma$ -algebra generated by the topology corresponding to the Euclidean metric. A function  $\phi : X \rightarrow \mathbb{R}$  is said to be simple if it is measurable and there exist a finite family of sets  $\{A_i\}_{i \in I}$  and a selection  $a = (a_i)_{i \in I} \in \times_{i \in I} A_i$  such that:

- $\cup_{i \in I} A_i = X$ ;
- for every  $x \in X$ ,  $\phi(x) = \sum_{i \in I} a_i \mathbb{I}_{A_i}(x)$ .

The Lebesgue integral of a simple function  $\phi : X \rightarrow \mathbb{R}$  on  $X$  with respect to  $\mu \in \Delta(X)$  is  $\int_X \phi(x) d\mu(x) := \sum_{i \in I} a_i \mu(A_i)$ . In order to extend the definition of Lebesgue integral to an arbitrary measurable function  $f : X \rightarrow \mathbb{R}$ , we let  $\mathcal{S}$  be the set of the simple functions defined on the measurable space  $(X, \mathcal{S})$ , define  $\mathcal{G} := \{g \in \mathcal{S} : \forall x \in X, 0 \leq g(x) \leq f(x)\}$  and consider the case of a non-negative function. The Lebesgue integral of a non-negative measurable function  $f : X \rightarrow \mathbb{R}$  with respect to  $\mu$  on  $X$  is  $\int_X f(x) d\mu(x) := \sup_{\phi \in \mathcal{G}} \int_X \phi(x) d\mu(x)$ . In other words, we consider the least upper bound of the Lebesgue integrals associated to the basic functions that "lie" below  $f$ . For a generic measurable function  $f : X \rightarrow \mathbb{R}$ , or a *random variable*, we let  $f^+ : X \rightarrow \mathbb{R}$ ,  $x \mapsto \max(f(x), 0)$  and  $f^- : X \rightarrow \mathbb{R}$ ,  $x \mapsto -\min(f(x), 0)$ , and define the Lebesgue integral of a function  $f : X \rightarrow \mathbb{R}$  with respect to  $\mu$  on  $X$  as  $\int_X f(x) d\mu(x) := \int_X f^+(x) d\mu(x) - \int_X f^-(x) d\mu(x)$ . Given a subset  $X'$  of  $X$ , and a function  $f : X \rightarrow \mathbb{R}$ , the Lebesgue integral of  $f$  on  $X' \subseteq X$  with respect to  $\mu$  is  $\int_{X'} f(x) d\mu(x) := \int_{X'} r(x) d\mu(x)$ , where  $r(x) := f(x) \cdot \mathbb{I}_{X'}(x)$ . The Lebesgue integral of a given measurable, real-valued, function is said to exist if it is a real number (i.e., it is finite), and the specific function considered is said to be Lebesgue integrable. We denote by  $L^1(X, \mathcal{S}, \mu)$  the subset of  $\mathbb{R}^X$  of all Lebesgue integrable functions. Given a measurable space  $(X, \mathcal{B}(X))$ , we let  $\mathcal{F}_X$  be the family of Borel-measurable real-valued functions and define the expectation functional as  $\mathbb{E} : \mathcal{F}_X \times \Delta(X) \rightarrow \mathbb{R}$  such that  $(f, \mu) \mapsto \int_X f(x) d\mu(x)$ . As a result, using the notation for sections of a function, for a given  $\mu \in \Delta(X)$ , the function  $\mathbb{E}_\mu : \mathcal{F}_X \rightarrow \mathbb{R}$  maps a random variable  $f \in \mathcal{F}_X$  to the corresponding expected value  $\int_X f(x) d\mu(x)$ . With a slight abuse of notation, we also write  $\mathbb{E}_\mu[f]$  to denote  $\mathbb{E}_\mu(f) = \int_X f(x) d\mu(x)$ .

Given a topological space  $(X, \tau_X)$ , let  $\mathcal{C}_X$  be the set of continuous and bounded function from  $X \rightarrow \mathbb{R}$ , under the usual assumption that  $\mathbb{R}$  is endowed with the topology generated by the Euclidean metric (i.e., its natural, or, usual, topology). We endow  $\Delta(X)$  with the corresponding weak\* topology, that is, the weak topology generated by the family of functions  $\{g_f : \mu \mapsto \int_X f(x) d\mu(x)\}_{f \in \mathcal{C}_X}$ . Under these topological assumptions, a net of Borel probability measures  $(\mu_i)_{i \in \mathbb{N}}$  on  $X$  converges to a measure  $\mu^* \in \Delta(X)$  if and only if for every  $f \in F$ ,

$$\left( \int_X f(x) d\mu_i(x) \right)_{i \in I} \xrightarrow{d_e} \int_X f(x) d\mu^*(x),$$

where the symbol  $\xrightarrow{d_e}$  denotes the convergence in the Euclidean metric  $d$  (i.e., the usual concept of limit in calculus). In other words, when  $\Delta(X)$  is endowed with the weak\* topology, the convergence of a net of measures can be evaluated using the classic notion of convergence in the Euclidean metric space. Another relevant example involving the weak\* topology, which motivates many results in probability, is the following. Let  $X$  be a finite set and  $\tau_X$  be the corresponding discrete topology, that is, the topology of  $X$  such that, for every  $x \in X$ ,  $\{x\}$  is open in  $\tau_X$ . There is a homeomorphism between the corresponding set of Borel probability distributions, equipped with the weak\* topology, and the set of non-negative vectors with entries adding up to one. Given a probability space  $(X, \mathcal{S}, \mu)$ , we say that  $\mu$  is Dirac if  $\exists x \in X$  and  $\mu(\{x\}) = 1$ ; if  $\Delta(X)$  is equipped with the weak\* topology,  $X$  can be topologically embedded in  $\Delta(X)$  through Dirac measures.

For functions of more variables, we generalize the concept of joint distributions by means of generalized Cartesian products, that is, a set of selections on (possibly flattened) index sets. Suppose that  $X = \prod_{i \in I} X_i$ , where  $I$  is an arbitrary countable set, and  $\{(X_i, \tau_{X_i})\}$  is a family of Polish spaces. Consider the corresponding probability space  $(X, \mathcal{B}(X), \mu)$ , where  $\mathcal{B}(X)$  is the Borel- $\sigma$  algebra obtained when  $X$  is endowed with the product topology. Since  $X$  is also Polish, we can show that  $\mathcal{B}(X) = \mathcal{B}(X_J) \otimes \mathcal{B}(X_{-J})$  for every  $J \in 2^I$ , where  $\mathcal{B}(X_J) \otimes \mathcal{B}(X_{-J})$  denotes the smallest  $\sigma$ -algebra containing the collection  $\{A \times B : A \in \mathcal{B}(X_J) \wedge B \in \mathcal{B}(X_{-J})\}$ . Given a subset  $J$  of  $I$ , the function  $\text{marg}_{X_J} \mu : \mathcal{B}(X_J)$  such that  $\text{marg}_{X_J} \mu(A) \mapsto \mu \circ \text{proj}_{X_J}^{-1}(A)$  maps an event  $A \in \mathcal{B}(X_J)$  to the probability of the corresponding cylinder set  $A \times X_{-J}$ , which is the standard definition of marginal probability; we also write  $\text{marg}_{X_J} \mu = \mu \circ \text{proj}_{X_J}^{-1}$  and refer to  $\text{marg}_{X_J} \mu$  as the push-forward measure of  $\mu$  under  $\text{proj}_{X_J}$ . Similarly, for a given  $i \in I$ , we let  $\text{marg}_{X_i} \mu := \text{marg}_{X_{\{i\}}} \mu$ . Given a subset  $J$  of  $I$ , the conditional measure of  $X_J$  on  $X_{-J}$ , if it exists, is a function  $\mu_{X_J|X_{-J}} : \mathcal{B}(X_{-J}) \rightarrow \Delta(X_J)$  such that, for every  $A \in \mathcal{B}(X_J)$ , the mapping  $x_{-J} \mapsto \mu_{X_J|X_{-J}}(x_{-j})(A)$  is measurable with respect

to  $\mathcal{B}(X_{-J})$  and, for every  $K \subseteq X$ ,  $\mu(K) = \int_{X_{-J}} \mu_{X_J|X_{-J}}(x_{-J})(K) d(\text{marg}_{X_{-J}}\mu(x))$ . In this regard, the requirement that the mapping  $x_{-J} \mapsto \mu_{X_J|X_{-J}}(x_{-J})(A)$  is measurable with respect to  $\mathcal{B}(X_{-J})$  is a necessary condition to ensure that  $\mu(K) = \int_{X_{-J}} \mu_{X_J|X_{-J}}(x_{-J})(K) d(\text{marg}_{X_{-J}}\mu(x))$  is well-formed, since the definition of the Lebesgue integral of a function requires the same function to be measurable with respect to the  $\sigma$ -algebra of the measure space considered. Conditional measures always exist when  $X$  is the product of a family of Polish spaces. Regarding marginal distributions, for a given  $i \in I$ , we let  $\mu_{X_i|X_{-i}} := \mu_{X_i|X_{I \setminus \{i\}}}$  for notational convenience. Given a subset  $J$  of  $I$ , we say that  $X_J$  and  $X_{-J}$  are independently distributed if  $\mu_{X_J|X_{-J}} = \text{marg}_{X_J}\mu$ ; similarly,  $X_i$  and  $X_{-i}$  are independently distributed if  $\mu_{X_i|X_{-i}} = \text{marg}_{X_i}\mu$ . By Fubini's theorem, if  $f \in L^1(X, \mathcal{B}(X), \mu)$ , for every  $i \in I$ ,  $\mathbb{E}_\mu(f) = \mathbb{E}_{\text{marg}_{X_i}\mu}\{\mathbb{E}_{\mu_{X_i|X_{-i}}}[f_{x_i}]\}$ , and, for every  $J \subseteq I$ ,  $\mathbb{E}_\mu(f) = \mathbb{E}_{\text{marg}_{X_J}\mu}\{\mathbb{E}_{\mu_{X_J|X_{-J}}}[f_{x_J}]\}$ . The notions of variance, covariance, and covariance matrix, as well as their conditional versions, are taken for granted, and we refer to Billingsley (1995) for additional details due to space reasons.

Finally, we state, without proof, the Kolmogorov extension theorem, which is central in the construction of type structures from hierarchies of beliefs in epistemic game theory (cf. **Section 1.3**). Suppose that  $X = \prod_{i \in I} X_i$ , where  $\{(X_i, \tau_{X_i})\}_{i \in \mathbb{N}}$  is a family of Polish spaces. Consider the corresponding measurable space  $(X, \mathcal{B}(X))$ , where  $\mathcal{B}(X)$  is the Borel  $\sigma$ -algebra obtained when  $X$  is endowed with the product topology. For every  $i \in I$ , let  $\mathcal{B}(X_i)$  be the Borel  $\sigma$ -algebra generated by  $\tau_{X_i}$ . The product  $\sigma$ -algebra, denoted by  $\otimes_{i \in \mathbb{N}} \mathcal{B}(X_i)$ , is the coarsest  $\sigma$ -algebra generated by the collection of cylinder sets

$$\{\prod_{i \in [m]} A_i \times X_{\mathbb{N} \setminus [m]} : m \in \mathbb{N}, \forall i \in [m], A_i \in \mathcal{B}(X_i)\}.$$

Since, for every  $i \in \mathbb{N}$ ,  $(X_i, \tau_{X_i})$  is Polish, it can be shown that  $\otimes_{i \in \mathbb{N}} \mathcal{B}(X_i) = \mathcal{B}(X)$ , generalizing the result presented in precedence. Now, consider a family of probability spaces  $\{(X_{[k]}, \mathcal{B}(X_{[k]}), \mu_k)\}_{k \in \mathbb{N}}$ . We say that the associated family of measures  $\{\mu_k\}_{k \in \mathbb{N}}$  is projectively consistent if, for every  $k \in \mathbb{N}$ ,  $\text{marg}_{X_{[k-1]}}\mu_k = \mu_{k-1}$ . In other words, every two measures agree on events that are measurable, considering appropriate cylinder sets, with respect to their  $\sigma$ -algebras. The Kolmogorov Extension theorem states that, under the assumption that every  $(X_i, \tau_{X_i})$  is Polish, if the family of measures  $\{\mu_k\}_{k \in \mathbb{N}}$  is projectively consistent, there exists a unique probability measure  $\mu$  defined on  $\mathcal{B}(X)$  such that, for every  $k \in \mathbb{N}$ ,  $\text{marg}_{X_{[k]}}\mu = \mu_k$ . In other words, a family of consistent probability measures on partial products  $X_{[m]} = \prod_{i \in [m]} X_i$ ,  $m \in \mathbb{N}$ , induces one and only one probability measure on the

full product  $X = \prod_{i \in I} X_i$ . Similarly, every probability measure on  $\mathcal{B}(X)$  induces a family of consistent probability measures on partial products  $X_{[m]} = \prod_{i \in [m]} X_i$ ,  $m \in N$ . The Kolmogorov extension theorem holds also for arbitrary countable index sets, but the previous formulation is sufficient for our needs. An accessible, yet rigorous, proof of the Kolmogorov extension theorem can be found in Chapter 15 of Aliprantis and Border (2006).

1.1.6. *Uncertainty relations, probabilities and statistics.* Why are probabilities so ubiquitous when dealing with social interactions?<sup>5</sup> In this section, we show that a binary relation, describing a collection of *qualitative statements* such as "I believe that a certain event to be more plausible than another," can be represented *quantitatively* by a probability measure under minimal consistency and regularity conditions.

Consider the following example. Alice and Bob are a nice happy couple, though both have a very bad memory. One day, Alice and Bob are at work and suddenly remember that they are supposed to meet at 8 pm on the same day, but they cannot remember where they agreed to meet. They work in two distant cities, their phones are broken, and they have really no chance to get in touch with each other. In a similar situation, it is clear that there is no objective probability, intended as a long-run frequency, of where Alice may go first to look for her partner.<sup>6</sup> Yet, based on what Bob knows about Alice and what he believes that Alice knows about him (and so on), he might start to list a number of alternatives where Alice may look for him, and then compare each two alternatives, or mix of alternatives, to say which one is more plausible than the other. It turns out that, if Bob's comparison of alternatives is complete and free from contradictions, it can be represented by a charge on a  $\sigma$ -algebra. We now go through the necessary formality.

Let  $(X, \mathcal{S})$  be a measurable space, and consider a binary relation  $\succsim$  on  $\mathcal{S}$ , such that, for any two sets  $A, B \in \mathcal{S}$ ,  $A \succsim B$  reads as the event  $A$  is considered more plausible, or more likely than the event  $B$ . A binary relation  $\succsim$  on  $\mathcal{S}$  is also known as an *uncertainty relation*, or *plausibility order*, as it can always be interpreted as a statement on the relative likelihood or plausibility of two events. In this regard, it is possible to prove that there exists a unique charge  $\mu : \mathcal{S} \rightarrow [0, 1]$  such that  $A \succsim B \iff \mu(A) \geq \mu(B)$  if and only if  $\succsim$  is reflexive, transitive, and, for every triplet  $A, B, C$  of disjoint sets in  $\mathcal{S}$ ,  $A \succsim B \implies A \cup C \succsim B \cup C$  (monotonicity under disjoint unions). If  $X$  is a finite set,  $\mu$  is also a probability measure. However, if  $X$  is infinite, we must ensure countable additivity

<sup>5</sup>The main references for this section are Chapters 1-5 in Bernardo and Smith (1994) and Savage (1954).

<sup>6</sup>The concepts of long-run frequency and objective probability will be introduced rigorously in a moment.

to obtain a probability measure. In this regard, we say that a binary uncertainty relation  $\succsim$  on  $\mathcal{S}$  is continuous if the following conditions hold:

- for every family  $\{A_n\}_{n \in \mathbb{N}}$ , such that  $\forall n \in \mathbb{N}$ ,  $A_n \in \mathcal{S}$ ,  $A_n \subseteq A_{n+1}$  (strictly increasing sets),  $\forall n \in \mathbb{N}$ ,  $A_n \succsim B \implies A \succsim B$ , where  $A = \bigcup_{n \in \mathbb{N}} A_n = \{x \in X : \exists n \in \mathbb{N}, x \in A_n\}$ ;
- for every family  $\{B_n\}_{n \in \mathbb{N}}$ , such that  $\forall n \in \mathbb{N}$ ,  $B_n \in \mathcal{S}$ ,  $B_{n+1} \subseteq B_n$ ,  $\forall n \in \mathbb{N}$ ,  $A \succsim B_n \implies A \succsim B$ , where  $B = \bigcap_{n \in \mathbb{N}} B_n = \{x \in X : \forall n \in \mathbb{N}, x \in B_n\}$ .

It is possible to show that, given a binary uncertainty relation  $\succsim$  on  $\mathcal{S}$ , there exists a unique probability measure  $\mu : \mathcal{S} \rightarrow [0, 1]$ , known as subjective probability, such that  $A \succsim B \iff \mu(A) \geq \mu(B)$  if and only if  $\succsim$  is reflexive, transitive, monotonic under disjoint unions and continuous.

Under the Bayesian paradigm, statistics is essentially a clever application of probability theory to data analysis. Unlike frequentist statistical methods, which are vague on the intrinsic nature of probabilities, Bayesian methods are explicit on the meaning of probability and provide a simple, general, receipt to address estimation and inference problems. We let  $\Omega$  be a Polish space, where  $\Omega$  is a set of the possible outcomes of a situation of interest or *an experiment*; we also generally refer to  $\Omega$  as the set of states of the world. To express opinions on the likelihood of different events, we introduce the measure space  $(\Omega, \mathcal{B}(\Omega))$  where  $\mathcal{B}(\Omega)$  is the Borel  $\sigma$ -algebra containing all the open sets in  $\Omega$ ; an event is an element of  $\mathcal{B}(\Omega)$ . As usual, we endow  $\Delta(\Omega)$  with the corresponding weak\* topology. Consider an individual or decision maker (DM) expressing an uncertainty relation  $\succsim$  on  $\mathcal{B}(\Omega)$  that is equivalent to a probability measure  $\eta \in \Delta(\Omega)$ , in the sense that for every  $A, B \in \mathcal{B}(\Omega)$ ,

$$A \succsim B \iff \eta(A) \geq \eta(B).$$

Fix an integer  $i \in \mathbb{N}$  and consider a function  $x_i : \Omega \rightarrow Y$ , where  $Y = \mathbb{R}^J$  and  $J$  is a finite set or  $Y = \mathbb{R}$ , which is measurable with respect to  $(\Omega, \mathcal{B}(\otimes))$ . We refer to  $x_i$  as a random variable and let  $p_{x_i} := \eta \circ x_i^{-1}$  be the push-forward measure of  $\eta$  through  $x_i$ ; it is immediately verified that  $p_{x_i} \in \Delta(Y)$  induces an uncertainty relation on  $\mathcal{B}(Y)$  consistent with  $\eta$ . As usual, we endow  $Y^{\mathbb{N}}$  with its natural topology, that is, the product topology, and we define a complete random sample as a profile  $x \in \bar{X}^{\mathbb{N}}$  of random variables, where  $\bar{X}$  is the set of all random variables from  $\Omega$  to  $Y$ ; a finite random sample is any projection of a profile  $x \in \bar{X}^{\mathbb{N}}$  onto a subset  $I$  of  $\mathbb{N}$ . A realization of a random sample, which is often called an observed sample, or a sample for short, is a collection of values assumed by a each variable in the same random sample for a given state of the world. In this

regard, given a random sample  $x$ , we denote by  $\hat{F}$  its limiting empirical distribution, that is,

$$\begin{aligned} \hat{F} : \mathcal{B}(Y) \times \Omega &\rightarrow [0, 1] \\ (a, \omega) &\mapsto \lim_{n \rightarrow \infty} \sum_{i \in [n]} \frac{\mathbb{I}_A(x_i(\omega))}{n}. \end{aligned}$$

It is immediate to verify that every random sample is a random variable, and, for every  $A \in \mathcal{B}(Y)$ ,  $\hat{F}_A$  is also a random variable. Hence, the push-forward measure  $p := \eta \circ x^{-1}$ ,  $p \in \Delta(Y^{\mathbb{N}})$  defines an uncertainty relation  $\preceq_{Y^{\mathbb{N}}}$  on  $\mathcal{B}(Y^{\mathbb{N}})$  for DM such that, for every  $A, B \in \mathcal{B}(Y^{\mathbb{N}})$ ,

$$A \preceq_{Y^{\mathbb{N}}} B \iff p(A) \geq p(B) \iff \eta(\cap_{i \in \mathbb{N}} x_i^{-1}(A_i)) \geq \eta(\cap_{i \in \mathbb{N}} x_i^{-1}(B_i)).$$

In other words,  $p$  allows DM to express uncertainty statements on random samples that are consistent with the "original" plausibility order  $\preceq$ . We let  $\Omega^*$  be the support of  $\eta$ , that is, the smallest closed set in  $\tau$  with probability one, and we say that a certain property holds for almost all P-almost every  $\omega \in \Omega$  if it is verified on  $\Omega^*$ . For every  $n \in \mathbb{N}$ , let  $\phi_n : [n] \rightarrow [n]$  be a bijection, and  $\Phi_n$  the set of all bijections included in  $[n]^{[n]}$ . A random sample  $(x_i)_{i \in [n]}$  is said to be exchangeable with respect to  $\eta$  if, for every  $\phi_n \in \Phi_n$ , and every  $A \in \mathcal{B}(X)$  we have that,

$$\eta(\cap_{i \in [n]} x_i^{-1}(A_i)) = \eta(\cap_{i \in [n]} x_{\phi_n(i)}^{-1}(A_i)).$$

A complete random sample  $x \in \bar{X}^{\mathbb{N}}$  is said to be exchangeable if, for every  $n \in \mathbb{N}$ ,  $(x_i)_{i \in [n]}$  is exchangeable. Interpreting the index of a sample as the order in which the realization of each variable is observed, a family of random variables is exchangeable if the order in which their realizations is observed is probabilistically irrelevant. With these definitions in mind, we have the following powerful representation theorem, which generalizes that of De Finetti's (1937) for 0–1 random variables (i.e. Bernoulli random variables).

**THEOREM.** Let  $\{x_i : i \in \mathbb{N}\}$  a family of exchangeable random variables with respect to  $\eta$ . For every  $I \in 2^{\mathbb{N}}$ , let  $p_{x_I} := \eta \circ x^{-1}$ . There exists a measurable function  $f : \Omega \rightarrow \Delta(\bar{X})$  such that:

- for every  $A_I \in \mathcal{B}(Y^I)$ ,  $p_{x_I}(A_I) = \int_{\Omega} \prod_{i \in I} f(\omega)(A_i) \, d\eta(\omega)$ ;
- for P-almost every  $\omega \in \Omega$  and for every  $A_i \in Y$ ,

$$\hat{F}_A(\omega) := \lim_{n \rightarrow \infty} \sum_{i \in [n]} \frac{\mathbb{I}_{A_i}(x_i(\omega))}{n} = f(\omega).$$

Hence, there exists a family of measurable functions  $F := \{f_\theta \in \bar{X}^\Omega : \theta \in \Theta\}$ , where  $\Theta$  is any valid index set, and a probability measure  $\mu \in \Delta(\Theta)$  such that:

- for every  $A_I \in \mathcal{B}(Y^I)$ ,  $p_{x_I}(A_I) = \int_\Theta \prod_{i \in I} f_\theta(A_i) \, d\mu(\theta)$ ;
- for every  $\theta \in \Theta$ ,  $\mu(\theta) = \eta(\{\omega \in \Omega : f(\omega) = f_\theta\})$
- for P-almost every  $\omega \in \Omega$ ,  $\exists \theta \in \Theta$ , such that, for every  $A_i \in Y$ ,

$$\hat{F}_A(\omega) := \lim_{n \rightarrow \infty} \sum_{i \in [n]} \frac{\mathbb{I}_A(x_i(\omega))}{n} = f_A.$$

The result contained in the previous theorem is of the utmost importance for a correct understanding of statistical procedures and their use in economics and social sciences. In particular, it provides a clear theoretical foundation for frequentist interpretations of probabilities. In this regard, we say that the DM believes that the data are consistent with an objective probability distribution if  $\Theta$  is a singleton. In such a case, the DM is certain that every sample has the same known limiting frequency  $f$ , and that  $\{x_i : i \in \mathbb{N}\}$  is a family of i.i.d. random variables, with a push-forward measure  $p_{x_i} = f$ . In other words, if the DM observed a sequence of finite samples of increasing length, suggesting a limiting frequency different from  $f$ , she would conclude only that an event she regarded as almost impossible, that is, a set  $E$  in  $\mathcal{B}(\Omega)$  such that  $\eta(E) = 0$  has occurred.

There are no such things as true objective probabilities, unless the data being considered are generated by a deterministic device that ensures that every generated sample has the same limiting frequency. In such a case, we say that there exists a data generating process (DGP), that is, there is a true value of  $\theta$  generating every observable sample, which the DM does not necessarily know. A DGP is also known as a pseudo-random number generator, of which there are many examples of algorithms in computer science. A vast literature in finance postulates the existence of a DGP for quantities such as assets' returns and macroeconomic shocks, essentially equivalent to the previously mentioned computer algorithms. We are of the opinion that such view may be motivated only within the context of a theoretical model, as in standard computer simulations, randomness, if and when necessary, is always introduced by deterministic algorithms known as pseudo-random number generators. In reality, care is needed before postulating the existence of a true DGP, unless there is reason to believe that a (suitable) theoretical model may be consistent with the data. In other words, there is always some degree of assumption in empirical works imposing the existence of DGP: since, in reality, the observation of a sample inconsistent with a given DGP is always a possibility, assuming the existence of a true DGP amounts to imposing a zero subjective probability to this

event. As a general remark, probability models are in practice always based on some unavoidable assumptions; this should come without any surprise, as probabilities are clever devices to express plausibility orders, which are intrinsically related to our knowledge, qualitative beliefs, and experience, as we discuss in detail in **Section 1.2.2** of this chapter. In this regard, in social sciences, such as economics and finance, the most sensible minimal assumption we can make is that, given their perception of the world, people do not act against their own interests, which may very well include altruistic or other psychological traits. Whether, in reality, this assumption may motivate the existence of a DGP is definitely up for debate. Despite this, in applying Bayesian statistics to economic and financial data, it is customary to posit uncertainty relations consistent with the certainty of the existence of a true, but unknown, DGP. Such an approach, as we have seen, is allowed by the representation theorem above. In practice, there are two practical ways of proceeding, which are known respectively as parametric and non-parametric statistical methods. We limit the discussion to the former, which essentially reduces to:

- introduce a function  $\pi : \mathcal{B}(\bar{X}) \times \Lambda \rightarrow [0, 1]$  such that, for every  $\lambda \in \Lambda$ ,  $\pi(\cdot|\lambda) \in \Delta(\bar{X})$ ;
- assign  $\eta$  such that for every  $I \subseteq \mathbb{N}$ , and every  $A_I \in \mathcal{B}(X_I)$ ,

$$\eta(\cap_{i \in I} x_i^{-1}(A_i)) = \int_{\Lambda} \prod_{i \in I} \pi(A_i|\lambda) \, d\pi(\lambda),$$

where  $\pi(\cdot) = \text{marg}_{\Lambda} \pi(\cdot)$ .

Under the additional hypothesis that  $\pi$  is continuous (resp. discrete), if we let  $\rho : \bar{X}^I \times \Lambda \rightarrow \mathbb{R}_+$  the corresponding density function (resp. probability mass function), we have, as an application of the Bayes theorem  $\rho(\lambda|x_I) \propto \rho(x_I|\lambda) \rho(\lambda)$ , that is,

$$\rho(\lambda|x_I) \propto \rho(\lambda) \cdot \prod_{i \in I} \rho(x_i|\lambda).$$

The left-hand side is the posterior density (resp. probability mass function) of  $\lambda$  given the observed sample  $x_I$ , while  $L(\theta, x_I) := \prod_{i \in I} \rho(x_i|\lambda)$  is the likelihood function in classical statistics. Estimation, hypothesis testing, and prediction are well-defined decision problems consistent with maximization of the expected value of a carefully chosen utility function, with respect to the measure generated by the posterior density (resp. probability mass function)  $\rho(\lambda|x_I)$ . When  $\rho(\lambda)$  is assumed to be equal to an appropriate constant, we say that it is a non-informative prior. Non-informative priors can be used to motivate, from an explicit decision-theoretic perspective, several classic statistical procedures, and, in particular, maximum likelihood estimation and standard hypothesis testing.

Since maximum likelihood can be used to motivate many other procedures, such as the generalized method of moments (cf. **Section 5.1.1** in **Chapter 2**), we may view classic statistics as a very specific case of the more general Bayesian methods.

We conclude with the obvious general remark that, in general, if we simulate the behavior of an economic model with a standard computer, every exogenous source of randomness, including additional measurement errors, is obtained from an algorithm ensuring the exact convergence of each sample to the distribution specified by the user. Hence, a true DGP always exists in the context of the simulated model. In particular, in **Chapter 2**, we use pseudo-random numbers generators to simulate samples from investment games under different solution concepts, and evaluate the properties of the generalized method of moments, which is an example of classic statistical procedure that finds wide application in the empirical asset pricing literature.

**1.1.7. Decision problems.** We conclude by presenting the general structure of *decision problems* that will be frequently encountered in the following sections and chapters. Let  $\{(X_i, \tau_{X_i}) : i \in I\}$  be a family of Polish spaces and,  $X := \prod_{i \in I} X_i$ , and, as usual, endow  $X$  with the corresponding product topology. Fix a generic element  $i \in I$ , and recall that, given a function  $f : X \rightarrow \mathbb{R}$ , the section of  $f$  at  $x_i \in X_i$  is a function  $f_{x_i} : X_{-i} \rightarrow \mathbb{R}$  such that  $x_{-i} \mapsto f(x_i, x_{-i})$ . Let  $\Delta(X_{-i})$  be the set of Borel probability distributions on  $X_{-i}$ , and let  $\mathcal{A}_i$  the set of measurable sections, that is,  $\mathcal{A}_i := \{\alpha \in \mathbb{R}^{X_{-i}} : \exists x_i \in X_i, \alpha = f_{x_i}\}$ . Every element  $f_{x_i}$  of  $\mathcal{A}_i$  is called a *Savage act*, and  $\mathbb{E}_\mu(f_{x_i})$  denotes the corresponding Lebesgue integral with respect to  $\mu \in \Delta(X_{-i})$ . We define as expected utility the function  $\mathcal{U} : X_i \rightarrow \mathbb{R}$  such that  $x_i \mapsto \mathbb{E}_\mu(f_{x_i})$ , and say that a point  $x_i^* \in X_i$  is optimal for  $i$  if  $x_i^* \in \operatorname{argmax}_{x_i} \mathcal{U}(x_i)$  or, equivalently, if  $x_i^* \in \operatorname{argmax}_{x_i} \mathbb{E}_\mu(f_{x_i})$ . The usual interpretation is that  $I$  is a set of individuals,  $X_{ij}$  the set of actions available to a given individual  $j \in I$ , and  $f$  is a utility function (see **Section 1.2**). For reasons of space, we do not expand on the conditions for the existence of an optimal point and on the elementary notions of differentiability in Euclidean spaces to solve "regular" optimization problems; we refer to Blume and Simon (1994) for the relevant definitions and results.

## 2. Games and economic models

An economic model is an artificial environment, or *thought experiment*, in which we imagine that a set of individuals, or *players*, interact with each other, defining a *situation of interest*, usually, but not necessarily, *comparable* with one experienced in reality. In short, given the objective difficulty to

perform controlled experiments in social sciences, the role of an economic model is to explain, often in conjunction with suitable statistical procedures, certain facts observed in the real world.<sup>7</sup> In this thesis we focus on static economic models of interactive situations, where by static we intend that all *players* are assumed to act simultaneously. Namely, a *static economic model* composed of:

- a static *game*, that is, an ordered tuple, including, at least, the set of players involved in the situation of interest, the actions available to each player, a set of relevant consequences for the players, a function linking actions to consequences and the players' preferences over consequences or the corresponding lotteries;
- an *epistemic model* of the game, specifying the set of *states of the world*, that is, a set of elements each corresponding to a full description of what may happen in the situation of interest, as well as an *information correspondence* and a *possibility correspondence* for every player, specifying what is *known* and *believed*, respectively, at a specific state of the world;
- for games with preferences over lotteries, a *Bayesian frame*, that is, a suitable mathematical structure to characterize players as *Bayesian decision makers* and *quantify* their *beliefs*;
- optionally, an *epistemic type structure*, providing a descriptive tool for *hierarchies of beliefs* under the assumption of preferences over lotteries and Bayesian decision makers;
- a *solution concept*, that is, a correspondence determining which profile of actions can be observed in the situation of interest, usually derived from explicit *behavioral and epistemic assumptions* formulated using the epistemic model of the game or, if available, an epistemic type structure.

We refer to a specific situation observed as a *play* of the *game*, while to the corresponding profile of actions observed as an outcome. As we shall see, most of the time we do not need to be explicit about all the elements characterizing a static economic model, and we may be able to collapse one or more components within a suitable mathematical structure. As a bibliographical note, the relevant references for this section are Chapters 1 and 5 in Osborne and Rubinstein (1994), Battigalli and Bonanno (1997, 1999), Bonanno (2015), and Chapters 2,5,6,8,9 and 10 in Bonanno (2024).

**2.1. Static games.** A *game frame* provides a description of the players involved in the situation of interest, as well as their available actions and how these translate into relevant consequences for

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<sup>7</sup>This is what is generally known as a *positive theory* in epistemology, the branch of philosophy that studies science.

the players. We denote by  $F$  the set of players with the typical element  $f$ ;<sup>8</sup> for reasons that will be clear in a moment, we require that  $0 \notin F$ . In a static game, every player  $f \in F$  chooses (i.e., executes) an action from a given set  $A_f$  with a typical element  $a_f$ . Since the game is static, all players choose (i.e. execute) simultaneously their actions at a certain point in time. Consequently, we let  $A = \times_{f \in F} A_f$  be the set of action profiles (selections) with a typical element  $a = (a_f)_{f \in F}$ .<sup>9</sup> As a matter of notation, the products of sets are always intended as sets of selections, with the exception of the sets containing binary relations. The profile of actions played (i.e. executed) determines the outcome of the game, possibly in conjunction with the realization of an exogenous state of nature. We denote by  $Y$  the set of all possible consequences with the typical element  $y$ , and by  $\Theta$  the set of exogenous states of nature, with the typical element  $\theta$ . The state of nature should be interpreted as a complete description of anything that may affect players consequences and that is not determined within the model (i.e. endogenously); it aims at introducing exogenous elements in the moment.<sup>10</sup> In this regard, we posit the existence of a *consequence function*  $g : \Theta \times A \rightarrow Y$  mapping, according to the rules of the game, a profile  $(\theta, a) \in \Theta \times A$  to the corresponding consequence  $g(\theta, a) \in Y$ . If, for every  $a \in A$ ,  $g_a$  is constant, the state of nature has no effect on the players' consequences, and we can equivalently specify  $g$  as a function of  $A$  only. We refer to the ordered tuple  $\langle F, \Theta, Y, (A_f)_{f \in F}, g \rangle$  as a *game frame*, which, in addition to the preferences of the players, provides a full description of the most primitive elements to describe an interactive situation.

As for the specification of the players' preferences, there are two alternative paradigms. The first paradigm, which is more general but less common in applications, consists in assuming that every player  $f \in F$  is endowed with a complete and transitive binary relation  $\succsim_f$  defined on  $Y$ . The second paradigm, which is often harder to motivate theoretically, but of easier practical implementation, assumes instead that every player  $f$  has a preference relation  $\succsim_f^*$  over  $\Delta(Y)$  admitting a subjective expected utility (SEU) representation; the most notable example is the case of a finite set  $Y$  and von Neumann-Morgenstern (vNM) preferences. Under the first paradigm we obtain a static game

<sup>8</sup>The notation differs from the usual convention of denoting by  $I$  the set of players. This choice is motivated by the model discussed in the next chapter, in which actions correspond to investments in production capacity made by different firms. The only relevant stylistic drawback of our notation is that we must avoid the use of  $f$  to denote a function, but this is a minor one compared to others that would emerge in the applications discussed in **Chapter 2** and **Chapter 3**.

<sup>9</sup>That is, a profile of actions  $a \in A$  is a function  $a : F \rightarrow \cup_{f \in F} A_f$  such that  $a(f) \in A_f$  for every  $f \in F$ .

<sup>10</sup>It is often said that *nature* chooses  $\theta$  from  $\Theta$ ; needless to say, the identification with the every day meaning of nature holds when  $\Theta$  is, for example, the weather or the motion of a planet and so on.

with *ordinal payoffs*, that is, an ordered tuple

$$\langle F, \Theta, Y, (A_f)_{f \in F}, g, (\succsim_f)_{f \in F} \rangle,$$

while under the second static game with *cardinal payoffs*, that is, an ordered tuple

$$\langle F, \Theta, Y, (A_f)_{f \in F}, g, (\succsim_f^*)_{f \in F} \rangle.$$

Given a game frame  $\langle F, \Theta, Y, (A_f)_{f \in F}, g \rangle$ , the corresponding games with ordinal and cardinal payoffs admit both a utility-based representation of the form

$$\langle F, \Theta, (A_f)_f, g, (U_f : Y \rightarrow \mathbb{R})_{f \in F} \rangle.$$

It is important to observe that, for every  $f \in F$ , in the case of a static game with ordinal payoffs,  $U_f$  is identified up to strictly increasing composition, while in that of a static game with cardinal payoffs,  $U_f$  is identified up to positive affine transformations. In the remainder of the thesis we always describe the structure of a game adopting a utility-based representation, and specify whether the players' utilities, or *payoffs*, should be interpreted as ordinal or cardinal when relevant. In fact, it is always possible to embed a preference over outcomes in a suitable preference over lotteries by means of Dirac measures. Similarly, it is always possible to derive a preference relation over outcomes consistent from a preference over lotteries. It is important to stress that we view a game as a set of primitive elements describing the situation of interest, rather than the situation itself; as we shall see in the following subsection, this remark is of particular importance from an epistemic perspective.

**2.2. Epistemic models and Bayesian frames.** Contrary to standard game theory, epistemic game theory provides a formal framework for modeling essential aspects, such as the knowledge and beliefs of the players. Consider a game

$$\Gamma = \langle F, \Theta, Y, (A_f)_{f \in F}, g, (U_f)_{f \in F} \rangle,$$

with ordinal or cardinal payoffs. We let  $\Omega$  be the set of all possible states of the world, or states for short, where each state  $\omega \in \Omega$  denotes a "complete" description of the situation of interest. Notice that, in general,  $\Omega$  may include states in which the game being played is not  $\Gamma$ , and it is exactly in this way that we may model a situation in which one or more players do not know the rules of

the game. Proceeding in order, we endow each player  $f \in F$  with an information correspondence  $\mathcal{K}_f : \Omega \rightrightarrows \Omega$ , known as the information function of  $f$  such that:

- $\forall \omega \in \Omega, \omega \in \mathcal{K}_f(\omega)$ ;
- $\forall \omega, \omega' \in \Omega, \omega' \in \mathcal{K}_f(\omega) \implies \mathcal{K}_f(\omega) = \mathcal{K}_f(\omega')$ .

For every  $\omega \in \Omega$ , we interpret  $\mathcal{K}_f(\omega)$  as the states of the world that the player  $f$  considers possible when the true state of the world is  $\omega$ . As such, the requirements above ensure that a generic player  $f$  does not have false or contradictory information, respectively. From a more formal perspective, for every  $f \in F$ , the graph of  $\mathcal{K}_f$  is a reflexive and euclidean, hence transitive, relation on  $\Omega$ . We also say that, at  $\omega$ , the player  $f$  is informed that the true state is in  $\mathcal{K}_f(\omega)$ . Equivalently, at  $\omega$ ,  $f$  is informed that the situation occurring is among those described by  $\mathcal{K}_f(\omega)$ .

**2.2.1. Knowledge.** An event  $E \subseteq \Omega$  is said to be known by  $f$  at  $\omega$  if  $\mathcal{K}_f(\omega) \subseteq E$ , and we also say that  $f$  knows  $E$  at  $\omega$ ; in other words, given the occurrence of a certain state of the world, a player knows that an event has been realized if it is consistent with her available information. We denote by  $K_f$  the knowledge operator of the player  $f \in F$ , that is,

$$K_f : 2^\Omega \rightarrow 2^\Omega$$

$$E \mapsto \{\omega \in \Omega : \mathcal{K}_f(\omega) \subseteq E\}.$$

Following Aumann (1976), an event  $E \subseteq \Omega$  is said to be common knowledge at  $\omega \in \Omega$  if:

- $\omega \in K^1(E) := \cap_{f \in F} K_f(E)$ ;
- $\forall m \in \mathbb{N} \setminus \{1\}, \omega \in K^m(E) := \cap_{f \in F} K_f(K^{m-1}(E))$ .

We also denote by  $CK(E) = \cap_{m \in \mathbb{N}} K^m(E)$  the event “ $E$  is common knowledge”. We assume that we can interpret the event  $K^2(E)$  as “everyone knows that everyone knows  $E$ ”, and so on. Similarly, we assume that  $CK(E)$  can be interpreted as the event in which every player knows  $E$ , knows that everyone knows  $E$ , knows that everyone knows that everyone knows  $E$ , and so on (*ad infinitum*). An important debate emerges in relation to this interpretation. Indeed, without additional assumptions,  $K^2(E)$  can be interpreted only as the event that everyone knows  $K(E)$ , see also Brandenburger and Dekel (1993) reference. The crux is that knowledge and beliefs relate to propositions and events are just introduced to provide a set-theoretic representation. A possible solution to the debate is provided by the syntactic approach, which consists in the adoption of a formal language, based on modal logic, to express propositions in a way that  $CK(\cdot)$  can be correctly interpreted as we did informally. In this regard, we refer to Dekel and Gul (1997), Battigalli and Bonanno (1999), and

Bonanno (2002) for the detailed construction of a language and propositions, and the corresponding derivation of an appropriate set of states of the world. The discussion would require an extensive digression on modal logic and it is omitted for reason of space; for a comprehensive treatment of modal logic, see Chellas (1995), while for a brief review in the context of games, see Bonanno (2002). It is worth mentioning that there are some inevitable limitations with the syntactic approach. On the one hand, Dekel and Gul (1997) and Heifetz (1999) show the existence of events that might not be represented in the same language. On the other hand, Brandenburger and Keisler (2006) show the existence of propositions that cannot be represented as events. However, in the context of this thesis, these limitations are acceptable. We also refer to Fagin et al. (1995) for a comprehensive treatment of knowledge in interactive situations.

**2.2.2. Beliefs.** Social interactions are driven by beliefs, that is, statements that are supposed to be true even if they could be false. Recall that, at a certain state  $\omega$ , every player is generally informed that  $\omega$  is one of the possible states, among others. However, for some reason, the same player may be fully convinced that the true state is  $\omega' \in \mathcal{K}_f(\omega)$ . Going back to the example of Alice and Bob in **Section 1.1.5**, Alice may be sure that Bob is going to a restaurant, even if Bob is actually going to a cinema and Alice knew about this possibility. To formalize the concept of *belief*, for every player  $f \in F$ , we introduce a possibility correspondence  $\mathcal{P}_f : \Omega \rightrightarrows \Omega$  such that:

- $\forall \omega \in \Omega, \mathcal{P}_f(\omega) \subseteq \mathcal{K}_f(\omega)$ ;
- $\forall \omega, \omega' \in \Omega, \omega' \in \mathcal{K}_f(\omega) \implies \mathcal{P}(\omega') = \mathcal{P}(\omega)$ ;
- $\forall \omega \in \Omega, \mathcal{P}_f(\omega) \neq \emptyset$ ;
- $\forall \omega, \omega' \in \Omega, \omega' \in \mathcal{P}_f(\omega) \implies \mathcal{P}_f(\omega') \subseteq \mathcal{P}_f(\omega)$ ;
- $\forall \omega, \omega' \in \Omega, \omega' \in \mathcal{P}_f(\omega) \implies \mathcal{P}_f(\omega) \subseteq \mathcal{P}_f(\omega')$ .

In other words,  $\mathcal{P}_f(\omega)$  denotes the set of states that are *doxastically* accessible to  $f$  at  $\omega$  (cf. Bonanno, 2015). Notice that the first three requirements serve to exclude beliefs contradicting the player's knowledge, while the remaining two are intended to exclude contradicting beliefs; see also Battigalli and Bonanno (1997) for a detailed discussion. Notice that, for every  $f \in F$ , the graph of  $\mathcal{P}_f$  is a serial (i.e. non-empty), transitive and euclidean relation on  $\Omega$ .

The belief operator for a generic player  $f \in F$ , denoted by  $B_f$ , is defined as

$$B_f : 2^\Omega \rightarrow 2^\Omega$$

$$E \mapsto \{\omega \in \Omega : \mathcal{P}_f(\omega) \subseteq E\},$$

since an event is said to be believed if it is supposed to be true. Similarly to the case of common knowledge, an event  $E \subseteq \Omega$  is said to be common belief at  $\omega$  if at the same state of the world every player believes  $E$ , believes that everyone else believes  $E$ , believes that everyone else believes that everyone believes  $E$ , and so on (*ad infinitum*). Namely, an event  $E \subseteq \Omega$  is said to be common knowledge at  $\omega \in \Omega$  if:

- $\omega \in B^1(E) := \cap_{f \in B} B_f(E)$ ;
- $\forall m \in \mathbb{N} \setminus \{1\}, \omega \in B^m(E) := \cap_{f \in F} B_f(B^{m-1}(E))$ .

We also denote the event “ $E$  is common belief” as  $CB(E) = \cap_{m \in \mathbb{N}} B^m(E)$ . The same interpretational issue discussed for common knowledge also emerges for the case of common belief. Again, we assume that  $\Omega$  has been specified in a way that allows us to view  $CB(E)$  as the event in which every player is certain of  $E$ , every player is certain that everyone else is certain of  $E$  and so on, *ad infinitum*.

2.2.3. *Epistemic models.* An epistemic model of  $\Gamma$  is an ordered tuple

$$\mathcal{G} = \langle F, \Theta, \Omega, \nu : \Omega \rightarrow \Theta, (\mathcal{K}_f)_{f \in F}, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$$

where:

- $\Omega$  is generated by a formal language, or *syntax*, that ensures our interpretation of  $CK$  and  $CB$ ;
- $\nu$  is a function that maps every state of the world  $\omega \in \Omega$  to the corresponding state of nature;
- for every  $f \in F$ , for every  $\omega \in \Omega$ ,  $\sigma_f(\omega)$  denotes the action taken by  $f$  at  $\omega$ , for every  $\omega' \in P(\omega)$ ,  $\sigma_f(\omega') = \sigma_f(\omega)$ .

The above requirement ensures that each player knows her action in every state, which also implies that every player has a correct belief regarding her own action. However, notice that, unless we assume that, for every  $\omega \in \Omega$ , the game being played is  $\Gamma$ , the action space of a player  $f \in F$  could be different from  $A_f$  at one or more states. In this regard, observe that, if an event is true at every state of the world, there must be common knowledge of the same event in every state of the world (cf. Osborne and Rubinstein, 1995, Chapter 5, pp. 81). The converse is also true, as an event is known only if it is true. Consequently, if, for every  $\omega \in \Omega$ , the game being played is  $\Gamma$ , we also have that, at every  $\omega \in \Omega$ , it is common knowledge that the game being played is  $\Gamma$  and vice versa. We refer to this epistemic condition as  $CK\Gamma$ . Similarly, if, for every state of the world, there is

no more informed player about the true state of nature, there is common knowledge of symmetric information in every state of the world and vice versa. We refer to this epistemic condition as SYM.

2.2.4. *Weak rationality.* A fundamental concept in game theory is that of rationality. Given a static game with ordinal or cardinal payoffs

$$\Gamma = \langle F, \Theta, Y, (A_f)_{f \in F}, g, (U_f)_{f \in F} \rangle,$$

equipped with an epistemic model  $\mathcal{G}$  consistent with  $\text{CK}\Gamma$  and SYM, we say that the player  $f$  is *weakly rational* at  $\omega$  if  $\nexists a_f \in A_f$  such that, for every  $\omega' \in \mathcal{P}(\omega)$ ,

$$U_f \left( g \left( \zeta(\omega'), a_f, (\sigma_j(\omega'))_{j \in F \setminus \{f\}} \right) \right) \geq U_f \left( g \left( \zeta(\omega'), \sigma_f(\omega'), (\sigma_j(\omega'))_{j \in F \setminus \{f\}} \right) \right).$$

Three important remarks are worth mention. First, the formula above makes sense to the extent that players know the consequence function characterizing the game, which is necessarily the case if it is common knowledge that the game being played is  $\Gamma$ . Nevertheless, even in the absence of common knowledge of  $\Gamma$ , the general definition of weak rationality remains valid. Second, weak rationality does not rule out the possibility of erroneous beliefs and therefore is a rather weak restriction. Third, while in games with ordinal payoffs the players' preferences are unaffected by any eventual *plausibility order* on the set of states deemed as possible, in games with cardinal payoffs players must express quantitative beliefs, that is, subjective probability measures, consistent with the assumption of preferences over lotteries. As a result, in games with cardinal payoffs, the notion of weak rationality is only a necessary requirement for a sensible definition of rational behavior.

**2.3. Bayesian frames, quantitative beliefs and rationality in games with cardinal payoffs.** Suppose that  $\Gamma = \langle F, \Theta, Y, (A_f)_{f \in F}, g, (U_f)_{f \in F} \rangle$  is a static game with cardinal payoffs equipped with an epistemic model  $\mathcal{G}$  consistent with  $\text{CK}\Gamma$  and SYM. In a given state of the world  $\omega \in \Omega$ , in order to evaluate lotteries over  $\Omega$ , every player  $f \in F$  must express a plausibility order on  $\Omega$  admitting a probabilistic representation, which we denote as quantitative beliefs. In this regard, we require that quantitative beliefs attribute non-zero probability to every state deemed possible and vice versa; in this way, we avoid quantitative beliefs to be inconsistent with  $\mathcal{G}$ . In addition, we require every player to know her quantitative beliefs. Consequently, for every  $f \in F$ , we introduce a function  $p_f : \Omega \rightarrow \Delta(\Omega)$  such that, for every  $\omega \in \Omega$ :

- $\text{supp } p_f(\omega) = \mathcal{P}_f(\omega)$ ;

$$\bullet \omega' \in \mathcal{P}(\omega) \implies p_f(\omega') = p_f(\omega).$$

We refer to the ordered tuple  $\mathcal{F} = \langle F, \Omega, (\mathcal{P}_f)_{f \in F}, (p_f)_{f \in F} \rangle$  as a Bayesian frame for  $\Gamma$ . In this regard, if there exists a measure  $q \in \Delta(\Omega)$  such that for every  $f \in F$ , every  $\omega \in \Omega$ ,  $p_f(\omega) = \frac{1}{q(\mathcal{P}(\omega))} \cdot q$ , we say that  $\mathcal{F}$  is generated by a common prior  $q$  on  $\Omega$ , or there is a common prior for short. Regardless of the existence of a common prior, a player  $f \in F$  believes in the event  $E \in 2^\Omega$  at  $\omega$  if and only if  $p_f(\omega)(E) = 1$ . Consequently, having assumed CK  $\Gamma$  and SYM, for every  $\omega \in \Omega$ , and every  $f \in F$ , it is immediate to show that  $U_f$  induces a preference relation  $\succsim_{f, p_f(\omega)}^*$ , on  $A_f$  at  $\omega$ , denoted as such that

$$a_f \succsim_{f, p_f(\omega)}^* a'_f \iff \mathbb{E}_{p_f(\omega)}(\pi_{f, a_f}) \geq \mathbb{E}_{p_f(\omega)}(\pi_{f, a'_f}),$$

where  $\pi_f := U_f \circ g \circ \phi_f$  is a composite utility function,

$$\begin{aligned} \phi_f : A_f \times \Omega &\rightarrow \Theta \times A \\ (a_f, \omega) &\mapsto \left( \zeta(\omega), a_f, (\sigma_j(\omega))_{j \in F \setminus \{f\}} \right) \end{aligned}$$

and  $z$  a function mapping every  $\omega \in \Omega$  in the corresponding state of nature. In other words, for every state of the world, players rank their actions according to the corresponding subjective expected utility (SEU). We also say that  $f$  is a *Bayesian decision maker* (DM), and a *rational Bayesian DM*, or *Bayesian rational*, at  $\omega$  if  $\forall a_f \in A_f$

$$\mathbb{E}_{p_f(\omega)}(\pi_{f, \sigma_f(\omega)}) \geq \mathbb{E}_{p_f(\omega)}(\pi_{f, a_f}).$$

In this regard, it is immediate to verify that, if  $f$  is a rational Bayesian DM at  $\omega \in \Omega$ , it must be the case that it is also weakly rational at  $\omega$ , in the sense that  $\nexists a_f \in A_f$  such that, for every  $\omega' \in \mathcal{P}(\omega)$ ,

$$U_f \left( g \left( \zeta(\omega'), a_f, (\sigma_j(\omega'))_{j \in F \setminus \{f\}} \right) \right) \geq U_f \left( g \left( \zeta(\omega'), \sigma_f(\omega'), (\sigma_j(\omega'))_{j \in F \setminus \{f\}} \right) \right).$$

Hence the definition of a Bayesian rationality in games with cardinal payoffs is consistent with that of rationality in games with ordinal payoffs but more specific. For this reason, we also say that  $f$  is *rational* at  $\omega \in \Omega$  if  $\forall a_f \in A_f$ ,  $\mathbb{E}_{p_f(\omega)}(\pi_{f, \sigma_f(\omega)}) \geq \mathbb{E}_{p_f(\omega)}(\pi_{f, a_f})$ . Furthermore, it is also immediate to verify that, for every  $\omega \in \Omega$ , there exists  $\mu_{f, \omega} \in \Delta(\Omega)$  such that

$$\mathbb{E}_{p_f(\omega)}(\pi_{f, a_f}) = \mathbb{E}_{\mu_{f, \omega}}(u_{f, a_f}),$$

where  $u_f := U_f \circ g$ . Consequently, the ordered tuple

$$\langle F, \Theta, (A_f, u_f : \Theta \times \prod_{f \in F} A_f \rightarrow \mathbb{R})_{f \in F} \rangle$$

is said to be the reduced form representation of  $\Gamma$  equipped with  $\mathcal{G}$  and  $\mathcal{F}$ , and we also write  $\Gamma = \langle F, \Theta, (A_f, u_f)_{f \in F} \rangle$  to save space. We also say that  $\Gamma = \langle F, \Theta, (A_f, u_f)_{f \in F} \rangle$  is a static game, and, when  $\Theta$  affects the utility of at least one player, that  $\Gamma$  is a static game with payoff uncertainty with symmetric information, or just a static game with payoff uncertainty for short. In the remainder of this thesis we shall adopt this notation, as we focus on static games with cardinal payoffs that are equipped with an epistemic model in which CK $\Gamma$  and SYM hold, as well as with a suitable Bayesian frame. Notice that if  $\Theta$  is a singleton, the state of nature must be common knowledge among players in every state of the world. As a result, if  $\Theta$  is a singleton it can be omitted from the description, treating the unique state of nature as a known parameter affecting the outcome function. In addition, it is also possible to omit  $\Theta$  when the state of nature is payoff-irrelevant, that is, it does not affect the consequence function; see Example 24 in Chapter 8 of Battigalli et al. (2024a) for a reference. Whenever  $\Theta$  is irrelevant, either because it i) is a singleton or ii) does not affect the reduced form utilities or iii) the state of nature is common knowledge among players in every state of the world, we can exclude it from the analysis without loss of generality. In this case, we obtain a *game of complete information*, which is an ordered tuple

$$\langle F, (A_f, u_f : \times_{f \in F} A_f \rightarrow \mathbb{R})_{f \in F} \rangle.$$

Note that utilities are unaffected by the state of nature if and only if, for every  $a \in A$ ,  $g_a$  is constant or  $g$  is a function of  $A$  only.

Finally, we acknowledge that the assumption of preference over lotteries is rather abstract. In this regard, for a more theoretically sound formalization of SEU in the context of a game, we refer to Gilboa and Schmeidler (2003), which, assuming probabilities as primitives, extends Savage (1954) SEU theory to interactive situations. For an even more general approach, which derives subjective probabilities and utilities in the context of a static game, we refer to Di Tillio (2008). We also notice that our specification of preferences excludes the possibility of psychological traits such as reciprocity, guilt, envy, etc. In general, it is possible to extend the analysis to include similar psychological aspects. A possible approach consists in extending preferences to take into account the beliefs of other players. In such a case, we would speak of psychological games. For space reasons,

we do not expand on this topic. For additional references on psychological games, see Battigalli et al. (2019) and Battigalli and Dufwenberg (2022).

### 3. Epistemic Type Structures

In the remainder of the chapter we shall focus on games with complete information for ease of exposition, leaving the more general case of games with payoff uncertainty in **Section 2** and **Section 4** of **Chapter 3**; as a matter of notation, we shall sometimes write  $-f$  as an abbreviation for the statement “all the other players different from player  $f$ ”. An economic model can be loosely identified with a thought experiment in which a set of "artificial" individuals interact with each other. In this regard, we have so far limited our attention to tools necessary to describe the situation of interest, but we remain to introduce a solution concept to characterize what may happen in a play. Under the assumption of rational Bayesian DMs, this amounts to specifying how players form their quantitative beliefs. The purpose of this is to introduce a more operational model of quantitative beliefs, known as *epistemic type structure*. The major advantage of this "tool" is that it simplifies enormously the description of strategic reasoning, that is, the mental process through which players try to take into account the beliefs of their opponents to predict their rational behavior.

**3.1. Hierarchies of Beliefs.** So far we have modeled knowledge and beliefs with a very abstract model of the possible states of the world, introducing for every player information and plausibility correspondences. Under the assumption of common knowledge of  $\Gamma$  and Bayesian DM, it seems intuitive to look for a more operational way to model quantitative beliefs. Let  $\Gamma = \langle F, (A_f, u_f)_{f \in F} \rangle$  be a static game of complete information in reduced form. It is evident that for every player the only element of uncertainty is the profile of actions played by her opponents. However, it is reasonable to expect that an intelligent player will realize that the actions of her opponents depend on their beliefs. This simple observation suggests the introduction of hierarchies of beliefs.

For every  $f \in F$ , we endow the set  $A_f$  with its usual topology, assuming that one exists, and  $A$  with the corresponding product topology. We assume that for every  $f \in F$ ,  $A_f$  is a Polish space, therefore  $A$  is also Polish in the product topology. Let  $Z_{f,1} = A_{-f}$  be the *basic uncertainty space* of the player  $f \in F$ . A first-order belief, or *conjecture*, of  $f$ , denoted by  $h_{f,1}$ , is a probability measure  $h_{f,1} \in \Delta(Z_{f,1})$ . A second-order belief of  $f$ , denoted by  $h_{2,f}$  is a probability measure on the Borel  $\sigma$ -field of  $Z_{f,1}$  and the space of the first-order beliefs of her opponents, that is,

$$h_{f,2} \in \Delta(Z_{f,1} \times (\times_{j \in F \setminus \{f\}} \Delta(Z_{j,1}))).$$

Setting  $Z_{f,2} = Z_{f,1} \times (\times_{j \in F \setminus \{f\}} \Delta(Z_{j,1}))$ , we have, trivially,  $h_{f,2} \in \Delta(Z_{f,2})$ . A third-order belief of  $f$ , denoted by  $h_{f,3}$ , of  $f$  is a probability measure on the Borel  $\sigma$ -field of  $Z_f^2$  and the space of the second-order beliefs of  $-f$ , that is,  $h_{f,3} \in \Delta(Z_{f,3})$  where  $Z_{f,3} = Z_{f,2} \times (\times_{j \in F \setminus \{f\}} \Delta(Z_{j,2}))$ . Continuing in this fashion, the definition of a belief of the  $k$ -th order, for a generic  $k \in \mathbb{N}$ , can be obtained by induction.

DEFINITION 1.1. Let  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  be a game of complete information. A function  $h = (h_f)_{f \in F}$  and a sequence  $Z = (\times_{k \in \mathbb{N}} Z_{f,k})_{f \in F}$  define a profile of belief hierarchies  $h$  defined on  $Z$ , if, for every  $f \in F$ :

- $Z_{f,1} = A_{-f}$  and  $h_{f,1} \in \Delta(A_{-f})$ ;
- for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $Z_{f,k} = Z_{f,k-1} \times (\times_{j \in F \setminus \{f\}} \Delta(Z_{j,k-1}))$  and  $h_{f,k} \in \Delta(Z_{f,k})$ .

For every  $f \in F$ , and every  $k \in \mathbb{N}$ , we endow  $Z_{f,k}$  with the associated product, and  $\Delta(Z_{f,k})$  with the resulting weak\* topology. We let  $H_f^0 := \times_{k \in \mathbb{N}} \Delta(Z_{f,k}^k)$  be the space of all belief hierarchies of  $f$ , and endow it with the associated product topology. It follows that  $Z_{f,k}$ ,  $\Delta(Z_{f,k})$ , and  $H_f^0$  are Polish. Recalling the notion of projective consistency given in **Section 1.1.2.5**, a belief hierarchy  $h_f \in H_f^0$  is said to be *consistent* if, for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $\text{marg}_{Z_{f,k-1}} h_{f,k} = h_{f,k}$ . From now on, we shall assume that, in every state of the world  $\omega \in \Omega$ , every player  $f \in F$  has a consistent belief hierarchy. In this regard, we denote by  $H_f^1 \subseteq H_f^0$  the space of consistent belief hierarchies of a given player  $f \in F$ , which is also Polish in the relative topology, that is, the topology inherited from  $(H_f^0, \tau_\Pi)$ , where  $\tau_\Pi$  denotes the product topology of  $H_f^0$ . Consider a probability measure  $\mu_f \in \Delta(Z_f^1 \times (\times_{j \in F \setminus \{f\}} H_j^0)) = \Delta(A_{-f} \times H_{-f}^0)$ ; clearly,  $\mu_f$  induces a hierarchy  $h_f \in H_f^1$  of consistent beliefs. Since belief hierarchies are rather sophisticated objects, it would be interesting to understand whether the converse is also true, that is, when a consistent belief hierarchy can be represented as a probability measure on  $A_{-f} \times H_{-f}^0$ . Indeed, in order to translate predicates such as “player  $f$  believes that player  $j \neq f$  believes that player  $k \neq j$  plays action  $a_j$ ” it would be rather convenient to work with a probability measure defined over the set  $A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j^0)$ , which is denoted as the *extended uncertainty space* of player  $f$ . Under the topological assumptions made so far, as an application of the Kolmogorov extension theorem, Brandenburger and Dekel (1993) generalize the results of Mertens and Zamir (1985) and show that for every  $f \in F$ , there exists a homeomorphism

$$\beta_f : H_f^1 \rightarrow \Delta(A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j^0))$$

such that for every  $k \in \mathbb{N}$ ,  $\text{marg}_{Z_{f,k}} \beta_f(h_f) = h_{f,k}$ . Additional references on the construction of belief hierarchies include Böge and Eisele (1979), Heifetz (1993), Heifetz and Samet (1998), and Friedenbergl (2010).

**3.2. Types structures.** Fix a game  $\langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$ , and consider a player  $f \in F$ , and let  $\mu_f = \beta_f(h_f)$  be a probability measure  $\mu_f \in \Delta(A_{-f} \times H_f^0)$  equivalent to a hierarchy of consistent beliefs  $h_f \in H_f^1$ , where  $H^0 = \times_{f \in F} H_f$ . By construction, a generic belief hierarchy for  $f$  may attribute a probability different from zero to belief hierarchies of the other players that are not consistent. For this reason, we assume that, in all  $\omega \in \Omega$ , players have belief hierarchies satisfying consistency and mutual belief in consistency. In general, we say that the player  $f \in F$  *believes* in the event  $E \subseteq A_{-f} \times H_{-f}^0$  under  $\mu_f \in \Delta(A_{-f} \times H_{-f}^0)$  if  $\mu_f(E) = 1$ , and that  $h_f \in H_f^0$  satisfies common belief in consistency if, for  $l \in \mathbb{N}$ ,  $h_f \in H_f^l$ , where, for every  $l \in \mathbb{N} \setminus \{1\}$ ,

$$H_f^l = \left\{ h_f \in H_f^{l-1} : \beta_f(h_f)(A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j^{l-1})) = 1 \right\}.$$

In other words, every player holds consistent belief hierarchies, believes that the other players hold consistent belief hierarchies, believes that every other player believes that everyone else has consistent belief hierarchies, so on *ad infinitum*. To minimize technical jargon, and for reasons that will become clear in a moment, this assumption is defined as *common belief in consistency*. That is, common belief in consistency requires that every  $f \in F$  forms a belief hierarchy  $h_f \in \cap_{l \in \mathbb{N} \setminus \{1\}} H_f^l$ , and it is again a minimum epistemic requirement since a player who does not contradict herself has no reason to believe that her opponents would. Hence, the assumption that we shall maintain from now on is that, for every  $f \in F$ , belief hierarchies are restricted to those satisfying consistency and common belief consistency, that is,  $f$  always hold a belief hierarchy  $h_f \in H_f$ , where  $H_f = H^0 \cap (\cap_{l \in \mathbb{N}} H_f^l)$ . For every  $f \in F$ , we endow  $H_f$  with the relative topology induced by  $H_f^0$ . We define  $H$  as the set  $H := \times_{f \in F} H_f$  and endow it with the product topology, which is Polish (cf. **Section 1.1.3**). As a result, Brandenburger and Dekel (1993) again show that, as an application of the Kolmogorov Extension Theorem, for every  $f \in F$ , there exists a canonical homeomorphism

$$\eta_f : H_f \rightarrow \Delta(A_{-f} \times H_{-f})$$

such that, for every  $k \in \mathbb{N}$ ,  $\text{marg}_{Z_{f,k}} \eta_f(h_f) = h_{f,k}$ . In other words,  $\beta_f$  is one-to-one and onto when restricted to  $H_f$ . Hence, every  $\beta_f \in \Delta(A_{-f} \times H_{-f})$  is induced by one and only one belief hierarchy  $h \in H_f$  that satisfies consistency and common belief in consistency. Therefore, every set  $T$  such that

$T \subseteq H$  can be associated with a mathematical structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ , where each mapping  $\eta_f : T_f \rightarrow \Delta(A_{-f} \times T_{-f})$  is continuous and corresponds to a probability distribution on the Borel  $\sigma$ -field of  $A_{-f} \times T_{-f}$ , which we refer to as the extended uncertainty space of  $f$ . We refer to  $\mathcal{T}$  as an epistemic type structure for  $\Gamma$ , or a type structure for short, given its analogy with the framework developed by Harsanyi (1967) for games of incomplete information. When  $\Gamma$  is equipped with a type structure  $\mathcal{T}$ , we assume that  $\mathcal{T}$  is common knowledge among players. Given a type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \eta_f)_{f \in F} \rangle$  for  $\Gamma$ :

- $t_f \in T_f$  is said to be *type* of  $f$ , and  $t = (t_f)_{f \in F} \in T$  a profile of types or an *epistemic state*;
- an element  $(a, t) \in A \times T$  is identified as a *state of the world* as it provides with a full characterization of the actions and *epistemic state* that may occur in the game;
- $A \times T$  is said to be the extended uncertainty space of the game;
- if, for every  $f \in F$ ,  $T_f = H_f$ ,  $\mathcal{T}$  is said to be the universal type structure of  $\Gamma$ , as it contains all the hierarchies of consistent beliefs that satisfy the requirement of common belief in consistency.

An epistemic model of  $\Gamma$ , in which the game being played is common knowledge and no interpretational issues arise in relation to common knowledge and common belief, can always be obtained starting from an epistemic type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ , and identifying  $\Omega$  as  $A \times (\times_{f \in F} T_f)$ . The identification of  $A \times T$  as  $\Omega$  is a natural consequence of our assumptions on players' knowledge and consistency of beliefs, which ensure the existence of a bijection between the two sets preserving payoff-relevant information. In fact, under the assumption of common knowledge of the game being played and complete information, every pair  $\omega, \omega' \in \Omega$  may differ only in terms of a combination of the players' actions, beliefs about actions, beliefs about beliefs about actions, and so on. In other words, every state of the world induces an outcome of the game, although the same outcome may be associated with different belief hierarchies, as, by assumption, there are no states of the world in which the primitive components of the game are not common knowledge. For a rigorous proof of these claims, see Tan and Werlang (1992) and Brandenburger and Dekel (1993).

**3.3. Belief operator and common belief in type structures.** Except for **Section 1.4.4**, in the remainder of this thesis we always start from an epistemic type structure and identify  $A \times T$  as the set of all possible states of the world. For this reason, in this section we revisit the belief operator within the context of an epistemic type structure. Let  $\Gamma = \langle F, (A_f, u_f : A \rightarrow \mathbb{R})_{f \in F} \rangle$  be

a game and  $\mathcal{T} = \langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \rangle$  be a type structure for  $\Gamma$ . An event  $E$  is now defined as an element of  $\mathcal{B}(A \times T)$ , where  $\mathcal{B}(A \times T)$  is the relevant Borel  $\sigma$ -algebra. The type  $t_f \in T_f$  of player  $f$  is said to *believe in the event*  $E_{-f}$  if  $\beta_f(t_f)(E_{-f}) = 1$ , where

$$E_{-f} = \times_{j \in F \setminus \{f\}} (A_j \times T_j) \subseteq A_{-f} \times T_{-f}.$$

Accordingly, the *belief operator* for the player  $f$ , is defined as

$$B_f : A_{-f} \times T_{-f} \rightarrow A_f \times T_f$$

$$E_{-f} \mapsto B_f(E_{-f}) = \{(a_f, t_f) \in A_f \times T_f : \beta_f(E_{-f}) = 1\}.$$

Given a state of the world  $(a, t) \in A \times T$ , the event  $E$  is said to be mutually believed if, for every  $f \in F$ ,  $\beta_f(t_f)(E_{-f}) = 1$ . Thus, the event “ $E$  is mutually believed” is  $B(E)$ , where

$$B : 2^{A \times T} \rightarrow A \times T$$

$$E \mapsto B(E) = \times_{f \in F} B_f(E_{-f}).$$

Let  $E_0, E_1$  be any two subsets of  $A \times T$ ; the belief operator satisfies two important properties that are easy to verify, *monotonicity*

$$E_0 \subseteq E_1 \implies B(E_0) \subseteq B(E_1),$$

and *conjunction*

$$B(E_0 \cap E_1) = B(E_0) \cap B(E_1).$$

These properties are particularly relevant in the definition of mutual belief of  $k$ th-order in  $E$  and common belief in  $E$ . The event “ $E$  is commonly believed” is defined as the set  $CB(E) = \cap_{k \in \mathbb{N}} B^k(E)$ , while  $kB(E) = \cap_{l \in [k]} B^l(E)$  is the event “ $E$  is  $k$ th-order mutually believed”. In other words, the event  $E$  is mutually believed up to order  $k \in \mathbb{N}$  if everyone believes in  $E$  ( $B(E) = (B^1(E))$ ), everyone believes that everyone believes in  $E$  ( $B(B(E)) = B^2(E)$ ), and so on, up to  $B^k(E)$ . Notice that the interpretational issue discussed in section 1.2.2 does not arise in this context since, by definition, types are belief hierarchies; see also Tan and Werlang (1992) for additional details. Finally, the event “ $E$  is transparent” is  $ECBE = E \cap CB(E)$ , that is, for every  $(a, t) \in ECBE$ , the event  $E$  is true and commonly believed. We refer to Dekel and Siniscalchi (2015) for additional details.

#### 4. Epistemic analysis and solution concepts

Consider a static game in reduced form

$$\Gamma = \left\langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle$$

endowed with an epistemic type structure  $\mathcal{T} = \left\langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \right\rangle$ . An outcome of  $\Gamma$  is a profile  $a \in A$  of actions that can be observed in a play, that is, in a situation described by  $\Gamma$ . For avoidance of doubts, by a *play* we mean the simultaneous choice of actions made by the players. Without any additional assumption on players' behavior and reasoning process, a play may result in any arbitrary outcome  $a \in A$ . Thus, in order to characterize what can happen in the model, it is necessary to endow  $\Gamma$  with a solution concept, that is, a correspondence<sup>11</sup>  $\Sigma : \{A\} \rightrightarrows A$  restricting  $A$  to a subset of possible outcomes. How do we choose which solution concept is more appropriate? Unfortunately, there is no universal answer. However, given some intuitive notions of behavioral and epistemic assumptions, we already noted that solution concepts can be obtained from the combination of these two. Hence, the question reduces to which assumptions are appropriate to model a certain social phenomenon of interest. As said, we argued in favor of behavioral assumptions that are consistent with self-interested behavior, or *rational behavior*.

In general, a behavioral assumption  $BA_f$  for the player  $f \in F$  is a *condition* that restricts the action-type pairs of  $f$  to the set  $BA_f = A'_f \times T_f$ , where  $A'_f$  is a strict subset of  $A_f$ , that is,  $A'_f \subset A_f$ . Notice the different use of characters to distinguish the assumption, denoted by roman  $BA_f$ , from the corresponding restriction on  $A_f \times T_f$ , denoted by italic  $BA_f$ . Similarly, an epistemic assumption  $EA_f$  for the player  $f \in F$  is a *condition* that restricts  $A_f \times T_f$  to  $EA_f = A_f \times T'_f$ , where  $T'_f$  is a subset of  $T_f$ . Evidently, behavioral assumptions restrict the actions that may be observed as a function of players' beliefs, whereas epistemic assumptions impose restriction on belief hierarchies, that is, how strategic thinking unfolds in a play. Hence, a profile  $\mathcal{S} = (BA_f, EA_f)_{f \in F}$  induces a solution concept, denoted by  $\Sigma^{\mathcal{S}}$ ,

$$\Sigma^{\mathcal{S}} : \{A\} \rightrightarrows A$$

$$A \mapsto \text{proj}_A ((\times_{f \in F} BA_f) \cap (\times_{f \in F} EA_f)).$$

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<sup>11</sup>We do not make the blanket assumption that a game must have a unique solution for a given set of hypotheses on players' reasoning and behavior.

An action pair  $(a_f, t_f)$  is consistent with rational behavior if,

$$a_f \in \operatorname{argmax}_{a'_f \in A_f} \left\{ \mathbb{E}_{\operatorname{marg}_{A_{-f}} \beta_f(t_f)} \left[ u_{a'_f} \right] \right\}.$$

In this regard, the correspondence  $r_f$ ,

$$\begin{aligned} r_f : \Delta(A_{-f}) &\rightrightarrows A_f \\ \mu_f &\mapsto \operatorname{argmax}_{a'_f \in A_f} \left\{ \mathbb{E}_{\mu_f} \left[ u_{a'_f} \right] \right\}, \end{aligned}$$

is said to be the best-reply, or best-response, correspondence of  $f$ , as it maps a conjecture to the corresponding set of rational actions. With abuse of notation, if, for every  $\mu_f \in \Delta(A_{-f})$ ,  $r_f(\mu_f)$  is a singleton, we say that there exists a best-reply, or best-response, function

$$\begin{aligned} r_f : \Delta(A_{-f}) &\rightarrow A_f \\ \mu_f &\mapsto \iota \operatorname{argmax}_{a'_f \in A_f} \left\{ \mathbb{E}_{\mu_f} \left[ u_{a'_f} \right] \right\}, \end{aligned}$$

where, for a generic singleton  $X$ ,  $\iota X$  denotes its unique element. The abuse of notation stands clearly in the use of the same letter, that is,  $r$ , to denote both a single-valued correspondence and a function returning its unique value. Since there is a clear isomorphism, such an abuse is legitimate when the nature of  $r$ , being a correspondence or a function, is evident from the context.

We denote by  $R_f$  the behavioral assumption that imposes the condition that the player  $f \in F$  is rational; hence,

$$R_f = \{(a_f, t_f) \in A_f \times T_f : a_f \in r_f(\beta_f(t_f))\}.$$

Every  $a_f \in \operatorname{proj}_{A_f} R_f$  is said to be justifiable. We denote by  $R$  the assumption that every player is rational and by  $R$  the set  $R = \times_{f \in F} R_f$ . Given that expected utility provides only an alternative representation of consistent preferences, we view the assumption of rationality as almost tautological. In addition, preferences may be extended to include psychological traits, such as guilt, envy, altruism, etc. The vast literature in psychological games has shown that this is essentially equivalent to extending the argument of players' utilities to including the types of their opponents; see Genakopolos et al. (1989), Battigalli and Dufwenberg (2009), and Battigalli et al. (2019) for a reference. For reasons of space, we shall refrain from seeking this level of generality, which is nevertheless of a certain relevance also for applications in finance. Generally speaking, we say that an economic model is of rational choice if it consists of a strategic-form game equipped with a solution concept consistent with  $R$ . In this regard, a solution concept  $\Sigma$  for  $\Gamma$  is said to be supported by the epistemic

conditions associated with the profile  $(EA_f)_{f \in F}$  if there exists  $\mathcal{S} = (R_f, EA_f)_{f \in F}$  such that

$$\Sigma : \{A\} \rightrightarrows A$$

$$A \mapsto \text{proj}_A ((\times_{f \in F} R_f) \cap (\times_{f \in F} EA_f)).$$

Clearly, not every epistemic assumption is equivalently reasonable, so care is needed when taking the indirect approach of specifying a solution concept and then looking for epistemic assumptions that, coupled with the minimal behavioral requirement of rationality, result in the same solution concept.

The most common epistemic assumption in epistemic game theory is common belief rationality (CBR), which essentially aims to capture the players' reasoning about the implications of their rationality in the absence of cognitive boundaries. Essentially, every player understands that everyone else is rational and that everyone else can understand the same fact for her opponents, and continues to iterate the implications of this type of reasoning *ad infinitum*, that is, performing countably infinite reasoning steps. Formally, common belief in rationality amounts to imposing epistemic assumptions such that, for every  $f \in F$ , action-type pairs must be in the set  $B(R_f) \cap \left( \bigcap_{k \in \mathbb{N} \setminus \{1\}} B^k \left( B_{-f}^{k-1}(R) \right) \right)$ .

By the monotonicity and conjunction properties of the belief operator, we have

$$\times_{f \in F} \left( B(R_f) \cap \left( \bigcap_{k \in \mathbb{N} \setminus \{1\}} B^k \left( B_{-f}^{k-1}(R) \right) \right) \right) = \times_{f \in F} CB_f(R_{-f}) = CB(R),$$

see footnote 11 in Dekel and Siniscalchi (2015) for a proof. By rationality and common belief in rationality (RCBR) we mean the collection of behavioral and epistemic assumptions that restrict the set of states of the world  $A \times T$  to the subset  $RCBR = R \cap CB(R)$ . Notice that rationality is transparent at  $RCBR$ , being true and commonly believed.

The epistemic assumption of common belief in rationality is based on two implicit axioms. The first is that every player can perform “infinite strategic reasoning steps”, which is equivalent to saying that players do not exhibit cognitive boundaries; rational players without cognitive boundaries are said to be boundlessly rational. The second axiom is that players are willing to believe in the unbounded rationality of their opponents. For this reason, RCBR is often considered the formal definition of unbounded rationality, as it requires the highest degree of strategic reasoning about rationality and mutual belief in rationality. Whether it is a sensible assumption in applications is, of course, up to debate. Certainly, it represents a fundamental benchmark against which other hypotheses should be tested or compared. However, in many applications, the introduction of cognitive boundaries can

support the development of better and more realistic models of rational choice. There are many ways to introduce cognitive boundaries and, given the relevance for the model presented in **Chapter 2**, we limit the attention to bounded reasoning about rationality, see the introduction of Brandenburger (2014) as well as Brandenburger et al. (2021) and Friedenberg and Kneeland (2024) for additional references. Bounded reasoning about rationality amounts to fixing an integer  $m \in \mathbb{N}$  and imposing the epistemic assumption that, for every  $f \in F$ ,  $T_f$  is restricted to  $\text{proj}_{T_f} \cap_{l \in [m]} B_f^l(R_{-f})$ . In other words, bounded reasoning about rationality is equivalent to  $m$ -th order mutual belief in rationality for some  $m \in \mathbb{N}$ . For this reason, rationality bounded reasoning about is generally denoted as  $\text{RmBR}$ , and, given the particular choice of  $m \in \mathbb{N}$ , the resulting restriction on the set of states of the world is denoted by  $RmBR$ , see also Brandenburger et al. (2020) for a reference. It is important to stress that for a given game  $\Gamma$ , there might exist a type structure such that, for every  $m > k$ ,  $k \in \mathbb{N}$ ,  $RmBR = RCBR$ . In such a case, it becomes impossible to distinguish whether a play conforms to the hypothesis of rationality of common belief in rationality or  $\text{RkBR}$ . That is clearly a potential empirical issue.

At this point, the definition of a static economic model should now be clear. The structure of  $\Gamma$  intends to describe the rules of the game and the preferences of the players, while  $\mathcal{T}$  provides a descriptive tool for strategic thinking; finally, to characterize a play, a solution concept can be imposed, starting from (reasonable) behavioral and epistemic assumptions. In this regard, it is important to recognize that every economic model is false by definition, as it does not exist as a true entity in the objective reality. Yet, a model can be used to explain with a reasonable margin of error certain patterns observed in reality. An open question is whether an economic model that allows multiple outcomes is *falsifiable*. We are of the opinion that there is not a unique answer, and much depends on the context. Rather, we strongly believe that economic models should be viewed as simulation devices through which we can possibly generate *artificial intelligence*. In fact, solution concepts, when properly designed, ensure that outcomes are the result of (more or less) sophisticated strategic thinking and purposeful decision making. The remainder of this chapter is left to the discussion of some notable solution concepts, starting from the ubiquitous Nash equilibrium.

**4.1. Nash equilibrium.** Consider a game  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  endowed with the type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \rangle$ , and suppose that there exists one and only one outcome consistent with  $\text{RCBR}$ , that is,  $\text{proj}_A \text{RCBR} = \{a^*\}$ . In such a case,  $\text{RCBR}$  ensures that,

for every player  $f \in F$ ,

$$a_f^* \in r_f \left( \delta_{a_{-f}^*} \right),$$

where  $\delta_{a_{-f}^*} : A_{-f} \rightarrow [0, 1]$  is the Dirac measure operator, that is, for  $a_{-f} \in A_{-f}$ ,

$$\delta_{a_{-f}^*} (a_{-f}) = \begin{cases} 1 & a_{-f} = a_{-f}^* \\ 0 & a_{-f} \neq a_{-f}^*. \end{cases}$$

In other words, if there is a unique outcome consistent with rationality and common belief in rationality, then it must be the case that every player best-responds to a belief assigning probability one to the actions taken by her opponents. That is, players are rational and their conjectures are correct. To prove this claim, observe that  $\delta_{a_{-f}^*} \in \Delta(A_f)$ . In addition, for every  $f \in F$ , we have that

$$(a_f, t_f) \in RCBR_f \implies (a_f, t_f) \in B_f(RCBR_{-f}),$$

since, by the monotonicity and conjunction property of the belief operator, it holds that

$$B_f(RCBR_{-f}) = B_f(R_{-f}) \cap B_f(CB_f(R)) = B(R_f) \cap \left( \bigcap_{k \in \mathbb{N} \setminus \{1\}} B^k \left( B_{-f}^{k-1}(R) \right) \right) = CB(R).$$

As a result, for every  $(a_f, t_f) \in RCBR_f$ ,  $\beta_f(t_f) \left( \left( \text{proj}_{A_{-f}} RCBR \right) \times T_f \right) = 1$ . Since, by hypothesis,  $\text{proj}_A RCBR = \{a^*\}$ , it follows that  $\text{marg}_{A_{-f}} \beta_f(t_f) = \delta_{a_{-f}^*}$ . Finally, RCBR implies that, for every  $f \in F$ ,  $(a_f^*, t_f) \in R_f$ , that is,  $a_f^* \in r_f \left( \text{marg}_{A_{-f}} \beta_f(t_f) \right)$ . Therefore, for every  $f \in F$ , we can conclude that  $a_f^* \in r_f \left( \delta_{a_{-f}^*} \right)$ .

A Nash equilibrium is a profile of action  $a^* \in A$  such that, for every  $f \in F$ ,  $a_f^* \in r_f \left( \delta_{a_{-f}^*} \right)$ . In other words, Nash Equilibrium is a solution concept

$$\Sigma^{NE} : \{A\} \rightrightarrows A$$

$$A \mapsto \{a^* : \forall f \in F, a_f^* \in r_f \left( \delta_{a_{-f}^*} \right)\}.$$

In a Nash equilibrium, players are rational and hold correct conjectures in the sense that every player assigns probability one to the profile of actions effectively played by her opponents. That, if  $a^* \in A$  is a Nash equilibrium, there exists a type structure for  $\Gamma$  such that  $a^* \in \text{proj}_A RCBR$ . The proof of this result will be provided in **Section 1.3.4**. We stress that a game may have more than one Nash equilibrium, as is evident from the following example known as *the Battle of the Sexes*.

EXAMPLE 1.2. Alice and Bob agreed to meet on a given day at 10 in the morning, but they do not remember whether they agreed to play tennis or golf together. They cannot communicate with each other, and it is common knowledge that tennis can be played only at a specific tennis club and golf at a specific golf club. Alice and Bob also have the following preferences on the set of outcomes resulting from their actions, that we restrict for both to go either to the tennis or to the golf club. They both prefer outcomes in which they meet at the same club to those in which they do not meet each other. However, Alice prefers to play tennis with Bob rather than golf, while Bob prefers to play golf with Alice rather than tennis. This is clearly an example of social interaction that can be described by a game in strategic form, which is known as a coordination game. For example, we can model the situation involving Alice and Bob as a game

$$\Gamma = \langle F = \{Alice, Bob\}, (A_f = \{Tennis, Golf\}, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in \{Alice, Bob\}} \rangle,$$

such that

$$u_{Alice} : A_{Alice} \times A_{Bob} \rightarrow \mathbb{R}$$

$$a \mapsto \begin{cases} 1 & a_{Alice} = a_{Bob} = Tennis \\ 0 & a_{Alice} = a_{Bob} = Golf \\ -1 & a_{Alice} \neq a_{Bob} \end{cases}$$

and

$$u_{Bob} : A_{Bob} \times A_{Alice} \rightarrow \mathbb{R}$$

$$a \mapsto \begin{cases} 1 & a_{Alice} = a_{Bob} = Golf \\ 0 & a_{Alice} = a_{Bob} = Tennis \\ -1 & a_{Alice} \neq a_{Bob}. \end{cases}$$

We also note that  $\Gamma$  could be represented graphically as in the table below.

|                | Tennis (Bob) | Golf (Bob) |
|----------------|--------------|------------|
| Tennis (Alice) | (1, 1)       | (-1, -1)   |
| Golf (Alice)   | (-1, -1)     | (0, 1)     |

Comparing the utility of different outcomes, we conclude that  $\Gamma$  has two Nash equilibria, that is,

$$(a_{Alice} = Tennis, a_{Bob} = Tennis)$$

and

$$(a_{Alice} = Golf, a_{bob} = Golf).$$

We say that an economic model is an equilibrium model, or an equilibrium theory, if it is equipped with the solution concept of Nash equilibrium. Most of economic theories that assume rational behavior adopt an equilibrium approach; implicitly, implicitly, they rely on the epistemic assumption of correct conjectures encoded in the definition. Such an assumption is often motivated by rationality and common belief in rationality; see, for example, pp. 206 in Chapter 15 of Varian (1992). However, this argument is not necessarily valid, as there may be non-equilibrium outcomes consistent with  $RCBR$ . However, if  $\text{proj}_A RCBR = \{a^*\}$ , that is, there exists a unique outcome consistent with  $RCBR$ , then it must necessarily be a Nash equilibrium. The popularity of equilibrium theories is their simplicity, as they effectively assume that players face no uncertainty on the actions taken by their opponents. In particular, equilibrium solution concepts are easier to test empirically and the related statistical procedures are known as structural methods.

4.1.1. *Nash equilibrium as a fixed-point.* We define as joint point best-reply (PBR), denoted by  $\hat{r}$ , the correspondence

$$\hat{r} : A \rightrightarrows A$$

$$a \mapsto \times_{f \in F} \hat{r}_f(a_{-f}),$$

where, for every  $f \in F$ ,

$$\hat{r}_f : A_{-f} \rightrightarrows A_f$$

$$a_{-f} \mapsto \text{argmax}_{a_f \in A_f} \{u_f(a_f, a_{-f})\}.$$

In other words,  $\hat{r}_f(a_{-f})$  returns the set of actions that are rational for  $f$  when  $f$  is certain. that is, attributes probability one, that her opponent will play the profile  $a_{-f}$ . As a result,  $a^* \in A$  is a Nash equilibrium of  $\Gamma$  if and only if  $a^* = \hat{r}(a^*)$ . Hence, a Nash equilibrium (NE) is a fixed point of  $\hat{r}$ . This observation suggests the application of fixed point theorems to prove the existence of NE. In addition, it provides a possible justification for the choice of the word "equilibrium", which in mathematics is often a synonym of fixed point. In this regard, it is important not to confuse the concept of equilibrium in economics with the homonymous in natural sciences, which denotes a steady-state of a dynamical physical system.

4.1.2. *Existence of a Nash Equilibrium and Mixed Extension of a Game.* The existence of a NE for a game  $\Gamma = \langle F, (A_f, u_f : A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  should not be taken for granted. The following proposition provides sufficient conditions.

PROPOSITION 1.3. Consider a game  $\Gamma = \left\langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle$  such that, for every player  $f \in F$ :

- (1)  $A_f$  is a compact and convex subset of  $\mathbb{R}^F$  in the usual topology;
- (2)  $u_f$  is a continuous function on  $A_f \times A_{-f}$ , in the usual topology of  $\mathbb{R}^F$  ;
- (3)  $u_f$  is quasi-concave in  $a_f \in A_f$  for every  $a_{-f} \in A_{-f}$ .

There exists a profile of actions  $a^* \in A$  such that, for every  $f \in F$ ,  $a_f^* \in \hat{r}_f(a_{-f}^*)$ . That is,  $\Gamma$  has at least a Nash equilibrium.

PROOF. The proof is an application of the Kakutani fixed-point theorem; see Chapter 5 in Battigalli et al. (2024a) for a reference.  $\square$

However, there are many examples of games that do not admit any NE, such as *rock, paper, scissors* or *matching pennies*; see Osborne and Rubinstein (1994) for a reference and additional examples. One way to overcome this limitation is to introduce a randomizing device for each player. Evidently, this is equivalent to considering a new game with the same set of players but different actions, which is denoted as the mixed extension of the original game considered. For space reasons, we limit the discussion to finite games, that is, games in which not only the number of players is finite, but also the action space of each player is finite, that is,  $|A_f| \in \mathbb{N}$  for every  $f \in F$ . Namely, for a given game  $\Gamma = \left\langle F, (A_f, u_f : A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle$ , the mixed extension of  $\Gamma$  is defined as,

$$\bar{\Gamma} = \left\langle F, (\Delta(A_f), \bar{u}_f : \Delta(A_f) \times (\times_{i \in F \setminus \{f\}} \Delta(A_i)) \rightarrow \mathbb{R})_{f \in F} \right\rangle,$$

where  $\Delta(A_f)$  is the action's space of a generic  $f \in F$ ,  $\bar{u}_f = \sum_{a \in A} u_f(a_f, a_{-f}) \alpha_f \prod_{i \in F \setminus \{f\}} \alpha_{-f}(a_i)$  and  $\alpha_f \in \Delta(A_f)$  for every  $f \in F$ . The actions of the players in  $\bar{\Gamma}$  are known as *mixed actions* and should not be confused with additional actions available in the original game  $\Gamma$ . In particular, a mixed action of the player  $f$  is interpreted as a data-generating process, or pseudo-random number generators; see **Section 1.1.6** for the related definition. In this regard,  $A$  can be *embedded* in  $\times_{f \in F} \Delta(A_f)$  using Dirac measures. Consequently, the set of Nash equilibrium points of  $\Gamma$  can be embedded in the set of NE of  $\bar{\Gamma}$ . Notice that the construction of preferences is rather arbitrary and we should refrain from the temptation to attribute a precise epistemic interpretation. The relevance of mixed extensions is provided by the following technical result.

PROPOSITION 1.4. Consider a finite game  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$ , and let

$$\bar{\Gamma} = \left\langle F, (\Delta(A_f), \bar{u}_f : \Delta(A_f) \times (\times_{i \in F \setminus \{f\}} \Delta(A_i)) \rightarrow \mathbb{R})_{f \in F} \right\rangle$$

be the corresponding mixed extension. There exists a profile of actions  $\alpha^* \in \times_{f \in F} \Delta(A_f)$  such that, for every  $f \in F$ ,  $\alpha_f^* \in \hat{r}_f(\alpha_{-f}^*)$ . That is,  $\bar{\Gamma}$  has at least a Nash equilibrium.

PROOF. The proof is a special case of the more general result in Glicksberg (1952), which provides a further generalization of the Kakutani fixed-point theorem.  $\square$

4.1.3. *Mixed Equilibrium.* In reality, individuals do not make use of random number generators to take actions. Hence, we may be tempted to conclude that mixed strategies are just a tantamount to proving the existence of a NE. However, it turns out that there is a much more profound interpretation of mixed strategies and Nash equilibria of  $\hat{\Gamma}$ . This interpretation is central to the identification of sufficient epistemic conditions for a NE outcome in the original game, other than the questionable assumption of correct beliefs. The core idea is that a mixed strategy of the player  $f$  could also be the conjecture of a player  $j \neq f$  on the behavior of  $f$ . This may suggest a broader definition of NE, which we denote by *mixed equilibrium*, involving the conjectures of the players rather than their actions. This intuition is actually a direct consequence of the following result.

PROPOSITION 1.5. Consider a finite game  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$ , and let

$$\bar{\Gamma} = \left\langle F, (\Delta(A_f), \bar{u}_f : \Delta(A_f) \times (\times_{i \in F \setminus \{f\}} \Delta(A_i)) \rightarrow \mathbb{R})_{f \in F} \right\rangle$$

be the corresponding mixed extension and  $\alpha^* \in \times_{f \in F} \Delta(A_f)$  a NE of  $\bar{\Gamma}$ . For every player  $f \in F$ ,

$$\alpha_f^{**} \in \operatorname{argmax}_{\alpha_f \in \Delta(A_f)} \bar{u}_f(\alpha_f, \alpha_{-f}^{**}) \iff \operatorname{supp} \alpha_f^{**} \subseteq r_f \left( \prod_{j \in F \setminus \{f\}} \alpha_j^{**} \right).$$

PROOF. The proof is a direct application of Lemma 1 in Chapter 3 of Battigalli et al. (2024a), which we refer to also for the related proof.  $\square$

The previous proposition shows that  $\alpha^{**} \in \times_{f \in F} \Delta(A_f)$  is a NE of  $\bar{\Gamma}$  if and only if there exists a profile of conjectures  $(\mu_f^{**})_{f \in F} \in \times_{f \in F} \Delta(A_f)$  in the original game  $\Gamma$  such that for every  $f \in F$ :

- (1)  $\mu_f^{**} = \prod_{j \in F \setminus \{f\}} \alpha_j^{**}$ ,
- (2)  $\operatorname{supp} \alpha_f^{**} \subseteq r_f(\mu_f^{**})$ .

This observation motivates the following definition of *equilibrium beliefs*, or *mixed equilibrium*.

DEFINITION 1.6. Consider a finite game  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$ . A profile of beliefs  $\alpha^{**} \in \times_{f \in F} \Delta(A_f)$  is a *mixed equilibrium* of  $\Gamma$  if it is a NE of its mixed extension

$$\bar{\Gamma} = \langle F, (\Delta(A_f), \bar{u}_f : \Delta(A_f) \times (\times_{i \in F \setminus \{f\}} \Delta(A_i)) \rightarrow \mathbb{R})_{f \in F} \rangle.$$

As a bibliographical note, Nash (1950) introduced the definition of mixed equilibrium to deal with games without Nash equilibria. To better understand the previous definition, consider two different players, say  $f$  and  $j$ , and a third player  $k$  different from the previous two; with a few algebraic steps, it is easy to show that  $\text{marg}_{A_k} \mu_f^{**} = \text{marg}_{A_k} \mu_j^{**} = \alpha_k^{**}$ . In other words, every two players share the same conjecture about the behavior of a common opponent. Whenever this condition is verified, we say that players' conjectures are generated by the profile of beliefs  $\alpha \in \times_{f \in F} \Delta(A_f)$ . In addition, the support of  $\alpha_f^{**}$  includes only actions that  $f$  views as optimal in response to her conjecture generated by  $\alpha^{**}$ . Since the set of Nash equilibria of  $\Gamma$  can be embedded in that of  $\bar{\Gamma}$ , it follows that every NE of  $\Gamma$  is associated with a mixed equilibrium with deterministic conjectures. Similarly, every mixed equilibrium with deterministic conjectures is a NE of  $\Gamma$ . Hence, a Nash equilibrium can be equivalently defined as a mixed equilibrium with deterministic beliefs. As we shall see in the following paragraph, viewing equilibrium concepts as restriction on players' beliefs rather than on their actions allows to establish deeper epistemic results.

4.1.4. *Epistemic conditions for Mixed Equilibrium.* In his Ph.D. dissertation, John Nash motivated the adoption of equilibrium as a necessary implication of the existence of a unique solution for a given game (Nash, 1950). In short, Nash argued that since an external observer could have predicted the outcome of the game, the same could have been argued for rational players, leading to an equilibrium outcome. However, the possibility of multiple solutions and, in particular, multiple Nash equilibria makes this argument quite fragile, despite its intuitive appeal. However, it did not prevent economic theory from adopting, almost dogmatically, an equilibrium approach until the mid 80s of the twentieth century. As for every solution concept, we should provide robust motivation based on explicit behavioral and epistemic assumptions. A trivial epistemic condition for a NE, at least when it is unique, is the joint hypothesis of rational players and correct conjectures. However, the epistemic assumption of correct conjectures is a shortcut, or perhaps even a circular reference, to motivate a NE, in fact its definition. Assuming correct beliefs offers no insight into the players' reasoning process. Whether there exist more primitive epistemic assumptions inducing correct conjectures is far from obvious, except for the case where RCBR is assumed and  $|\text{proj}_A \text{RCBR}| = 1$ . For

this reason, in the epistemic game literature, Nash equilibrium is sometimes defined as a restriction on belief hierarchies rather than as a solution concept, equivalent to the concept of a *mixed equilibrium* given above. In such a case, there are well-known epistemic conditions. The idea of considering the set of mixed equilibria, which embeds, if exists, also the set of NE, goes back to Brandenburger and Dekel (1987b) and Tan Werlang (1988), and was later exploited by Aumann and Brandenburger (1995), Polak (1999), and Perea (2007). Namely, in every finite game endowed with an epistemic model in which the game being played is common knowledge, if, in a given state of the world, the players are Bayesian rational, there is a common prior, rationality is mutual knowledge, and there is common knowledge of conjectures, then the profile of players' conjectures in the same state of the world is a mixed equilibrium. Evidently, these conditions are rather strong (cf. Chapter 4 in Perea, 2012) and do not necessarily imply an NE outcome in the original game, unless the equilibrium conjecture of every player is a point-conjecture. In any case, when multiple (mixed) equilibria exist, their selection remains an open issue despite the large body of literature on the topic; for additional details on equilibrium selection, we refer to the monograph of Harsanyi and Selten (1988). An alternative epistemic characterization of mixed equilibrium is discussed in **Section 1.4.4**.

**4.2. Rationalizability.** In this section, we present one of the most relevant solution concepts in game theory, namely rationalizability, which was originally developed by Pearce (1984) and Bernheim (1984) in two seminal papers. In short, rationalizability consists in iteratively removing actions that progressively become not justifiable. Our treatment of rationalizability follows Tan and Werlang (1988), Epstein (1997), Arieli (2010), and Chapter 4 of Battigalli et al. In this regard, a game

$$\Gamma = \left\langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle,$$

is said to be compact-continuous if, for every  $f \in F$ ,  $A_f$  is a compact and non-empty Polish space, and  $u_f$  is a continuous function on  $A = \times_{f \in F} A_f$  when  $A$  is endowed with the product topology. Notable examples of compact-continuous games are finite games, when the action space of each player is endowed with the discrete topology, and games in which, for every  $f \in F$ ,  $A_f$  is a compact subset of a common Euclidean space. We let  $\mathcal{K}_{A_f}$  be the collection of closed subsets of  $A_f$ , and define

$$(4.1) \quad \mathcal{C} = \left\{ C \subseteq 2^A : \exists (C_f)_{f \in F} \subseteq \times_{f \in F} \mathcal{K}_{A_f}, C = \times_{f \in F} C_f \right\},$$

as the collection of compact Cartesian subsets of  $A$ . Following Battigalli et al. (2024a), we denote by  $\rho$  the rationalization operator, defined as

$$\begin{aligned} \rho : \mathcal{C} &\rightarrow \mathcal{C} \\ C &\mapsto \times_{f \in F} r_f(\Delta(C_{-f})). \end{aligned}$$

For a given set  $C \in \mathcal{C}$ , if  $a \in \rho(C)$ , it must be the case that, for every  $f \in F$ , there exists a conjecture  $\mu_f \in \Delta(C_{-f})$  such that  $a_f \in r_f(\mu_{-f})$ . In other words,  $\rho(C)$  identifies the set of justifiable action profiles if, for every  $f \in F$ , conjectures are restricted to assign probability one to  $C_{-f}$ . It is immediate to verify that:

- $\forall C \in \mathcal{C} \setminus \emptyset \implies \rho(C) \neq \emptyset$  (non-emptiness);
- $C, C' \in \mathcal{C}, C' \subseteq C \implies \rho(C') \subseteq \rho(C)$  (monotonicity).

Given a set  $C \in \mathcal{C}$ , we also define  $\rho^0(C) := C$ , and, for every  $k \in \mathbb{N}$ ,  $\rho^k(C)$  inductively as  $\rho^k(C) := \rho(\rho^{k-1}(C))$ . In addition, we define  $\rho^\infty(A) = \bigcap_{k \in \mathbb{N}} \rho^k(A)$  as the set of iteratively justifiable strategies.

DEFINITION 1.7. Let  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  be a game with cardinal payoffs. A set  $C \subseteq \mathcal{C}$  is said to be a best-reply, or a best-response, set (BRS) if  $C \subseteq \rho(C)$ . A profile of actions  $a \in A$  is said to be rationalizable if there exists a BRS set  $C \in \mathcal{C}$  such that  $a \in C$ . The set of rationalizable actions is the union of best-response sets.

To understand the meaning of definition, suppose that  $a \in A$  is rationalizable; since  $\rho(\cdot)$  is weakly monotone, for every  $k \in \mathbb{N}$ , we have  $a \in \rho^k(C) \subseteq \rho^k(A)$ . Thus, a rationalizable profile of  $a$  can be justified at every iteration of the rationalization operator. In fact, we have the following result.

PROPOSITION 1.8. Let  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  be a compact-continuous game, and let  $\rho^\infty(A) = \bigcap_{k \in \mathbb{N}} \rho^k(A)$ .

- (1) For every  $k \in \mathbb{N}$ ,  $\rho^k(A)$  is a compact and non-empty subset of  $A$ .
- (2)  $\rho^\infty(A)$  is non-empty.
- (3) An action  $a \in A$  is rationalizable if and only if  $a \in \rho^\infty(A)$ .
- (4) If  $\Gamma$  is finite, there exists  $m \in \mathbb{N}$  such that  $\rho^m(A) = \rho^{m+1}(A) = \rho^\infty(A)$ .

PROOF. 1) Since  $A$  is non-empty and, for every  $f \in F$ ,  $r_f(A)$  is non-empty, since  $f$  must at least weakly prefer one action to the others available, we have  $\rho(A) = \rho^1(A)$  is non-empty. By

induction, it follows that for every  $k \in \mathbb{N}$ ,  $\rho^k(A)$  is a closed and non-empty subset of  $A$ , and since  $A$  is compact,  $\rho^k(A)$  is also compact. 2) For every  $k \in \mathbb{N}$ ,  $\rho^{k+1}(A) \subseteq \rho^k(A)$ , therefore, for every  $l, m \in \mathbb{N}$ ,  $\rho^l(A) \cap \rho^m(A)$  is a closed and non-empty subset of  $A$ . Since  $A$  is compact, for every  $l, m \in \mathbb{N}$ ,  $\rho^l(A) \cap \rho^m(A)$  is compact and non-empty, therefore, by the finite intersection property,  $\bigcap_{k \in \mathbb{N}} \rho^k(A) := \rho^\infty(A)$  is non-empty. 3) Omitted for space reasons; see the Appendix in Chapter 4 in Battigalli et al. (2024a). 3) For the second part, assume that  $\Gamma$  is finite. Since  $A$  is non-empty, it follows that, for every  $k \in \mathbb{N}$ ,  $\rho^k(A)$  is non-empty. Since  $\rho$  is weakly monotone, it follows that  $\rho^\infty(A)$  contains at least one profile of actions. Therefore, the maximum number of iterations to obtain  $\rho^\infty(A)$  is  $m = \prod_{f \in F} (|A_f| - |\text{proj}_f \rho^\infty(A)|)$ . As a result,  $\rho^m(A) = \rho^{m+1}(A) = \rho^\infty(A)$   $\square$

It is important to stress that, for games that are not compact-continuous, the equivalence between the set of rationalizable strategies and  $\rho^\infty$  is not granted. In fact, contrary to the belief operator, the rationalization operator does not satisfy the conjunction property. Although the monotone property of  $\rho(\cdot)$  ensures that  $\rho(\rho^\infty(A)) \subseteq \rho^\infty(A)$ ,  $\rho^\infty(A) \subseteq \rho(\rho^\infty(A))$  is not necessarily true, unless we consider compact-continuous games, and it is exactly for this reason that we can establish the equivalence between the general definition of rationalizability above and  $\rho^\infty$ . The following theorem, which is adapted from Werlang (1986) and Tan and Werlang (1988), provides necessary and sufficient epistemic conditions for rationalizability in compact-continuous games. In particular, if  $\Gamma$  is endowed with the corresponding universal type structure,  $\rho^\infty(A) = \text{proj}_A \text{RCBR}$ , that is, a profile of actions is rationalizable if and only if it is supported by RCBR. Hence, rationalizability can be seen as an algorithm to identify the set of outcomes consistent with RCBR.

PROPOSITION 1.9. *Let  $\Gamma = \langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  be a compact-continuous game.*

- (1) *Given a type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \rangle$  for  $\Gamma$ , for every  $k \in \mathbb{N} \cup \{0\}$ ,  $\text{proj}_A(RkBR) \subseteq \rho^{k+1}(A)$ . If  $\mathcal{T}$  is universal, for every  $k \in \mathbb{N} \cup \{0\}$ ,  $\text{proj}_A(RkBR) = \rho^{k+1}(A)$ .*
- (2) *Given a type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \rangle$  for  $\Gamma$ ,  $\text{proj}_A(\text{RCBR}) \subseteq \rho^\infty(A)$ . If  $\mathcal{T}$  is universal,  $\text{proj}_A(\text{RCBR}) = \rho^\infty(A)$ .*
- (3) *Given a type structure  $\mathcal{T} = \langle F, (A_{-f}, T_f, \beta_f)_{f \in F} \rangle$ ,  $\text{proj}_A \text{RCBR}$  is a BRS.*

PROOF. Deferred to the **Appendix** to this chapter.  $\square$

The next proposition puts in relation Nash equilibrium with rationalizability. In particular, if there is a unique rationalizable outcome, it must be a Nash equilibrium. In addition, we prove the

claim made in the previous section that for every NE  $a^*$  of  $\Gamma$  there exists a type structure such that  $a^* \in \text{proj}_A RCBR$ .

PROPOSITION 1.10. *Let  $\Gamma = \langle F, (A_f, u_f : A \rightarrow \mathbb{R})_{f \in F} \rangle$  be a compact-continuous game.*

- (1)  *$a^*$  is a NE of  $\Gamma$  if and only if  $\{a^*\}$  is a BRS;*
- (2) *Every Nash equilibrium of  $\Gamma$  is rationalizable.*
- (3) *If  $\Gamma$  is equipped with its universal type space, if  $a^*$  is a NE of  $\Gamma$ , then  $a^* \in \text{proj}_A RCBR$ ;*
- (4) *If  $\rho^\infty(A) = \{a^*\}$ , then  $a^*$  is a Nash equilibrium of  $\Gamma$ .*

PROOF. By the definition of joint PBR,  $a^* \in \hat{r}(a^*) \iff a^* \in \rho(\{a^*\})$ . Therefore,  $a^*$  is a NE of  $\Gamma$  if and only if  $a^* \in \rho(\{a^*\})$ . Since  $a^* \in \rho(\{a^*\})$  is equivalent to  $\{a^*\} \subseteq \rho(\{a^*\})$ , we find that  $a^*$  is a NE of  $\Gamma$  if and only if  $\{a^*\} \subseteq \rho(\{a^*\})$ . Hence,  $a^*$  is a NE of  $\Gamma$  if and only if  $\{a^*\}$  is a BRS. It follows that, by definition of rationalizable outcome, every NE is rationalizable, since it can be included in a singleton with the BRS. If  $\Gamma$  is endowed with the corresponding universal type structure, the set of rationalizable outcomes coincides with  $\text{proj}_A RCBR$ , and, since every NE is rationalizable, we have that every NE is consistent with  $RCBR$ . Finally,  $\rho^\infty(A) = \{a^*\}$ , for every  $f \in F$ ,  $a_f^* \in r_f(\delta_{a_{-f}^*})$ , which is the definition of a Nash equilibrium.  $\square$

REMARK 1.11. There are many interesting games in which the action space is not compact, utilities are discontinuous, or both. Under these circumstances, the set of rationalizable actions can still be defined as in **Definition 1.11**. However, there is no guaranty that **Propositions 1.12-13** hold.

4.2.1. *Level- $k$  thinking.* An interesting application of the rationalization operator is *level- $k$  thinking* (Nagel, 1995; Costa-Gomes et al., 2001; Crawford and Iriberri, 2007), a solution concept that allows for irrational players and heterogeneous bounded reasoning about rationality. Consider a finite game  $\Gamma = \langle F, (A_f, u_f : A \rightarrow \mathbb{R})_{f \in F} \rangle$ , equipped with a suitable epistemic type structure, and suppose that  $A \neq \rho(A)$ . Recall that, given a set  $X$ , a partition of  $X$  is a family of disjoint subsets that covers  $X$ , in the sense that their union includes all  $x \in X$ . Level- $k$  thinking is a solution concept defined as follows. First, the players are divided into  $k + 1$ ,  $k \in \mathbb{N}$ , disjoint subsets of  $F$ , forming a partition  $\mathcal{F} = \{F_l : l \in L\}$ , where  $L := \{0\} \cup [k + 1]$ . Second, for every  $l \in L$ ,  $A'_L$  is set equal to

$\rho^l(A_{F \setminus L})$ . Hence, level- $k$  thinking is a solution concept

$$\mathcal{L}_k : \{A\} \rightarrow A$$

$$A \mapsto \times_{f \in F} \rho^{\lambda(f)}(F),$$

where  $\lambda : F \rightarrow L$ ,  $L = \{0\} \cup [k+1]$ , maps  $f$  to the unique value of  $l \in L$  such that  $f \in F_l$ . In other words, level- $k$  describes an epistemic state in which players are divided into several different groups with different sophistication. Except for the players in group zero, which behaves in a naive way, those in every other group  $l \in [k]$  behave rationally conditional on being certain to be the most sophisticated players in terms of strategic thinking. By construction, such belief is necessarily correct only for level- $k$  players. Similarly to RkBR, level- $k$  thinking describes a situation of bounded reasoning about rationality, furthermore, with possibly irrational level 0 players.

#### 4.3. Point-Rationalizability. A compact-continuous game

$$\Gamma = \left\langle F, (A_f, u_f : A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle$$

is said to be *nice* if, for every  $f \in F$ ,  $A_f$  is a closed and bounded interval of  $\mathbb{R}$  and  $u_f(a_f, a_{-f})$  is continuous on  $A = \times_{f \in F} A_f$  and quasi-concave on  $A_{-f}$  (Moulin, 1984). Nice games are common in several interesting applications, the most notable of which is certainly that of Cournot competition, discussed below. They also have the nice property that the set of rationalizable outcomes can be obtained considering only deterministic conjectures, that is, conjectures corresponding to Dirac measures. In this regard, a profile of actions  $a \in A$  is said to be point-rationalizable if  $a \in r^\infty(A) := \bigcap_{k \in \mathbb{N}} r^k(A)$ , where  $r^1 := r$  is the joint PBR function of  $\Gamma$ , and for every  $k \in \mathbb{N}$ ,  $r^{k+1}(A) = r(r^k(A))$ .

**PROPOSITION 1.12.** *Let  $\Gamma = \left\langle F, (I_f, u_f)_{f \in F} \right\rangle$  be a nice game.*

- (1) *For every  $k \in \mathbb{N}$ ,  $a \in r^k(A) \iff \rho^k(A)$ .*
- (2)  *$a \in r^\infty(A) \iff \rho^\infty(A)$ , that is, a profile of actions  $a \in A$  is rationalizable if and only if  $a$  is point-rationalizable.*
- (3) *If  $\Gamma$  is equipped with its universal type structure, every  $k \in \mathbb{N}$ ,  $\text{proj}_A \text{RkBR} = \hat{r}^k(A)$  and  $\text{proj}_A \text{RCBR} = \hat{r}^\infty(A)$ .*

**PROOF.** The proofs of the first two points are straightforward but tedious and can be found in Chapter 4 of Battigalli et al. (2024a). See also Moulin (1984) for the original proof of the result and the related discussion. The third point follows from the second and **Proposition 1.13**.  $\square$

REMARK 1.13. In general, the results of **Proposition 1.16** cannot be extended to the case where  $A$  is the product of compact subsets of  $\mathbb{R}^L$  where  $L$  is a finite set with at least two elements. In other words, the equivalence between rationalizability and point-rationalizability breaks when the action's space of at least one player has more than one dimension. See Zimper (2006a) for a reference.

A nice game is said to be symmetric if the players have the same set of actions ( $A_f = A_0$ ) and have symmetric utility functions. Every nice symmetric game has a symmetric Nash equilibrium, that is, there exists an action  $a_0^* \in A_0$  such that  $a_0^* \in r\left(\{a_0^*\}_{j \in F \setminus \{f\}}\right)$  for every  $f \in F$ .<sup>12</sup> As a corollary, if a nice symmetric game has a unique rationalizable solution, it must be a symmetric Nash equilibrium.

EXAMPLE 1.14. An interesting application of point-rationalizability is found in static games of Cournot oligopolistic competition. The model presented is adapted from Börgers and Janssen (1995), and additional details can be found also in Chapter 4 of Battigalli et al. (2024a). There are  $n \in \mathbb{N}$  identical companies, or firms, indexed by  $f \in F$ ,  $F = [n]$ , each producing the same good and with the same production function. The products produced are sold on a common market at the price  $p(q) = \max\left\{0, \alpha - \frac{\beta}{2n} \sum_{f \in F} q_f\right\}$ , where, for every  $f \in F$ ,  $q_f \in Q_f = \left[0, \frac{2\alpha}{c + \frac{\beta}{n}}\right]$  is the quantity produced by  $f$  and  $q = (q_f)_{f \in F}$ . For every  $f \in F$ , producing  $q_f$  entails a cost of  $\frac{c}{2} q_f^2$ , therefore, given the quantity produced by the other firms  $q_{-f}$ ,  $f$  obtains a profit equal to

$$\pi_f(q_f, q_{-f}) = p(q_f, q_{-f}) q_f - \frac{c}{2} q_f^2 = \begin{cases} \alpha q_f - \frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} q_j q_f - \left(c + \frac{\beta}{n}\right) \frac{q_f^2}{2} & \alpha > \frac{\beta}{2n} \sum_{f \in F} q_f \\ -\frac{c}{2} q_f^2 & \alpha \leq \frac{\beta}{2n} \sum_{f \in F} q_f. \end{cases}$$

Setting  $Q = \times_{f \in F} Q_f = \left[0, \frac{4\alpha}{\beta}\right]$ , it is immediately verified that  $\pi_f(q_f, q_{-f})$  is continuous and quasi-concave in  $q_f \in \mathbb{R}$  for every  $q_{-f} \in Q_{-f}$ . We assume that the utilities of the companies are equal to their respective profits, obtaining a nice game  $\Gamma_1 := \langle F, (Q_f, \pi_f(q_f, q_{-f}))_{f \in F} \rangle$ . By **Proposition 1.16**, the set of rationalizable actions coincides with that of point-rationalizable actions. The joint point best-reply function,  $\hat{r} : (\hat{r}_f(q_{-f}))_{f \in F} \rightarrow \left[0, \frac{4\alpha}{\beta}\right]^n$ , is obtained from the point best-reply function of each firm  $f$ ,

$$\hat{r}_f(q_{-f}) = \max \left\{ 0, \min \left\{ q_0(q_{-f}), \frac{2\alpha}{c + \frac{\beta}{n}} \right\} \right\},$$

<sup>12</sup>For a more rigorous definition of a symmetric game and the proof of the previous claim see Section 5.2 in Chapter 5 of Battigalli et al. (2024a).

where  $q_0(q_{-f})$  is the unique solution of  $\frac{\partial}{\partial x}\pi(x, q_{-f}) = 0$ , that is,  $q_0(q_{-f}) = \frac{\alpha}{c + \frac{\beta}{n}} - \gamma \sum_{j \in F \setminus \{f\}} q_j$  where  $\gamma := \frac{\frac{\beta}{2n}(n-1)}{c + \frac{\beta}{n}}$  and  $\frac{\alpha}{c + \frac{\beta}{n}}$  is the optimal production level under monopolistic competition. Since  $q_0(q_{-f})$  strictly decreases in  $q_j$  for every  $j \in F \setminus \{f\}$ , it obtains a maximum at  $q_{-f} = (0)_{j \in F \setminus \{f\}}$ , that is,  $\max_{q_{-f} \in Q_{-f}} q_0(q_{-f}) = \frac{\alpha}{c + \frac{\beta}{n}}$ . To make the analysis the most interesting, assume that  $\frac{\alpha}{c + \frac{\beta}{n}} < \gamma \frac{2\alpha}{c + \frac{\beta}{n}}$ , so that a firm conjecturing that all its competitors are operating at monopolistic capacity, if rational, decides to remain inactive. In this case,  $\hat{r}_f(q_{-f}) = \max \left\{ 0, \frac{\alpha}{c + \frac{\beta}{n}} - \gamma \sum_{j \in F \setminus \{f\}} q_j \right\}$  and

$$\times_{f \in F} \hat{r}(Q) = \left[ 0, \frac{\alpha}{c + \frac{\beta}{n}} \right].$$

Let  $\underline{q}^k = \min \left\{ \hat{r}_f^k(Q) : f \in F \right\}$  and  $\bar{q}^k = \max \left\{ \hat{r}_f^k(Q) : f \in F \right\}$ , respectively. Since, for every  $f \in F$ , and every  $j \in F \setminus \{f\}$ ,  $\hat{r}_f(q_{-f})$  strictly decreases on  $Q_j$ , we have, by induction,  $\underline{q}^k = \max \left\{ 0, \frac{\alpha}{c + \frac{\beta}{n}} - \frac{\frac{\beta}{2n}\bar{q}^{k-1}}{c + \frac{\beta}{n}} \right\}$  and  $\bar{q}^k = \max \left\{ 0, \frac{\alpha}{c + \frac{\beta}{n}} - \frac{\frac{\beta}{2n}\underline{q}^{k-1}}{c + \frac{\beta}{n}} \right\}$ . Therefore, if  $\gamma \geq 1$ , we can easily verify that  $\hat{r}(Q) = \hat{r}(\hat{r}(Q))$  and every profile of actions  $q \in Q$  is rationalizable. In contrast, if  $\gamma < 1$ , there is a unique rationalizable outcome. To see this, observe that  $\gamma < 1$  implies that, for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $\underline{q}^k = \frac{\alpha}{c + \frac{\beta}{n}} - \gamma \bar{q}^{k-1} < \frac{\alpha}{c + \frac{\beta}{n}}$  and, consequently,

$$\begin{bmatrix} \underline{q}^{k+1} \\ \bar{q}^{k+1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{c + \frac{\beta}{n}} \\ \frac{\alpha}{c + \frac{\beta}{n}} \end{bmatrix} + \begin{bmatrix} 0 & -\gamma \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} \underline{q}^k \\ \bar{q}^k \end{bmatrix}.$$

Proceeding by recursive substitutions, for every  $k \in \mathbb{N} \setminus \{1\}$ , we obtain

$$\underline{q}^k = \begin{cases} \frac{\alpha}{c + \frac{\beta}{n}} \sum_{l=0}^k (-\gamma)^l & k \text{ is odd} \\ \underline{q}^{k-1} & k \text{ is even,} \end{cases}$$

and,

$$\bar{q}^k = \begin{cases} \frac{\alpha}{c + \frac{\beta}{n}} \sum_{l=0}^k (-\gamma)^l & k \text{ is even} \\ \bar{q}^k & k \text{ is odd.} \end{cases}$$

Since  $\gamma < 1$ , we also have  $\lim_{k \rightarrow \infty} \underline{q}^k = \lim_{k \rightarrow \infty} \bar{q}^k = \frac{\alpha}{c + \frac{\beta}{n}} \frac{1}{1 + \gamma}$ , that is, the set of point-rationalizable outcomes shrinks to a singleton

$$\hat{r}^\infty(Q) = \times_{f \in F} \left[ \frac{\alpha}{c + \frac{\beta}{n}} \frac{1}{1 + \gamma}, \frac{\alpha}{c + \frac{\beta}{n}} \frac{1}{1 + \gamma} \right] = \left\{ \left( \frac{\alpha}{c + \frac{\beta}{2n} + \frac{\beta}{2}} \right)_{f \in F} \right\}.$$

Therefore, we conclude that  $\Gamma$  has a unique rationalizable outcome if and only if  $\gamma < 1$ .

**4.4. Correlated equilibrium.** Consider a game

$$\Gamma = \langle F, \{\theta\}, (A_f)_f, g, (U_f : Y \rightarrow \mathbb{R})_{f \in F} \rangle$$

in which every player has a finite set of actions, that is, for every  $f \in F$ ,  $|A_f| \in \mathbb{N}$  and let

$$\mathcal{G} = \langle F, \{\theta\}, \Omega, (\mathcal{K}_f)_{f \in F}, (\mathcal{B}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$$

and  $\mathcal{F} = \langle F, \Omega, (\mathcal{P}_f)_{f \in F}, (p_f)_{f \in F} \rangle$  be, respectively, the corresponding epistemic model and Bayesian frame. Suppose that  $\mathcal{G}$  is consistent with  $\text{CK}\Gamma$ , and, for every  $f \in F$ , let  $u_f := U_f \circ g$ . Since there is a unique state of nature, SYM is necessarily verified and we obtain a finite game of complete information,

$$\langle F, (A_f, u_f : \times_{f \in F} A_f \rightarrow \mathbb{R})_{f \in F} \rangle.$$

If  $\Omega$  is finite, there exists a common prior  $q \in \Delta(\Omega)$  and, for every  $\omega \in \Omega$ , every  $f \in F$  is Bayesian rational at  $\omega$ , the ordered tuple  $\langle \Omega, q, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$  is said to be an objective correlated equilibrium of  $\Gamma$ . That is,  $\langle \Omega, q, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$  is an objective correlated equilibrium if, for every  $\omega \in \Omega$ , every  $f \in F$  and every  $a_f \in A_f$ ,

$$\begin{aligned} \sum_{\omega' \in \mathcal{P}_f(\omega)} p_f(\omega)(\{\omega'\}) u_f(\sigma_f(\omega'), \sigma_{-f}(\omega')) &= \\ \sum_{\omega' \in \mathcal{P}_f(\omega)} \frac{q(\{\omega'\})}{q(\mathcal{P}_f(\omega))} u_f(\sigma_f(\omega), \sigma_{-f}(\omega')) &\geq \sum_{\omega' \in \mathcal{P}_f(\omega)} \frac{q(\{\omega'\})}{q(\mathcal{P}_f(\omega))} u_f(a_f, \sigma_{-f}(\omega')) \\ &= \sum_{\omega' \in \mathcal{P}_f(\omega)} p_f(\omega)(\{\omega'\}) u_f(a_f, \sigma_{-f}(\omega')). \end{aligned}$$

Hence, if  $\langle \Omega, q, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$  is an objective correlated equilibrium, it must be the case that for every  $f \in F$  and every  $a_f \in A_f$ ,

$$\sum_{\omega \in \Omega} q(\omega)(\{\omega\}) u_f(\sigma_f(\omega), \sigma_{-f}(\omega)) \geq \sum_{\omega \in \Omega} q(\omega)(\{\omega\}) u_f(a_f, \sigma_{-f}(\omega)).$$

Our definition of an objective correlated equilibrium is based on the equivalence result with the original of Aumann (1974) provided in Aumann (1987). Strictly speaking, an objective correlated equilibrium is a joint restriction on the set of states of the world and the players' beliefs, and not a solution concept; see also Chapter 3 and Chapter 5 in Osborne and Rubinstein (1994) for a reference. A common prior can be interpreted as a "probabilistic theory" on the possible outcome of the game

that the players share before the game is played. By definition, an objective correlated equilibrium requires that:

- (1) ex-ante, that is, before playing the game, every player has a plausibility order on the (Borel field of)  $\Omega$  which is equivalent to  $q$ ;
- (2) once players obtain their knowledge and form their beliefs, the so-called *interim* phase of the game, there is no state in which at least one player behaves irrationally from a Bayesian perspective;
- (3) rationality is commonly known and believed.<sup>13</sup>

Clearly, the ex-ante and interim phases are only metaphors, as in a static game everything happens simultaneously. Knowledge, beliefs, and actions manifest all at the same time, as every state of the world is, by definition, a full description of one of the possible outcomes of the situation of interest, and there is no actual ex-ante phase in which the players discuss what may happen in the game. It is immediate to show that every mixed equilibrium  $\alpha^{**} \in \times_{f \in F} \Delta(A_f)$  of  $\Gamma$  can be obtained in an objective correlated equilibrium such that:

- the set of states is (isomorphic to)  $A$ ;
- $q$  is such that for every  $a \in A$ ,  $q(\{a\}) = \prod_{f \in F} \alpha_f^{**}(a_{-f})$  (prior independence);
- there is no state of the world in which at least a player can exclude, based on her knowledge, any profile of actions of her opponents.

As argued in Brandenburger and Dekel (1987b), this result, which is rather easy to prove (see Chapter 3 in Osborne and Rubinstein, 1994), provides an epistemic characterization of mixed equilibrium, which is closely related to other results in the literature. In this regard, if we remove the requirement of a common prior, the ordered tuple  $\langle \Omega, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$  is said to be a subjective correlated equilibrium if for every  $\omega \in \Omega$ , every  $f \in F$  is Bayesian rational at  $\omega$ .<sup>14</sup> That is,  $\langle \Omega, q, (\mathcal{P}_f)_{f \in F}, (\sigma_f)_{f \in F} \rangle$  is a subjective correlated equilibrium of  $\Gamma$  if, for every  $\omega \in \Omega$ , every  $f \in F$  and every  $a_f \in A_f$ ,

$$\sum_{\omega' \in \mathcal{P}_f(\omega)} p_f(\omega)(\{\omega'\}) u_f(\sigma_f(\omega'), \sigma_{-f}(\omega')) \geq \sum_{\omega' \in \mathcal{P}_f(\omega)} p_f(\omega)(\{\omega'\}) u_f(a_f, \sigma_{-f}(\omega')).$$

<sup>13</sup>Recall that if an event is true in every state of the world, then it must be commonly known. Since we required qualitative beliefs to be consistent with information sets, hence with knowledge, and quantitative beliefs to be consistent with qualitative beliefs, the result follows.

<sup>14</sup>Aumann and Brandenburger (1987b) refers to an a *posteriori* equilibrium. Their definition of objective and subjective correlated equilibrium are slightly more general, as they rely on a weaker notion of rationality that applies to an hypothetical ex-ante phase of the game as in Aumann (1974). We followed the definitions Dekel and Siniscalchi (2015) and avoided this additional complication.

By definition, a subjective correlated equilibrium requires "only" that in the *interim* phase of the game players never behave irrationally, and, therefore, rationality is commonly known and believed in every state of the world. From the previous observations, it is possible to show that there exists a subjective correlated equilibrium of  $\Gamma$  such that, for every  $f \in F$ ,  $\text{proj}_{A_f} RCBR = \sigma_f(\Omega)$ . Similarly, for every subjective correlated equilibrium and every  $f \in F$ ,  $\sigma_f(\Omega) \subseteq \text{proj}_{A_f} RCBR$ . Both results are due to Brandenburger and Dekel (1987b), although with a slightly different original formulation.

### 5. Aggregative games and competitive equilibrium

Let  $\Gamma = \left\langle F, (A_f \subseteq \mathbb{R}^J, u_f : A \rightarrow \mathbb{R})_{f \in F} \right\rangle$  where  $J$  is a finite set. If, for every  $f \in F$ , there exist a function  $v_f \in A_f \times \mathbb{R}$  and  $\nu_f \in \mathbb{R}^J, G_f \in \mathbb{R}^{J \times I}$  such that, for every  $a \in A$ ,

$$u_f(a) = v_f(a_f, \nu_f + G_f a),$$

$\Gamma$  is said to be *quasi-aggregative*. Furthermore, if there exist  $\nu_0 \in \mathbb{R}^J$  and  $G_0 \in \mathbb{R}^{J \times I}$  such that, for every  $f \in F$ ,  $\nu_f = \nu_0$  and  $G_f = G_0$ ,  $\Gamma$  is also said to be *aggregative*. Prominent examples are, among others, the model of Cournot oligopolistic competition (cf. **Section 4.7** in this chapter), investment games (cf. **Section 2** in **Chapter 2**), network games with linear best-replies (cf. **Section 2.4** in **Chapter 2**) as well as many trading games, of which notable examples are Shapley and Shubik (1977), Dubey and Shubik (1978) and Sahi and Yao (1989). Aside from their general interest, the structure of these games is intrinsically connected with the model analyzed in **Chapter 2**. In this regard, the purpose of this section is not to describe the properties of these class of games, but to characterize the notion of *competitive equilibrium* (CE) as a solution concept for aggregative games, which is closely related to that of Walrasian Equilibrium in General Equilibrium models. In this regard, for every  $f \in F$ , we introduce the function

$$\Pi_f : A \rightarrow \mathbb{R}$$

$$a \mapsto \nu_f + G_f a,$$

and define a competitive equilibrium as a pair  $(a^{**}, p^{**}) \in A \times \mathbb{R}^J$  such that:

- (1) for every  $f \in F$ ,  $a_f^{**} \in \text{argmax}_{a \in A_f} u_f(a_f, p_f^{**})$ ;
- (2)  $p^{**} = \left( \Pi_f(a_{-f}^{**}) \right)_{f \in F}$ .

In other words, for every player  $f$ ,  $a_f^{**}$  is rational for  $f$  when  $f$  is certain that her opponents will choose a profile  $a_{-f}$  such that  $\nu_f + G_f(a_f^{**}, a_{-f}) = p^{**}$ . In other words, in a competitive equilibrium,

every  $f \in F$  is certain that her choice does not affect the realized value of  $G_f a$ . In this regard, observe that, if, for every  $f \in F$ ,  $G_f(f, f) = 0$ , the players are always correct in conjecturing that  $\Pi_f$  does not depend on  $a_f$ . Consequently, in this specific case, it is immediate to verify that  $a^{**}$  is a NE of  $\Gamma$  if and only if there exists  $p^{**} \in \mathbb{R}^J$  such that  $(a^{**}, p^{**})$  is a competitive equilibrium of  $\Gamma$ . Otherwise, the set of Nash and Competitive equilibria may be different and also have an empty intersection. To clarify this aspect, consider again the standard model of Cournot competition discussed in the previous section. In an interior competitive equilibrium, we see that every firm  $f \in F$  invests up to the same level  $q_0^{**}$  which is the solution of the first order condition

$$\frac{\partial}{\partial q} \left( pq_f - \frac{c}{2} q_f^2 \right) = 0,$$

that is,  $q_0^{**} = \frac{p^{**}}{c}$ , and, by aggregate consistency,  $p^{**}$  must be the solution of  $p^{**} = \alpha - \frac{\beta}{2} \frac{np^{**}}{nc}$ , obtaining  $p^{**} = \frac{\alpha c}{c + \frac{\beta}{2}}$  (aggregate demand at  $p^{**}$ ). Consequently,  $q_0^{**} = \frac{\alpha}{c + \frac{\beta}{2}}$ , which is different from the corresponding NE level,

$$q_0^* = \frac{\alpha}{c + \frac{\beta}{2n} + \frac{\beta}{2}}$$

for every  $n \in \mathbb{N}$ . However, for a sufficiently large value of  $n$ , the difference between  $q_0^*$  and  $q_0^{**}$  is negligible; therefore, in games with a large number of players, a competitive equilibrium may be in fact a good approximation of the corresponding NE. Such an approximation is attractive for two reasons. On the one hand, it avoids discussing strategic effects, viewing individuals at  $\Pi$ -takers (e.g.; price-takers consumers or firms). On the other hand, for large games, it might be computationally harder to compute NE rather than CE.

## 6. Conclusion

In this chapter, we provide a formal definition of a static economic model and its components. Considering the case of complete information, we reviewed the construction of a universal type structure *à la* Mertens and Zamir (1985), following the approach of Brandenburger and Dekel (1993). We discussed the notion of Nash equilibrium and its epistemic limitations. We also reviewed the solution concepts of rationalizability and showed its relation with the joint behavioral-epistemic assumption of rationality and common belief in rationality, a milestone of game theory due to Tan and Werlang (1988). Using the equivalence between rationalizability and point-rationalizability in nice games of complete information, in **Chapter 2** we study rationalizability in network games with linear best-replies and, particularly, in a strategic version of the celebrated Investment CAPM (Zhang,

2017). The construction of a universal type structure, as well as the definition of rationalizability and its relation to RCBR, extend seamlessly to games with payoff uncertainty, and are discussed in **Chapter 3**, in which we also review General Equilibrium models and Walrasian Equilibrium (WE). Nevertheless, the actual purpose of **Chapter 3** is to identify sufficient epistemic conditions to observe WE prices in a trading game describing an auction with limit orders and price priority for financial assets. Despite the theoretical nature of this thesis, many empirical considerations emerge from the discussion of the results obtained. In this regard, we would like to conclude this chapter with the following remark. Despite the fact that most of the actual economic phenomena are dynamic, nothing prevents us from assessing whether a theory emerging from a static model may provide useful insights. This is a far-from-obvious aspect that is too often neglected. A good model provides a falsifiable prediction that is not contradicted by the data, as well as a reasonable and intuitive explanation for the phenomenon being studied. Whether its primitive description adheres only partially to reality is of secondary importance and should not be the primary metric for judging the scientific or practical utility of a model. In fact, a model is false by definition; only the reality is "real".

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### Appendix to Chapter 1

PROOF OF PROPOSITION 1.13. The proof is adapted from those of Theorem 5.1 and 5.2 in Tan and Werlang (1988) and is also based on the results of Brandenburger and Dekel (1993) and Friedenberg (2010). The only difference is that we allow for arbitrary type structures, while Tan and Werlang (1988) consider the case of a universal type structure.

The assumption of a Polish and compact action space ensures that a universal type structure exists, which includes any subset of hierarchies of beliefs satisfying the requirement of consistency and common belief in consistency. For every  $f \in F$  and every  $T'_f \subseteq T_f$ , the image set of  $T'_f$  through  $\text{marg}_{A_{-f}}\beta_f$ , that is,  $\text{marg}_{A_{-f}}\beta_f(T'_f)$ , is the set of conjectures of  $f$  induced by a type in  $t_f \in T'_f$ . In addition, observe that, for a given topological space  $(X, \tau)$ , if  $D_1, D_2$  are Borel subsets of  $X$ ,  $D_1 \subseteq D_2 \implies \Delta(D_1) \subseteq \Delta(D_2)$  in the relative topology.

Recall that for a given player  $f \in F$ ,

$$R_f = \left\{ (a_f, t_f) \in A_f \times T_f : \exists \mu_f \in \text{marg}_{A_{-f}}\beta_f(T_f), a_f \in r_f(\mu_f) \right\}.$$

By construction,  $\beta_f(T_f) \subseteq \Delta(A_{-f} \times T_{-f})$ , therefore,

$$\text{marg}_{A_{-f}}\beta_f(T_f) \subseteq \text{marg}_{A_{-f}}\Delta(A_{-f} \times T_{-f}) \subseteq \Delta(A_{-f}),$$

which implies that  $\text{proj}_{A_f}R_f \subseteq \rho_f(A_{-f})$ . However, if  $\mathcal{T}$  is universal, for every  $\mu \in \Delta(A_{-f})$ , there exists  $t_f \in T_f$  such that  $\text{marg}_{A_{-f}}\beta_f(t_f) = \mu_f$ , and therefore

$$\text{marg}_{A_{-f}}\beta_f(T_f) = \text{marg}_{A_{-f}}\Delta(A_{-f} \times T_{-f}) = \Delta(A_{-f}).$$

Consequently, therefore  $\text{proj}_A R \subseteq \rho(A)$ , or equivalently,  $\text{proj}_A RCBR^0 \subseteq \rho(A)$ , holding as equal if  $\mathcal{T}$  is universal. Next, we claim that, by induction, for every  $k \in \mathbb{N}$ ,

$$\text{proj}_A RkBR \subseteq \rho^{k+1}(A),$$

holding as equality if  $\mathcal{T}$  is universal. Since this holds for every player  $f \in F$ , we conclude that,

$$\text{proj}_A R \subseteq \rho(A),$$

holding as equality if  $\mathcal{T}$  is universal. We claim that, by induction, for every  $k \in \mathbb{N}$ ,

$$\text{proj}_A(R(k)BR) \subseteq \rho^{k+1}(A),$$

holding as equality if  $\mathcal{T}$  is universal.

- (*Basis step*) Set  $k = 1$ . Recall that  $R0BR := R$ , and consider the event

$$R1BR = R \cap B(R) = \times_{f \in F} (R_f \cap B_f(R_{-f})).$$

For a given player  $f \in F$ , let  $T_f^1 = \text{proj}_{T_f}(R \cap B(R)) = \text{proj}_{T_f}(R_f \cap B_f(R_{-f}))$ , and notice that

$$\beta_f(T_f^1) = \{\delta_f \in \beta_f(T) : \forall j \in F \setminus \{f\}, \delta_f(R_{-j}) = 1\}.$$

Therefore,  $\text{marg}_{A_{-f}}\beta_f(T_f^1) \subseteq \{\mu \in \Delta(A) : \mu(\text{proj}_{A_{-f}}R) = 1\}$ , holding as a strict equality if  $\mathcal{T}$  is universal. Recalling that,

$$\rho_f^2(A_{-f}) = \{a_f \in A_f : \exists \mu_f \in \Delta(\rho(A)_{-f}), a_f \in r_f(\mu_f)\},$$

we conclude that  $\Delta(\text{proj}_{A_{-f}}R0BR) \subseteq \Delta(\rho_f(A_{-f}))$ , holding as equality if  $\mathcal{T}$  is universal. Consequently,

$$\text{proj}_{A_f}(R_f \cap B_f(R_{-f})) \subseteq \rho_f(\rho(A)_{-f}),$$

holding as equality if  $\mathcal{T}$  is universal. Since this holds for every  $f \in F$ , we can conclude that  $\text{proj}_A RCBR^1 \subseteq \rho^2(A)$ , holding as equality if  $\mathcal{T}$  is universal.

- (*Induction step*) Fix  $k \in \mathbb{N} \setminus \{1\}$ , suppose that  $\text{proj}_A R(k-1)BR \subseteq \rho^k(A)$ , holding as equality if  $\mathcal{T}$  is universal. Consider the event  $RkBR = \times_{f \in F} (R_f \cap kB_f(R_{-f}))$ . By the conjunction property of the belief operator, we have that, for every  $k \in \mathbb{N} \setminus \{1\}$ ,

$$RkBR = R \cap B(R \cap (\cap_{l \in [k-1]} B(R))) =$$

$$R \cap B(R(k-1)BR) = \times_{f \in F} (R_f \cap B_f(R(k-1)BR_{-f})).$$

A proof of the previous claim can be easily adapted from footnote 11 in Dekel and Siniscalchi (2015), and is omitted for reasons of space. For a given player  $f \in F$ , define  $T_f^k$  as,

$$T_f^k := \text{proj}_{T_f}(R \cap CB(R(k-1)BR)),$$

and observe that

$$\beta_f(T_f^k) = \{\delta_f \in \beta_f(T) : \delta_f(R(k-1)BR_{-f}) = 1\}.$$

Since

$$\rho_f^{k+1}(A_{-f}) = \left\{a_f \in A_f : \exists \mu_f \in \Delta\left(\rho^k(A)_{-f}\right), a_f \in r_f(\mu_f)\right\},$$

and, by hypothesis,  $\text{proj}_{A_{-f}}RCBR_{-f}^{k-1} \subseteq \rho^k(A)_{-f}$ , holding as equality if  $\mathcal{T}$  is universal, we can conclude that  $\Delta\left(\text{proj}_{A_{-f}}RCBR_{-f}^{k-1}\right) \subseteq \Delta\left(\rho^k(A)_{-f}\right)$ , holding also as equality if  $\mathcal{T}$  is universal. Consequently,

$$\text{proj}_{A_f}\left(R_f \cap B_f\left(R(k-1)BR_{-f}^{k-1}\right)\right) \subseteq \rho_f\left(\rho^k(A)_{-f}\right),$$

holding as an equality if  $\mathcal{T}$  is universal. Since this holds for every  $f \in F$ , we conclude that  $\text{proj}_ARCB R^k \subseteq \rho^{k+1}(A)$ , holding as an equality if  $\mathcal{T}$  is universal.

As a consequence, since  $\rho(\cdot)$  and  $B(\cdot)$  satisfy monotonicity and conjunction on their respective domains, and, for every  $l \in \mathbb{N} \cup \{0\}$ ,  $\text{proj}_ARlBR \subseteq \rho^{l+1}(A)$ , we also have that, for every  $k \in \mathbb{N}$

$$\text{proj}_ARkBR = \text{proj}_A\left[R \cap \left(\bigcap_{l \in [k]} B^l(R)\right)\right] \subseteq \bigcap_{l \in [k]} \rho^k(A),$$

holding as equality if  $\mathcal{T}$  is universal, since, in that case, for every  $l \in \mathbb{N} \cup \{0\}$ ,  $\text{proj}_ARlBR = \rho^{l+1}(A)$ .

Hence, by the finite intersection property,

$$\text{proj}_ARCB R = \text{proj}_A\left[R \cap \left(\bigcap_{l \in \mathbb{N}} B^l(R)\right)\right] \subseteq \bigcap_{l \in \mathbb{N}} \rho^k(A) = \rho^\infty(A),$$

holding as equality if  $\mathcal{T}$  is universal, since, in that case, for every  $k \in \mathbb{N}$ ,  $\text{proj}_ARkBR = \bigcap_{l \in [k]} \rho^k(A)$ .

Since  $A$  is compact and, for every  $f \in F$ ,  $u_f$  is continuous, by **Proposition 1.12** we have that a subset of  $A \subseteq \rho^\infty(A)$  if and only if it is a BRS. Since  $\text{proj}_ARCB R \subseteq \rho^\infty(A)$ , we have that  $\text{proj}_ARCB R$  is a BRS.  $\square$

## Rationalizability in Investment Games

### 1. Introduction

In this chapter, we study rationalizability and bounded reasoning about rationality in a game-theoretic formulation of the static Investment CAPM (Zhang, 2017). Despite its simplicity, the static investment CAPM motivates the  $q$ -factor model (Hou et al., 2015), which outperforms the traditional Fama-French (Fama and French, 1993) and the Carhart-Fama-French (Carhart, 1997) factor models in cross-sectional regressions of stock returns. As with many economic theories, the Investment CAPM has its own limitations. First, being a Walrasian equilibrium model, the Investment CAPM is axiomatic in the sense that no motivation is offered for assuming competitive equilibrium behavior in the specific situation of interest. Second, the Investment CAPM predicts a perfect relation between stock returns and marginal returns on assets, as well as a perfect linear relation between Tobin  $Q$  and investment rates with unit intercept. Considering characteristics sorted portfolios, a vast literature has tested the validity of these relations, with mixed results. Liu et al. (2009) showed that the Investment CAPM holds in the cross-section of stock returns but fails in the time-series, a finding that has been recently confirmed in Belo et al. (2024) with methodological improvements, while Belo et al. (2013) showed that it holds simultaneously both in the cross section of stock returns and valuations (Tobin  $Q$ ). Nevertheless, the estimates for the parameter governing investment adjustment costs are unstable, depending on the specific factor used to sort stocks, and are likely to overestimate the size of investment adjustment costs (i.e., they are too high). Goncalves et al. (2020) introduces capital heterogeneity and improves certain methodological aspect, and show that both issues become less severe; yet, the explicative power of the Investment CAPM in the time-series of returns remain considerably worse than in the cross-section, and far from the levels obtained for the time-series of valuations. In this regard, Belo et al. (2002) show that the introduction of capital heterogeneity is sufficient to obtain a very good performance of the Investment CAPM both in the cross-section and time-series of valuations. In short, a puzzle emerges as production based asset pricing models seem to explain valuations, both cross-sectionally and in the time-series, but

not the time-series of portfolio returns.

The purpose of this chapter is to address both the theoretical and empirical issues of the Investment CAPM using the tools of epistemic game theory. We consider a static investment-based asset pricing model, with a unique productive asset and a linear stochastic discount factor (SDF), formulated as a static game in which the payoffs of Bayesian firms are set equal to the financial value created for their shareholders. We refer to this framework as an investment game, which is presented in **Section 2**. Despite we allow firms' assets to obtain random payoffs in the future, if the stochastic discount factor is common knowledge among firms, an investment game is i) of complete information, ii) compact-continuous, iii) nice, and iv) a network game with linear best-replies (Bramoullé et al., 2014). In **Section 3** we study rationalizability in network games with linear best-replies, as, in sufficiently rich epistemic type structures, an outcome is rationalizable if and only if it is consistent with rationality and common belief in rationality. We provide necessary and sufficient conditions for the existence of a unique rationalizable outcome and, in **Section 4**, we apply the results obtained to investment games. In this regard, if the number of firms in an investment game is sufficiently large, the sufficient conditions identified ensure that RCBR yields a Nash equilibrium that is basically indistinguishable from a competitive equilibrium. In other words, we provide sufficient epistemic conditions for the Walrasian equilibrium of the Investment CAPM to hold with an arbitrary approximation. In **Section 5** we address the empirical puzzles of the Investment CAPM by departing from the epistemic assumption of common belief in rationality among firms, which in large games may be too demanding in terms of required reasoning skills. To do this, we replicate the behavior of an econometrician estimating the Investment CAPM with the Generalized Method of Moments (GMM) on simulated data consistent with the assumption rationality and mutual belief in rationality (i.e. first-order belief in rationality), which is the most extreme form of bounded reasoning about rationality. We use a pseudo-random numbers generator to jointly simulate the firms' conjectures and the future productivity of their assets. In particular, conjectures are simulated with a procedure reminiscent of level- $k$  thinking (Nagel, 1995; Costa-Gomes et al., 2001; Crawford and Iriberri, 2007). Calibrating the model to the top one thousand companies by stock market capitalization listed on U.S. markets, with the exclusion of Financial, Real Estate, and Utility firms, we obtain a simulated Monte Carlo distribution for the returns and characteristics of value and size sorted portfolios, both built as in Fama and French (1993). Our simulations provide very realistic stock market returns, while generating a low  $TS - R2$  for the relationship between stock and investment returns and a

high  $TS - R2$  for that between stock and investment valuations, consistent with the findings of Belo et al. (2024) and Belo et al. (2022), respectively. The reason for this apparent puzzle is that firms often fail to align their expected marginal return on assets with the actual cost of capital. Despite this bias, firms with higher expected profitability invest more and vice versa, and, since the model generates an actual cost of capital which is similar across characteristics sorted portfolios, Tobin Q appears basically linear in investment rates. Since firms are rational, they invest only if their perceived Tobin Q is greater than one. Consequently, a regression with fixed unit intercept of Tobin Q on investment rates has a very high R-squared, that is, a high  $TS - R2$ . However, we also find that the parameter controlling investment adjustment costs is overestimated to any reasonable degree of confidence. In addition, several firms, particularly value and small companies, overestimate their cost of capital, consistent with surveys on capital budgeting practices (cf. Graham, 2022). Since the calibration of the model implies the existence of a unique rationalizable outcome, our simulations suggest that including bounded reasoning about rationality is a promising direction to explain the empirical puzzles documented in the literature; to put it in the words of Lu Zhang (2025), *"firms seek value via heuristics"*. Adopting an epistemic game theoretical approach, we show that a bridge between traditional (i.e. neoclassical) and behavioral finance is possible and advisable.

## 2. Investment Games and Network Games with Linear best-replies

**2.1. Investment Games.** In this section, we consider a version of the static Investment CAPM (Zhang, 2017), formulated as a static game with cardinal payoffs, which we denote as an investment game. Namely, an investment game is an ordered tuple

$$\langle F, X, Y, (I_f)_{f \in F}, g, (U_f)_{f \in F} \rangle,$$

where:

- $F$  is a finite set of players;
- $X$  is the set of possible states of nature;
- $I_f$  is the set of actions available to the player  $f \in F$ ;
- $Y = \mathbb{R}^F$  is the set of relevant consequences for the players;
- $g : \times_{f \in F} I_f \rightarrow Y$  is the consequence function;
- $U_f$  is the cardinal utility function of the player  $f \in F$ .

Every investment game  $G$  is endowed with an epistemic model consistent with  $CKG$ , that is, we assume that it is common knowledge among players that the game being played is  $G$ , as well as with a suitable Bayesian frame. Therefore, having assumed that the consequence function is unaffected by the state of nature, we can represent an investment game as a static game of complete information in reduced form

$$\Gamma = \langle F, (I_f, u_f := U_f \circ g)_{f \in F} \rangle$$

that we equip with the corresponding universal type structure.

**2.1.1. The players and their actions.** An element  $f$  of  $F$  is understood as the manager of a company, or *firm*, which, with abuse of language, is identified with the company itself. The set of actions available to a company  $f \in F$  is  $I_f = [0, \bar{i}_f]$ , where  $\bar{i}_f$  is a positive real number. An element  $i_f \in I_f$  is understood as the total productive assets of the firms; we also refer to  $i_f$  as the *investment level* or *production capacity* of  $f$ . The action space of the game is  $I = \times_{f \in F} I_f$ , endowed with the usual topology of  $\mathbb{R}^F$ . There is a unique type of productive assets, which is perfectly divisible and can be purchased at the price of one dollar<sup>1</sup> per unit. Each  $f \in F$  is endowed with an amount of cash  $e_f$  to purchase productive assets; the initial endowment of productive assets is zero for every firm. For every  $f \in F$ , the total expense in dollars to install  $i_f$  units of productive assets is equal to  $i_f + \frac{\theta_f}{2} i_f^2$ , where  $\theta_f > 0$  is a constant; the term  $\frac{\theta_f}{2} i_f^2$  corresponds to accessory costs, also known as *investment adjustment costs*, in which  $f$  incurs to install  $i_f$  units of productive assets. We assume that companies have no access to capital markets in  $t$ , and that, for every  $f \in F$ ,  $e_f$  is large enough to support the installment of  $\bar{i}_f$  units of productive assets, namely,  $e_f > \bar{i}_f + \frac{\theta_f}{2} \bar{i}_f^2$ . We also denote by  $n \in \mathbb{N}$  the number of firms in the game, that is,  $n := |F|$ . For convenience, we denote by  $t$  the exact instant at which firms choose simultaneously their respective investment levels, that is, the moment when the particular situation described by the game takes place. An outcome of the game is a profile of investments  $i \in I$ .

**2.1.2. Consequences and preferences.** The situation described by the model is the firms' simultaneous choice of their respective investment levels. However, this does not exclude modeling consequences and payoffs as if there were common knowledge of other exogenous events taking place afterwards. In particular, we assume that there are dates  $t'$  and  $t''$ ,  $t' < t''$ , denoting, respectively, the time at which trading in the firms' shares takes place and the time at which firms are liquidated and

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<sup>1</sup>We use the term *dollar* to denote an hypothetic currency. However, the introduction of a currency is not strictly necessary. In this regard, notice that, having set the price of productive assets equal to one, it would be equivalent to assume that productive assets serve as numéraire.

the corresponding proceeds are distributed to shareholders. In this regard, the set of consequences is  $Y = \mathbb{R}^F$  with the typical element  $y = (y_f)_{f \in F}$ , where  $y_f$  is interpreted as the *cum-dividend market value* of  $f$  in  $t$ . We also refer to  $y_f$  as the shareholder value of  $f$ , which is the sum of dividends paid in  $t$  to shareholders and the market value of their shares in  $t'$ .<sup>2</sup> We assume that every firm  $f \in F$  has preferences over lotteries on  $Y_f$ , described by a linear and increasing utility function  $U_f : Y_f \rightarrow \mathbb{R}$ . That is, every company is a risk-neutral Bayesian decision maker that benefits from the financial value created for its initial investors (i.e those invested in the company at date  $t$ ). For the sake of clarity, we stress that shareholders, or investors, are exogenous in the model; in investment games, the players are firms.

**2.1.3. The consequence function.** The consequence function of the game maps every profile of actions into the corresponding consequence for the players, hence, in an investment game, it describes how the firms decisions affect their respective shareholder value. We let  $X$  be a compact subset of  $\mathbb{R}^F$  in the usual topology, that is,  $X \subseteq \times_{f \in F} [0, \bar{x}_f]$  where  $\forall f \in F, \bar{x}_f \in \mathbb{R}_+$ . Firms are assumed to be listed on a perfectly competitive financial market,<sup>3</sup> and, in  $t'$  the ex-dividend market value of  $f$ 's is assumed to be equal to  $p_f(i_f, i_{-f}) = \int_X m(x, i) x_f i_f d\pi(x)$ ,<sup>4</sup> where  $\pi \in \Delta(X)$  is interpreted as the common prior on  $X$  of the investors trading in firms' shares at date  $t'$ . Interpreting  $x_f$  as the liquidation value of  $f$ , the function  $m$  can be interpreted as a stochastic discount factor (SDF). In this regard, we assume a positive linear SDF, that is,  $m(x, i_f, i_{-f}) = \alpha - \frac{\beta}{2n} \sum_{j \in F} i_j x_j$ , where  $\alpha$  and  $\beta$  are two positive real numbers ensuring that, for every  $(x, i) \in X \times I$ ,  $m(x, i_f, i_{-f}) > 0$ . Without loss of generality,  $x_f$  can be identified as the sum of a random net operating profit per unit of asset and the resale price of a unit of productive asset in  $t''$ . Since the price of productive assets serves as numéraire,  $x_f$  can be also identified as the profitability of  $f$  at date  $t''$ .<sup>5</sup> Trading in shares may affect the shareholders base, and the liquidation value of firm  $f$ , which is equal to  $x_f i_f$ , will be paid to those investors holding shares of  $f$  after the trading session. Hence, the consequence function  $g : I \rightarrow Y$  is such that, for every  $i \in I$ ,

$$i \mapsto \left( e_f - i_f - \theta_f \frac{i_f^2}{2} + p_f(i_f, i_{-f}) \right)_{f \in F}.$$

<sup>2</sup>As usual, we assume implicitly that  $t$  and  $t'$  are close enough to abstract from any discounting effect.

<sup>3</sup>See the Preface and Chapter 3 for the related definition

<sup>4</sup>In **Chapter 3**, I analyze the possibility of competitive financial prices from an epistemic perspective.

<sup>5</sup>From an accounting perspective,  $x_f - 1$  can be viewed either as return on equity or return on invested capital of  $f$ , as in  $t'$   $f$  is financed only with equity and holds no cash.

Therefore, without loss of generality, for every  $f \in F$ , we fix  $U_f = -e_f + g(i)(f)$  and, consequently,  $u_f = -i_f - \theta_f \frac{i_f^2}{2} + p_f(i_f, i_{-f})$ . The assumption of a linear SDF can be motivated assuming that assets are priced as in an interior Walrasian Equilibrium (WE) of a market economy (cf. **Sections 3.1-3.3 of Chapter 3**), under the following assumptions:

- the assets traded in the economy are the firms' shares and, for every  $f \in F$ ,  $|X_f| = |I|$ ;
- complete asset markets, that is,  $\det \left( (X_f)_{f \in F} \right) \neq 0$ ;
- risk-averse investors with preferences consistent with SEU and quadratic utility functions of their future wealth;
- a common prior  $\pi \in \Delta(X)$ .

Under these assumptions, there exists a representative investor (RI) with marginal utility consistent with a linear SDF; additional restrictions on the parameters characterizing utilities remain necessary to obtain a positive SDF over the range of possible values for the RI's wealth generated by  $X \times I$ . In **Section 3.3 of Chapter 3** we describe a realistic trading game for a similar market economy, and, in **Section 5 of Chapter 3**, we provide sufficient epistemic conditions for WE asset prices.

**2.1.4. The reduced form.** By construction,  $X$  is the set of possible states of nature, since it contains the description of exogenous events related to the situation of interest. Hence, the game described so far is a static game with cardinal payoffs  $\langle F, X, Y, (I_f)_{f \in F}, g, (U_f)_{f \in F} \rangle$ . As said, we equip the game with an epistemic model in which it is common knowledge among firms that the game being played is  $\langle F, X, Y, (I_f)_{f \in F}, g, (U_f)_{f \in F} \rangle$ . Notice that, since the specific realization of the state of nature has no impact on the consequence function, the firms' preferences are unaffected by the state of nature. Hence, having assumed common knowledge of the game among firms,  $X$  can be excluded from the components of the game without loss of generality, as we argued in **Section 1.2.3 of Chapter 1**. The economic intuition is that the preferences of the firms are defined over the lotteries on their respective shareholder value in  $t$  and, since  $g$  is common knowledge, for every  $f \in F$ , the set of lotteries on shareholder value is spanned by  $\Delta(I_{-f})$ . Hence, an investment game  $\Gamma$  is a static game of complete information with Bayesian players in reduced form

$$\Gamma = \langle F, (I_f, u_f := U_f \circ g)_{f \in F} \rangle$$

which we equip with the corresponding universal type structure  $\mathcal{T} = \langle F, (I_{-f}, H_f, \eta_f)_{f \in F} \rangle$ . Consequently, the type  $t_f$  of firm  $f$  is rational if

$$a_f \in \operatorname{argmax}_{i_f \in T_f} \left\{ -i_f - \theta_f \frac{i_f^2}{2} + \mathbb{E}_{\eta_f(t_f)} [p_{f,i_f}(i_{-f})] \right\}.$$

**2.1.5. Comments.** Every investment game formalizes the setting of the classic two-period investment CAPM under the following additional hypotheses: i) the initial capital stock is set equal to zero, ii) the SDF is linear, and iii) firms are financially constrained. The first hypothesis is without loss of generality and serves to save on the notation, while the second can be seen as a reasonable approximation of more sophisticated SDFs. The inclusion of financing constraints is just technical, to ensure that  $I$  is the product of compact intervals of the non-negative real line. In this regard, in the remainder of the chapter we assume that the structure of the model's parameters  $\langle n, \alpha, \beta, (\theta_f, \bar{i}_f)_{f \in F} \rangle$  ensures that financing constraints are inactive for *rational* firms, thus making them irrelevant in practice. The assumption on firms' preferences is the same as in the static Investment CAPM (cf. Zhang, 2017), and it is standard in corporate finance models; see also Chapter 3 of Saggiaschi and Savona (2021) for a reference. In short, it reflects the idea that every firm is managed to reflect the financial interests of its initial shareholders, who are going to sell their shares in  $t'$  on a perfectly competitive financial market. In other words, given the static nature of the model, shareholder value can be loosely identified as the exit value of initial shareholders.

**2.2. Solution concepts for investment games.** In this section, we briefly review the solution concepts that will be considered in the analysis. Before proceeding, we observe that, given a first-order belief  $\mu_f \in \Delta(I_{-f})$ , action  $i_f$  is rational for firm  $f \in F$  if

$$i_f \in \operatorname{argmax}_{i'_f \in [0, \bar{i}_f]} \left\{ -i_f - \frac{\theta_f}{2} i_f'^2 + \mathbb{E}_{\mu_f} [p_{i_f}(i_{-f})] \right\}.$$

Hence, the decision problem of a generic rational firm, say  $f$ , is

$$(2.1) \quad \max_{i_f \in [0, \bar{i}_f]} e_f - i_f - \frac{\theta_f}{2} i_f^2 + \mathbb{E}_{\mu_f} \left\{ \mathbb{E}_{\pi} \left[ \left( \alpha - \frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} i_j x_j \right) i_f x_f - \frac{\beta}{2n} (i_f x_f)^2 \right] \right\}.$$

**2.2.1. Rationalizability.** We already noted that  $\Gamma$  is a compact-continuous game of complete information, since  $I$  is a compact metric space and, for every  $f \in F$ ,  $u_f$  is continuous on  $I$ . Hence, having equipped  $\Gamma$  with the corresponding universal type structure, by **Theorem 1.2 of Chapter 1** we have  $\operatorname{proj}_I RCBR = \rho^\infty(I)$ , and  $\rho^\infty(I)$  is the set of rationalizable outcomes. In addition,

since  $u_f(i_f, i_{-f})$  is quasi-concave in  $i_f$ ,  $\Gamma$  is also nice and, by **Lemma 1.4** and **Proposition 1.5** of **Chapter 1**,  $\rho^\infty(I) = \hat{r}^\infty(I) = \hat{r}(\hat{r}^\infty(I))$ . Therefore, if the set of point-rationalizable solutions is a singleton, that is,  $\hat{r}^\infty(I) = \{i^*\}$ , the unique rationalizable outcome is also the unique Nash equilibrium of the game. It follows that, in our context, the existence of a unique point-rationalizable outcome is equivalent to the existence of a unique outcome consistent with rationality and common belief in rationality (RCBR), which is also a Nash equilibrium.

2.2.2. *Point-rationalizability.* As in **Section 1.4.2**, we let

$$\mathcal{C} = \left\{ C \subseteq 2^I : \exists (C_f)_{f \in F} \subseteq \times_{f \in F} \mathcal{K}_{I_f}, C = \times_{f \in F} C_f \right\}$$

be the collection of all closed Cartesian subsets of the action space  $I$ , which is compact in its usual topology, that is, the product topology. The point-rationalization operator  $\hat{r}(C)$  maps every set  $C \in \mathcal{C}$  to  $\times_{f \in F} \hat{r}_f(C_{-f}) \subseteq I$ , where,

$$(2.2) \quad \hat{r}_f(C_{-f}) = \left\{ i_f \in I_f : \exists i_{-f} \in C_{-f}, i_f \in \operatorname{argmax}_{i'_f \in [0, \bar{i}_f]} \left[ u_f(i'_f, i_{-f}) \right] \right\}.$$

Fix a firm  $f \in F$ . Suppose that  $f$  believes that the other firms will choose the investment profile  $\hat{i}_{-f} \in I_{-f}$ , and given this conjecture,  $f$  rationally play the action  $i_f^* \in (0, \bar{i}_f)$ . Since  $u_f$  is continuous, differentiable, and concave in  $i_f$  over  $(0, \bar{i}_f)$ , it must be the case that  $\frac{\partial}{\partial i_f} u_f(i_f, i_{-f}) = 0$  at  $(i_f^*, \hat{i}_{-f})$ , that is,

$$(2.3) \quad 1 + \left[ \theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2) \right] i_f^* = \mathbb{E}_\pi \left[ \left( \alpha - \frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} \hat{i}_j x_j \right) x_f \right].$$

By the usual properties of the expectation functional  $\mathbb{E}(\cdot, \cdot)$ , we obtain

$$(2.4) \quad i_f^* = \frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} - \frac{\frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} \mathbb{E}_\pi(x_f x_j) \hat{i}_j}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)}.$$

Thus, for every  $f \in F$  and every  $i_{-f} \in I_{-f}$ ,  $\operatorname{argmax}_{i_f \in [0, \bar{i}_f]} \{u_f(i_f, i_{-f})\}$  is a singleton, and let the joint point best-reply (PBR) function be

$$\hat{r} : I \rightarrow I$$

$$(i_f)_{f \in F} \mapsto (\hat{r}_f(i_{-f}))_{f \in F}$$

where, for every  $f \in F$ ,

$$(2.5) \quad \hat{r}_f(i_{-f}) = \max \left\{ 0, \min \left\{ \frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} - \frac{\frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} \mathbb{E}_\pi(x_f x_j) i_j}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)}, \bar{i}_f \right\} \right\}.$$

Since, for every  $f \in F$ ,  $\hat{r}_f(i_{-f})$  is continuous and, for every  $j \in F \setminus \{f\}$ ,  $\hat{r}_f(i_{-f})$  is strictly decreasing in  $i_j$ , it follows that  $\hat{r}(i_{-f}) \leq \hat{r}((0)_{j \in F \setminus \{f\}})$  and  $\hat{r}(i_{-f}) \geq \hat{r}((\bar{i}_j)_{j \in F \setminus \{f\}})$ .<sup>6</sup> Therefore, to ensure inactive financing constraints, we must impose that the structure of the model parameters

$$\Xi = \left\langle \alpha, \beta, (\theta_f, \mathbb{E}_\pi(x_f), \mathbb{E}_\pi(x_f^2))_{f \in F} \right\rangle$$

satisfies the condition  $\frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} < \bar{i}_f$  for every firm  $f \in F$ . In addition, the strict positivity of SDF requires that for every  $f \in F$ ,

$$\alpha > \frac{\beta}{2} \sum_{f \in F} \bar{x}_f \bar{i}_f,$$

while the existence of a non-null rationalizable investment profile requires instead that for every  $f \in F$ ,  $\alpha \mathbb{E}_\pi(x_f) > 1$ . Together, these conditions are said to be the minimal hypotheses on  $\Xi$ , and an investment game satisfying these conditions is said to be *regular*.

Finally, we recall that, given two vectors  $x, y \in \mathbb{R}^F$ ,  $\max(x, y) := (\max(x(i), y(i)))_{f \in F}$  and  $\min(x, y) := (\min(x(i), y(i)))_{f \in F}$ ; accordingly, for every  $i \in I$ ,

$$\hat{r}(i) = \max(0_F, \min(a - Bi, \bar{i})),$$

where  $0_F := (0)_{f \in F}$  and  $B = (b_{f,j})_{F \times F}$  is a matrix with typical element  $b_{f,j}$ , such that, for every  $f, j \in F$ ,

$$(2.6) \quad b_{f,j} = \begin{cases} \frac{\frac{\beta}{2n} \mathbb{E}_\pi(x_f x_j)}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} & f \neq j \\ 0 & f = j, \end{cases}$$

and

$$a = \left( \frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} \right)_{f \in F}.$$

Furthermore, if  $\Gamma$  is regular, we also have that, for every  $i \in I$ ,

$$\hat{r}(i) = \max(0_F, a - Bi).$$

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<sup>6</sup>Note the similarity with the model of Cournot oligopolistic competition discussed in **Section 1.4.4** of **Chapter 1**.

**2.2.3. Nash equilibrium.** In a Nash equilibrium, firms are rational and have correct conjectures about their opponents' actions, that is, a profile of actions  $i^* \in I$  is a Nash equilibrium (NE) if, for every  $f \in F$ ,  $i_f^* \in \operatorname{argmax}_{i_f \in [0, \bar{i}_f]} \left\{ u_f(i_f, i_{-f}^*) \right\}$ . Thus,  $i^* \in I$  is a Nash equilibrium if and only if it is a fixed point of the joint PBR function  $\hat{r}$ . Every Nash equilibrium is point-rationalizable. Arguably, a Nash equilibrium is a *refinement* rationalizability, under the additional epistemic assumption that players have correct conjectures, which is not necessarily realistic. More specifically, if  $\Gamma$  is dominant solvable, the unique rationalizable action profile is also the unique Nash equilibrium of the game. Every nice game of complete information has at least one Nash equilibrium, and hence every investment game has a Nash equilibrium. In the case of a regular investment game, it is easy to show that only one firm  $f$  can play the corresponding action  $a_f$  in a NE. Consequently, any equilibrium investment profile  $i^*$  satisfies  $0 \leq i^* < a \leq \bar{i}$ . In this regard, a NE is said to be *interior* if, for every  $f \in F$ ,  $i_f^* \in (0, i_f)$ . Since  $\Gamma$  is nice, an interior equilibrium requires that, for every  $f \in F$ , the first-order condition  $\frac{\partial}{\partial i_f} u_f(i_f, i_{-f}) = 0$  is verified at  $(i_f^*, i_f^*)$ , that is,

$$(2.7) \quad i_f^* = \frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} - \frac{\frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} \mathbb{E}_\pi(x_f x_j) \tilde{i}_j}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)},$$

for every  $f \in F$ .

**2.3. Investment-based asset pricing and bounded rationality.** Consider an investment game with a unique rationalizable outcome and with a very large number of players. In this case, the unique rationalizable outcome  $i^*$ , which is a NE, is well approximated by the competitive equilibrium profile  $i^{**}$  which, for every  $f \in F$ , satisfies the first order condition

$$1 + \theta_f i_f^{**} = \mathbb{E}_\pi[M(x) x_f],$$

where  $M(x) = m(i^{**}, x)$ . By the approximation made,  $p_f = (1 + \theta_f i_f^{**}) i_f^{**}$ , that is, for every  $f \in F$ , the price-to-book ratio  $q_f := \frac{p_f}{i_f^{**}}$ , or Tobin Q, is, in (a competitive) equilibrium, linear in investment with unit intercept. In this regard, the quantity  $tr_f^i := \frac{x_f}{1 + \theta_f i_f^{**}} - 1$  is the marginal profitability of a dollar of invested capital (*investment return*), which, in (a competitive) equilibrium, is equal to the financial return of a dollar invested in the firm's shares in  $t'$ , that is,  $tr_f^s = \frac{x_f}{p_f} - 1$  (*stock return*). In other words, in a competitive equilibrium, investment returns  $(tr_f^i)$  are always equal to stock returns  $(tr_f^s)$ , and valuation ratios are fully explained by the observed investment levels. A large body of literature has tested the validity of this view with mixed results, as we anticipated in the

introduction. Here, only observe that testing for the validity of the (static) Investment CAPM in the time-series is essentially a test of (static) competitive equilibrium behavior. Despite the fact that the number of listed firms is usually high enough to approximate a Nash equilibrium with a competitive one, there might be no valid reason to assume that only equilibria can be observed in the data, either because there is more than one rationalizable outcome or because common belief in rationality is unlikely to occur. We will return on this point at the end of **Section 5**, which is one of the possible applications of the analysis presented in the next section.

**2.4. Network games with linear best-replies.** A network game with linear best-reply, is a nice static game of complete information in reduced form with Bayesian players,

$$\Gamma = \left\langle F, (I_f)_{f \in F}, (u_f : \times_{f \in F} I_f \rightarrow \mathbb{R})_{f \in F} \right\rangle,$$

such that, for every player  $f \in F$ , there exists a PBR function satisfying the following equation

$$(2.8) \quad \hat{r}_f(i_{-f}) = \begin{cases} a_f - \sum_{j \in F} b_{fj} i_j & a_f > \sum_{j=1} b_{fj} i_j \\ 0 & a_f = \sum_{j=1} b_{fj} i_j, \end{cases}$$

where:

- $a_f = \hat{r}_f((0)_{j \in F \setminus \{f\}}) > 0$  is a positive vector such that, for every  $f \in F$ ,  $\max\{I_f\} \geq a_f$ ;
- $b_{fj} \geq 0$  for every  $(f, j) \in F \times F$ .

The matrix  $B \in \mathbb{R}^{F \times F}$  of elements  $b_{fj}$  is known as *adjacency matrix*, or *network matrix*, since it connects the players of the game through a *graph*, or *network*, and allows to write the joint PBR function as

$$\hat{r}(i) = \max(0_F, a - Bi).$$

Since every network game with linear best-replies is, by definition, nice, we have that, for every  $k \in \mathbb{N}$ ,  $\hat{r}^k(I) = \rho^k(I)$ , and  $\hat{r}^\infty(I) = \rho^\infty(I)$ . Every regular investment game is a network game with linear best-replies with network matrix

$$B : F \times F \rightarrow \mathbb{R}$$

$$(f, j) \mapsto b_{f,j} = \begin{cases} \frac{\frac{\beta}{2n} \mathbb{E}_\pi(x_f x_j)}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} & f \neq j \\ 0 & f = j, \end{cases}$$

and

$$a = \left( \frac{\alpha \mathbb{E}_\pi(x_f) - 1}{\theta_f + \frac{\beta}{n} \mathbb{E}_\pi(x_f^2)} \right)_{f \in F}.$$

For this reason, in the following section we study rationalizability in the more general case of network games with linear best-replies.

### 3. Rationalizability in network games with linear best-replies

Let  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  be a network game with linear best-reply with network matrix  $B$ . When  $\Gamma$  admits a unique rationalizable outcome, we also say that it is dominant solvable. Since  $\Gamma$  is nice, rationalizability is equivalent to point-rationalizability, and, with a slight abuse of language, we can use the two terms interchangeably. Given a profile of actions  $i \in I$ , a player  $f \in F$  is said to be active at  $i$  if  $i_f > 0$ . For every  $k \in \mathbb{N}$ , we define the profile of actions  $\bar{i}^k = (\bar{i}_f^k)_{f \in F}$  and  $\underline{i}^k = (\underline{i}_f^k)_{f \in F}$ , where  $\bar{i}_f^k = \max \{ \hat{r}_f^k(I_{-f}) \}$  and  $\underline{i}_f^k = \min \{ \hat{r}_f^k(I_{-f}) \}$ , respectively. Since  $\Gamma$  is nice and, for all  $f$  and  $j \neq f$ ,  $\frac{\partial r_f}{\partial i_j} < 0$ , it follows that, for every  $k \in \mathbb{N}$ ,

$$\rho^k(I) = \hat{r}^k(I) = \times_{f \in F} \left[ \hat{r}_f \left( \bar{i}_{-f}^{k-1} \right), \hat{r}_f \left( \underline{i}_{-f}^{k-1} \right) \right].$$

In this regard, a player  $f \in F$  is said to be *possibly active* at  $k \in \mathbb{N}$  if  $\bar{i}_f^k > 0$  and inactive otherwise. Similarly, a player  $f \in F$  is said to be *completely active* at  $k \in \mathbb{N}$  if  $\underline{i}_f^k > 0$ , that is, if it always chooses a positive investment level after  $k$  iterations of  $\hat{r}$ . A game is said to be fully active at  $k \in \mathbb{N}$  if every player is completely active at  $k$ . It is immediate to verify that, since the point-rationalization operator is monotone, if  $\Gamma$  is fully active at  $k$ , then it is also fully active at  $k + 1$ . Finally,  $\Gamma$  is said to be fully active if it is fully active at  $k = 2$ . It is immediate to verify that  $\Gamma$  is fully active if and only if  $a < Ba$ .

**3.1. Fully active games.** As a first step towards characterizing the necessary and sufficient conditions for dominance solvability in network games with linear best-replies, we consider the special case of fully active games. In particular, we show that every fully active network game with linear best-replies has a unique rationalizable outcome. To prove this result, we rely on the following lemma.

LEMMA 2.1. *Let  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  be a network game with linear best-replies, with network matrix  $B$ . If  $\Gamma$  is fully active, then  $\varrho(B) < 1$ .*

PROOF. If  $\Gamma$  is fully active, then it must be the case that  $a > Ba$ . Since  $B$  is a square non-negative matrix, according to the Perron-Frobenius theorem,  $\varrho(B)$  is the largest positive eigenvalue of  $B$ . Let  $y \in \mathbb{R}^F$  be the eigenvector of  $B$  associated with the eigenvalue of  $B$  that is equal to  $\varrho(B)$ , that is,  $yB = \varrho(B)y$ ; in a fully active game, it must be the case that  $A > Ba$ , hence we have that, so that  $B^\top y = \varrho(B)y$ ; by definition, in a fully active game  $(a - Ba) > 0$ , and consequently,

$$y(a - Ba) > 0 \iff y \cdot a - yBa > 0.$$

Since  $yB = \varrho(B)y$ , we have  $yBa = \varrho(B)y \cdot a$ . Therefore, we can write the above equivalence as

$$y(a - Ba) > 0 \iff y > \varrho(B)y.$$

By the Perron-Frobenius theorem,  $y$  is a non-negative vector with at least a strictly positive entry, therefore,

$$y > \varrho(B)y \iff \varrho(B) < 1.$$

Hence, we conclude that if  $\Gamma$  is fully active, it must be the case that  $\varrho(B) < 1$ .  $\square$

Notice that the converse of the previous lemma is not necessarily true. In fact, we can easily build examples of games that are not fully active but have a matrix  $B$  with spectral radius strictly less than one. We also recall the following result from linear algebra and the theory of systems of difference equations.

**THEOREM 2.2.** *Let  $B_0 : F \times F \rightarrow \mathbb{R}$  be a matrix and  $c \in \mathbb{R}^F$  be a vector. Every sequence  $(x_k)_{k \in \mathbb{N}_0}$ , such that, for every  $k \in \mathbb{N}$ ,  $x_{k+1} = c + B_0 x_k$ , converges to  $x^* = (\mathcal{I}_n - B)^{-1}c$  in the Euclidean metric space.*

PROOF. See Theorems 1.10, 1.11, 4.1, and Corollary 4.2 in Saad (2003).  $\square$

As a straightforward application of **Lemma 2.1** and **Theorem 2.2**, every fully active game is dominant solvable.

**PROPOSITION 2.3.** *Every fully active network game with linear best-replies  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  admits a unique rationalizable outcome that is strictly positive, that is,  $\hat{r}^\infty(I) = \{i^* > 0\}$ .*

PROOF. Since  $\Gamma$  is fully active, for every  $f \in F$ , a null investment level is never a best-response for and  $\hat{r}_f(i_{-f}) = a_f - \sum_{j \in F \setminus \{f\}} b_{f,j} i_j$ . Therefore, it must be the case that, for every  $k \geq 1$ ,

$\underline{i}^k = a - B\bar{i}^{k-1}$  and  $\bar{i}^k = a - B\underline{i}^{k-1}$ , that is,

$$\begin{bmatrix} \bar{i}^k \\ \underline{i}^k \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} + B_1 \begin{bmatrix} \bar{i}^{k-1} \\ \underline{i}^{k-1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & -B \\ -B & 0 \end{bmatrix},$$

as well as, for every  $f \in F$ ,  $\underline{i}_f^0 = 0$  and  $\bar{i}_f^0 = \bar{i}_f$ . By **Lemma 2.1**, the spectral radius of  $B$  must be less than one; therefore,  $\varrho(-B) < 1$ . Since  $B_1$  is block-diagonal with blocks  $(-B, -B)$ , its eigenvalues coincide with those of  $-B$  and, therefore,  $\varrho(B_1) = \varrho(-B) < 1$ . By **Theorem 2.2**,

$$\lim_{k \rightarrow \infty} \underline{i}^k = \lim_{k \rightarrow \infty} \bar{i}^k = (\mathcal{I}_n + B)^{-1} a,$$

that is,  $\hat{r}^\infty(I) = \{i^*\}$ , where  $i^* = (\mathcal{I}_n + B)^{-1} a > 0$ .  $\square$

**3.2. Sufficient conditions for dominance solvability.** In this section, we show that, for a network game with linear best-replies  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  with network matrix  $B$ , the condition  $\varrho(B) < 1$  is sufficient for the existence of a unique rationalizable outcome. Before proving this result, we recall that, given metric spaces  $(X, \delta)$ , a function  $f : X \rightarrow X$  is a contraction on  $(X, \delta)$  if there exists  $c \in [0, 1)$  such that, for every  $x, y \in X$ ,  $d(f(x), f(y)) \leq c \cdot d(x, y)$ , or equivalently, if for every  $x, y \in X$ ,  $d(f(x), f(y)) < d(x, y)$ . Extending earlier results in Moulin (1984), Zimper (2006b) shows that, in every nice game of complete information, if the joint PBR mapping is a contraction, there exists a unique rationalizable investment profile. Hence, the proof of our claim reduces to showing that, in every network game with linear best-replies,  $\varrho(B) < 1$  implies that  $\hat{r}(\cdot)$  is a contraction on  $(I, d)$ , where  $d$  denotes the Euclidean metric. To do this, we rely on the following lemma.

**LEMMA 2.4.** *Let  $(X, d)$  be a subset of an Euclidean metric space, and  $f : X \rightarrow X$  a contraction on  $(X, d)$ , and  $s \in \mathbb{R}_+^F \cup \{0\}$  a vector with nonnegative entries. The function*

$$g : X \rightarrow X$$

$$x \mapsto \max(0_F, \min(f(x), s))$$

*is a contraction on  $X$ .*

**PROOF.** We introduce the auxiliary function  $\phi : X \rightarrow X$  such that  $x \mapsto \min(f(x), s)$ , and write  $g(x) = \max(0, \phi(x))$ . We claim that for every  $x, y \in X$ ,

$$d(\phi(x), \phi(y)) \leq d(f(x), f(y)).$$

To see this, recall that for every  $u, v \in \mathbb{R}^F$ ,  $d(u, v)^2 = \sum_{j \in F} [u(j) - v(j)]^2$ . Thus, to prove the claim made above, it is sufficient to show that, for every  $x, y \in X$ , and for every  $j \in F$ ,

$$|\phi(x)(j) - \phi(y)(j)| \leq |f(x)(j) - f(y)(j)|.$$

For a given pair of vectors  $x, y \in X$ , and a generic value  $j \in F$ , we have

$$|\phi(x)(j) - \phi(y)(j)| = \begin{cases} |f(x)(j) - f(y)(j)| & f(y)(j), f(x)(j) \leq s \\ 0 & f(y)(j), f(x)(j) > s \\ |f(x)(j) - s| & f(x)(j) \leq s < f(y)(j) \\ |s - f(y)(j)| & f(y)(j) \leq s < f(x)(j). \end{cases}$$

Therefore, in every possible case of the equation above, we obtain

$$|\phi(x)(j) - \phi(y)(j)| \leq |f(x)(j) - f(y)(j)|,$$

and since this holds for every  $j \in F$ , we conclude  $d(\phi(x), \phi(y)) \leq d(f(x), f(y))$ . In addition, since  $f$  is a contraction, we have that, for every  $x, y \in X$ ,

$$d(\phi(x), \phi(y)) \leq d(f(x), f(y)) < d(x, y),$$

that is,  $\phi(\cdot)$  is a contraction on  $(X, d)$ . Now, observe that if  $\phi(\cdot)$  is a contraction on  $(X, d)$ , also the function  $\Phi : X \rightarrow X$  such that  $x \mapsto \Phi(x) = -\phi(x)$  is a contraction on  $(X, d)$ . Hence, we can conclude that  $g(\cdot)$  is a contraction on  $(X, d)$ , since, for every  $x \in X$ , we have

$$g(x) = -\min(0_F, -\phi(x)),$$

and, as showed in precedence, for every contraction  $h : X \rightarrow X$ , the associated composite function  $h' : X \rightarrow X$  such that  $x \mapsto -\min(0_F, h(x))$  is a contraction on  $(X, d)$ .  $\square$

We can now state and prove the main result of this section.

**PROPOSITION 2.5.** *Let  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  be a network game with linear best-replies, and let  $B$  the corresponding network matrix. If  $\varrho(B) < 1$ , there exists a unique rationalizable outcome.*

**PROOF.** Since  $\Gamma$  is a nice game of complete information, the set of rationalizable outcomes coincides with the set of point-rationalizable outcomes (cf. **Section 2.2.1**). For every  $i \in I$ , recall

that

$$\hat{r}(i) = \max(0_F, \min(a - Bi, \bar{i})).$$

By standard results in linear algebra, the function  $h : \mathbb{R}^F \rightarrow \mathbb{R}^F$  such that  $i \mapsto a - Bi$  is a contraction on  $(\mathbb{R}^F, d)$  if and only if  $\varrho(B) < 1$ . Since we have by hypothesis that  $\varrho(B) < 1$ ,  $h(\cdot)$  is a contraction on  $(\mathbb{R}^F, d)$  and, consequently, a contraction on  $(I, d)$ . Hence, by **Lemma 2.4** we conclude that  $\hat{r}(\cdot)$  is a contraction on  $(I, d)$ , and, as an application of Theorem 3 in Zimper (2006b), that there exists a unique point-rationalizable outcome. Since a network game with linear best-replies is nice, it follows that there exists a unique rationalizable outcome.  $\square$

In general, condition  $\varrho(B) < 1$  is not always necessary for the existence of a unique rationalizable outcome, as we show in the next section. In fact, the "clipping" role played by the min and max operators in the joint PBR function may allow a non-contractive linear mapping  $i \mapsto a - Bi$  to induce a contractive joint PBR. In this regard, Bramoullé et al. (2014) consider the class of network games with linear best-replies satisfying the following conditions:

- (1)  $a_f = 1$  for every  $f \in F$ ,
- (2)  $g_{fj} \in \{0, 1\}$  for every pair of distinct players  $f \neq j$ ,
- (3)  $\sum_{j=1}^n g_{fj} = k \leq n - 1$  for every  $f \in F$ ,

which find several applications in economics, and show that there exists a unique Nash equilibrium if and only if  $\varrho(G) < \frac{1}{\delta}$ . Furthermore, they show that the same condition is necessary and sufficient to ensure stability under Nash tâtonnement. However, Nash tâtonnement is not a proper solution concept and neither an epistemic condition for Nash equilibrium. Nevertheless, **Proposition 2.6** shows that  $\varrho(G) < \frac{1}{\delta}$  is necessary and sufficient for the existence of a unique rationalizable outcome in this particular class of games.

**PROPOSITION 2.6.** *Suppose  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$  is a network game with linear best-replies with network matrix  $B = \delta G$  where  $\delta$  is a positive real number. If  $\Gamma$  satisfies the following hypotheses:*

- (1) *for every  $f \in F$ ,  $a_f = 1$ ;*
- (2) *for every  $f, j \in F$ ,  $g_{fj} \in \{0, 1\}$  and  $g_{ff} = g_{jj} = 0$ ;*
- (3) *for every  $f \in F$ ,  $\sum_{j=1}^n g_{fj} = k \leq n - 1$  ;*

*there exists a unique rationalizable outcome if and only if  $\varrho(G) < \frac{1}{\delta}$ .*

PROOF. Since  $\Gamma$  is a network game with linear best-replies, **Proposition 2.5** establishes that  $\varrho(B) < 1$ , or equivalently,  $\varrho(G) < \frac{1}{\delta}$ , is a sufficient condition for the existence of a unique rationalizable outcome. Corollary 1 in Bramoullé et al. (2014) establishes that  $\varrho(G) > \frac{1}{\delta}$  implies the existence of multiple Nash equilibria. Since every Nash equilibrium is rationalizable, it follows that it is also necessary  $\varrho(G) < \frac{1}{\delta}$  for the existence of a unique rationalizable outcome.  $\square$

**3.3. Necessary conditions for dominance solvability.** The following proposition provides the necessary conditions for the existence of a unique rationalizable outcome.

**PROPOSITION 2.7.** *Consider a network game with linear best-replies  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R}) \rangle$  with network matrix  $B$ . For every  $k \in \mathbb{N}$ , let  $L^k \subseteq F$  be the subset of inactive players at  $k$ . Suppose that there exists a unique rationalizable investment profile, denoted by  $i^*$ , and let  $L^\infty \subseteq F$  be the set of inactive players at  $i^*$ .*

- (1)  $\exists f \in F$  such that  $a_f > \sum_{j \in F \setminus \{f\}} b_{fj} a_j$ .
- (2) The subset of active players in  $i^*$  is obtained with a number  $k_0 \in \mathbb{N}$  of iterations of  $\hat{r}(\cdot)$ .
- (3) If  $L^\infty = \emptyset$ , it must be the case that  $\varrho(B) < 1$ .
- (4) If  $L^\infty \neq \emptyset$ , it must be the case that  $\varrho(B^{**}) < 1$ , where  $B^{**} := (b_{\varphi^{-1}(s)\varphi^{-1}(t)})_{[\alpha] \times [\alpha]}$ ,  $\alpha := |F| - |L^\infty|$  and  $\varphi : F \setminus L^\infty \rightarrow [\alpha]$  is a bijection re-indexing the set of active players.

**PROOF. Point 1.** By contradiction, if  $\nexists f \in F$  such that  $a_f > \sum_{j \in F \setminus \{f\}} b_{fj} a_j$ , the two extreme investment profiles  $i = (0)_{j \in F}$  and  $i = (a_j)_{j \in F}$  are both justifiable. As a result, the set  $\times_{f \in F} [0, a_f]$  has the best-reply property, and every  $i \in \times_{f \in F} [0, a_f]$  is rationalizable.

**Point 2.** First, observe that it must be the case that there exists  $k' \in \mathbb{N}$  such that, for every  $k \geq k'$ ,  $\bar{i}_{F \setminus L^\infty}^k > 0$ . Otherwise, since  $\hat{r}(\cdot)$  is weakly monotone, there exist  $f \in F \setminus L^\infty$  and  $k'' \in \mathbb{N}$  such that, for every  $k \geq k''$ ,  $\bar{i}_f^k = 0$ , and consequently  $\hat{r}_f^\infty(I) = \{0\}$ , reaching a contradiction. Now, suppose, by contradiction, that there exists  $f \in F \setminus L^\infty$  such that, for every  $k \in \mathbb{N}$ ,  $\bar{i}_f^k = 0$ . In this case, for every  $k \geq k'$ , we would have  $\hat{r}_f^k(I) = [0, \bar{i}_f^k]$  for some  $\bar{i}_f^k \geq i_f^*$ , and consequently  $\hat{r}_f^\infty(I) = [0, c]$  for some  $c \geq 0$ . Since, by hypothesis, there is a unique rationalizable outcome  $i^*$  and, by the definition of  $L^\infty$ ,  $i_f^*$  is strictly greater than zero, we have reached a contradiction. Hence, there exists no  $f \in F \setminus L^\infty$  such that, for every  $k \in \mathbb{N}$ ,  $\bar{i}_f^k = 0$ . As a consequence, there exists  $k''' \in \mathbb{N}$  such that, for every  $k \geq k'''$ , and every  $f \in F \setminus L^\infty$ ,  $\bar{i}_f^k > 0$ . Setting  $k_0 = \max(k', k''')$  completes the proof of the claim.

**Point 3.** From the considerations made at point 2), it follows that if  $L^\infty = \emptyset$ , for every  $k \geq k_0$ , we

have

$$\hat{r}^k(I) = \times_{f \in F} \left[ r_f \left( \bar{i}_{-f}^{k-1} \right) > 0, \hat{r}_f \left( \underline{i}_{-f}^{k-1} \right) \right].$$

By the proof of **Theorem 2.2**,  $\left( \bar{i}^k \right)_{k \in \mathbb{N} \setminus [k_0-1]}$  and  $\left( \underline{i}^k \right)_{k \in \mathbb{N} \setminus [k_0-1]}$  converge to  $i^*$  if and only if  $\varrho(B) < 1$ . In addition, if  $\left( \bar{i}^k \right)_{k \in \mathbb{N} \setminus [k_0-1]}$  and  $\left( \underline{i}^k \right)_{k \in \mathbb{N} \setminus [k_0-1]}$  converge to  $i^*$ , then  $i^* > 0$ , as well as  $i^* \leq \bar{i}$  since  $\bar{i}^1 < \bar{i}$  and the operator  $\hat{\rho}(\cdot)$  is weakly monotone.

**Point 4.** Suppose that  $L^\infty \neq \emptyset$ . Since  $\hat{\rho}(\cdot)$  is weakly monotone, there exists  $k_1 \geq k_0$  such that, for every  $k \geq k_1$ ,

$$\hat{r}^\infty(I) = \left( \times_{f \in L^\infty} [0, 0] \right) \times \left( \times_{f \in F \setminus L^\infty} \left[ \hat{r}_f \left( \bar{i}_{-f}^{k-1} \right) > 0, \hat{r}_f \left( \underline{i}_{-f}^{k-1} \right) \right] \right),$$

or, in alternative, for every  $k \geq k_0$ ,

$$\hat{r}^k(I) = \left( \times_{f \in L^\infty} [0, \hat{r}_f(\bar{i}_{-f}^k)] \right) \times \left( \times_{f \in F \setminus L^\infty} \left[ \hat{r}_f \left( \bar{i}_{-f}^{k-1} \right) > 0, \hat{r}_f \left( \underline{i}_{-f}^{k-1} \right) \right] \right).$$

We consider the two cases separately for simplicity, which we denote by **case a** and **case b**, respectively. Before examining the two cases, observe that  $i_{F \setminus L^\infty}^*$  is a restricted selection (subvector) such that, for every  $f \in F \setminus L^\infty$ ,  $i_f^* = a_f - \sum_{j \in (F \setminus L^\infty) \setminus \{f\}} b_{fj} i_j^*$ . In this regard, we let  $\varphi : F \setminus L^\infty \rightarrow [\alpha]$  be a bijection re-indexing the set of active players, where  $\alpha := |F| - |L^\infty|$ , as well as  $a^{**} := (a_{\varphi^{-1}(s)})_{s \in [\alpha]}$  and  $B^{**} := (b_{\varphi^{-1}(s)\varphi^{-1}(t)})_{[\alpha] \times [\alpha]}$ .

**Case a.** If there exists  $k_1 \geq k_0$  such that, for every  $k \geq k_1$ ,

$$\hat{r}^k(I) = \left( \times_{f \in L^k} [0, 0] \right) \times \left( \times_{f \in F \setminus L^{k_0}} \left[ \hat{r}_f \left( \bar{i}_{-f}^{k-1} \right) > 0, \hat{r}_f \left( \underline{i}_{-f}^{k-1} \right) \right] \right).$$

it must also be the case that for every  $k \geq k_1$ ,

$$\begin{bmatrix} \left( \bar{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} \end{bmatrix} = \begin{bmatrix} a^{**} \\ a^{**} \end{bmatrix} + B_1 \begin{bmatrix} \left( \bar{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0 & -B^{**} \\ -B^{**} & 0 \end{bmatrix}.$$

Since, by hypothesis, there exists a unique rationalizable outcome, it must be the case that

$$\left( \bar{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]}, \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} \xrightarrow{d} \left( i_{\varphi^{-1}(s)}^{**} \right)_{s \in [\alpha]},$$

and, by **Theorem 2.2**, it follows that it is necessary and sufficient that  $\varrho(B^{**}) < 1$ .

**Case b.** Suppose, instead, that for every  $k \geq k_0$ ,

$$\hat{r}^k(I) = \left( \times_{f \in L^\infty} [0, \hat{r}_f(\underline{i}_{-f}^k)] \right) \times \left( \times_{f \in F \setminus L^\infty} \left[ \hat{r}_f(\underline{i}_{-f}^{k-1}) > 0, \hat{r}_f(\underline{i}_{-f}^{k-1}) \right] \right).$$

Then, for every  $k \geq k_0$ ,

$$\left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} = a^{**} - B^{**} \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]},$$

and

$$\left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} = a^{**} - B^{**} \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} - H^{**} \left( \underline{i}_{\phi^{-1}(s)}^{k-1} \right)_{s \in [n] \setminus [\alpha]},$$

where  $n := |F|$ ,  $\phi : L^\infty \rightarrow [n] \setminus [\alpha]$  is a bijection re-indexing the set of inactive players,  $H^{**} := (b_{\phi^{-1}(s)\phi^{-1}(t)})_{[\alpha] \times [n] \setminus [\alpha]}$  for  $s \in [\alpha]$  and  $t \in [n] \setminus [\alpha]$ . In addition, setting  $G^* := (b_{\phi^{-1}(s)\phi^{-1}(t)})_{[n] \setminus [\alpha] \times [\alpha]}$  for  $s \in [n] \setminus [\alpha]$  and  $t \in [\alpha]$ , we have that, for every  $k \geq k_0$ ,

$$\left( \underline{i}_{\phi^{-1}(s)}^k \right)_{s \in [n] \setminus [\alpha]} = a^* - G^* \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]},$$

where  $a^* := (a_{\varphi^{-1}(s)})_{s \in [n] \setminus [\alpha]}$ . Since there is a unique rationalizable outcome by hypothesis, the sequence  $\iota := \left( \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]}, \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} \right)_{k \in \mathbb{N} \setminus [k_0]}$  must converge in the Euclidean metric space.

Combining the previous recursive equations, we have that, for every  $k \in \mathbb{N} \setminus [k_0 - 1]$ ,

$$\begin{bmatrix} \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [n] \setminus [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^k \right)_{s \in [\alpha]} \end{bmatrix} = \begin{bmatrix} a^{**} \\ a^{**} \\ a^* \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ B^{**} & 0 \\ 0 & 0 \\ G^* & 0 \\ B^{**} & H^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [n] \setminus [\alpha]} \\ \left( \underline{i}_{\varphi^{-1}(s)}^{k-1} \right)_{s \in [\alpha]} \end{bmatrix}.$$

The equation above defines a linear system of difference equation in  $y^k \in \mathbb{R}^{\{1,2,3\} \times [n]}$  such that, for every  $k \in \mathbb{N} \setminus [k_0 - 1]$ ,

$$y^k = c - Dy^{k-1},$$

where  $D := \begin{bmatrix} 0 & 0 & B^{**} \\ 0 & 0 & G^* \\ B^{**} & H^* & 0 \end{bmatrix}$  and  $c := \begin{bmatrix} a^{**} \\ a^{**} \\ a^* \end{bmatrix}$ . Since, by hypothesis, there is a unique rationalizable outcome  $i^*$ , it must be the case that the previous system converges for  $k \rightarrow \infty$  with respect to  $d$  to a point  $y^* \in \mathbb{R}^{\{1,2,3\} \times [n]}$ . Hence, by **Theorem 2.2**, the spectral radius of  $D$  must be less than

one. Since  $D$  is non-negative, by standard linear algebra results we find that the spectral radius of every square sub-matrix of  $D$  must be less than one. Hence, we conclude that  $\varrho(B^{**}) < 1$ .  $\square$

In summary, for the existence of a unique rationalizable outcome, it is sufficient that  $\varrho(B) < 1$ , and necessary that  $\varrho(B^{**}) < 1$ , where  $B^{**}$  is the matrix obtained by removing from  $B$  the columns and rows associated with the inactive players in the unique Nash equilibrium of a dominant solvable game. In addition, if there is a unique rationalizable outcome  $i^{**}$ , there exists  $k_0 \in \mathbb{N}$  such that, for every  $k \geq k_0$ , the players active at  $i^{**}$  are completely active at  $k$ .

#### 4. Rationalizability in investment games

Since every regular investment game is a network game with linear best-replies, we have the following proposition, which summarizes the results of **Proposition 2.5** and **Proposition 2.7**.

**PROPOSITION 2.8.** *Consider a regular investment game  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$  with network matrix  $B$ .*

- (1) *If  $\varrho(B) < 1$ ,  $\Gamma$  is dominant solvable.*
- (2) *If  $\Gamma$  has a unique rationalizable outcome  $i^*$ , there exists  $k_0 \in \mathbb{N}$  such that, for every  $k \geq k_0$ , every firm active at  $i^{**}$  is active at  $k$*
- (3) *If  $\Gamma$  has a unique rationalizable outcome  $i^*$ , and  $L^\infty \subseteq F$  is the set of inactive firms at  $i^*$ , for every bijection  $\varphi : F \setminus L^\infty \rightarrow [\alpha]$  re-indexing the set of active players from 1 to  $\alpha := |F| - |L^\infty|$ , it must be the case that  $\varrho(B^{**})$  for  $B^{**} := (b_{\varphi^{-1}(s)\varphi^{-1}(t)})_{[\alpha] \times [\alpha]}$ .*

**PROOF.** Every regular investment game is a network game with linear best-replies, hence the result follows as an application of **Proposition 2.5** and **Proposition 2.7**.  $\square$

To get the intuition behind the formal results of **Proposition 2.8**, consider an investment game that is not fully active. If firms are sufficiently heterogeneous, it may be possible that inactive firms at  $k = 1$  become possibly active after several iterations of the point-rationalization operator, provided that the upper limit of justifiable investment profiles shrinks sufficiently fast. If at some point  $k_0 \in \mathbb{N}$  of the iterated application of  $\hat{r}(\cdot)$  all firms are completely active, the same argument as in the proof of **Theorem 2.2** requires that  $\varrho(B) < 1$  for the existence of a unique rationalizable investment profile. However, there may be firms, usually with low profitability, that, as a consequence of the weakly increasing pattern of  $\underline{i}^k$ , remain inactive or become completely inactive. If, at some point  $k_1 \in \mathbb{N}$  of the iteration process,  $F$  can be divided into two sets, one with completely active firms and

one with completely inactive firms, it is again necessary and sufficient that the matrix obtained by removing from  $B$  the columns and rows associated with completely inactive firms in  $k_1$  has a spectral radius less than one. However, it is also possible that the inactive firms in the unique rationalizable outcome  $\Gamma$  are possibly active for every  $k \in \mathbb{N}$ . In such a case, it is necessary but, in general, not sufficient, that the matrix obtained by removing from  $B$  the columns and rows associated with completely inactive firms at  $i^*$  has a spectral radius less than one.

**4.1. Rationalizability and the Investment CAPM.** Every investment game is aggregative according to the definition provided in **Section 1.5 of Chapter 1**. Hence, a competitive equilibrium can be properly introduced as a solution concept. Namely, a competitive equilibrium for an investment game  $\Gamma$  is a tuple  $(i^{**}, M)$ , where  $i^{**} \in I$  and  $M \in \mathbb{R}_+^X$ , such that:

- (1) for every  $f \in F$ ,  $i_f^{**} \in \operatorname{argmax}_{i_f \in [0, \bar{i}_f]} \{-i_f - \frac{\theta_f}{2} i_f^2 + \mathbb{E}_\pi [M(x) x_f i_f]\}$ ;
- (2) for every  $x \in X$ ,  $M(x) = m(i^{**}, x)$

. A competitive equilibrium is equivalent to the Walrasian equilibrium of the Investment CAPM,<sup>7</sup> and it is said to be interior if, for every  $f \in F$ ,  $i_f^{**} \in (0, \bar{i}_f)$ . If  $(i^{**}, M)$  is an interior competitive equilibrium, by standard argument of calculus, it must be the case that, for every  $f \in F$ , the following first-order condition is verified,

$$(4.1) \quad 1 + \theta_f i_f^{**} = \mathbb{E}_\pi [M(x) x_f].$$

Imposing the aggregate consistency condition  $M(x) = m(i^{**}, x)$ , we obtain the following equation

$$(4.2) \quad 1 + \left[ \theta_f + \frac{\beta}{2n} \mathbb{E}_\pi (x_f^2) \right] i_f^{**} = \mathbb{E}_\pi \left[ \left( \alpha - \frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} x_j i_j^{**} \right) x_f \right],$$

or, equivalently,

$$(4.3) \quad i_f^{**} = \frac{\alpha \mathbb{E}_\pi (x_f) - 1}{\theta_f + \frac{\beta}{2n} \mathbb{E}_\pi (x_f^2)} - \frac{\frac{\beta}{2n} \sum_{j \in F \setminus \{f\}} \mathbb{E}_\pi (x_f x_j) i_j^{**}}{\theta_f + \frac{\beta}{2n} \mathbb{E}_\pi (x_f^2)}.$$

Hence, if there is a unique and interior Nash equilibrium, it cannot be a competitive equilibrium. Thus, if there is a unique rationalizable outcome, it cannot exactly coincide with the Walrasian equilibrium of the Investment CAPM. However, for sufficiently large values of  $n$ , an interior Nash equilibrium of  $\Gamma$  is very well approximated by the corresponding competitive equilibrium. In this

<sup>7</sup>Recall that we define a competitive equilibrium as a solution concept of an aggregative game, while a Walrasian equilibrium is an axiomatic characterization of what may happen in a price-mediated economy, which is a more essential structure compared to a static game, being tacit on the players actions and the consequence function

case, stock returns should be basically indistinguishable from investment returns. In contrast, if the number of firms is relatively small, the identification of adjustment costs in empirical application becomes problematic, since, in an interior NE, the marginal investment cost is equal to  $1 + \left[ \theta_f + \frac{\beta}{n} \mathbb{E}_\pi \left( x_f^2 \right) \right] i_f$  and the term  $\frac{\beta}{n} \mathbb{E}_\pi \left( x_f^2 \right)$  is no longer negligible. In other words, in a hypothetically small financial marketplace, the effects of equilibrium strategic competition between firms on capital markets are not distinguishable from the dead weight imposed by adjustment costs. However, most, if not all, empirical studies consider markets with a very large number of firms. Consequently, the equilibrium strategic effects cannot explain the empirical puzzles mentioned in the introduction.

**4.2. Tobin Q, cost of capital and inactivity.** Desgranges and Gauthier (2016) characterize the set of rationalizable outcomes in a deterministic model of Cournot competition featuring heterogeneous firms, which is another example of network game with linear best-replies. Although their game is similar to an investment game, additional assumptions on the payoff structure<sup>8</sup> ensure that  $\varrho(B)$  is always necessary and sufficient for dominance solvability. As a result, in Desgranges and Gauthier (2016), if there is a unique rationalizable outcome  $i^{**}$ , it must be strictly positive, that is, there cannot be inactive firms at  $i^{**}$ . In contrast, investment games allow for the existence of a unique rationalizable outcome with inactive firms. In other words, the results of **Proposition 2.5** and **Proposition 2.7** are more general and include the specific case of Desgranges and Gauthier (2016).

The possibility of inactive firms in a Nash equilibrium is consistent with the application of the *positive net present value rule* (NPV) in capital budgeting, which, in this case, is also equivalent to the “internal rate of return” (IRR) rule; see Zhang (2017) and Chapter 3 in Sagliaschi and Savona (2021) for a reference. Without loss of generality, fix a firm  $f \in F$ . By hypothesis,  $f$  holds only cash on its books before investing in productive assets. Consequently, the investment in productive assets is equal to the stock of assets at the beginning of the period in which  $f$  is liquidated. In addition, the marginal “cost” of adding a unit of capital is equal to 1 if  $i_f = 0$ , which implies that installment costs are negligible for sufficiently small investments. Clearly, for rational firms to become active, it must be the case that the investment in an infinitesimal amount of capital  $di$  is value accretive for its shareholders. Since installment costs are virtually null, it follows that  $f$  becomes active if and

<sup>8</sup>The payoff structure of an investment game is far more general than the one considered by Desgranges and Gauthier (2016). First, contrary to the setting of Desgranges and Gauthier (2016), firms’ payoffs are affected by the heterogeneity in  $x_f$ . Second, there is no unique way to determine the size of a firm based on its parameters, which is another key aspect of the specific class of games considered in Desgranges and Gauthier (2016).

only if she conjectures that the market value of an amount  $di$  of assets is greater than  $di$ ,

$$(4.4) \quad \frac{\hat{p}_f}{di} = \frac{\mathbb{E}_\pi(x_f)}{1 + \mathbb{E}_\pi(tr_f^s)} > 1,$$

where  $\hat{p}_f$  denotes the conjectured market value of the company's shares. For an infinitesimal amount of future output capacity, we also have that the corresponding investment return is equal to  $\mathbb{E}(x_f) - 1$ . Hence,  $f$  becomes active if and only if  $\mathbb{E}_\pi(x_f) - 1 > \mathbb{E}_\pi(tr_f^s)$ , that is, if and only if the cost of capital demanded by the market is lower than the profitability expected by secondary market investors. From this perspective, an important interpretational question emerges. By assumption, it is common knowledge among firms that investors have a common prior  $\pi$  on  $X$ , and since  $X$  does not enter the consequence function of the game, we obtain a game of complete information. However, in the real-world, expectations on future profitability are largely affected by the *guidance* offered to investors during earnings calls. Implicitly, we are assuming that profitability is the outcome of a data generating process, which is common knowledge among firms and investors. This observation also shows that the introduction of incomplete and asymmetric information requires a dynamic extension of the model. We leave this extension for future research.

**4.3. Symmetric investment games and market risk-aversion.** A nice game is said to be symmetric if the players' utility functions are symmetric and each player has the same set of actions; by standard results, every symmetric investment game has a symmetric Nash equilibrium. The converse is not necessarily true, unless there is a unique rationalizable outcome. Consider a regular investment game  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$  such that for every firm  $f \in F$ :

- (1)  $\bar{i}_f = \bar{i} > 0$ ;
- (2)  $\theta_f = \theta > 0$ ;
- (3)  $\mathbb{E}_\pi(x_f^o) = \mathbb{E}_\pi(x_j^o) = c_o > 0$  for  $o = 1, 2$ ;
- (4)  $\mathbb{E}(x_f x_j) = c_3 > 0$  for every  $j \in F \setminus \{f\}$ .

Under these hypotheses, it is immediate to verify that  $\Gamma$  is a nice symmetric game. Since every nice symmetric game has a symmetric Nash equilibrium, if  $\Gamma$  is symmetric and dominant solvable, the unique rationalizable investment profile, denoted by  $i^*$ , must satisfy  $i^* = (i_0^*)_{f \in F}$  for some  $i_0^* \in [0, \bar{I}]$ . Since  $\hat{r}_f(i_{-f})$  is also symmetric among firms, and  $\Gamma$  is regular, we have  $\underline{i}^1 > 0$ . By the weak monotonicity of the point-rationalizability operator, we reach the conclusion that  $i_0^* \in (0, \bar{i})$ . The network matrix  $B$  associated to a symmetric investment game is symmetric. Namely, for every

$f \in F$ ,  $b_{ff} = 0$  and  $b_{fj} = b_{jf} = \frac{\frac{\beta}{2n}c_3}{\theta + \frac{\beta}{n}c_2}$  for  $j \in F \setminus \{f\}$ . In addition, the vector  $a$  is equal to  $(a_0)_{f \in F}$ , where  $a_0 = \frac{\alpha \mathbb{E}(x) - 1}{\frac{\beta}{n}\theta + c_2}$ . Based on these considerations, the following result establishes some additional interesting properties of symmetric investment games.

PROPOSITION 2.9. *Consider a regular symmetric investment game  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$ . The following statements are equivalent:*

- (1)  $\Gamma$  has a unique rationalizable outcome;
- (2)  $\varrho(B) < 1$ ;
- (3)  $\Gamma$  is fully active.

PROOF. A necessary condition for the existence of a unique rationalizable outcome is that  $\hat{r}(I) = [0, \bar{i}]^F$  does not satisfy the best-reply property, that is,  $[0, a_0]^F \neq \hat{r}([0, a_0]^F)$ . Since  $r_f(\cdot)$  is monotonic decreasing, the previous condition is equivalent to  $\hat{r}_f((a_0)_{f \in F}) > 0$ . Hence, a symmetric investment game is dominant solvable if and only if it is fully active. Moreover  $\hat{r}_f((a_0)_{f \in F}) > 0$  if and only if  $a_0 - \frac{\beta \frac{n-1}{n} \frac{c_3}{2}}{\theta + \frac{\beta}{n} c_2} a_0 > 0$ , that is,  $\hat{r}_f((a_0)_{f \in F}) > 0 \iff \varrho(B) < 1$ .  $\square$

It is worth observing that, in a symmetric investment game,  $\varrho(B)$  is strictly increasing in the coefficient  $\beta$ ,  $\frac{\partial \varrho(B)}{\partial \beta} = \frac{(n-1) \frac{c_3}{2}}{(\frac{\beta}{n}\theta + c_2)^2} \frac{n\theta}{\beta^2} > 0$ , which can be interpreted as a measure of investors' risk aversion, see Chapter 3 in Cochrane (2005), for a reference. Hence, an increase in risk-aversion slows the convergence rate of  $(\hat{r}^k(I))_{k \geq 1}$  and, if large enough, it may even compromise dominance solvability, that is, the existence of a unique rationalizable outcome. This result carries over to all regular investment games. In fact, according to the Perron-Frobenius theorem, for any two nonnegative and symmetric  $F \times F$  matrices  $B'$  and  $B''$ , if  $B'(f, j) \leq B''(f, j)$  for every  $f, j \in F$  then  $\varrho(B') \leq \varrho(B'')$ , holding as strict equality if  $B' \neq B''$ . Thus, dominance solvability is harder when risk-aversion is higher on the market. Hence, it is reasonable to conclude that the relation between stock and investment returns is more likely to hold when market risk-aversion is low. This is a notable aspect for empirical studies that deserves some additional considerations to clarify the underlying economic intuition. In this regard, observe that, since we always maintain the hypothesis that firms are risk-neutral, this result does not relate to any change in the players' risk aversion. *Ceteris paribus*, an increase in investors' risk aversion negatively affects Tobin Q for every firm. Hence, once we fix a prior  $\pi \in \Delta(X)$ , for every  $f \in F$ , the higher  $\beta$ , the lower the value of  $\bar{i}_{-f}$  at which  $f$  decides rationally to become active. In other words, the higher  $\beta$ , the easier it becomes to justify, in a game-theoretic sense, a zero investment level for every firm. Consequently, it also

becomes easier to justify very high investment levels for every firm. Thus, the higher  $\beta$ , the slower the exclusion of very high and very low investment levels in the iterated deletion of strictly dominated actions, possibly preventing the convergence of  $\left(\bar{i}_f^k\right)_{f \in F}$  and  $\left(\underline{i}_f^k\right)_{f \in F}$  to  $i^*$ .

## 5. Bounded reasoning about rationality and the Investment CAPM

In the previous section, we motivated the Investment CAPM as a possible consequence of rationality and common rationality when the number of firms is large. However, the larger the number of players, the more difficult it becomes for each firm to think strategically. In other words, common belief in rationality may not be the most appropriate way to provide a good description of how real-world corporations think. In this regard, Brandenburger et al. (2021) showed that, for a given game equipped with its universal type structure, the outcomes consistent with rationality and common belief in rationality  $m$ -order (RmBR) can be significantly different from those implied by RCBR, even for large values of  $m \in \mathbb{N}$ . The purpose of this section is to investigate the implications of bounded reasoning about rationality for the relationship between stock and investment returns (resp. valuations) predicted by the investment CAPM. In particular, we wish to replicate the behavior of an econometrician attempting to estimate the Investment CAPM on simulated data consistent with rational firms that mutually believe in the rationality of their opponents (R1BR). Despite the fact that an investment game describes a two-period situation, we show that it can be used to simulate panel data of stock and investment returns (resp. valuations), provided that we focus on portfolios of stocks rather than on single companies. Since the majority of the literature places emphasis on characteristics-sorted portfolios, we believe that this is an acceptable compromise. In extreme synthesis, a simulation consists of a set of dates or epochs. At the beginning of each epoch a new generation of firms is born, and we simulate the outcome of an investment game with a parameter structure calibrated to match certain properties of the real-world. In particular, we draw an outcome of the game from  $\rho^2(I)$ , through a pseudo-random extraction process reminiscent of level- $k$  thinking (Nagel, 1995; Costa-Gomes et al., 2001; Crawford and Iriberri, 2007). The firms active in a given epoch are liquidated in the following one, and we obtain their total return on the secondary market. More specifically, we assume that firms are born unlisted and that the initial shareholders sell their shares to secondary market investors, consistent with the specification of the firms' preferences in an investment game (cf. **Section 2.1**). Accordingly, in every period we form *value* and *size* sorted portfolios, and we track their respective performance and characteristics. Finally, we use the data

generated in the simulation to estimate the relations for returns and valuation multiples predicted by the static Investment CAPM. Combining the results of a large number of simulations, we obtain a Monte Carlo distribution of the relevant statistics to assess the validity of the Investment CAPM under (R1BR).

**5.1. Simulation methodology.** For ease of exposure, the timing notation will be slightly different from that of **Section 2.1**, but this should not be a source of confusion, as the order of events is made explicit. A simulation consists in repeating for a number of times  $T \in \mathbb{N}$  a regular investment game  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$ , following a given set of rules for every date or epoch  $t \in T$ . Namely, for every  $t \in T$ , we assume that the following events occur in virtual order:

- (1) A profile  $(x_{tf})_{f \in F}$  is set equal to a vector extracted from a multivariate normal distribution, with location  $v > 0$  and scale  $\mathbb{C}$ , and successively truncated to zero below and 3 standard deviations above;  $\mathbb{C}$ .
- (2) the firms active in  $t - 1$  pay a profile of liquidation values  $(x_{tf} \cdot i_{t-1,f})_{f \in F}$  to their shareholders and cease to exist, provided that  $t > 1$ ;
- (3) a new generation of firm is born, and each firm is endowed with a certain amount of capital  $(e_{tf})$ , provided by a group of entrepreneurs (“founders”), to invest in production capacity;
- (4) for every  $f \in F$ , its founders form a point-conjecture  $\hat{i}_{t,-f}$  and decide (irreversibly) how much to invest  $(i_{tf})$  in production capacity, with the purpose of maximizing shareholder value (dividends paid plus ex-dividend market value of shares), thus setting  $i_{tf} = \hat{r}_f(\hat{i}_{t,-f})$ ;
- (5) for every  $f \in F$ , the dividends  $(e_{tf} - i_{tf} - \frac{\theta_f}{2} i_{tf}^2)_{f \in F}$  resulting from the capital held in excess of investment needs are returned founders (incumbent shareholders);
- (6) inactive firms at  $t$  cease to exist;
- (7) the trading of shares takes place, determining the base of shareholders entitled to receive the total liquidation value  $(x_{t+1,f} \cdot i_{tf})_{f \in F}$  at the beginning of the date  $t + 1$ .

As for point 4), we introduce the following epistemic assumption to determine firms’ conjectures. Namely, we suppose that all  $f \in F$  believe that all  $j \in F \setminus \{f\}$  invest up to a certain percentile, denoted by  $z_{tf}$ , of their respective justifiable investment levels. That is, we assume that for every  $f \in F$ , there exists  $z_{tf} \in [0, 1]$ , such that  $\hat{i}_{t,-f} := \times_{j \in F \setminus \{f\}} \{\hat{i}_{j,t}^1 + z_{tf}(\bar{i}_{j,t}^1 - \hat{i}_{j,t}^1)\}$ . This assumption is consistent with the logic of level-k thinking; having endowed  $\Gamma$  with the corresponding capital structure, we assume that every player believes that her opponents are rational but do not engage in

strategic thinking. The profile  $(z_{tf})_{f \in F}$  is set equal to a vector extracted from a multivariate normal distribution, with location  $(0.5)_{f \in F}$  and scale  $0.1 \cdot \mathcal{I}_n$ , and successively truncated to zero below and one above. The choice of location and scale parameters is instrumental in limiting the number of firms with extreme, and less realistically, conjectures.

We let  $S \subset \mathbb{N}$  be the set of simulations with the typical element  $s$ . For every  $s \in S$ , and every  $t \in T$ , we rank firms either by deciles of market capitalization  $(p_{tf})$  and or price-to-book ratio  $\left(q_{tf} = \frac{p_{tf}}{i_{tf}}\right)$  respectively, obtaining, respectively, *size* and *value* sorted portfolios as in Fama and French (1993). In both cases, the firms included in the first decile are those with the highest value of the specific factor considered, and vice versa. More specifically, a portfolio is defined by a factor or characteristic, denoted by  $c \in C$ ,  $C := (\text{value}, \text{size})$ , to sort stocks in descending order, a decile  $d \in D := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  of the corresponding ranking and a weighting scheme  $W \in (EW, MW)$ . The weighting scheme denoted by *EW* assigns to each firm, or stock, a weight in the portfolio equal to one divided by the number of companies that belong to the same portfolio (equal weights); *EW* denotes an equally weighted portfolio, while *MW* stands for a market-cap weighted one. We also assume that every portfolio  $(c, d, w) \in C \times D \times W$  is rebalanced after dividends are paid; therefore, a dollar invested in  $t$  in the portfolio  $(c, d, w) \in C \times D \times W$  will be equal in  $t + 1$  to the weighted average of the financial returns of the stocks included in the portfolio, which is a function of the liquidation values realized.

**5.1.1.1. *Simulating the behavior of an econometrician.*** The purpose of simulations is to replicate the behavior of an econometrician attempting to estimate the Investment CAPM on simulated data, in order to investigate the implication of first order mutual belief in rationality. We introduce the following notation to facilitate the discussion. We let  $o = (s, t, c, d, w)$  be an element, known as a tag, in  $S \times T \times C \times D \times W$ . A tag  $o = (s, t, c, d, w)$  is used to identify the portfolio  $(c, d, w) \in C \times D \times W$  on the date  $t \in T$  of the simulation  $s \in S$ , and we denote by  $w_{of}$  the weight of a stock  $f \in F$  at  $o$ . Following Belo et al. (2013), we assume that, for every  $f \in F$ ,  $\theta_f = \theta > 0$ , and, for every  $s$ , and  $(c, d, w) \in C \times D \times W$ , we obtain an estimate, denoted by  $\hat{\theta}_{scw}$ , of  $\theta$  with the Generalized Method of Moments (GMM). In particular, we merge the approach of Belo et al. (2013) and Goncalves et al. (2020). Namely, we include both return and valuation errors in the specification of moments, which, in turn, are built using firm-level data rather than portfolio aggregates, thus avoiding aggregation bias. Following Belo et al. (2013), given a value  $\theta_0 \in \mathbb{R}$  for  $\theta$ , the corresponding valuation error for

the firm  $f$  on the date  $t$  of the simulation  $s$  is

$$(5.1) \quad \varepsilon_{stf}^v(\theta_0) := q_{tf} - \hat{q}_{tf}(\theta_0) = \frac{p_{stf}}{i_{stf}} - \hat{q}_{tf}(\theta_0),$$

whereas the return error is

$$(5.2) \quad \varepsilon_{stf}^r(\theta_0) := trs_t - \hat{tr}i_{tf}(\theta_0) = \frac{x_{t,f}}{p_{s,t-1,f}} - \frac{x_{t,f}}{\hat{q}_{t-1,f}(\theta_0)},$$

where  $\hat{q}_{tf}(\theta_0) := 1 + \hat{r}_f(\varsigma_f(z_{tf}))$  and  $\hat{tr}i_{tf}(\theta_0) := \frac{x_{t,f}}{\hat{q}_{t-1,f}(\theta_0)}$  would be the investment valuation and return of  $f$  at  $(s, t)$  if  $\theta = \theta_0$ . Following the methodology of Goncalves et al. (2020), for every  $o \in S \times T \times C \times D \times W$ , investment and return errors are averaged, obtaining the valuation error

$$\varepsilon_o^v(\theta_0) := \sum_{f \in F} w_{of} \varepsilon_{stf}^v(\theta_0),$$

and the return error

$$\varepsilon_o^r(\theta_0) := \sum_{f \in F} w_{of} \varepsilon_{stf}^r(\theta_0).$$

Accordingly, we let the  $w$ -weighted valuation and return moments for  $c$ -sorted portfolios be respectively,

$$\varepsilon_{scw}^v = \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{(s,t,c,d,w)}^v(\theta_0) \right)_{d \in D},$$

and,

$$\varepsilon_{scw}^r = \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{(s,t,c,d,w)}^r(\theta_0) \right)_{d \in D}.$$

Finally, for each simulation  $s \in S$ , we obtain the GMM estimate  $\hat{\theta}_{scw}$  of  $\theta$  as

$$(5.3) \quad \hat{\theta}_{scw} = \operatorname{argmin}_{\theta_0 \geq 0} \left\{ \begin{bmatrix} \varepsilon_{scw}^r \\ \varepsilon_{scw}^v \end{bmatrix}^\top \begin{bmatrix} \mathcal{I}_{10} & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} \varepsilon_{scw}^r \\ \varepsilon_{scw}^v \end{bmatrix} \right\},$$

where  $\Lambda$  is a diagonal matrix of size  $10 \times 10$ . In particular,  $\Lambda$  is obtained as in Belo et al. (2013), in order to avoid the predominance of valuation moments in the estimation procedure. Namely, we first obtain the  $\hat{\theta}_{scw}^r$  and  $\hat{\theta}_{scw}^v$ , respectively, as

$$\hat{\theta}_{scw}^r = \operatorname{argmin}_{\theta_0 \geq 0} \left\{ [\varepsilon_{scw}^r(\theta_0)]^\top \mathcal{I}_{10} \varepsilon_{scw}^r(\theta_0) \right\},$$

and,

$$\hat{\theta}_{scw}^v = \operatorname{argmin}_{\theta_0 \geq 0} \left\{ [\varepsilon_{scw}^v(\theta_0)]^\top \mathcal{I}_{10} \varepsilon_{scw}^v(\theta_0) \right\}.$$

Second, each diagonal entry of  $\Lambda$ , say  $\lambda_{dd}$ ,  $d \in D$ , is determined as  $\lambda_{dd} = \frac{\frac{1}{T} \sum_{t=1}^T \varepsilon_{(s,t,c,d,w)}^r (\hat{\theta}_{scw}^r)}{\frac{1}{T} \sum_{t=1}^T \varepsilon_{(s,t,c,d,w)}^v (\hat{\theta}_{scw}^v)}$ , to compare the scale of the average valuation error with that of the average return error.

5.1.2. *Portfolio analytics.* For every tag  $o = (s, t, c, d, w) \in S \times T \times C \times D \times W$ , we obtain the following portfolio analytics of interest:

- $tr_o := \sum_{f \in F_o} \frac{w_{of} x_{s,t+1,f}}{p_{s,t,f}} - 1$ , the total return of portfolio  $(c, d, w)$  at the date  $t$  of simulation  $s$ ;
- $\hat{tr}_o := \sum_{f \in F_o} \frac{w_{of} x_{s,t+1,f}}{1 + \hat{\theta}_{scw} i_{s,t,f}} - 1$ , the estimated investment return of the portfolio  $(c, d, w)$  at the date  $t$  of the simulation  $s$ ;
- $q_o := \sum_{f \in F_o} \frac{p_{s,t,f}}{i_{s,t,f}}$ , the Tobin Q, or valuation ratio, of the portfolio  $(c, d, w)$  at the date  $t$  of simulation  $s$ ;
- $\hat{q}_o := \sum_{f \in F_o} \left(1 + \hat{\theta}_{scw} i_{s,t,f}\right)$ , the estimated investment Q of the portfolio  $(c, d, w)$  at date  $t$  of the simulation  $s$ .

Following Belo et al. (2024), we measure the empirical performance of the Investment CAPM in simulated samples. That is, for every  $a \in (tr, q)$  and its corresponding “investment CAPM” counterpart  $\hat{a} \in (\hat{tr}, \hat{q})$ , the following statistics are obtained:

- $XS - R2_{(a,s,c,w)} = 1 - \frac{\sum_{d=1}^{10} \left( \sum_{t=1}^T a_{(s,c,d,w,t)} - \sum_{t=1}^T \hat{a}_{(s,c,d,w,t)} \right)^2}{\sum_{d=1}^{10} \left( \sum_{t=1}^T a_{(s,c,d,w,t)} - \bar{a}_{scw} \right)^2}$ , that is, cross-sectional pooled OLS R-squared, under the null hypothesis that the Investment CAPM is true,
- $TS - R2_{(a,s,c,w)} = 1 - \frac{\sum_{d=1}^{10} \sum_{t=1}^T \left( a_{(s,c,d,w,t)} - \hat{a}_{(s,c,d,w,t)} \right)^2}{\sum_{d=1}^{10} \sum_{t=1}^T \left( a_{(s,c,d,w,t)} - \bar{a}_{scw} \right)^2}$ , that is, the time-series pooled OLS R-squared, under the same null hypothesis of the previous point,

where  $\bar{a}_{scw} := \frac{1}{10T} \sum_{d=1}^{10} \sum_{t=1}^T a_{(s,c,d,w,t)}$ ,  $\bar{a}_{scdw} := \frac{1}{T} \sum_{t=1}^T a_{(s,c,d,w,t)}$  and  $\bar{\hat{a}}_{scdw} := \frac{1}{T} \sum_{t=1}^T \hat{a}_{(s,c,d,w,t)}$ .

Assuming a large number of firms, if the Investment CAPM holds, both the statistics  $XS - R2$  and  $TS - R2$  should not close to one. Just to offer a bit more of insight, for a given analytic  $a \in (tr, q)$ , the statistic  $XS - R2$  corresponds to the R-squared of a linear regression of  $\frac{1}{T} \sum_{t=1}^T a_{(s,c,d,w,t)}$  against  $\frac{1}{T} \sum_{t=1}^T \hat{a}_{(s,c,d,w,t)}$  (cross-sectional regression of historical average portfolio values), under the hypothesis that the Investment CAPM is true.<sup>9</sup> Likewise,  $TS - R2$  is the R-Squared of a linear regression of  $a_{(s,c,d,w,t)}$  against  $\hat{a}_{(s,c,d,w,t)}$  (panel regression of portfolio values), under the same hypothesis that the simulated data are consistent with the Investment CAPM.

**5.2. Calibration.** We calibrate the parameter structure  $\Xi$  considering the top 1000 companies by market capitalization listed on a regulated US stock market as of September 30, 2024,

<sup>9</sup>Notice that, if this hypothesis was true, the GMM estimator would return, with sufficient good approximation, the actual value of  $\theta$ .

having excluded Financial, Real Estate, and Utilities companies in the first place. Following Hou et al. (2015),  $x_{tf}$  is identified as one plus the 12-month trailing return on common equity, net of extraordinary items (ROE). The parameters  $v$  and  $\mathbb{C}$  are set to match the empirical distribution of ROE (Compustat Capital IQ API item: *IQ\_RETURN\_EQUITY\_CS*) observed annually during the period from September 30, 2019 to September 30, 2024. The choice of a 5-year observation period appeared as a good trade-off between the necessity to maintain the composition of the universe as much stable as possible and having enough observations to compute pairwise correlations. Data points with negative ROE were excluded from the analysis. ROE data were obtained from Compustat, as well as those for the investment-to-asset ratio (IA), which is defined as the annual growth rate of net property, plants and equipment (Compustat - Capital IQ API item: *IQ\_NPPE\_1YR\_AN\_GROWTH\_CS*) based on the last four reported fiscal quarters. Needless to say, IA represents the empirical counterpart of an outcome  $i \in I$  of the model. After dropping companies with missing or insufficient observations, 889 firms were included in the calibration sample, and we set  $F = [889]$ ; **Table 1** below provides a synthetic overview of the (marginal) distributions of ROE and IA in the calibration sample. Interestingly, we note that the operating

TABLE 1. Profitability and Investment: calibration sample

|                 | <i>ROE</i> | <i>std(ROE)</i> | <i>corr(ROE<sub>f</sub>, ROE<sub>j</sub>)</i> | <i>IA</i> |
|-----------------|------------|-----------------|-----------------------------------------------|-----------|
| Mean            | 0.2889     | 0.1255          | 0.0370                                        | 0.2220    |
| 25th percentile | 0.0825     | 0.0326          | -0.4252                                       | -0.0134   |
| Median          | 0.1622     | 0.0631          | 0.0464                                        | 0.0612    |
| 75th percentile | 0.2970     | 0.1276          | 0.5062                                        | 0.1911    |

*Legend:* *ROE* is Return-on-Equity (Compustat Capital IQ API item: *IQ\_RETURN\_EQUITY\_CS*), *std(ROE)* is the 5-year standard deviation of Return-on-Equity, *corr(ROE<sub>f</sub>, ROE<sub>j</sub>)* is the 5-year correlation between the Return-on-Equity of two different companies and *IA* is the investment-to-asset-ratio (Compustat - Capital IQ API item: *IQ\_NPPE\_1YR\_AN\_GROWTH\_CS*).

performances of the companies included in the calibration sample were limitedly volatile during the period considered and, on average, almost uncorrelated. This is quite remarkable considering the shocks faced during the pandemic and post-pandemic period.

Interpreting the SDF as the marginal utility of a representative investor, the parameter  $\alpha$  is normalized to one without loss of generality. Consequently,  $\beta$  is set equal to one as well, which is the minimum value ensuring a relative risk aversion coefficient always greater than one. We also set  $\theta_f = \theta = 1.5$  following the results presented in Table 3 of Belo et al. (2022). As a result, we obtain a parameter structure  $\Xi$  that induces a network matrix  $B$  with a spectral radius less than one (see

**Section 3).** Consequently,  $\Gamma = \langle F, (I_f, u_f : I \rightarrow \mathbb{R})_{f \in F} \rangle$  has at most one rationalizable outcome  $i^{**}$ . Finally, we set  $T = 30$ , interpreting epochs as years, and  $S = [100]$ .<sup>10</sup>

**5.3. Benchmark data.** In this section, we analyze the characteristic of characteristic-sorted portfolios considering data for the period 30/09/1994 to 30/09/2024, assuming annual rebalancing at the end of September of every year included in the sample. At each rebalancing date, we obtain the top 1000 companies by market capitalization listed on a regulated US stock market as of the same date considered and form characteristic sorted portfolios. Rebalancing dates are distant one year from the other and lie in the range 30/09/1994 to 30/09/2024, consistent with the length of the simulated data. The total returns and analytics are obtained from Bloomberg FTST. In particular,  $p$  is obtained as the product of  $EQY\_SH\_OUT$  (field DS124: total outstanding shares) times  $PX\_LAST$  (field PR005: closing share price) while  $q$  as  $PX\_LAST$  divided by  $BOOK\_VAL\_PER\_SH$  (field RR020: book value per share). Total returns take into account the effects of corporate actions and dividend reinvestment. **Table 2** provides the historical average total return of each portfolio considered in the analysis. As in the simulated data, the first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values. The value (rep. size) premium is defined as the difference between the average performance of the 10th decile portfolio and the first decile portfolio by  $q$  (resp.  $p$ ). For the case of equally weighted portfolios, the sample features marginally negative value and size premia, equal to -1.60% and -2.08%, respectively, against an average universe return of 11.51%. A standard t-test shows that both premia are not statistically different from zero. Considering market cap-weighted portfolios, both factor premia become even more negative. Although the total return to the universe is just slightly higher and equal to 12.32%, the observed value and size premia reach even more negative levels, -3.32% and -2.94%, respectively. Besides, there is no monotone pattern in decile portfolios' returns, regardless of the specific characteristic and (or) weighting scheme considered. These features, despite being in contrast to the original findings of Fama and French (1993), are easy to explain. Except for 2022, a year of high inflation compared to the previous two decades, several large growth companies have been a consistent source of superior stock market returns.<sup>11</sup>

**5.4. Results.** In this section, we discuss the results of our simulations. We find that the simulated samples are, on average, consistent with the main features of the benchmark data. As

<sup>10</sup>The Python code used to obtain data from Compustat and simulate the model is available upon request.

<sup>11</sup>Notable examples are Alphabet, Apple, Broadcom, Microsoft, and Nvidia. They all have been able to steadily grow their business while reaching and maintaining profitability levels well above the median.

TABLE 2. Performance of value and size sorted portfolios

| Factor           | Equal weights |               | Market weights |               |
|------------------|---------------|---------------|----------------|---------------|
|                  | Size ( $p$ )  | Value ( $q$ ) | Size ( $p$ )   | Value ( $q$ ) |
| Universe         | 0.1168        | 0.1168        | 0.1232         | 0.1232        |
| Decile 1         | 0.1149        | 0.1474        | 0.1270         | 0.1549        |
| Decile 2         | 0.1106        | 0.1120        | 0.1152         | 0.1289        |
| Decile 3         | 0.1266        | 0.1277        | 0.1315         | 0.1286        |
| Decile 4         | 0.1172        | 0.1116        | 0.1194         | 0.1352        |
| Decile 5         | 0.1228        | 0.1082        | 0.1248         | 0.1071        |
| Decile 6         | 0.1275        | 0.1054        | 0.1324         | 0.1208        |
| Decile 7         | 0.1168        | 0.1174        | 0.1202         | 0.1128        |
| Decile 8         | 0.1231        | 0.1153        | 0.1264         | 0.1134        |
| Decile 9         | 0.1120        | 0.1236        | 0.1137         | 0.1007        |
| Decile 10        | 0.0937        | 0.1314        | 0.0976         | 0.1217        |
| Factor Premium   | -0.0212       | -0.0160       | -0.0294        | -0.0332       |
| t-stat premium   | -1.1400       | -0.0093       | -1.3478        | -0.7070       |
| p-Value (t-stat) | 0.2636        | 0.9926        | 0.1881         | 0.4852        |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values.

shown in **Table 3**, the equally weighted investible universe earned, on average, 11.45% per year. Similarly, the market-weighted investible universe earned on average 11.53% per year. However, the dispersion of the simulated returns is much lower than in the benchmark data. For the equal weighted universe, the interquantile range in the model is on average 3.45%, which is well below the level observed in reality, namely, 25.79%. Considering the market cap-weighted universe, which is the traditional way to build stock market indices, the average simulated interquantile range increases to 4.93%, while the one observed on the US stock market shrinks to 20.78%. The inability to generate a larger dispersion in stock returns is a rather unsurprising result, which stems from having assumed one-period-lived firms, with the consequence that capital gains on the stock market are fully determined by liquidation values, whose dispersion is calibrated to match that of profitability (ROE). As said, we compare the simulated distribution of  $i$  with the empirical one of IA; for this

TABLE 3. Annual returns, simulated data

| Universe returns | Equal weights | Market weights |
|------------------|---------------|----------------|
| Mean             | 0.1145        | 0.1153         |
| 25th percentile  | 0.0972        | 0.0906         |
| Median           | 0.1151        | 0.1153         |
| 75th percentile  | 0.1317        | 0.1399         |

reason, we refer to  $i$  also as investment rate, with abuse of definition. In this regard, **Table 4** shows

that the model is capable of matching the size of the average and median investment rates observed in the calibration sample. Hence, the simulation model does a very good job of reproducing the average characteristics of stocks returns for the period considered, but fails to capture the volatility witnessed during the last 30 years. A possible explanation is the presence of tail events such as the *dot-com* crisis of early 2000s, the *great financial crisis* of 2008, and the 2019-2020 pandemic. Clearly, a simple model as the one considered here has no chance to reproduce this type of episodes. **Tables**

TABLE 4. Distribution of profitability and investment intensity, simulated data

| Universe        | $x - 1$ | $i$    |
|-----------------|---------|--------|
| Mean            | 0.4268  | 0.1620 |
| 25th percentile | 0.1715  | 0.0258 |
| Median          | 0.2636  | 0.0650 |
| 75th percentile | 0.4370  | 0.1486 |

*Legend:*  $x$  is the liquidation value per unit of asset, while  $i$  is the capacity installed. Empirically, we identify  $x - 1$  with Return-on-Equity and  $i$  with the investment-to-assets ratio.

**5** and **6** provide the average returns and valuations of the portfolios considered in the analysis, whereas **Table 7** reports other characteristics of possible interest. In particular, we observe that the cost of capital is basically the same for every portfolio, while firms that are more profitable tend to invest more and have a higher Tobin Q.

The number of firms in the model is large enough to consider strategic effects negligible, that is,

TABLE 5. Decile portfolios, average returns, simulated data

| Factor    | Equal weights |        |               |        | Market weights |        |               |        |
|-----------|---------------|--------|---------------|--------|----------------|--------|---------------|--------|
|           | Size ( $p$ )  |        | Value ( $q$ ) |        | Size ( $p$ )   |        | Value ( $q$ ) |        |
| Analytic  | $\hat{tr}$    | $tr$   | $\hat{tr}$    | $tr$   | $\hat{tr}$     | $tr$   | $\hat{tr}$    | $tr$   |
| Decile 1  | 0.1084        | 0.1156 | 0.1081        | 0.1155 | 0.1099         | 0.1155 | 0.1098        | 0.1155 |
| Decile 2  | 0.1245        | 0.1153 | 0.1246        | 0.1151 | 0.1256         | 0.1154 | 0.1257        | 0.1154 |
| Decile 3  | 0.1317        | 0.1144 | 0.1337        | 0.1148 | 0.1330         | 0.1145 | 0.1343        | 0.1149 |
| Decile 4  | 0.1368        | 0.1145 | 0.1372        | 0.1146 | 0.1379         | 0.1146 | 0.1374        | 0.1146 |
| Decile 5  | 0.1389        | 0.1143 | 0.1398        | 0.1145 | 0.1396         | 0.1144 | 0.1393        | 0.1144 |
| Decile 6  | 0.1408        | 0.1147 | 0.1420        | 0.1144 | 0.1412         | 0.1148 | 0.1410        | 0.1144 |
| Decile 7  | 0.1426        | 0.1142 | 0.1432        | 0.1141 | 0.1430         | 0.1143 | 0.1415        | 0.1141 |
| Decile 8  | 0.1438        | 0.1138 | 0.1446        | 0.1140 | 0.1441         | 0.1138 | 0.1417        | 0.1141 |
| Decile 9  | 0.1439        | 0.1141 | 0.1452        | 0.1141 | 0.1438         | 0.1141 | 0.1402        | 0.1139 |
| Decile 10 | 0.1473        | 0.1141 | 0.1397        | 0.1140 | 0.1469         | 0.1142 | 0.1355        | 0.1140 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values. Column  $\hat{tr}$  is the portfolio average estimated investment return, while column  $tr$  is the portfolio total return on the stock market.

TABLE 6. Decile portfolios, average valuations, simulated data

| Factor    | Equal weights |        |               |        | Market weights |        |               |        |
|-----------|---------------|--------|---------------|--------|----------------|--------|---------------|--------|
|           | Size ( $p$ )  |        | Value ( $q$ ) |        | Size ( $p$ )   |        | Value ( $q$ ) |        |
| Analytic  | $\hat{q}$     | $q$    | $\hat{q}$     | $q$    | $\hat{q}$      | $q$    | $\hat{q}$     | $q$    |
| Decile 1  | 2.3581        | 2.3575 | 2.3403        | 2.3403 | 2.5793         | 2.5791 | 2.5640        | 2.5639 |
| Decile 2  | 1.4844        | 1.4844 | 1.4964        | 1.4962 | 1.5222         | 1.5220 | 1.5357        | 1.5354 |
| Decile 3  | 1.2465        | 1.2473 | 1.2674        | 1.2666 | 1.2525         | 1.2525 | 1.2741        | 1.2733 |
| Decile 4  | 1.1724        | 1.1724 | 1.1960        | 1.1959 | 1.1737         | 1.1729 | 1.1975        | 1.1974 |
| Decile 5  | 1.1247        | 1.1250 | 1.1501        | 1.1498 | 1.1262         | 1.1255 | 1.1513        | 1.1510 |
| Decile 6  | 1.0888        | 1.0894 | 1.1156        | 1.1148 | 1.0904         | 1.0897 | 1.1163        | 1.1155 |
| Decile 7  | 1.0635        | 1.0640 | 1.0912        | 1.0911 | 1.0658         | 1.0644 | 1.0919        | 1.0918 |
| Decile 8  | 1.0419        | 1.0418 | 1.0704        | 1.0698 | 1.0456         | 1.0425 | 1.0715        | 1.0709 |
| Decile 9  | 1.0217        | 1.0224 | 1.0501        | 1.0497 | 1.0269         | 1.0233 | 1.0510        | 1.0507 |
| Decile 10 | 1.0108        | 1.0074 | 1.0341        | 1.0373 | 1.0166         | 1.0097 | 1.0361        | 1.0394 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values. Column  $\hat{q}$  is the portfolio average Tobin Q predicted by the estimated Investment CAPM, while column  $q$  is the actual portfolio average Tobin Q.

$$\frac{\beta}{n} \mathbb{E}_{\pi} (x_f^2) \approx \frac{\beta}{2n} \mathbb{E}_{\pi} (x_f^n) \approx 0,$$

therefore, for every simulation  $s \in S$ , and every date  $t \in T$ , if  $f$  is active, it must be the case that,

$$\theta_f i_f = \frac{\mathbb{E}(x_f)}{p_f(i_{s,t,f}, \hat{i}_{s,t,-f})} - 1 =: \hat{wacc}_{s,t,f}.$$

The right-hand side of the above equation is the perceived cost of capital of  $f$ , based on its conjectures, which may be wrong. It follows that the validity of the Investment CAPM in simulations depends on the dispersion of the dispersion of  $i$  around  $i^*$ . Unfortunately, the interquantile range of simulated investment rated is too large, therefore, under the epistemic assumption of mutual belief in rationality, we must reject the Investment CAPM as a "perfect" causal theory of the cross-section of stock returns. In particular, we find that value and small stocks are usually firms with low profitability and conjecturing a very high level of investment activity for their opponents. That is, small and value firms are firms with low profitability and conjecturing an excessively higher cost of capital, compared to the true expected return of their shares once stock prices are determined on the market. Similarly, growth and large companies are usually the opposite. We also find that these effects have no impact on the returns of characteristics-sorted portfolios, that is, there is no value or size premium in the simulations, consistent with the benchmark data. In addition, we find that investment and stock returns are misaligned the most at the upper and lower deciles of value

and size portfolios, consistent with the results in Figure 2 of Goncalves et al. In this regard, several listed firms have been documented to adopt fixed hurdle rates to decide the financial convenience of investment projects, often above realistic average stock market returns; see Graham (2022) for a review of the relevant literature. Our model provides a possible explanation based on the lack of common belief in rationality. **Table 8** provides the average, median, and interquantile range

TABLE 7. Decile portfolios, additional analytics of interest, average values

| Equal weights |              |                         |         |        |               |                         |         |        |
|---------------|--------------|-------------------------|---------|--------|---------------|-------------------------|---------|--------|
| Factor        | Size ( $p$ ) |                         |         |        | Value ( $q$ ) |                         |         |        |
| Analytic      | $i$          | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    | $i$           | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    |
| Decile 1      | 0.8455       | 0.1144                  | 1.6108  | 0.4993 | 0.8454        | 0.1144                  | 1.6107  | 0.5001 |
| Decile 2      | 0.3017       | 0.1142                  | 0.6687  | 0.4953 | 0.3015        | 0.1141                  | 0.6687  | 0.4996 |
| Decile 3      | 0.1541       | 0.1143                  | 0.4116  | 0.4856 | 0.1535        | 0.1143                  | 0.4129  | 0.4999 |
| Decile 4      | 0.1074       | 0.1143                  | 0.3328  | 0.4970 | 0.1073        | 0.1143                  | 0.3331  | 0.4997 |
| Decile 5      | 0.0779       | 0.1142                  | 0.2813  | 0.4935 | 0.0776        | 0.1143                  | 0.2817  | 0.4999 |
| Decile 6      | 0.0557       | 0.1143                  | 0.2427  | 0.4870 | 0.0553        | 0.1143                  | 0.2433  | 0.5002 |
| Decile 7      | 0.0399       | 0.1142                  | 0.2157  | 0.4933 | 0.0396        | 0.1142                  | 0.2157  | 0.4996 |
| Decile 8      | 0.0260       | 0.1142                  | 0.1915  | 0.4944 | 0.0261        | 0.1142                  | 0.1924  | 0.4994 |
| Decile 9      | 0.0140       | 0.1142                  | 0.1695  | 0.4803 | 0.0135        | 0.1142                  | 0.1699  | 0.4916 |
| Decile 10     | 0.0046       | 0.1142                  | 0.1557  | 0.5008 | 0.0067        | 0.1142                  | 0.1520  | 0.4371 |

| Market weights |              |                         |         |        |               |                         |         |        |
|----------------|--------------|-------------------------|---------|--------|---------------|-------------------------|---------|--------|
| Factor         | Size ( $p$ ) |                         |         |        | Value ( $q$ ) |                         |         |        |
| Analytic       | $i$          | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    | $i$           | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    |
| Decile 1       | 0.9896       | 0.1144                  | 1.8599  | 0.4979 | 0.9896        | 0.1144                  | 1.8598  | 0.4982 |
| Decile 2       | 0.3271       | 0.1142                  | 0.7127  | 0.4949 | 0.3272        | 0.1141                  | 0.7130  | 0.4965 |
| Decile 3       | 0.1583       | 0.1143                  | 0.4190  | 0.4869 | 0.1582        | 0.1143                  | 0.4205  | 0.4947 |
| Decile 4       | 0.1084       | 0.1143                  | 0.3346  | 0.4971 | 0.1088        | 0.1143                  | 0.3348  | 0.4928 |
| Decile 5       | 0.0787       | 0.1142                  | 0.2827  | 0.4933 | 0.0791        | 0.1142                  | 0.2830  | 0.4904 |
| Decile 6       | 0.0562       | 0.1143                  | 0.2435  | 0.4862 | 0.0566        | 0.1143                  | 0.2440  | 0.4874 |
| Decile 7       | 0.0403       | 0.1142                  | 0.2166  | 0.4930 | 0.0412        | 0.1142                  | 0.2165  | 0.4823 |
| Decile 8       | 0.0266       | 0.1142                  | 0.1927  | 0.4954 | 0.0286        | 0.1142                  | 0.1937  | 0.4736 |
| Decile 9       | 0.0146       | 0.1142                  | 0.1705  | 0.4794 | 0.0168        | 0.1142                  | 0.1707  | 0.4508 |
| Decile 10      | 0.0061       | 0.1142                  | 0.1580  | 0.4981 | 0.0104        | 0.1142                  | 0.1542  | 0.4027 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values.  $\mathbb{E}(r_{t+1}^s)$  is the actual expected stock return, or actual cost of capital,  $x$  is the liquidation value per unit of asset and  $i$  is the capacity installed. Empirically, we identify  $x - 1$  with Return-on-Equity and  $i$  with the investment-to-assets ratio.

of statistics related to the estimation of the Investment CAPM on simulated data. The 5%-95% confidence interval for  $\hat{\theta}$  is 1.6038-1.6071, which is clearly unable to include the true value of  $\theta$ , which is equal to 1.5. Hence, the GMM estimator over-estimates the size of adjustment costs with any reasonable degree of confidence, consistent with the overestimation problem documented in Table 2 of Liu et al. (2009) for value sorted portfolios. The exaggerated negative values of  $XS - R2$  for

the cross-section of stock returns are of limited interest. As the model does not generate a size or value premium, the variance of average returns for decile portfolios is close to zero. Consequently, small errors result in very large values of the  $XS - R2$  statistic. However, the sign of  $XS - R2$  suggests a structural bias, as investment returns are on average higher than stock market returns (cf. **Table 5**), which is consistent with evidence that firms tend to overestimate the cost of capital (Graham, 2022). The  $TS - R2$  for stock returns is more interesting and, although higher than the value found in actual data by Belo et al. (2024), and closer to those implied by Goncalves et al. (2020), it appears too low to support the empirical validity of the Investment CAPM, as the Pearson correlation between stock and investment returns is close to one. The latter is evidently much higher than the levels documented in Goncalves et al. (2020), but this is a natural consequence of assuming one-period-lived firms, as capital gains are fully determined by liquidation values. Since there is no omitted capital input in our GMM estimations, our results reconcile, in a sense, the different findings of Goncalves et al. (2020) and Belo et al (2024). Hence, we conclude that there is a wedge between stock and investment returns which prevents the validity of the Investment CAPM both in the time-series of returns. As for valuations, the Investment CAPM holds almost exactly in simulated data, both in the cross-section and the time-series. Belo et al. (2022) document a high  $XS - R2$  and  $TS - R2$  for the relation between stock and investment valuations of U.S. listed companies, once intangible assets are included in the production function. Since in our simulation model there is a unique productive asset, our results are consistent with those of Belo et al. (2022). The reason why the Investment CAPM holds for valuations is the following. Since the imperfect estimation of the cost of capital is relatively self-contained, we still observe that the more profitable firms invest more than less profitable firms. As the cost of capital is similar among decile portfolios, firms with higher profitability obtain a higher Tobin Q, and the correlation between Tobin Q and investment rates result is very high. Besides, rational firms become active only if their Tobin Q is greater than one (cf. **Section 4.2**) and investment rates are restricted to non-negative numbers. Consequently, a regression with a fixed unit intercept of Tobin Q on investment rates necessarily has a very high R-squared, which coincides with the  $TS - R2$  statistic. For completeness, in the **Appendix** to this chapter, the analysis is repeated for the case of RCBR, in which the Investment CAPM holds (almost) perfectly.

TABLE 8. Investment CAPM, GMM estimates for  $\theta$  and goodness of fit statistics, simulated data (Average value and 5%-95% confidence interval)

|                   |                | Equal weights        |                      |
|-------------------|----------------|----------------------|----------------------|
|                   |                | Size ( $p$ )         | Value ( $q$ )        |
| Return moments    | $\hat{\theta}$ | 1.6055               | 1.6064               |
|                   |                | [1.6038,1.6071]      | [1.6046,1.6078]      |
|                   | $TS - R2$      | 0.3927               | 0.4199               |
|                   |                | [0.1351,0.5634]      | [0.1838,0.5741]      |
|                   | $XS - R2$      | -95.0902             | -101.8305            |
|                   |                | [-213.9135,-22.1539] | [-265.0921,-19.7749] |
| Valuation moments | $\hat{\theta}$ | 1.6055               | 1.6064               |
|                   |                | [1.6038,1.6071]      | [1.6046,1.6078]      |
|                   | $TS - R2$      | 0.9959               | 0.9959               |
|                   |                | [0.9958,0.9959]      | [0.9959,0.996]       |
|                   | $XS - R2$      | 0.9959               | 0.9959               |
|                   |                | [0.9959,0.996]       | [0.9959,0.996]       |
|                   |                | Market weights       |                      |
|                   |                | Size ( $p$ )         | Value ( $q$ )        |
| Return moments    | $\hat{\theta}$ | 1.5957               | 1.5959               |
|                   |                | [1.5939,1.5972]      | [1.5941,1.5973]      |
|                   | $TS - R2$      | 0.3564               | 0.4882               |
|                   |                | [0.0801,0.5419]      | [0.3126,0.6293]      |
|                   | $XS - R2$      | -90.0736             | -73.0516             |
|                   |                | [-199.2018,-24.0005] | [-234.6109,-13.5638] |
| Valuation moments | $\hat{\theta}$ | 1.5957               | 1.5959               |
|                   |                | [1.5939,1.5972]      | [1.5941,1.5973]      |
|                   | $TS - R2$      | 0.9969               | 0.9974               |
|                   |                | [0.9969,0.997]       | [0.9973,0.9974]      |
|                   | $XS - R2$      | 0.9969               | 0.9974               |
|                   |                | [0.9969,0.997]       | [0.9974,0.9974]      |

Legend:  $p$  stands for market cap, while  $q$  for Tobin Q.  $\hat{\theta}$  is the GMM estimate for the parameter  $\theta$ , which determines the impact of investment adjustment costs  $TS - R2$  is the time-series pooled OLS R-squared.  $XS - R2$  is the cross-sectional pooled OLS R-squared.

## 6. Conclusion

We formulated the static Investment CAPM (Zhang, 2017) as a network game with linear best-replies (Bramoullé et al., 2014), which we denoted as an *investment game*. We provided necessary and sufficient conditions for the existence of a unique rationalizable outcome in network games with linear best-replies, and applied them to investment games. In addition, we showed that, when the number of firms is large and there is a unique rationalizable investment profile, the competitive equilibrium of the Investment CAPM approximates the unique rationalizable outcome, which is a Nash equilibrium. Since rationalizability is motivated epistemically by rationality and common

belief in rationality, we provided sufficient epistemic conditions for the Walrasian equilibrium of the Investment CAPM to hold with arbitrary approximation. Nevertheless, when the number of firms is large, as on actual stock markets, common belief in rationality may require non-realistic reasoning skills. Assuming rationality and mutual belief in rationality (i.e. first-order belief in rationality), we simulated the returns and characteristics of *value* and *size* sorted portfolios and estimated the Investment CAPM with the Generalized Method of Moments. Calibrating the parameters of the model to match the distribution of the trailing 12 months Return-on-Equity of the top 1000 US listed firms, having excluded those in the Financial, Real Estate, and Utility sector in the first place, we obtain simulated data consistent with the stylized facts of the same investible universe observed in the past 30 years. In particular, our simulations replicate several empirical puzzles documented in the investment-based asset pricing literature:

- the GMM overestimates the parameter governing the size of investment adjustment costs (cf. Liu et al. 2009);
- the Investment CAPM holds in the cross-section and time-series of valuations (cf. Belo et al. 2022);
- the Investment CAPM does not hold in the time-series of stock returns, as there is a wedge between stock and investment returns (cf. Belo et al. 2024).

The correlation between stock and investment returns is very high in simulations, much higher than that documented in Goncalves et al. (2020), but this comes as a natural consequence of assuming one-period-lived firms, which implies that capital gains are fully determined by liquidation values. Since there is no omitted capital input in our GMM estimations, our results reconcile, in a sense, the different findings of Goncalves et al. (2020) and Belo et al (2024). In addition, we find that many firms overestimate their cost of capital, consistent with survey on capital budgeting practices (cf. Graham, 2022). Despite this bias, firms with higher expected profitability invest more and vice versa, and, since the model generates an actual cost of capital which is similar across characteristics sorted portfolios, Tobin Q results are highly correlated to investment rates. Since rational firms invest only if their perceived Tobin Q is greater than one, we find that a regression with fixed unit intercept of Tobin Q on investment rates has a very high R-squared, which coincides, by definition, with the time-series R-squared statistic of the GMM. Nevertheless, this result requires the existence of an SDF that prices future cash flows distributed to shareholders, which is also common knowledge among firms. In the next chapter, we shall examine under which epistemic conditions we may observe

a neoclassical SDF in a trading game consistent with the microstructure of modern stock exchanges. To conclude, we note that, in the introductory chapter of his monograph, Brandenburger (2014) advocated for an empirical program in epistemic game theory. To the best of our knowledge, this chapter is the first example of an empirical approach based on epistemic game theory to address open issues in financial economics. Since the calibration of the model implies the existence of a unique rationalizable outcome, our simulations suggest that including bounded reasoning about rationality is a promising direction for explaining the empirical puzzles documented in the literature, as implicitly advocated in Zhang (2025). In other words, the results of our simulations could be interpreted, in light of the empirical puzzles documented in the literature, as evidence against the common belief in rationality. In this regard, we believe that epistemic game theory provides the tools to build a bridge between neoclassical and behavioral finance, enhancing our understanding of financial markets.

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## Appendix to Chapter 2

**Simulations under RCBR.** We repeat the analysis presented in **Section 5** of this chapter assuming RCBR instead of R1BR. As the calibrated game has a unique rationalizable outcome, every simulation results in a time series of equilibrium outcomes. The following tables provide the analog of **Table 3,4,5,6,7, and 8**.

TABLE 9. Annual returns, simulated data

| Universe returns | Equal weights | Market weights |
|------------------|---------------|----------------|
| Mean             | 0.1235        | 0.1234         |
| 25th percentile  | 0.1081        | 0.0996         |
| Median           | 0.1228        | 0.1228         |
| 75th percentile  | 0.1394        | 0.1476         |

TABLE 10. Distribution of profitability and investment intensity, simulated data

| Universe        | $x - 1$ | $i$    |
|-----------------|---------|--------|
| Mean            | 0.3852  | 0.1550 |
| 25th percentile | 0.1509  | 0.0249 |
| Median          | 0.2341  | 0.0659 |
| 75th percentile | 0.3925  | 0.1436 |

*Legend:*  $ROE$  is Return-on-Equity (Compustat Capital IQ API item:  $IQ\_RETURN\_EQUITY\_CS$ ),  $std(ROE)$  is the 5-year standard deviation of Return-on-Equity,  $corr(ROE_f, ROE_j)$  is the 5-year correlation between the Return-on-Equity of two different companies and  $IA$  is the investment-to-asset-ratio (Compustat - Capital IQ API item:  $IQ\_NPPE\_1YR\_AN\_GROWTH\_CS$ ).

TABLE 11. Decile sorted portfolios, average returns, simulated data

| Factor    | Equal weights |        |            |        | Market weights |        |            |        |
|-----------|---------------|--------|------------|--------|----------------|--------|------------|--------|
|           | $p$           | $q$    | $p$        | $q$    | $p$            | $q$    | $p$        | $q$    |
| Analytic  | $\hat{tr}$    | $tr$   | $\hat{tr}$ | $tr$   | $\hat{tr}$     | $tr$   | $\hat{tr}$ | $tr$   |
| Decile 1  | 0.1228        | 0.1227 | 0.1228     | 0.1227 | 0.1229         | 0.1229 | 0.1229     | 0.1229 |
| Decile 2  | 0.1249        | 0.1254 | 0.1249     | 0.1254 | 0.1255         | 0.1262 | 0.1255     | 0.1262 |
| Decile 3  | 0.1226        | 0.1230 | 0.1226     | 0.1230 | 0.1225         | 0.1231 | 0.1225     | 0.1231 |
| Decile 4  | 0.1231        | 0.1234 | 0.1231     | 0.1234 | 0.1230         | 0.1234 | 0.1230     | 0.1234 |
| Decile 5  | 0.1226        | 0.1228 | 0.1226     | 0.1228 | 0.1225         | 0.1228 | 0.1225     | 0.1228 |
| Decile 6  | 0.1231        | 0.1232 | 0.1231     | 0.1232 | 0.1230         | 0.1232 | 0.1230     | 0.1232 |
| Decile 7  | 0.1231        | 0.1233 | 0.1231     | 0.1233 | 0.1231         | 0.1233 | 0.1231     | 0.1233 |
| Decile 8  | 0.1234        | 0.1235 | 0.1234     | 0.1235 | 0.1235         | 0.1236 | 0.1235     | 0.1236 |
| Decile 9  | 0.1236        | 0.1237 | 0.1236     | 0.1237 | 0.1236         | 0.1236 | 0.1236     | 0.1236 |
| Decile 10 | 0.1238        | 0.1238 | 0.1238     | 0.1238 | 0.1238         | 0.1238 | 0.1238     | 0.1238 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values. Column  $\hat{tr}$  is the portfolio average estimated investment return, while column  $tr$  is the portfolio total return on the stock market.

TABLE 12. Decile sorted portfolios, average valuations, simulated data

| Factor<br>Analytic | Equal weights |        |           |        | Market weights |        |           |        |
|--------------------|---------------|--------|-----------|--------|----------------|--------|-----------|--------|
|                    | $p$           |        | $q$       |        | $p$            |        | $q$       |        |
|                    | $\hat{q}$     | $q$    | $\hat{q}$ | $q$    | $\hat{q}$      | $q$    | $\hat{q}$ | $q$    |
| Decile 1           | 2.2254        | 2.2254 | 2.2258    | 2.2258 | 2.4899         | 2.4899 | 2.4904    | 2.4904 |
| Decile 2           | 1.4049        | 1.4049 | 1.4043    | 1.4043 | 1.4363         | 1.4363 | 1.4353    | 1.4353 |
| Decile 3           | 1.2196        | 1.2196 | 1.2192    | 1.2192 | 1.2225         | 1.2225 | 1.2220    | 1.2220 |
| Decile 4           | 1.1587        | 1.1587 | 1.1584    | 1.1584 | 1.1604         | 1.1604 | 1.1600    | 1.1600 |
| Decile 5           | 1.1140        | 1.1140 | 1.1138    | 1.1138 | 1.1151         | 1.1151 | 1.1148    | 1.1148 |
| Decile 6           | 1.0842        | 1.0842 | 1.0840    | 1.0840 | 1.0850         | 1.0850 | 1.0848    | 1.0848 |
| Decile 7           | 1.0599        | 1.0599 | 1.0598    | 1.0598 | 1.0609         | 1.0609 | 1.0608    | 1.0608 |
| Decile 8           | 1.0380        | 1.0380 | 1.0380    | 1.0380 | 1.0386         | 1.0386 | 1.0385    | 1.0385 |
| Decile 9           | 1.0235        | 1.0235 | 1.0235    | 1.0235 | 1.0245         | 1.0245 | 1.0245    | 1.0245 |
| Decile 10          | 1.0062        | 1.0062 | 1.0062    | 1.0062 | 1.0089         | 1.0089 | 1.0088    | 1.0088 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values. Column  $\hat{q}$  is the portfolio average Tobin Q predicted by the estimated Investment CAPM, while column  $q$  is the actual portfolio average Tobin Q.

TABLE 13. Decile portfolios, additional analytics of interest, averages of simulated data

| Equal weights |        |                         |         |        |        |                         |         |        |
|---------------|--------|-------------------------|---------|--------|--------|-------------------------|---------|--------|
| Factor        | $p$    |                         |         |        | $q$    |                         |         |        |
| Analytic      | $i$    | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    | $i$    | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    |
| Decile 1      | 0.8148 | 0.1246                  | 1.4989  | 0.5005 | 0.8148 | 0.1246                  | 1.4989  | 0.5005 |
| Decile 2      | 0.2692 | 0.1244                  | 0.5806  | 0.4999 | 0.2692 | 0.1244                  | 0.5806  | 0.4999 |
| Decile 3      | 0.1460 | 0.1245                  | 0.3692  | 0.5001 | 0.1460 | 0.1245                  | 0.3692  | 0.5001 |
| Decile 4      | 0.1055 | 0.1245                  | 0.3014  | 0.5005 | 0.1055 | 0.1245                  | 0.3014  | 0.5005 |
| Decile 5      | 0.0758 | 0.1246                  | 0.2506  | 0.4999 | 0.0758 | 0.1246                  | 0.2506  | 0.4999 |
| Decile 6      | 0.0560 | 0.1245                  | 0.2176  | 0.4999 | 0.0560 | 0.1245                  | 0.2176  | 0.4999 |
| Decile 7      | 0.0398 | 0.1245                  | 0.1904  | 0.5002 | 0.0398 | 0.1245                  | 0.1904  | 0.5002 |
| Decile 8      | 0.0253 | 0.1245                  | 0.1662  | 0.4999 | 0.0253 | 0.1245                  | 0.1662  | 0.4999 |
| Decile 9      | 0.0156 | 0.1245                  | 0.1500  | 0.5003 | 0.0156 | 0.1245                  | 0.1500  | 0.5003 |
| Decile 10     | 0.0041 | 0.1245                  | 0.1308  | 0.5000 | 0.0041 | 0.1245                  | 0.1308  | 0.5000 |

| Market weights |        |                         |         |        |        |                         |         |        |
|----------------|--------|-------------------------|---------|--------|--------|-------------------------|---------|--------|
| Factor         | $p$    |                         |         |        | $q$    |                         |         |        |
| Analytic       | $i$    | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    | $i$    | $\mathbb{E}(r_{t+1}^s)$ | $x - 1$ | $z$    |
| 1              | 0.9901 | 0.1246                  | 1.7969  | 0.5004 | 0.9901 | 0.1246                  | 1.7969  | 0.5004 |
| 2              | 0.2899 | 0.1244                  | 0.6168  | 0.5000 | 0.2899 | 0.1244                  | 0.6168  | 0.5000 |
| 3              | 0.1479 | 0.1245                  | 0.3723  | 0.5000 | 0.1479 | 0.1245                  | 0.3723  | 0.5000 |
| 4              | 0.1066 | 0.1245                  | 0.3031  | 0.5006 | 0.1066 | 0.1245                  | 0.3031  | 0.5006 |
| 5              | 0.0765 | 0.1246                  | 0.2518  | 0.5000 | 0.0765 | 0.1246                  | 0.2518  | 0.5000 |
| 6              | 0.0565 | 0.1245                  | 0.2185  | 0.4998 | 0.0565 | 0.1245                  | 0.2185  | 0.4998 |
| 7              | 0.0405 | 0.1245                  | 0.1916  | 0.5002 | 0.0405 | 0.1245                  | 0.1916  | 0.5002 |
| 8              | 0.0256 | 0.1245                  | 0.1668  | 0.4999 | 0.0256 | 0.1245                  | 0.1668  | 0.4999 |
| 9              | 0.0163 | 0.1245                  | 0.1511  | 0.5002 | 0.0163 | 0.1245                  | 0.1511  | 0.5002 |
| 10             | 0.0059 | 0.1245                  | 0.1337  | 0.5001 | 0.0059 | 0.1245                  | 0.1337  | 0.5001 |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q. The first decile of each factor corresponds to the portfolio with the highest value of the same factor, whereas the 10th includes the stocks with the lowest values.  $\mathbb{E}(r_{t+1}^s)$  is the actual expected stock return, or actual cost of capital,  $x$  is the liquidation value per unit of asset and  $i$  is the capacity installed. Empirically, we identify  $x - 1$  with Return-on-Equity and  $i$  with the investment-to-assets ratio.

TABLE 14. Investment CAPM, GMM estimates for  $\theta$  and goodness of fit statistics, simulated data (average value and 5%-95% confidence interval)

| Equal weights  |                |                           |                           |
|----------------|----------------|---------------------------|---------------------------|
|                |                | $p$                       | $q$                       |
| Return moments | $\hat{\theta}$ | 1.5049<br>[1.5049,1.5049] | 1.5049<br>[1.5049,1.5049] |
|                | $TS - R2$      | 0.9999<br>[0.9999,0.9999] | 0.9999<br>[0.9999,0.9999] |
|                | $XS - R2$      | 0.975<br>[0.8633,0.9974]  | 0.975<br>[0.8633,0.9974]  |
|                | $\hat{\theta}$ | 1.504<br>[1.504,1.504]    | 1.504<br>[1.504,1.504]    |
|                | $TS - R2$      | 1.0<br>[1.0,1.0]          | 1.0<br>[1.0,1.0]          |
|                | $XS - R2$      | 1.0<br>[1.0,1.0]          | 1.0<br>[1.0,1.0]          |
| Market weights |                |                           |                           |
|                |                | $p$                       | $q$                       |
| Return moments | $\hat{\theta}$ | 1.5049<br>[1.5049,1.5049] | 1.5049<br>[1.5049,1.5049] |
|                | $TS - R2$      | 0.9998<br>[0.9998,0.9999] | 0.9998<br>[0.9998,0.9999] |
|                | $XS - R2$      | 0.964<br>[0.8386,0.9963]  | 0.964<br>[0.8386,0.9963]  |
|                | $\hat{\theta}$ | 1.5049<br>[1.5049,1.5049] | 1.5049<br>[1.5049,1.5049] |
|                | $TS - R2$      | 1.0<br>[1.0,1.0]          | 1.0<br>[1.0,1.0]          |
|                | $XS - R2$      | 1.0<br>[1.0,1.0]          | 1.0<br>[1.0,1.0]          |

*Legend:*  $p$  stands for market cap, while  $q$  for Tobin Q.  $\hat{\theta}$  is the GMM estimate for the parameter  $\theta$ , which determines the impact of investment adjustment costs  $TS - R2$  is the time-series pooled OLS R-squared.  $XS - R2$  is the cross-sectional pooled OLS R-squared.

## Epistemic Conditions for Competitive Prices

### 1. Introduction

Modern asset pricing is an application of General equilibrium theory (GET), which makes use of Walrasian equilibria to characterize the outcome of a market economy. As argued in the preface, while intuitively appealing, general equilibrium models are rather abstract, and Walrasian equilibrium lacks epistemic foundations. In fact, a *market economy*, the essential descriptive component of any general equilibrium model, does not specify the actions taken by the agents involved and how these translate into prices and allocations of consumption goods or financial assets. Heuristically, it is often argued that every agent can buy (resp. sell) as much as she wishes of any traded good or asset at its market price, despite prices being effectively determined by these choices through the necessary market-clearing conditions. However, even if we accepted that agents could be modeled as if they chose their preferred allocations, it would remain unclear why they could be able to anticipate the vector of market prices resulting from their choices. In this sense, Walrasian equilibrium suffers from the same epistemic limitation of Nash equilibrium, which does not come with a formal (i.e. epistemic) justification for the implicit assumption of correct conjectures. The purpose of this chapter is to close this gap, considering a realistic trading protocol for financial assets, namely, an auction with limit orders and price priority in which the market price of every asset traded is determined as the bid price of the lowest filled buyer. In this regard, Dubey (1982) provided a game theoretical description of this market mechanism for the case of trading in consumption goods and showed the existence of Nash equilibria consistent with Walrasian prices and allocations. However, as in most of the literature on trading games, no motivation is offered for the adoption of a Nash equilibrium (NE) as a solution concept. In this regard, we already argued in **Section 4.1** of **Chapter 1** that Nash equilibrium is not necessarily a "natural" solution concept. To the best of our knowledge, the epistemic analysis of trading games has received very limited attention in the literature, perhaps implicitly relying on the sufficient epistemic conditions for NE of Aumann and Brandenburger (1995) and Polak (1999). In particular, Aumann and Brandenburger (1995) showed that in every finite

game endowed with an epistemic model in which the game being played is common knowledge, if the players are Bayesian rational, there is a common prior, rationality is mutual knowledge, and there is common knowledge of conjectures, then the players' conjectures constitute a NE of the mixed extension of the game considered. Evidently, these conditions are rather strong and do not necessarily imply an NE outcome in the original game, unless the equilibrium conjecture of every player is a point-conjecture; see **Section 1.4.1.3-4** for a reference. Given the fundamental role of competitive equilibrium in modern asset pricing (i.e., existence of a stochastic discount factor, non-arbitrage theories), we adapt the model of Dubey (1982) to the case of trading in financial assets and provide sufficient epistemic conditions for competitive asset prices that are less demanding and more intuitively appealing compared to those of Aumann and Brandenburger (1995). For the avoidance of doubts, asset prices are said to be competitive if they are consistent with the price vector associated with a Walrasian equilibrium (WE) of the reference economy; we also say, for short, that prices or allocations are Walrasian if they are consistent with a Walrasian equilibrium.<sup>1</sup>

**1.1. Relation with the literature.** At the end of the 1970s, Shapley and Shubik (1977) questioned the foundations of general equilibrium theory (GET).<sup>2</sup> Their critique provided a voice to a general feeling of dissatisfaction with the metaphor of a Walrasian auctioneer and the *tâtonnement* process. In short, the quick development of game theoretical models exposed the axiomatic nature of GET, which neither specifies which actions are available to agents in GE models and how these are linked to determine prices and allocations, nor why rational individuals should necessarily coordinate so as to reach a Walrasian equilibrium. The model of Shapley and Shubik (1977) was a milestone in the process of showing that Walrasian equilibria are closely related to Nash Equilibria in trading games, that is, static games in which a set of players choose actions that allow them to modify their initial allocations through an explicit market mechanism. However, an important limitation of their setting was exactly the lack of realism of the market mechanism considered, which featured monetary bids in exchange of an uncertain<sup>3</sup> amount of the commodity traded. Barring the case of currency swaps, financial markets usually work in a very different way, as traders need to specify the maximum quantity they intend to buy (resp. sell) of an asset and the maximum (resp. minimum) execution price that they are willing to accept. In this regard, Shubik (1977), Dubey and Shubik (1980), and Dubey (1982) considered a trading protocol resembling the way in which the majority

<sup>1</sup>Hence, by a Walrasian asset allocation we effectively mean a Walrasian equilibrium asset allocation.

<sup>2</sup>To avoid confusion with the solution concept of competitive equilibrium in quasi-aggregative games, we refrain from using the same term as a synonym for a Walrasian equilibrium.

<sup>3</sup>This uncertainty resolves in Nash equilibria, which are the unique solutions considered in their model.

of centralized exchanges are organized nowadays, that is, an auction with limit orders and price priority. The results presented in Dubey (1982) were generalized further by Benassy (1986). In addition, during the same decade, Wilson (1978), Hurwicz (1979), and Schmeidler (1980) addressed closely related topics in a sequence of highly influential papers. Nevertheless, despite the vast literature on trading games, there are very few examples of epistemic analysis and, to our knowledge, none considering a realistic trading mechanism for financial assets. Desgranges and Ghosal (2010) consider a version of the Shapley window model (Sahi and Yao, 1989; Codognato and Ghosal, 2000) with a continuum of players and show that, if traders are rational and attach a sufficiently large probability to an outcome consistent with a Walrasian equilibrium, there is common belief in the corresponding event, then every outcome consistent with these assumptions will support the same Walrasian equilibrium. However, their analysis has two limitations: i) the Shapley window model is a fairly unrealistic trading protocol for financial assets, and, ii) their results hinge on the equivalence between Nash and Walrasian equilibria, which is verified only in the limiting case of a continuum of players. As shown in Sahi and Yao (1989), with finitely many traders, the intersection between the set of competitive and Nash equilibria includes only trivial outcomes, such as those in which no asset is being traded. Ben-Porath and Heifetz (2011) consider instead the case of a continuum of rational players with common belief in rationality and market clearing. In their model, the description of the trading mechanism is absent and the analysis is a rare, if not unique, example of epistemic analysis of Walrasian equilibrium. More recently, Zhou (2022) introduced level- $k$  thinking in the Kyle (1989) model for the price determination of a single risky asset with informed trading, which is based on the “share auction” trading protocol of Wilson (1979). Again, a major limitation of this study lies in the considered trading mechanism, which presumes that a trader, say  $f$ , submits a function  $a_f : \mathbb{R} \rightarrow \mathbb{R}$  to an auctioneer, corresponding to her commitment to buy (sell, if negative) a certain amount of the risky asset depending on the price that will be subsequently fixed by the auctioneer. In contrast, the model presented in this chapter features a more realistic trading protocol, in which traders submit limit orders to a centralized exchange, that is, an instruction to buy (resp. sell) up to a certain quantity of each asset provided that the price paid (resp. received) is not greater (resp. lower) than a certain limit.

**1.2. Contribution to the literature.** In this chapter, we consider a trading game describing an auction for digital monetary claims on the realization of an exogenous state of nature. In particular, we consider an auction with limit orders and price priority, in which the market price of

each asset is determined as the bid price of the lowest filled buyer. In this class of games, players submit buy and sell limit orders for finitely many different assets, each traded at a different trading post. The trading protocol considered is essentially the same as in Dubey (1982), barring for minor technical variations which ensure that the total traded volume is maximized. For every asset traded, the corresponding buy orders are sorted starting from the highest bid to the lowest, while sell orders from the lowest ask to the highest; clearly, a buy and a sell order can be matched if and only if the limit price of the former is equal to or higher than that of the latter. Buy and sell orders are matched based on their *price priority*, that is, the highest bidding buyers are served first and matched against the lowest asking sellers, and the process continues until no additional buy and sell order can be matched, therefore maximizing the total tradable volume. The market price is determined as the limit price of the lowest active buyer, that is, the maximum price that the last "served" buyer is willing to pay. In case the quantity demanded results different from the quantity supplied, the traders rationed are the lowest active buyers if there is an excess of demand, while the highest active sellers if there is an excess of supply. In this regard, we consider two alternative payment or *settlement* rules: a) buyers pay the price they quote. In the former case, we speak of *pay-as-bid* auctions while in the latter, of *pay-as-clear* auctions. Under the assumption of risk-averse Bayesian traders, with preferences on monetary lotteries, common knowledge of the game, and symmetric information on the state of nature, we obtain a static game with payoff uncertainty. We introduce the notion of strategic equilibrium (SE), which essentially reflects a situation of rationality as well as common knowledge among traders of the actions being played and of their subjective priors on the state of nature, and show that every interior Walrasian equilibrium is supported by an SE. However, we also find that every profile of limit orders (i.e. actions) is rationalizable regardless of the traders' priors on the state of nature, and therefore we could never motivate an SE as a necessary consequence of rationality and common belief in rationality and transparent priors on the state of nature. For this reason, assuming that markets are complete and there is a unique, and interior, Walrasian equilibrium, we study the implication of rationality and mutual  $\pi$ -belief in Walrasian markets,<sup>4</sup> where by the latter we intend an epistemic states in which players: i) do not correlate their conjectures (i.e., beliefs about their opponents' actions) with the state of nature, ii) believe that, in their absence, the market price of each asset would be competitive and ii) believe that there are

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<sup>4</sup>The notation is similar to that of  $p$ -belief in game of complete information, that is the, attribution of probability  $p$  by the type of each player to the relevant projection of a given event. Here  $\pi$  is a profile of priors on the state of nature affecting the payoffs of the risky assets traded, hence the reason for the notation adopted.

always sufficient volumes to implement the Walrasian allocation of every risky-asset at Walrasian limit prices. Although these epistemic hypotheses may appear restrictive or ad-hoc, they provide a precise epistemic description of *price-taking behavior* in an auction with limit orders and price priority. Furthermore, these epistemic assumptions imply that no trader believes that it is possible to affect asset prices to create profitable arbitrage opportunities. In pay-as-bid auctions, that is, when buyers pay the price they quote, we show that rationality and mutual  $\pi$ -belief in Walrasian markets support competitive asset prices, although the resulting asset allocations may not be consistent with a Walrasian equilibrium (WE). In contrast, in pay-as-clear auctions, that is, when buyers pay the market price, we find that the assumption of rationality and mutual  $\pi$ -belief in Walrasian markets is no longer sufficient to ensure competitive asset prices. Indeed, asset prices may be equal to or greater than their corresponding Walrasian levels. From a normative perspective, we find that pay-as-bid auctions are more likely to coordinate buyers and sellers on outcomes consistent with Walrasian equilibrium, with better price discovery. This result echoes the conclusion of Woodward and Pycia (2025) in (a non-epistemic) equilibrium analysis of a trading protocol similar to the one considered in this chapter. In particular, when buyers pay the market price, there is a common prior on the state of nature and markets are complete, our results show that asset pricing bubbles can be observed even when players are rational and believe in Walrasian markets, and, in particular, as a consequence of choosing "incautious actions" akin to "market-orders" (i.e., orders without a limit price). From an empirical perspective, we suspect that momentum and short-term reversal may be explained from this perspective, but we have not so far developed any methodology to test this hypothesis concretely, which would also call for a dynamic extension of our model. As a technical result, we stress that most of the results obtained hold under the additional assumption that the players' initial asset allocations (endowments) are not part of a WE allocation, in which case there might be instead out-of-market orders preventing competitive prices also in pay-as-bid auctions. In other words, we escape no-trade situations à la Milgrom and Stockey (1982) by imposing non-Walrasian endowments of assets.<sup>5</sup> Despite being a theoretical work, we believe that our contribution could be levered both empirically and from a normative perspective.

**1.3. Structure of the chapter.** We view financial assets as digital contingent claims on the realization of an exogenous state of nature, which could be interpreted also as the outcome of an exogenous data generating process (see **Section 1.1.6 of Chapter 1** for a reference). Regardless

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<sup>5</sup>See **Section 6** for additional details.

of a true DGP being postulated, we allow players to disagree on their priors on the (Borel  $\sigma$ -field of the) state of nature and we rule out the possibility of private signals; that is, we work under the assumption of symmetric information. Unlike the model in **Chapter 2**, the state of nature is payoff relevant in the model, hence it cannot be left out of the epistemic analysis. For this reason, in **Section 2** we discuss the construction of belief hierarchies for games with payoff uncertainty. **Section 3** describes a market economy, as intended in GET, and the corresponding trading game featuring an auction with limit orders and price priority. In **Section 4** we define a Strategic Equilibrium (SE) of the game, which is essentially a special case of Bayesian Nash Equilibrium (BNE), and we show that every interior WE of the economy can be sustained in an SE, extending the results of Dubey (1982). Contrary to Dubey (1982), since we assume standard preferences over lotteries on monetary outcomes, there is no need to introduce ad-hoc penalties representing exogenous financing constraints. In **Section 5** we show that every profile of actions is rationalizable and consistent with rationality and common belief rationality. For this reason, we consider an economy with a unique and interior Walrasian equilibrium, and, after introducing the notion of mutual  $\pi$ -beliefs in Walrasian markets, we provide the main results of the paper. Finally, **Section 6** is left for conclusions.

## 2. Type structures for games with payoff uncertainty

In this section, we review the construction of a universal epistemic type structure in the case of payoff uncertainty, that is, in the presence of a payoff relevant state of nature. Consider a static game with cardinal payoffs equipped with an epistemic model in which there is common knowledge of the game being played and symmetric information about the state of nature, as well as a suitable Bayesian frame, and let

$$\Gamma = \left\langle F, S, (u_f : S \times A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \right\rangle$$

be the corresponding game in reduced form. As usual,  $F$  is the set of players with a typical element  $f$ , and for every  $f \in F$ ,  $A_f$  is the set of actions available while  $u_f$  is a vNM utility function. As the utility of each player is a function of the state of nature, the construction of belief hierarchies presented in **Section 1.3** of **Chapter 1** must be extended to include  $S$  in the basic uncertainty space of each player. In this regard, it is important to recall that beliefs represent uncertainty relations. As such, we do not necessarily require that players know the true DGP of the state of nature, assuming one exists, or that players agree on a *common prior* over the Borel  $\sigma$ -field of  $S$ .

For the rest, the construction of belief hierarchies and type structures is essentially the same as in **Section 1.3** of **Chapter 1**, as is evident from the following definition.

**DEFINITION 3.1.** Let  $\Gamma = \langle F, S, (A_f, u_f : S \times A_f \times A_{-f} \rightarrow \mathbb{R})_{f \in F} \rangle$  be a game with payoff uncertainty. A profile  $h = (h_f)_{f \in F}$  is a belief hierarchy associated with the space  $Z = (\times_{k \in \mathbb{N}} Z_{f,k})_{f \in F}$  if, for every  $f \in F$ :

- $Z_{f,1} = S \times A_{-f}$  and  $h_{f,1} \in \Delta(S \times A_{-f})$ ;
- for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $Z_{f,k} = Z_{f,k-1} \times (\times_{j \in F \setminus \{f\}} \Delta(Z_{f,k-1}))$  and  $h_{f,k} \in \Delta(Z_{f,k})$ .

For every  $f \in F$ , and every  $k \in \mathbb{N}$ , we endow  $Z_{f,k}$  with the product topology and  $\Delta(Z_{f,k})$  with the weak\* topology. We let  $H_f^0 := \times_{k \in \mathbb{N}} \Delta(Z_{f,k}^0)$  be the space of all belief hierarchies of  $f$ , and endow it with the product topology. It follows that  $Z_{f,k}$ ,  $\Delta(Z_{f,k})$ , and  $H_f^0$  are Polish with respect to the topologies adopted. Recalling the notion of projective consistency given in **Section 1.1.2.5** of **Chapter 1**, a belief hierarchy  $h_f \in H_f^0$  is said to be *consistent* if, for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $\text{marg}_{Z_{f,k-1}} h_{f,k} = h_{f,k}$ . Hence, the results of Brandenburger and Dekel (1993) hold, namely, as an application of the Kolmogorov extension theorem (**Section 1.1.2.5** of **Chapter 1**), there exists a homeomorphism  $\eta'_f : H_f^1 \rightarrow \Delta(S \times A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j^0))$  such that, for every  $k \in \mathbb{N} \setminus \{1\}$ ,  $\text{marg}_{Z_{f,k-1}} \eta'_f(h_f) = h_{f,k}$ . In addition, restricting attention to belief hierarchies satisfying *common belief in consistency*, that is,  $h_f \in H_f = \cap_{l \in \mathbb{N}} H_f^l$ , where, for every  $l \in \mathbb{N}$ ,

$$H_f^l = \left\{ h_f \in H_f^{l-1} : \eta_f(h_f)(S \times A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j^{l-1})) = 1 \right\}$$

we obtain a homeomorphism  $\eta_f : H_f \rightarrow \Delta(S \times A_{-f} \times (\times_{j \in F \setminus \{f\}} H_j))$ , see Dekel and Siniscalchi (2015) for a reference. As a result, any  $\beta_f \in \Delta(S \times A_{-f} \times H_{-f})$  is induced by one and only one belief hierarchy  $t_f \in H_f$ . Consistent with the nomenclature adopted in complete information games, a point  $t_f \in H_f$ , that is, a belief hierarchy that satisfies the requirements of consistency and common belief in consistency, is said to be a *type* of player  $f$ ; the product space  $H = \times_{f \in F} H_f$ , which we endow with the topology of the product, is the *universal type space* of  $\Gamma$ , as it includes all possible belief hierarchies of  $f$  satisfying the requirements of consistency and common belief in consistency. Therefore, every collection of belief hierarchies  $T \subseteq H$  can be represented according to an epistemic type structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \beta_f)_{f \in F} \rangle$ , where  $\beta_f : T_f \rightarrow \Delta(S \times A_{-f} \times T_{-f})$  is a continuous mapping. The mathematical structure  $\mathcal{T} = \langle F, (A_{-f}, H_f, \eta_f)_{f \in F} \rangle$  is said to be the *universal type structure* of  $\Gamma$ , and by the same considerations made in **Section 1.3.2** of **Chapter 1**, we identify

$S \times A \times T$  with the set of possible *states of the world*. An event is a Borel subset of  $S \times A \times T$  or  $A \times T$ ; abusing notation, a cylinder set  $S \times P$ , where  $P \subseteq A \times T$ , is sometimes identified by the corresponding projection on  $A \times T$ , that is, by  $P$ . The event “player  $f$  is rational” is defined as

$$R_f = \left\{ (a_f, t_f) \in A_f \times T_f : a_f \in r_f(\delta_f), \delta_f = \text{marg}_{S \times A_{-f}} \eta_f(t_f) \right\},$$

where  $r_f(\nu) = \text{argmax}_{a_f \in A_f} \mathbb{E}_\nu[u_{f,(a_f, a_{-f})}]$  and  $\nu \in \Delta(S \times A_{-f})$ , while the event “*players are rational*” is  $R = \times_{f \in F} R_f$ . The event “*player  $f$  believes in the event  $E_{-f}$* ”, where  $E_{-f} \subseteq S \times A_{-f} \times T_{-f}$ , is

$$B_f(E_{-f}) = \{(a_f, t_f) \in A_f \times T_f : \eta_f(t_f)(E_{-f}) = 1\},$$

and the event “ $E = \times_{f \in F} E_f$  is mutually believed” is  $B(E) = S \times (\times_{f \in F} B_f(E_{-f}))$ . The belief operator  $B : S \times A \times T \rightarrow S \times A \times T$  has the properties of monotonicity and conjunction (cf. **Section 3.3 of Chapter 1**). The event “ $E$  is commonly believed” is

$$CB(E) = \cap_{k \geq 1} B^k(E),$$

where  $B^k$  is defined inductively as in **Section 1.3.2**, and an event  $E$  is transparent if it is true and commonly believed. The event “*players are rational and there is common belief in rationality*” is  $RCBR = (S \times R) \cap CB(S \times R)$  and is transparent.

**2.1. Rationalizability.** Let  $\langle F, S, (A_f, u_f : S \times A \rightarrow \mathbb{R})_{f \in F} \rangle$  be a static game with payoff uncertainty, where  $S \cup \{A_f : f \in F\}$  is a family of Polish topological spaces, and  $\mathcal{C}$  the set of closed subsets of  $S \times (\times_{f \in F} A_f)$  in the product topology. A set  $C \in \mathcal{C}$  is said to have the best-reply, or best-response, property if  $C \subseteq \rho(C)$ , where

$$\rho : \mathcal{C} \rightarrow \mathcal{C}$$

$$C \mapsto S \times (\times_{f \in F} r_f(C_{-f})),$$

is the rationalization operator, which is weakly monotone, and for every  $f \in F$ ,

$$r_f : S \times A_{-f} \rightrightarrows A_f$$

$$(s, a_{-f}) \mapsto \{a_f \in A_f : \exists \mu_f \in \Delta(S \times A_{-f}), a_f \in \text{argmax}_{a'_f \in A_f} \mathbb{E}_{\mu_f}(u_{a'_f})\}$$

is the best-reply, or best-response, correspondence of the player  $f$ . A profile of actions  $a \in A$  is said to be rationalizable if there exists a set  $C \in \mathcal{C}$  with the best-reply property such that  $a \in C$ . The

set of rationalizable strategies is the union of the sets with the best-reply property. Let

$$\rho^\infty(S \times A) := \bigcap_{k \in \mathbb{N}} \rho^k(S \times A)$$

be the set of actions that survive the iterated deletion of strictly dominated actions. If  $S \times A$  is compact and connected, and  $u$  is continuous, it is possible to show that:

- the set of rationalizable actions coincides with  $\rho^\infty(S \times A)$  and  $\rho^\infty(S \times A) = \rho(\rho^\infty(S \times A))$ ;
- if  $\Gamma$  is equipped with the corresponding universal type structure,  $\text{proj}_A RCBR = \rho^\infty(S \times A)$ .

We refer to Battigalli and Siniscalchi (2003), Dekel and Siniscalchi (2015) and Chapter 8 in Battigalli et al. (2024) for the details.

### 3. Auctions with limit orders and price priority

This section is dedicated to modeling a situation in which a group of individuals trade in financial assets with binary payouts depending on the realization of an exogenous state of nature. We begin with analyzing this situation considering a market economy and its Walrasian equilibria, that is, a General Equilibrium model. Next, we adopt a game theoretical approach and describe the same situation with a trading game. The trading protocol is an auction with limit orders and price priority similar to those considered in Shubik (1977) and Dubey (1982, 1994). At the end of this section, we introduce the solution concept of strategic equilibrium, which is essentially an extension of Nash equilibrium in games with payoff uncertainty, and prove that every interior Walrasian equilibrium of the market economy is supported by at least a strategic equilibrium of the trading game.

#### 3.1. General Equilibrium.

3.1.1. *Market economy.* A market economy for the situation of interest is an ordered tuple

$$\mathcal{E} = \left\langle F, S, J, r, (\theta_f, \mathcal{U}_f)_{f \in F} \right\rangle,$$

where:

- $F$  is a finite set of individuals with typical element  $f \in F$ ;
- $S = [m]$ ,  $m \in \mathbb{N}$ , is a set of exogenous states, or states of nature, with the typical element  $s \neq 0$ ;
- $J = \{0\} \cup [n]$ ,  $n \in [m-1]$ , is a set of perfectly divisible assets, with the typical element  $j$  being a function mapping  $S$  to  $\{0, 1\}$ ;
- $(\theta_f, \mathcal{U}_f)_{f \in F}$  is a tuple that describes the endowments and preferences of the individuals.

We also refer to the elements of  $F$  as agents, players, or traders. The true state of the world is unknown to traders and is revealed only after traders choose their asset allocations. As usual, we let  $\Delta(S)$  be the set of probability measures on the Borel  $\sigma$ -field of  $S$  equipped with its natural topology (i.e. the discrete topology). For every asset  $j \in J$ , we denote by  $y_j : S \rightarrow \mathbb{R}$  the corresponding payout function, that is, the function that denotes the payment made by a unit of the corresponding asset to its holder. In particular, we assume that for every  $j \in J \setminus \{0\}$ ,

$$y_j(s) = \begin{cases} 1 & s = j \\ 0 & s \neq j, \end{cases}$$

while for every  $s \in S$ ,  $y_0(s) = 1$ . The asset 0 is intended as a risk-free monetary deposit or bond. Except for asset 0, the remaining  $m - 1$  assets are digital state-contingent claims, or Arrow-Debreu securities. Due to the linear independence of assets payoffs, the presence of the risk-free asset ensures that markets are complete if and only if  $|J| = |S|$ . The implicit assumption of a zero risk-free rate is without loss of generality, since the model is static, and simplifies the framework. The choice of  $S$  and  $J$  is also without loss of generality, as it simply amounts to attribute a higher index to the states of nature that are not associated to an Arrow-Debreu security, if any. For every  $f \in F$ ,  $\theta_f \in \mathbb{R}_+^J$  is a vector with a typical element  $\theta_{f,j}$  that denotes the initial endowment of the asset  $j \in J$  for the player  $f$ . In General Equilibrium (GE) models, players are assumed to be able to adjust their asset allocations through an unspecified trading protocol. In this regard, for every  $f \in F$ , we denote by  $x_f \in \mathbb{R}^J$  an asset allocation of the player  $f$ . We denote by  $(x_f)_{f \in F} \in \mathbb{R}^{F \times J}$  the profile of asset allocations obtained by the players at the end of the (unspecified) market process, where  $x_{f,j} = x(f, j)$  is the number of units of asset  $j$  held by the player  $f$  at the end of the market process; we also refer to  $x_f$  as the *asset allocation* of the player  $f$ . In this regard, every player  $f \in F$  has a family  $(\succsim_{f, \pi_f})_{\pi_f \in \Delta(S)}$  of preference relations on the set of her possible asset allocations such that

$$x'_f \succsim_f x_{f, \pi_f} \iff \mathbb{E}_{\pi_f}(U_{f, x'_f}) \geq \mathbb{E}_{\pi_f}(U_{f, x_f}),$$

where

$$U_f : S \times \mathbb{R}^J \rightarrow \mathbb{R}$$

$$(s, x_f) \rightarrow \mathcal{U}_f \left( \sum_{j \in J \setminus \{0\}} x_{f,j} y_j(s) + x_{f,0} \right)$$

and  $\mathcal{U}_f$  is a continuous, strictly monotone, and concave function of its argument. For every player  $f \in F$ , we interpret the argument of  $\mathcal{U}_f$  as the future wealth of the player  $f$ , which is the sum of the monetary payments obtained from her assets.

3.1.2. *Walrasian equilibrium.* We still have to specify a solution concept for  $\mathcal{E}$ , that is, a correspondence that assigns the set of possible asset allocations and asset prices that may be observed. As in the majority of GE models, solution concepts restrict outcomes to (normalized) Walrasian equilibria, which is based on the following assumptions:

- the market process assigns a price equal to 1<sup>6</sup> to the risk-free asset and a vector  $p^{**} \in \mathbb{R}^{J \setminus \{0\}}$  to the risky assets;<sup>7</sup>
- every player  $f \in F$  forms a belief  $\pi_f \in \Delta(S)$  on the state of nature and chooses her preferred asset allocation within those in  $BS_f(p^{**})$ , where

$$BS_f : \mathbb{R}^{J \setminus \{0\}} \rightrightarrows \mathbb{R}^J$$

$$p \mapsto \left\{ x_f : x_{f0} = \theta_{f0} + \sum_{j \in J_{-0}} p_j (\theta_{fj} - x_{fj}) \right\}.$$

is the *budget set correspondence* of  $f$ , or the budget constraint.

Informally, the assumptions above require that i) the trades to implement the desired asset allocations are compatible with each other (i.e., the total amount bought of each asset equals the total amount sold), and ii) the event "prices are equal to  $p^{**}$ " is common belief among traders; we say informally because i) the market mechanism is unspecified and ii) there is no epistemic model of  $\mathcal{E}$ . As we said above, this axiomatic characterization of the outcome of  $\mathcal{E}$  is known as *Walrasian equilibrium* (WE), which is defined more rigorously below.

DEFINITION 3.2. Consider an economy  $\mathcal{E} = \langle F, S, J, r, (\theta_f, \mathcal{U}_f)_{f \in F} \rangle$ . A Walrasian equilibrium of  $\mathcal{E}$  for the profile of priors  $\pi \in \Delta(S)^F$  is a pair  $(x^{**}, p^{**}) \in \mathbb{R}^{F \times J} \times \mathbb{R}^{J \setminus \{0\}}$ , where  $X := \times_{f \in F} X_f$ , such that:

- (1) for every  $f \in F$ ,

$$x_f^{**} \in \operatorname{argmax}_{x_f \in BS_f(p^{**})} \{ \mathbb{E}_{\pi_f}(U_{f, x_f}) \};$$

<sup>6</sup>This choice avoids the issue of price indeterminacy in Walrasian equilibria, as a consequence of the Walras' law; See Chapter 4 in Back(2017) for a reference.

<sup>7</sup>The "(normalized)" is related to the assumption that the risk-free asset has unit price, hence serving as numéraire, which is a sufficient condition to ensure price determinacy. For the avoidance of doubts,  $p_j^{**}$  is the price assigned to asset  $j \in J \setminus \{0\}$ .

$$(2) \sum_{f \in F} x_{fj}^{**} = \sum_{f \in F} \theta_{fj} \text{ for every } j \in J.$$

An allocation  $x \in X$  is said to be *interior* if, for every  $f \in F$ , and every  $j \in J$ ,  $x_{f,j} > 0$ , and a WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$  is said to be interior if  $x^{**}$  is interior and  $p^{**}$  is strictly positive. In general, once we fix a profile of priors  $\pi \in \Delta(S)^F$ , standard results in GET apply (cf. Chapter 19 in Mas-Colell et al., 1995). In this regard, by the assumption on the agents' preferences, if  $(x^{**}, p^{**})$  is a WE of  $\mathcal{E}$ , we note that  $p^{**} \in [0, 1]^{J \setminus \{0\}}$ , as every WE must be free from arbitrage opportunities, see Cochrane (2005) for a reference. In addition if the *gross substitution* property holds, there exists a unique WE for  $\mathcal{E}$ , see Section F in Chapter 17 of Mas-Colell et al. (1995) for a reference. In this regard, Hens and Loeffler (1995) provided sufficient conditions to ensure that the gross substitution property holds for  $\mathcal{E}$ , of which an example is the assumption of complete markets and individuals with a constant absolute risk aversion. Furthermore, if markets are complete, the *first welfare theorem* holds, that is, the allocation associated to a WE is Pareto efficient.

**3.1.3. The Stochastic Discount Factor.** A vector  $p \in \mathbb{R}_+^{J \setminus \{0\}}$  of market prices is said to be *fundamental* if there exists a vector  $m_p \in \mathbb{R}^S$  such that:

- $\sum_{s \in S} m_p(s) = 1$ ;
- for every  $j \in J \setminus \{0\}$ ,  $p(j) = \sum_{s \in S} m_p(s) y_j(s) = m_p(j)$ .

In other words, the existence of  $m_p$  ensures that the price of each asset "discounts" the corresponding future cash-flows. Given a pricing kernel  $m_p$ , the stochastic discount factor (SDF) associated with a measure  $\mu \in \Delta(S)$  is a vector  $M_{p,\mu} \in \mathbb{R}^S$ , such that for every  $s \in S$ ,  $m_p(s) = M_{p,\mu}(s) \mu(\{s\})$ . By standard results in GET, we have the following proposition.

**PROPOSITION 3.3.** *Consider a market economy  $\mathcal{E} = \langle F, S, J, r, (\theta_f, \mathcal{U}_f)_{f \in F} \rangle$ , consistent with the assumptions made in **Section 3.1.1**. Let  $\pi_1 \in \Delta(S)$ , and suppose that there exists an interior Walrasian equilibrium*

$$(x^{**} = (x_f^{**})_{f \in F}, p^{**} = (p_j^{**})_{j \in J \setminus \{0\}}).$$

*for the profile of priors  $(\pi_1)_{f \in F} \in \Delta(S)^F$  (i.e.  $\pi_1$  is a common prior on the state of nature). If  $|S| = |J|$  and  $\sum_{f \in F} \theta_f$  is a strictly positive vector, there exists an economy*

$$\mathcal{E}' = \langle \{1\}, S, J, r, (\nu_1, \theta_1, \mathcal{U}_1) \rangle,$$

that admits  $(\{\sum_{f \in F} x_f^{**}\}, p^{**})$  as a competitive equilibrium. In particular, for every  $j \in J \setminus \{0\}$ ,

$$p_j^{**} = \mathbb{E}_{\pi_1} [My_j]$$

where  $M := \frac{d\mathcal{U}(w)}{dw}|_{w=w^{**}}$ ,

$$w_1^{**} := \sum_{j \in [m]} \sum_{f \in F} x_{f,j}^{**} y_j(s) + \sum_{f \in F} x_{f,0}$$

and  $\mathbb{E}_{\pi_{\mu}} [My_0] = 1$ . Consequently, agent 1 is said to be a representative agent of  $\mathcal{E}$ , while  $M$  is a stochastic discount factor.

PROOF. Omitted for space reasons. See Theorem 1.6 in Chapter 1 of Ma (2011).  $\square$

In other words, when markets are complete and there is a common prior on the state of nature, Walrasian equilibrium prices imply the existence of an SDF. In this regard, much has been written to address how issues such as market incompleteness and heterogeneous priors may prevent the existence of a representative agent, and the related consequences for establishing a link between Walrasian equilibrium and the existence of a SDF; see Magill and Quinzi (1996) for a reference. In particular, with heterogeneous priors or incomplete markets, or both, every interior competitive equilibrium of  $\mathcal{E}$  supports the existence of a SDF by the absence of arbitrage opportunities; however, there is not necessarily a link between the SDF and the marginal utility of a properly defined representative investor, since the latter may not exist. As a result, the definition of the fundamental value of an asset becomes problematic, if not entirely arbitrary.

**3.1.4. Discussion.** Although intuitively appealing, Walrasian equilibrium lacks strategic and epistemic foundations. Firstly, there is no explicit mention of the players' actions and how these translate into asset allocations and prices. Secondly, in choosing their preferred asset allocations, it is not explained why players are able to anticipate the vector of prices that will result from their actions and the market process, just as in the Nash equilibrium of a static game no explanation is offered for the assumption of correct conjectures. Surprisingly, despite the large scientific production in the field of asset pricing, very little has been written in terms of which market mechanisms for  $\mathcal{E}$  (i.e., well-defined trading games) may admit outcomes consistent with Walrasian equilibria as a consequence of players' rationality and reasonable epistemic assumptions. Consequently, the existence of an SDF lacks strategic and epistemic foundations. In the remainder of this chapter, we try to fill this gap, considering a realistic trading protocol. For the sake of clarity, we stress that we are not arguing that GET is wrong or useless. Rather, recognizing the importance of GET in modern asset pricing,

we believe that investigating its strategic and epistemic foundations is of particular relevance for a correct application of its results.

**3.2. Trading Game.** In this section, we model the situation of interest as a static game with cardinal payoffs. Consequently, the game inherits several elements from the market economy  $\mathcal{E}$  described in the previous section, starting from the set of players, that is,  $F$ . To some extent, the only difference between a game and a market economy is the presence of a set of actions and an explicit mechanism linking actions to outcomes. Namely, we consider a game in which:

- trade in risky assets through an auction with limit orders and strict price priority;
- the market price of each asset is determined as the bid price of the lowest filled buyer;
- players have preferences over lotteries defined on their future wealth as in  $\mathcal{E}$ .

By strict price priority we mean that, in case buyers (resp. sellers) must be rationed, buyers (resp. sellers) with lower (resp. higher) limit prices are rationed first. The game is endowed with an epistemic model in which there is common knowledge among players of the game being played, and symmetric information on the state of nature. In addition, we assume that the game is endowed with a suitable Bayesian frame. For this reason, we focus on the reduced form of the game, and eventually describe the situation of interest as a static game with payoff uncertainty

$$\Gamma = \left\langle F, S, (A_f, u_f = U_f \circ g_f^{\mathcal{P}})_{f \in F} \right\rangle,$$

which we equip with the corresponding universal type structure  $\mathcal{T} = \left\langle F, S, (A_{-f}, T_f, \eta_f)_{f \in F} \right\rangle$ . The consequence function is  $g^{\mathcal{P}}$ , where  $\mathcal{P}$  denotes the payment rule of the game, for which consider the alternative case of i) buyers pay the price they quote ( $\mathcal{P} = \text{pay} - \text{as} - \text{bid}$ ) and ii) buyers pay the market price ( $\mathcal{P} = \text{pay} - \text{as} - \text{clear}$ ). We also say that  $\mathcal{E}$  is the reference economy and refer to  $\Gamma$  as a trading game for  $\mathcal{E}$ , or just a trading game for short. Every player  $f \in F$  takes an action  $a_f \in A_f$ , consisting of a collection of limit orders, one for each risky financial asset  $j \in J \setminus \{0\}$  submitted on a hypothetical date  $t < t' < t''$ . Observe that there are no orders for the risk-free asset, that is, the asset 0, which is identified as electronic money held in bank accounts. The reason for this choice is that trades are settled in electronic money. For the avoidance of doubts, we also recall that, in a static game, dates can be interpreted as a virtual ordering of sequential events occurring at a certain date or as distinct dates, or as a combination of the two interpretations. In applications, it might be convenient to assume that  $t$  and  $t'$  are two sequential events that occur on the same date, while  $t''$  is a future point in time. In particular, on date  $t$  traders submit simultaneously their limit orders to

an electronic exchange, while on  $t'$  the resulting trades are settled according to the exchange rules to match buy and sell proposals. Finally, in  $t''$  the state of nature is revealed, and the payments of the risky assets are distributed to their respective holders.

**3.2.1. Actions.** Fix a generic player  $f \in F$ . The set of  $f$ 's actions is  $A_f$  with the typical element  $a_f : J \setminus \{0\} \times [4] \rightarrow \mathbb{R}$ . For every  $j \in J \setminus \{0\}$ ,  $a_{fj} = (a_{fjk})_{k \in [4]}$  is a limit order, that is, a selection isomorphic to the ordered 4-tuple  $(q_{fj}^d, p_{fj}^d, q_{fj}^s, p_{fj}^s)$  carrying the following interpretation: “player  $f$  is willing to buy up to  $q_{fj}^d = a_{fj1}$  units (*quantity demanded*, or, *individual demand*) of  $j$  at an execution price less or equal than the *buy limit price*  $p_{fj}^d = a_{fj2}$  (*bid price*), and is willing to sell up to  $q_{fj}^s = a_{fj3}$  units (*quantity supplied*, or, *individual supply*) of  $j$  at an execution price greater or equal than the *sell limit price*  $p_{fj}^s = a_{fj4}$  (*ask price*).<sup>8</sup> Trades are settled in electronic money, say *dollars*, and players are endowed with an amount  $\theta_{fj} \geq 0$  of each asset  $j \in J$ . Short sales ( $q_{fj}^s > \theta_{fj}$ ) are not allowed, limit prices  $(p_{fj}^d, p_{fj}^s)$  must be non-negative. Unlike Dubey (1982), we do not allow a player to be simultaneously a buyer and a seller of a risky asset  $j \in J \setminus \{0\}$ , that is, for every  $j \in J$ ,  $q_{fj}^d q_{fj}^s = 0$ .<sup>9</sup> In addition, we restrict the individual demand of each player to be less than the total availability of the same asset in the economy. This constraint intends to capture the idea that, for many assets, such as stocks, there are regulatory limits imposing, either implicitly or explicitly, a maximum order size. We also restrict bid and ask prices to be less than or equal to one to ensure a compact action space. This assumption is with limited loss of generality, as the monetary payout of every risky asset is 1 or 0. Hence, for every  $f \in F$ , and every  $j \in J \setminus \{0\}$ , we can introduce the family of sets  $\{A_{fjk} : k \in [4]\}$  such that:

- $A_{fj1} = [0, \sum_{f \in F} \theta_f]$ , with the typical element  $a_{fj1}$  corresponding to  $q_{fj}^d$  ;
- $A_{fj2} = [0, 1]$ , with the typical element  $a_{fj2}$  corresponding to  $p_{fj}^d$  ;
- $A_{fj3} = [0, \theta_f]$ , with the typical element  $a_{fj3}$  corresponding to  $q_{fj}^s$  ;
- $A_{fj4} = [0, 1]$ , with the typical element  $a_{fj4}$  corresponding to  $p_{fj}^s$  .

and write  $A_f = \times_{j \in J \setminus \{0\}} A_{fj}$ , where, for every  $j \in J \setminus \{0\}$ ,

$$A_{fj} = \left\{ a_{fj} \in \times_{k \in [4]} A_{fjk} : a_{fj1} \cdot a_{fj3} = 0 \right\}.$$

Notice that, for a given  $j \in J \setminus \{0\}$ , the set  $A_{fj}$  is not a Cartesian product, but a strict subset of  $\times_{k \in [4]} A_{fjk}$ . As usual, every subset of the real line is endowed with the topology generated by the

<sup>8</sup>Recall that, since assets are perfectly divisible, any real amount can be traded in principle.

<sup>9</sup>Including this possibility would introduce additional complications to the epistemic analysis that follows.

Euclidean metric. For every  $j \in J \setminus \{0\}$ , we endow  $\times_{k \in [4]} A_{fjk}$  with the topology of the product, and  $A_{fj}$  with the relative topology. The sets  $A_f$ ,  $A = \times_{f \in F} A_f$  and  $S \times A$  are endowed with their respective product topologies, which are Polish. For every  $f \in F$ , we let  $\Delta(S \times A_{-f})$  be the family of probability measures on the Borel  $\sigma$ -field of  $S \times A_{-f}$  endowed with the relative topology. Finally, a player  $f \in F$  is said to be an active buyer (resp. seller) of the asset  $j$  if she receives (resp. delivers) a non-null amount of asset  $j$ , denoted by  $\tilde{q}_{fj}^d$  (resp.  $\tilde{q}_{fj}^s$ ), once orders are matched by the exchange.

**3.2.2. Order matching protocol.** We now turn to describe the set of rules to match orders, that is, *order matching protocol*, which contributes to define the outcome function of the game. In a trading game, the latter is also known as *trading protocol* or *market-mechanism*. Every contingent claim  $j \in J \setminus \{0\}$  is traded on a dedicated virtual post, based on the same trading protocol. The trading protocol is completed by a payment, or settlement, rule, which, for ease of exposure, we discuss in the following subsection. For a given risky-asset every  $j \in J \setminus \{0\}$ , buy orders are arranged in descending order by the corresponding limit price, while sell orders are sorted in ascending order, obtaining a demand and a supply curve, respectively, intended as the cumulative amount that traders are willing to buy and sell at a certain price according to the profile of orders  $(a_{fj})_{f \in F}$  submitted to the exchange. Namely, the demand curve, or demand for short, of the asset  $j$ , is

$$Q_j^d : A \times (\mathbb{R}^+ \cup \{0\}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

$$(a, p) \mapsto \sum_{f \in F} a_{fj1} \mathbb{I}(a_{fj2} \geq p_j)$$

where  $a \in A$  and  $p \geq 0$ . Similarly, the supply curve, or supply for short, of the asset  $j$ , is

$$Q_j^s : A \times (\mathbb{R}^+ \cup \{0\}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

$$(a, p) \mapsto \sum_{f \in F} a_{fj3} \mathbb{I}(a_{fj4} \leq p_j)$$

where  $a \in A$  and  $p \geq 0$ . Orders are matched according to *price priority*: the highest bidding buyers are served by the lowest asking sellers, and, if the quantity of the latter (resp. former) is insufficient to complete the trade, rationing proportional to demanded (resp. supplied) quantities applies, and the residual is attempted to be matched against the second lowest sellers (resp. buyers), and so on. The *market price*, or *market-clearing price*, of asset  $j$ , denoted by  $p_j(a)$ ,<sup>10</sup> where  $a \in A$ , is determined as the the limit price of the lowest buyer filled, or lowest active buyer, that is, the active buyer with the lowest bid. Accordingly, only buyers (resp. sellers) with a limit price higher (lower) or equal to

<sup>10</sup>Note that the market price of  $j$  is a function  $p_j : A \rightarrow \mathbb{R}_+$ .

the market price are active.<sup>11</sup> That is, the lowest active buyers act as *marginal* traders, effectively setting the market price of  $j$ . In other words, orders are matched until the demand and supply curves intersect at a single price or at a closed interval of possible prices, and no additional buyer and seller can be matched based on their respective buy and sell limit prices. If at  $p_j(a)$  the demand for  $j$  exceeds the supply, the lowest bidding buyers are rationed proportionally to the respective quantities demanded. Likewise, if supply exceeds demand at the market price, the marginal sellers of  $j$ , that is, the highest asking sellers, are rationed proportionally to the respective quantities demanded. In the event that a market price for  $j$  cannot be determined, no trade is executed for the asset  $j$ , without preventing trades in the other risky assets. The following algorithm formalizes the order matching protocol, which is also known as *the* process. It is worth observing that, since bid and ask prices must lie in the closed unit interval, the market price must be in the same range. By construction, **Algorithm 3.1** maximizes the total quantity traded of every risky asset while preserving strict price priority; this is an additional difference compared to the game denoted as  $\Gamma_1$  in Dubey (1982). In other words, for every  $j \in J \setminus \{0\}$ , the minimum between the corresponding aggregate demand and supply schedules is reached at  $p_j(a)$ . To see this, suppose by contradiction that for some  $j \in J \setminus \{0\}$   $p_j(a)$  does not maximize the traded volume. Then, there exists a buyer with a bid price lower than  $p_j(a)$  and a seller with an ask price higher than  $p_j(a)$  that have not been matched, despite the limit price of the former was not lower than the limit price of latter. But this is clearly impossible, since, even in the case that no residual demand or supply emerged at  $p_j(a)$ , **Algorithm 3.1** would attempt to match the next available buyers and sellers.

**3.2.3. Settlement rules.** Given the orders received ( $a = \times_{f \in F} a_f$ ), the electronic exchange applies **Algorithm 3.1** to assign an *execution print*  $e_f(a) = (e_{fj}(a))_{j \in [m]}$  to every player  $f \in F$ . For every  $f \in F$ , every  $j \in J \setminus \{0\}$ ,  $e_{fj}(a) = (e_{fjk}(a))_{k \in [4]}$  is a selection isomorphic to the ordered 4-tuple  $(\tilde{q}_{fj}^d(a), \tilde{p}_{fj}^d(a), \tilde{q}_{fj}^s(a), \tilde{p}_{fj}^s(a))$  which carries the following interpretation "in  $t'$ , player  $f$  obtains  $\tilde{q}_{fj}^d(a)$  in exchange of a monetary payment equals to  $\tilde{q}_{fj}^d(a) \cdot \tilde{p}_{fj}^d(a)$  if she is a buyer of asset  $j$ , or, if she is a seller of  $j$ , obtains a monetary payments equals to  $\tilde{q}_{fj}^s(a) \cdot \tilde{p}_{fj}^s(a)$  in exchange of the immediate delivery of  $\tilde{q}_{fj}^s(a)$  units of  $j$ ". Consequently, given an execution  $e_f(a) = (e_{fj}(a))_{j \in [m]}$ , the amount of risk-free asset, that is, electronic money held in a bank account, of  $f$  is equal to

$$x_{f0}(a) = \theta_{f0} + \sum_{j \in [m]} (\tilde{p}_{fj}^s(a) \tilde{q}_{fj}^s(a) - \tilde{p}_{fj}^d(a) \tilde{q}_{fj}^d(a)).$$

<sup>11</sup>In **Section 6** we discuss the implications of different rules for pay-as-bid auctions.

**Algorithm 3.1** Order Matching Protocol (Market-Clearing process)

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$\bar{p} \leftarrow \max(a_{fj2})_{\{f \in F: a_{fj1} > 0\}}$  is the bid price of the buyers that we are considering  
 $p \leftarrow \min(a_{fj4})_{\{f \in F: a_{fj3} > 0\}}$  is the ask price of the sellers that we are considering  
 $\bar{q} \leftarrow Q_j^d(a, \bar{p})$  is the quantity of asset  $j$  for the buyers being considered  
 $q \leftarrow Q_j^s(a, p)$  is the quantity of asset  $j$  for the sellers being considered  
 $v \leftarrow 0$  is the quantity of  $j$  for which we have found a match between buyer and sellers  
**while**  $\bar{p} \geq p$  **do**  
    Update the total amount tradable  
     $v \leftarrow v + \min\{q, \bar{q}\}$   
    and set the provisional market price to  $\bar{p}$   
     $p_j(a) \leftarrow \bar{p}$ .  
    **if**  $\bar{q} > q$  **then**:  
        Apply proportional rationing on buyers with bid price equal to  $\bar{p}$  and update their quantity,  
         $\bar{q} \leftarrow \bar{q} - q$   
        and then move to the sellers with the next lowest ask price.  
         $p \leftarrow \min(a_{fj4})_{\{f \in F: a_{fj3} > 0, a_{fj4} > p\}}$   
         $q \leftarrow Q_j^s(a, p) - v$ .  
    **else if**  $q < \bar{q}$  **then**:  
        Apply proportional rationing on sellers with ask price equal to  $p$  and update their quantity,  
         $q \leftarrow q - \bar{q}$   
        and then move to the buyers with the next highest bid price  
         $\bar{p} \leftarrow \min(a_{fj2})_{\{f \in F: a_{fj1} > 0, a_{fj2} > \bar{p}\}}$   
         $q \leftarrow Q_j^d(a, p) - v$ .  
    **else if**  $q = \bar{q}$  **then**:  
        Move to the sellers with the next lowest ask price,  
         $p \leftarrow \min(a_{fj4})_{\{f \in F: a_{fj3} > 0, a_{fj4} > p\}}$   
         $q \leftarrow Q_j^s(a, p) - v$ .  
        and move to the buyers with the next highest bid price.  
         $\bar{p} \leftarrow \min(a_{fj2})_{\{f \in F: a_{fj1} > 0, a_{fj2} > \bar{p}\}}$   
         $q \leftarrow Q_j^d(a, p) - v$ .  
    **end if**  
**end while**  
**if**  $v > 0$  **then**:  
    There exists a market for asset  $j$ . A total amount of  $v$  units of assets will be traded.  
    The active buyers with the lowest bid price post orders at  $\bar{p}$ . Hence,  $p_j(a) = \bar{p}$ .  
**end if**

---

In this regard, the bank does not pay or charge interests, consistent with the specification of the reference economy. This assumption is without loss of generality, since the proof of all the results presented in the following sections would be unaffected by the introduction of a positive risk-free rate. In addition, the bank is also implicitly assumed to have deep pockets to fund every amount that traders need to borrow to settle their trades. Namely, if  $x_{f0}$  is negative, the player  $f$  is borrowing from the bank, and we assume that she has outside wealth to cover her debts in the event that the monetary payments obtained from her asset allocation are insufficient. Finally, execution prices are determined by the settlement rule, or payment rule, of the game, which we denote by  $\mathcal{P}$ . In

particular, we consider two different types of payment rule: i) buyers pay the price they quote ( $\mathcal{P} = \text{pay} - \text{as} - \text{bid}$ ) and ii) buyers pay the market price ( $\mathcal{P} = \text{pay} - \text{as} - \text{clear}$ ). In case i), the resulting trading protocol is also known as *pay-as-bid* or *discriminatory price auction*, and it is frequently adopted in treasury sales; as the name suggests, buyers pay their limit price. In case ii), buyers pay the market price, and the resulting trading protocol is known as *pay-as-clear* or *uniform price auction*, which is usually adopted by regulated stock exchanges to organize opening and closing auctions. Hence, for every  $f \in F$ , and every  $j \in J \setminus \{0\}$ , if  $\mathcal{P} = \text{pay} - \text{as} - \text{bid}$ ,  $\tilde{p}_{fj}^d = a_{fj2}$ , while if  $\mathcal{P} = \text{pay} - \text{as} - \text{clear}$ ,  $\tilde{p}_{fj}^d = p_j(a)$ . Under both payment rules, the sellers of an asset obtain the price paid by the buyers against which their orders have been matched; hence, if  $\mathcal{P} = \text{pay} - \text{as} - \text{clear}$ , the sellers of a risky asset obtain the corresponding market price per unit sold.

REMARK 3.4. The choice of determining the market price as the bid price of the lowest active sellers ensures consistency between the two payment rules, as, in the case of a pay-as-bid auction, there cannot be execution prices below such level. In addition, note that traders must always choose a limit price for every asset traded. In reality, it is common to observe market-orders, that is, orders intended to be executed regardless of the resulting execution price. Since risky assets can be traded and settled only at a price lying in the unit interval, a buy (resp. sell) order with bid (resp. ask) price equal to one (resp. zero) can be identified as a market order. Finally, notice that if a player  $f \in F$  does not want to buy or sell any amount of the asset  $j \in J \setminus \{0\}$ , it is sufficient to choose  $a_{fj1} = a_{fj3} = 0$ .

3.2.4. *Consequences, consequence function and preferences.* We assume that:

- the set of consequences corresponds to the set of possible profiles of financial wealth of the players in  $t''$ ;
- every player has risk-averse preferences on the lotteries over  $S \times \mathbb{R}^{F \times J}$ , consistent with the preferences in  $\mathcal{E}$ , where  $\mathbb{R}^{F \times J}$  identifies the set of asset allocations.

For every  $f \in F$ , we let  $g_f^{\mathcal{P}} : A_f \times A_{-f} \rightarrow \mathbb{R}_+^J$  be a function that maps the profile of the actions that are played  $(a_f, a_{-f})$  to the corresponding allocation  $x_f = (x_{fj})_{j \in J}$ , that is,

$$a \mapsto g_f^{\mathcal{P}}(a) = \left( x_{f0}(a) = \theta_{f0} + \sum_{j \in J \setminus \{0\}} (\tilde{p}_{fj}^s(a) \tilde{q}_{fj}^s(a) - \tilde{p}_{fj}^d(a) \tilde{q}_{fj}^d(a)), x_{f,-0} = e_f(a) \right).$$

Accordingly, the consequence function of the game is

$$g^{\mathcal{P}} : S \times A \rightarrow \mathbb{R}^{F \times J}$$

$$(s, a) \mapsto \left( s, (g_f^{\mathcal{P}}(a))_{f \in F} \right).$$

Since the game is endowed with an epistemic model in which the game being played is common knowledge among traders, the consequence function, as well as preferences and endowments, is also common knowledge. In addition, there is symmetric information and traders are Bayesian decision makers, as the game is also equipped with a suitable Bayesian frame. Therefore, the preferences of a generic player  $f \in F$  are described by the utility function

$$u_f : S \times A_f \times A_{-f} \rightarrow \mathbb{R}$$

$$(s, a_f, a_{-f}) \mapsto U_f(s, g_f^{\mathcal{P}}(a_f, a_{-f})),$$

where

$$U_f : S \times \mathbb{R}^J \rightarrow \mathbb{R}$$

$$(s, x_f) \mapsto \mathcal{U}_f \left( \sum_{j \in J \setminus \{0\}} x_{f,j} y_j(s) + x_{f,0} \right)$$

and  $\mathcal{U}_f$  are the same as in **Section 3.1.1**. Consequently, for every  $f \in F$ , we obtain a family of preference relations on  $A_f$  with typical elements  $\succsim_{f, \mu_f}$  such that

$$a_f \succsim_{f, \mu_f} a'_f \iff \int_{S \times A} u(s, a_f, a_{-f}) d\mu_f(s, a_{-f}) \geq \int_{S \times A} u(s, a'_f, a_{-f}) d\mu_f(s, a_{-f}),$$

where  $u_f = U_f \circ g_f^{\mathcal{P}}$  and  $\mu_f \in \Delta(S \times A_{-f})$ . Notice that the Lebesgue integrals above are well-defined. By the hypotheses on  $\mathcal{U}_f$ , since  $A$  is connected, Theorem 11.6 in Chapter 11 of Aliprantis and Border (2006) ensures that  $\int_{S \times A} u(s, a_f, a_{-f}) d\mu_f(s, a_{-f})$  exists for every  $\mu_f \in \Delta(S \times A_{-f})$ , even though  $u(\cdot, \cdot, \cdot)$  is not continuous on  $S \times A_f \times A_{-f}$ .

**3.2.5. The reduced form game.** Combining the elements described so far, we obtain a static game with payoff uncertainty

$$\Gamma = \left\langle F, S, (A_f, U_f \circ g_f^{\mathcal{P}})_{f \in F} \right\rangle,$$

that we equip with the universal type structure  $\mathcal{T} = \left\langle F, S, (A_{-f}, T_f, \eta_f)_{f \in F} \right\rangle$ , as the assumed epistemic model ensures symmetric information on the state of nature. Notice that the construction of a universal type structure is always possible since  $S \cup \{A_f : f \in F\}$  is a family of Polish spaces.

We refer to  $\Gamma$  as a trading game for  $\mathcal{E}$ , or a trading game for short.<sup>12</sup> For future reference, we also denote by  $X = \times_{f \in F} X_f$  the set of asset allocations observable in the game, where, for every  $f \in F$ ,

$$X_f = \{x_f \in \mathbb{R}^J : \exists a \in A \ g_f^{\mathcal{P}}(a) = x_f\}.$$

Clearly, given a generic player  $f \in F$  and a profile of orders  $a_{-f} \in A_{-f}$  of her opponents,  $f$  may be able to achieve only a small subset of  $X_f$  through her available orders. This brings us to extend the notion of Nash equilibrium to the case of a trading game. Before proceeding, two important remarks are in order. First, differently from Dubey (1982), the utility function of every player is free from penalty terms in the case of a negative cash balance. The reason is that the risk-free asset, which serves as a means of payment, has a strictly positive marginal utility contribution for every player. Second, the consequence function depends on the state of nature, and  $S$  could not be a singleton, otherwise the action space would be empty in the absence of any risky asset to trade. In other words, different from the case of an investment game (**Section 2.1 of Chapter 2**), the set of states of nature cannot be omitted from the game.

#### 4. Strategic Equilibrium

In this section, we introduce the solution concept of *strategic equilibrium* (SE) of a trading game

$$\Gamma = \left\langle F, S, (A_f, U_f \circ g_f^{\mathcal{P}})_{f \in F} \right\rangle.$$

To motivate its definition, suppose that traders have priors  $\pi \in \Delta(S)^F$  on the possible state of nature and that there is common belief in such an event, that is, the event

$$\times_{f \in F} \{(a_f, t_f) \in A_f \times T_f : \text{marg}_S \eta_f(t_f) = \pi_f\}$$

is *transparent*. In a similar situation, an “equilibrium” solution concept should capture the idea that the “theories” of rational players are verified, that is, traders behave optimally while being certain of their opponents’ actions (cf. Chapter 8 in Battigalli et al., 2024). An SE is meant to represent an outcome consistent with this epistemic state.

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<sup>12</sup>In principle, other trading games, characterized by a different trading protocol, may be considered. Since we consider only auctions with limit orders and price priority, the use of the expression “trading game” to identify  $\Gamma$  should not be source of confusion.

DEFINITION 3.5. Given a profile  $\pi = (\pi_f)_{f \in F} \in \Delta(S)^F$  of subjective beliefs, a strategic equilibrium (SE) of  $\Gamma$  is as a profile of actions  $a^{**} \in A$  such that for every  $f \in F$ ,

$$a_f^* \in \operatorname{argmax}_{a_f \in A_f} \mathbb{E}_{\pi_f} \left( u_{f, a_f, a_{-f}^{**}} \right).$$

Notice the evident similarity with the definition of a Nash equilibrium in games with complete information since, in an SE, players are certain of the actions played by their opponents. However, it is worth stressing that a Nash equilibrium is properly defined only in games of complete information. The concept of an SE should not be confused with the more general of a Bayesian Nash equilibrium of a Bayesian game, in which players choose actions as a function of their information types (see Chapter 8 in Battigalli et al. 2024a). However, if we modeled a trading game as a simple Bayesian game with symmetric information, in which the players' priors on the state of nature are common belief, the corresponding definition of a Bayesian Nash equilibrium would be equivalent to that of an SE of  $\Gamma$ . The following lemma provides additional clarity on the epistemic hypotheses behind the definition of an SE.

LEMMA 3.6. *The following results are true for a trading game  $\Gamma = \left\langle F, S, \left( A_f, u_f = U_f \circ g_f^P \right)_{f \in F} \right\rangle$ , equipped with its universal epistemic structure  $\mathcal{T} = \left\langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \right\rangle$ .*

- (1) *For every  $(a_f, t_f) \in R_f \cap B_f \left( S \times \{a_{-f}^{(f)}\} \right)$ ,  $a_f \in \operatorname{argmax}_{a'_f \in A_f} \mathbb{E}_{\pi_f} \left( u_{f, a'_f, a_{-f}^{(f)}} \right)$  where  $\pi_f = \operatorname{marg}_{S_{-f}} \eta_f(t_f)$  and  $a_{-f}^{(f)} \in A_{-f}$ ;*
- (2) *for every  $a^* \in \operatorname{proj}_A \{R \cap B(S \times \{a^*\} \times T_{-f})\}$ , there exists a collection of  $\pi = (\pi_f)_{f \in F} \in \Delta(S)^F$  such that, given  $\pi$ ,  $a^*$  is a strategic equilibrium of  $\Gamma$ .*

Before providing a proof of the previous lemma, it is convenient to introduce the concept of residual demand and supply faced by a given trader  $f \in F$ . The residual demand for a risky asset  $j \in J \setminus \{0\}$  for the player  $f \in F$ , is

$$Q_j^{d,f} : A_{-f} \times (\mathbb{R}^+ \cup \{0\}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

$$(a_{-f}, p) \mapsto \sum_{i \in F \setminus \{f\}} a_{ij1} \mathbb{I}(a_{ij2} \geq p)$$

where  $a_{-f} \in A_{-f}$  and  $p \geq 0$ . Similarly, the residual supply of the asset  $j \in J \setminus \{0\}$  for the player  $f \in F$ , is

$$Q_j^{s,f} : A_{-f} \times (\mathbb{R}^+ \cup \{0\}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

$$(a_{-f}, p) \mapsto \sum_{i \in F \setminus \{f\}} a_{ij} 2\mathbb{I}(a_{ij} \leq p)$$

where, as before,  $a_{-f} \in A_{-f}$  and  $p \geq 0$ . Consequently, the residual excess demand for the asset  $j \in J \setminus \{0\}$  for  $f \in F$  is

$$Q_j^{z,f} : A_{-f} \times (\mathbb{R}^+ \cup \{0\}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

$$(a_{-f}, p) \mapsto Q_j^{d,f}(a_{-f}, p) - Q_j^{s,f}(a_{-f}, p)$$

PROOF. 1. From the definition of  $R$ ,

$$R_f = \left\{ (a_f, t_f) \in A_f \times T_f : a_f \in r_f(\delta_f), \delta_f = \text{marg}_{S \times A_{-f}} \eta_f(t_f) \right\},$$

where  $r_f(\nu) = \text{argmax}_{a'_f \in A_f} \mathbb{E}_\nu(u_{f,a'_f})$  for a given  $\nu \in \Delta(S \times A_{-f})$ . For every player  $f \in F$ , and every  $(a_f, t_f) \in R_f \cap B_f(S \times \{a_{-f}^{(f)}\})$ , let  $\pi_f$  be  $\text{marg}_S \eta_f(t_f)$ ; by hypothesis, for every  $E_{-f} \subseteq S \times A_{-f}$ ,

$$\text{marg}_{S \times A_{-f}} \eta_f(t_f) = \begin{cases} \pi_f(\text{proj}_S E_{-f}) & \text{proj}_{A_{-f}} E_{-f} = \{a_{-f}^{(f)}\} \\ 0 & \text{proj}_{A_{-f}} E_{-f} \neq \{a_{-f}^{(f)}\}, \end{cases}$$

where  $\pi_f := \text{marg}_S \eta_f(t_f)$ . Consequently, for every  $f \in F$  and every  $(a_f, t_f) \in R_f \cap B_f(S \times \{a_{-f}^{(f)}\})$ ,

$$a_f \in \text{argmax}_{a'_f \in A_f} \mathbb{E}_{\pi_f}(u_{f,a'_f, a_{-f}^{(f)}}).$$

2. Consider a point  $(a_f^*, t_f^*) \in A \times T$  such that, for every player  $f \in F$ ,

$$(a_f^*, t_f^*) \in R_f \cap B_f(S \times \{a_{-f}^*\} \times T_{-f}).$$

From point 1) above, it must be the case that, for every  $f \in F$ ,

$$a_f^* \in \text{argmax}_{a'_f \in A_f} \mathbb{E}_{\pi_f}(u_{f,a'_f, a_{-f}^*}),$$

where  $\pi = \left( \text{marg}_S \eta_f(t_f^*) \right)_{f \in F} \in \Delta(S)^F$ , which is the definition of an SE of  $\Gamma$ .  $\square$

An outcome  $a \in A$  of  $\Gamma$  is said to support competitive prices if, for every  $j \in J \setminus \{0\}$ ,  $p_j(a) = p_j^{**}$  where  $(x^{**}, p^{**}) \in \mathbb{R}_+^{J \setminus \{0\}}$  is an interior WE of  $\mathcal{E}$  for some profile of priors  $\pi \in \Delta(S)^F$ . Equivalently, we say that prices are competitive if players' limit orders induce an outcome of the game such that,

for every risky asset  $j \in J \setminus \{0\}$ , the lowest active buyer of  $j$  is bidding  $p_j^{**}$ , where  $(x^{**}, p^{**})$  is an interior WE of  $\mathcal{E}$ . Using the previous lemma, the next proposition establishes that every interior Walrasian equilibrium of  $\mathcal{E}$  is supported by at least an SE of  $\Gamma$ , independently of the specific payment rule (i.e. the "value" of  $\mathcal{P}$ ). Hence, we extend the results of Dubey (1982) to the case of financial assets and in the absence of an *ad hoc* penalty function in the players' utilities.

**PROPOSITION 3.7.** *Let  $(x^{**}, p^{**})$  be a WE of  $\mathcal{E} = \langle S, F, (U_f, \theta_f)_{f \in F} \rangle$  for a profile of priors  $\pi = (\pi_f)_{f \in F} \in \Delta(S)^F$  such that  $x_f^{**} > 0$  for every  $f \in F$ , and consider a trading game  $\Gamma = \langle F, S, (A_f, u_f = U_f \circ g_f^{\mathcal{P}})_{f \in F} \rangle$ . There exists an SE  $a^{**} \in A$  for the profile of priors  $\pi = (\pi_f)_{f \in F}$  such that:*

- $a_{fj2}^{**} = p_j^{**}$  for every  $(f, j) \in F \times J \setminus \{0\}$  such that  $a_{fj1} > 0$ ;
- $a_{fj4}^{**} = p_j^{**}$  for every  $(f, j) \in F \times J \setminus \{0\}$  such that  $a_{fj3} > 0$ .

Hence, there exists an SE  $a^{**} \in A$  that induces an outcome of  $\Gamma$  consistent with  $(x^{**}, p^{**})$ .

**PROOF.** Let  $\mathbb{I}(\cdot)$  be an indicator function of its argument. Consider an action profile  $a^{**} \in A$  such that for every  $(f, j) \in F \times J \setminus \{0\}$ :

- $a_{fj1}^{**} = (x_j^{**} - \theta_{fj}) \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$  ;
- $a_{fj3}^{**} = (\theta_{fj} - x_j^{**}) \mathbb{I}(x_{fj}^{**} < \theta_{fj})$  ;
- $a_{fj2}^{**} = p_j^{**} \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$  ;
- $a_{fj4}^{**} = p_j^{**} \mathbb{I}(x_{fj}^{**} < \theta_{fj})$ .

The action profile above has the features enumerated in the proposition. In fact, if  $a_{fj1}^{**} > 0$ , then it must be the case that  $x_{fj}^{**} \geq \theta_{fj}$ , and, consequently,  $a_{fj2}^{**} = p_j^{**}$ . Similarly, if  $a_{fj2}^{**} > 0$ , then it must be the case that  $x_{fj}^{**} < \theta_{fj}$ , and, consequently,  $a_{fj4}^{**} = p_j^{**}$ . In order to show that  $a^{**}$  is an SE of  $\Gamma$  for the profile of priors  $\pi = (\pi_f)_{f \in F}$ , we must show that, for every player  $f \in F$ , and every  $a_f \in A_f$ ,

$$\mathbb{E}_{\pi_f} (u_{f, a_f^{**}, a_{-f}^{**}}) \geq \mathbb{E}_{\pi_f} (u_{f, a_f, a_{-f}^{**}}).$$

Without loss of generality, fix a player  $f \in F$  and let  $X'_f = \{x_f \in \mathbb{R}^J : \exists a_f \in A_f, x_f = g_f^{\mathcal{P}}(a_f, a_{-f}^{**})\}$  be the set of asset allocations that the same player can implement if her opponents choose the profile of orders  $a_{-f}^{**}$ . For convenience, we write  $x_f = (x_{f0}, x_{fJ \setminus \{0\}})$  for every  $x_f \in X'_f$ , with the usual abuse of notation. By definition,  $a_{-f}^{**}$  implies that the residual demand and supply

curves faced by the player  $f$  for a given risky asset  $j \in J \setminus \{0\}$  are respectively equal to

$$Q_j^{d,f}(a_{-f}^{**}, p) = \begin{cases} \sum_{\{i \in F \setminus \{f\} : x_i^{**} \geq \theta_{ij}\}} a_{ij1} = \sum_{\{i \in F \setminus \{f\} : x_i^{**} \geq \theta_{ij}\}} (x_{ij}^{**} - \theta_{ij}) & p = p^{**} \\ 0 & p \neq p^{**}, \end{cases}$$

and

$$Q_j^{s,f}(a_{-f}^{**}, p) = \begin{cases} \sum_{\{i \in F \setminus \{f\} : x_i^{**} < \theta_{ij}\}} a_{ij3} = \sum_{\{i \in F \setminus \{f\} : x_i^{**} < \theta_{ij}\}} (\theta_{ij} - x_{ij}^{**}) & p = p^{**} \\ 0 & p \neq p^{**}. \end{cases}$$

Consider an allocation  $x_f \in X'_f$  such that  $x_{fj} > \theta_{fj}$  for a given  $j \in J \setminus \{0\}$ . In this case,  $f$  would be a buyer of  $j$  and, given the shape of the residual demand and supply for the same asset, she must bid a price higher than or equal to  $p_j^{**}$ . It follows that, depending on whether  $f$  will be the only active buyer and the payment rule of the game,  $f$  will pay a price equal to or greater than  $p_{fj}^{**}$  for every unit of  $j$  acquired. In contrast, to implement  $x_{fj} < \theta_{fj}$ ,  $f$  must ask a price lower than or equal to  $p_j^{**}$ . Regardless of the payment rule and the amount sold, since all buyers are bidding  $p_j^{**}$  and the market price is determined as the bid price of the lowest active buyer,  $f$  will receive a price equal to  $p_j^{**}$  for every unit of asset  $j$  sold. Thus, since a player can be either a buyer or a seller of any risky asset, given an action  $a_f \in A_f$  that implements  $x_f \in X'_f$ , there exists  $\epsilon_{fj} \geq 0$  for every  $j \in \{k \in J \setminus \{0\} : x_{fk} > \theta_{fk}\}$  such that

$$(4.1) \quad \begin{aligned} x_{f0} - \theta_{f0} &= \sum_{\{j \in J' : x_{fj} < \theta_{fj}\}} p_j^{**} (x_{fj} - \theta_{fj}) - \sum_{\{j \in J' : x_{fj} > \theta_{fj}\}} (p_j^{**} + \epsilon_{fj}) (x_{fj} - \theta_{fj}) = \\ &= \sum_{j \in J'} p_j^{**} (x_{fj} - \theta_{fj}) - \sum_{\{j \in J' : x_{fj} > \theta_{fj}\}} \epsilon_{fj} (x_{fj} - \theta_{fj}) \end{aligned}$$

Therefore, for every  $x'_f \in X'_f$  it must be the case that  $x'_{0f} \leq \sum_{j \in J'} p_j^{**} (x'_{fj} - \theta_{fj})$ . Now fix an allocation  $x'_f \in X'_f$  and consider the alternative allocation  $x_f \in \mathbb{R}^J$ , not necessarily included in  $X'_f$  or  $X_f$ , defined as

$$x_f = \begin{cases} x_{fj} = x'_{fj} & j \neq 0 \\ x_{fj} = \theta_{f0} + \sum_{j \in J'} p_j^{**} (x'_{fj} - \theta_{fj}) & j = 0, \end{cases}$$

that is, the asset allocation that  $f$  would obtain if she paid a price equal to  $p_j^{**}$  for every asset  $j$  purchased. Since, for every  $j \in J$ ,  $U_f(s, x''_f)$  is strictly increasing in  $x''_{fj}$  by hypothesis, we have

that  $U_f(s, x_f) \geq U_f(s, x'_f)$  for every  $s \in S$ , and thus

$$\mathbb{E}_{\pi_f}(U_{f,x_f}) \geq \mathbb{E}_{\pi_f}(U_{f,x'_f}).$$

By construction,  $x_f \in BS_f(p^{**})$ , and, consequently, by the definition of a WE, for every  $x''_f \in X_f$ ,

$$\mathbb{E}_{\pi_f}(U_{f,x_f^{**}}) \geq \mathbb{E}_{\pi_f}(U_{f,x''_f}),$$

which, jointly with the previous inequality, implies that, for every  $x'_f \in X_f$ ,

$$\mathbb{E}_{\pi_f}(U_{f,x_f^{**}}) \geq \mathbb{E}_{\pi_f}(U_{f,x'_f}).$$

Since  $X'_f$  is the image of  $A_f \times \{a^{*-f}\}$  through  $g_f^P$ , we conclude that, for every  $a_f \in A_f$ ,

$$\mathbb{E}_{\pi_f}(u_{f,a_f^{**},a_{-f}^{**}}) \geq \mathbb{E}_{\pi_f}(u_{f,a_f,a_{-f}^{**}}),$$

as  $g_f^P(a_f^{**}, a_{-f}^{**}) = x_f^{**}$ . □

**Proposition 3.7** establishes that allocations and prices consistent with an interior WE of  $\mathcal{E}$  can always be sustained in an SE of  $\Gamma$ , extending the earlier results of Dubey (1982). In this regard, the proof in Dubey (1982) is based on the pay-as-bid payment rule, and it is somewhat partial, as it leaves some non-trivial steps to the reader. In this regard, the proof of **Proposition 3.7** puts under the spotlight several relevant features of auctions with limit orders and price priority. In particular, the decision problem faced by each trader is highly discontinuous, and marginal changes to bid (resp. ask) prices can have a large impact on the set of asset allocations achievable. As we shall see, this is a crucial aspect for epistemic analysis of the game.

## 5. Epistemic analysis

In this section, we study sufficient epistemic conditions to observe competitive asset prices in a trading game. Competitive prices are "efficient", in the sense that they reflect some appropriate discounting of future fundamentals, and are at the core of many theoretical empirical models, see Cochrane (2005) for a reference. For this reason, it is very important to identify which epistemic conditions can support competitive prices in the trading game presented in the previous section, which provides a good approximation of many actual trading venues. To understand the epistemic uncertainty in a trading game, observe that, in addition to being uncertain about the "true" state

of nature, traders do not necessarily know whether their orders will be executed in full and at what prices. This is a crucial aspect of the *market process* upon which the general equilibrium paradigm remains silent. In general, every player could face significant uncertainty about the asset allocation that results from the market process. Consequently, even in the case of an objective prior on the state of nature that is common knowledge among traders, the expected returns of the assets remain uncertain at the time that orders are sent to the market.<sup>13</sup> Clearly, in an SE, this type of uncertainty unfolds, as, by definition, players are certain of the actions being simultaneously played by their opponents. Thus, it may be tempting to argue that an outcome consistent with a WE could be motivated following the same strategy as the case of investment games (cf. **Sections 3-4** of **Chapter 2**). This would require to:

- prove that an SE is rationalizable;
- identify sufficient conditions for a unique rationalizable outcome for a given profile of priors on the state of nature;
- show that, despite  $u$  being discontinuous,  $\text{proj}_A RCBR$  is the set of rationalizable outcomes when the game is equipped with its universal type structure.

Unfortunately, it turns out that every profile of actions is rationalizable and consistent with rationality and common belief in rationality, regardless of any restriction on the players' priors on the state of nature. Before stating and proving this result, we recall the following definition, which is based on Battigalli and Siniscalchi (2003) and Chapter 8 in Battigalli et. al (2024).

**DEFINITION 3.8.** Let  $\langle F, S, (A_f, u_f : S \times A \rightarrow \mathbb{R})_{f \in F} \rangle$  be a static game with payoff uncertainty, where  $S \cup \{A_f : f \in F\}$  is a family of Polish topological spaces, and  $\mathcal{C}$  the set of closed subsets of  $S \times (\times_{f \in F} A_f)$  in the product topology. Let  $\Delta \subseteq \times_{f \in F} \Delta(S \times A_{-f})$  be a restriction on the players' first order beliefs. The set  $C \in \mathcal{C}$  is said to have the best-reply, or best-response, property directed by  $\Delta$  if  $C \subseteq \rho^\Delta(C)$ , where

$$\rho^\Delta : \mathcal{C} \rightarrow \mathcal{C}$$

$$C \mapsto S \times (\times_{f \in F} r_f^\Delta(C_{-f})),$$

and for every  $f \in F$ ,

$$r_f^\Delta : S \times A_{-f} \rightrightarrows A_f$$

$$(s, a_{-f}) \mapsto \{a_f \in A_f : \exists \mu_f \in \Delta(S \times A_{-f}) \cap \Delta_f, a_f \in \text{argmax}_{a'_f \in A_f} \mathbb{E}_{\mu_f}(u_{a'_f})\}$$

<sup>13</sup>This fact is well known by asset allocators. When adjusting tactically the exposure of an investment fund to the stock market, the execution of the price of the corresponding trade may be far from the one originally conjectured.

is the best-reply, or best-response, correspondence of the player  $f$  directed by  $\Delta$ . A profile of actions  $a \in A$  is said to be  $\Delta$ -rationalizable if there exists a set  $C \in \mathcal{C}$  with the best-reply property directed by  $\pi$  such that  $a \in \text{proj}_A C$ .

In other words, the set of  $\Delta$ -rationalizable strategies is the set of rationalizable strategies when first order beliefs are restricted to a subset of  $\times_{f \in F} \Delta(S \times A_{-f})$ .

PROPOSITION 3.9. *Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game*

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

*equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . For every  $\pi \in \Delta(S)$ , let  $D^\pi \subseteq A \times T$  be the event corresponding to "players have a profile of priors  $\pi$  on the state of nature", that is,*

$$D^\pi = A \times \{t \in T : \forall f \in F, \text{marg}_S \eta_f(t_f) = \pi_f\}$$

*and consider the restriction  $\Delta^\pi \subseteq \times_{f \in F} \Delta(S \times A_{-f})$  defined as*

$$\Delta^\pi := \times_{f \in F} \{\mu_f \in \Delta(S \times A_{-f}) : \text{marg}_S \mu_f = \pi_f\}.$$

- (1) *Every  $a \in A$  is a rationalizable profile of actions.*
- (2) *For every  $\pi \in \Delta(S)$ , every  $a \in A$  is  $\Delta^\pi$ -rationalizable.*
- (3)  $\text{proj}_A RCBR = A$ .
- (4) *For every  $\pi \in \Delta(S)^F$ ,*

$$\text{proj}_A((S \times R) \cap (S \times D^\pi) \cap CB((S \times R) \cap (S \times D^\pi))) = A,$$

*that is, every profile of actions is consistent with rationality, common belief in rationality and transparency of  $D^\pi$ .*

PROOF. **Points 1 and 2.** Fix a player  $f \in F$ . For every belief  $\mu_f \in \Delta(S \times A_{-f})$ , such that for every  $(i, j) \in (F \setminus \{f\}) \times (J \setminus \{0\})$ ,

$$\text{marg}_{A_{ij1}} \mu_f(\{0\}) = \text{marg}_{A_{ij3}} \mu_f(\{0\}) = 1$$

we have

$$\forall (a_f, a_{-f}) \in A_f \times \text{supp marg}_{A_{-f}}(\mu_f), g_f^{\mathcal{P}}(a_f, a_{-f}) = \theta_f, ;$$

consequently,

$$\forall a_f \in A_f, \mathbb{E}_{\mu_f}(u_{a_f}) = \mathbb{E}_{\text{marg}_S \mu_f}(U_{f, \theta_f}),$$

where

$$\begin{aligned} U_{f, \theta_f} : S &\rightarrow \mathbb{R} \\ s &\mapsto U_f(s, \theta_f) \end{aligned}$$

As a result,  $r_f(\mu_f) = A_f$  and  $\rho_f(S \times A_f) = A_f$ . In addition, for every  $\pi_f \in \Delta(S)$ , there exists a belief  $\mu_f^\pi \in \Delta(S \times A_{-f})$  such that, for every  $(i, j) \in (F \setminus \{f\}) \times (J \setminus \{0\})$ ,

$$\text{marg}_{A_{ij1}} \mu_f^\pi(\{0\}) = \text{marg}_{A_{ij3}} \mu_f^\pi(\{0\}) = 1,$$

since it is sufficient to consider the product measure  $\pi_f \cdot \text{marg}_A \mu_f^\pi$ . Therefore,  $r_f(\mu_f') = A_f$  and  $\rho_f^{\Delta^\pi}(S \times A_f) = A_f$ . Since the previous results hold for every  $f \in F$ , we conclude that:

- $\rho(S \times A) = S \times (\times_{f \in F} A_f) = A$ , which implies that  $C := S \times A$  has the best-reply property, and, consequently, every  $a \in A$  is rationalizable as  $\text{proj}_A C = A$ ;
- $\rho^{\Delta^\pi}(S \times A) = S \times (\times_{f \in F} A_f) = A$ , which implies that  $C := S \times A$  has the best-reply property directed by  $\pi$ , and, consequently, every  $a \in A$  is  $\Delta^\pi$ -rationalizable as  $\text{proj}_A C = A$ .

**Points 3 and 4.** For every  $f \in F$ , recall the definition of  $\mu_f$  and  $\mu_f^\pi$  above. Since we assumed a universal type structure, for every  $f \in F$ , there exists

$$t_f \in \text{proj}_{T_f} CB_f(S \times R_{-f})$$

such that  $\text{marg}_{A_{-f}} \eta_f(t_f) = \mu_f$ . Hence, for every  $f \in F$ ,

$$\text{proj}_{A_f}(R_f \cap CB_f(S \times R_{-f})) = A_f,$$

which implies  $\text{proj}_A RCBR = A$ . Similarly, having assumed a universal type structure, for every  $\pi \in \Delta(S)^F$  and, for every  $f \in F$ , there exists

$$t_f \in \text{proj}_{T_f} CB_f((S \times R_{-f}) \cap (S \times D_{-f}^\pi))$$

such that  $\text{marg}_{S \times A_{-f}} \eta_f(t_f) = \mu_f^\pi$ . Hence, for every  $\pi \in \Delta(S)^F$ , and every  $f \in F$ ,

$$\text{proj}_{A_f}(R_f \cap CB_f((S \times R_{-f}) \cap (S \times D_{-f}^\pi))) = A_f,$$

which implies

$$\text{proj}_A((S \times R) \cap (S \times D^\pi) \cap CB((S \times R) \cap (S \times D^\pi))) = A.$$

□

While striking at first, this result follows from the trivial observation that, if a player is sure that the others have zero demand and zero supply for a given risky asset, then any action has the same expected payoff. As a straightforward corollary of **Proposition 3.9**, despite the players' utilities are discontinuous, we have  $\rho^\infty(A) = \rho(\rho^\infty(A)) = \text{proj}_A RCBR = A$ . In addition, any SE is just one of the many uncountable outcomes consistent with rationalizability and *RCBR*. The crux is that rationalizability allows for weakly dominated actions supported by "extreme" conjectures. Consequently, we need a different approach to provide sufficient epistemic conditions for competitive prices. In this regard, in the following section, we start by identifying a sufficiently large set of "epistemic states" (i.e., a set of types) that do not contradict the validity of General Equilibrium Theory for a given player, unless the same player intends to act as a price-setter for one or more risky assets. This approach will lead us to the definition of mutual  $\pi$ -belief in Walrasian markets. For the case of a game in which buyers pay the price they quote, we will show that rationality and mutual  $\pi$ -belief in Walrasian markets ensure competitive asset prices. Nevertheless, in the case that buyers pay the market price, we will not be able to exclude the existence of a profile of weakly dominated actions that would prevent prices from being competitive. One remarkable feature of this particular profile of actions is the "incautious" attitude taken by traders.

**5.1. Definitions and preliminary considerations.** We consider a trading game

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle$$

equipped with the corresponding universal type structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . To simplify the analysis, we fix a generic profile of priors on the state of nature,  $\pi \in \Delta(S)^F$ , and assume that  $\mathcal{E}$  has a unique and interior WE associated with  $\pi$ , denoted by  $(x^{**}, p^{**})$ .<sup>14</sup> In this regard, we say that  $x^{**}$  is the 'Walrasian allocation of the players, while  $p^{**}$  is a vector of Walrasian prices, or

<sup>14</sup>Given our assumption on preferences, it is sufficient that the gross substitution property holds.

competitive prices. The remainder of this sub-section is organized as follows. First, we provide a more convenient representation of expected utilities. Second, we classify a generic *action-type* pair  $(a_f, t_f) \in A_f \times T_f$  by the intended effect of the action  $a_f$  according to  $t_f$ . Finally, we discuss the relevant epistemic assumptions, providing the definition of (mutual)  $\pi$ -belief in Walrasian markets.

5.1.1. *Expected utility.* Without loss of generality, fix a player  $f \in F$ . With a slight abuse of language, we say that an action-type pair  $(a_f, t_f)$  of the player  $f$  attributes a probability  $p \in [0, 1]$  to the event  $E_{-f} \subseteq A_{-f} \times T_{-f}$  if  $\eta_f(t_f)(S \times E_{-f}) = p$ . Consequently, given the measurable mapping  $o_f : S \times A \times T \rightarrow I$ , where  $I$  is a generic set, we say that  $(a'_f, t_f)$  attributes the probability  $p \in [0, 1]$  to  $I' \subseteq I$  if  $\eta_f(t_f)(o_{a'_f}^{-1}(I')) = p$ , where  $o_{a'_f}^{-1}$  denotes the inverse mapping of  $o_{f,a'_f}$ . In other words,  $(a_f, t_f)$  attributes a certain probability to  $I'$  through the push-forward measure  $\eta_f(t_f) \circ o_{a_f}^{-1}$ . In particular, every  $(a'_f, t_f) \in A_f \times T_f$  assigns a probability  $C_{a_f, t_f}(S', X') = \eta_f(t_f)(S', g_{a'_f}^{-1}(X'_f))$  to every  $S' \times X' \in \mathcal{B}(S \times X_f)$ ; therefore,

$$\mathbb{E}_{\eta_f(t_f)}(u_{f,a_f}) = \sum_{s \in S} \text{marg}_S \eta_f(t_f)(s) \int_{X_f} U(s, x_f) dC_{a_f, t_f}(x_f|s),$$

where the conditional measure  $C_{a_f, t_f}(x_f|s)$  satisfies the standard measure theoretic definition (cf. **Section 1.1.5 of Chapter 1**), that is,  $C_{a_f, t_f}(X' \times S') = \sum_{s \in S'} \text{marg}_S \eta_f(t_f)(s) \int_{X_f} dC_{a_f, t_f}(x_f|s)$  for every  $S' \times X' \in \mathcal{B}(S \times X_f)$ . Therefore, for every  $f \in F$ , we have

$$(5.1) \quad (a_f, t_f) \in R_f \iff a_f \in \text{argmax}_{a'_f \in A_f} \left[ \sum_{s \in S} \text{marg}_S \eta_f(t_f)(s) \int_{X_f} U(s, x_f) dC_{a'_f, t_f}(x_f|s) \right].$$

5.1.2. *Orders classification.* Without loss of generality, fix a player  $f \in F$  and a risky asset  $j \in J \setminus \{0\}$ ; the pair  $(a_f, t_f) \in A_f \times T_f$  denotes:

- a buy (resp. sale) *over-subscription* of the asset  $j$ , if  $t_f$  is certain to buy less than  $a_{fj1}$  (resp. sell than  $a_{fj3}$ );
- an *over-bid* (resp. under-ask) for the asset  $j$ , if  $t_f$  is certain to pay less than  $a_{fj2}$  (resp. to obtain more than  $a_{fj4}$ ) for every unit of  $j$  bought (resp. sold);
- a buy (resp. sale) *penny jump* of the asset  $j$ , if  $t_f$  is certain to be the only active buyer (resp. seller) of  $j$ ;
- an "out-of-market" order for asset  $j \in J \setminus \{0\}$ , if  $t_f$  is certain not to buy (resp. sell) any amount of  $j$ .

As an example, suppose  $t_f$  is certain that her opponents are all selling the asset  $v$  at the ask price  $p$ ; any action for  $f$  that demands a positive quantity of  $v$  at a bid price less than  $p$  is an out-of-market order for asset  $v$ . These notions will prove to be useful at a later stage.

5.1.3. *Epistemic assumptions.* Let  $a^{**} \in A$  be an action profile that satisfies, for every  $f \in F$  and every  $j \in J \setminus \{0\}$ , the following requirements:

- $a_{fj1}^{**} = (x_j^{**} - \theta_{fj}) \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$  ;
- $a_{fj3}^{**} = (\theta_{fj} - x_j^{**}) \mathbb{I}(x_{fj}^{**} < \theta_{fj})$  ;
- $a_{fj2}^{**} = p_j^{**} \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$  ;
- $a_{fj4}^{**} = p_j^{**} \mathbb{I}(x_{fj}^{**} < \theta_{fj})$  .

By the proof of **Proposition 3.7**,  $a^{**}$  is an SE for the profile of priors  $\pi = (\pi_f)_{f \in F} \in \Delta(S)^F$  supporting the corresponding WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ . A trivial epistemic condition for  $a^{**}$  is that: i) the players are rational, ii) they are certain of the action that their opponents are playing, and iii) they have first-order beliefs such that the corresponding marginal over  $S$  is consistent with  $\pi$ . Hence, if a pair  $(a_f^{**}, t_f^{**}) \in A_f \times T_f$  satisfies the previous epistemic conditions, it must be the case that

$$\text{marg}_{S \times A_{-f}} \eta_f(t_f^*) = \pi_f \delta_f,$$

where  $\delta_f \in \Delta(\{a_{-f}^{**}\})$ . Thus, to extend the analysis to the case in which players might not be necessary certain of the action profile being played by their opponents, we shall restrict the attention to types with first-order independent beliefs and priors  $\pi$  on the state of nature. Namely, the event where the players have first-order independent beliefs and priors  $\pi = (\pi_f)_{f \in F}$  over  $S$  is defined as

$$V^\pi = S \times \left( \times_{f \in F} \{(a_f, t_f) \in A_f \times T_f : \exists \mu_f \in \Delta(A_{-f}), \text{marg}_{S \times A_{-f}} \eta_f(t_f) = \pi_f \mu_f\} \right).$$

We also say that players have  $\pi$ -independent beliefs to refer to  $V^\pi$ . To clarify the meaning of the previous definition, observe that, at  $V^\pi$ , the conditional probability that the type  $t_f$  of player  $f \in F$  attributes to a given state of nature is independent of the profile of actions played by her opponents. By the Bayes theorem, this is equivalent to saying that conjectures are independent from the state of nature. Restricting the analysis to  $V^\pi$  can be motivated by assuming that players are certain that their opponents do not have a better ability to predict the state of nature, that is, they agree to disagree on their prior beliefs on  $\mathcal{B}(S)$ .

For every  $f \in F$ , and every  $(a_f, t_f) \in V_f^\pi$ , we can represent the expected utility of  $f$  as,

$$\begin{aligned} \mathbb{E}_{\eta_f(t_f)}[u_f(a_f)] &= \sum_{s \in S} \pi_f(s) \int_{X_f} U(s, x_f) d\mu_{a_f, t_f}(x_f) = \\ &= \int_{X_{a_f, t_f}} \left[ \sum_{s \in S} \pi_f(s) U(s, x_f) \right] d\mu_{a_f, t_f}(x_f), \end{aligned}$$

where  $\mu_{a_f, t_f} := \text{marg}_{X_f} C_{a_f, t_f}$  and  $X_{a_f, t_f}$  is defined as the support of  $\mu_{a_f, t_f}$ .

Hence, at  $V^\pi$  all players  $f \in F$  rank their available actions by the expected value of their *Walrasian payoff*  $\sum_{s \in S} \pi_f(s) U(s, x_f)$ . As said, while (the type of) a player may be uncertain of the total volume that is traded on the market, general equilibrium may be still considered as a useful theory to predict execution of prices for some reason. For example, (the type of) a player may be sure that her opponents do not attempt at manipulating the price of any risky assets, or are too "small" to do that, and that market interactions allow implementing Pareto efficient allocations.<sup>15</sup> This is, essentially, the idea of price-taking behavior in the General Equilibrium Theory of Arrow and Debreu (1954). In fact, the notion of price-taking behavior consists of two minimum requirements:

- (1) at a WE, agents take the Walrasian equilibrium prices as given;
- (2) at a WE, except for budget constraints, agents are not subject to trading constraint in the implementation of their optimal asset allocations (i.e. the Walrasian equilibrium allocations).

As said, GE models are not equipped with an epistemic model. However, we can model both requirements through appropriate restrictions on the beliefs of the players in the trading game  $\Gamma$ . In this regard, recall that, for every risky asset  $j \in J \setminus \{0\}$ , there is a market price for  $j$  if there exists at least one buy order for  $j$  which can be matched against a sell order for the same asset.

Price taking behavior: requirement 1. In a trading game, players can always act as price-setters. Hence, we model requirement 1) restricting, for every  $f \in F$ , the types of  $f$  to those being certain that the bid and ask prices of her opponents are competitive. Hence, for every  $f \in F$ , we consider the event  $B_f \left( S \times L_{-f}^{p**} \right)$ , where, for a given  $p \in \mathbb{R}^{J \setminus \{0\}}$

$$L^{(p)} := \{(a, t) \in A \times T : \forall f \in F, a_{fj2} = a_{fj4} = p\},$$

is the event "players have buy and sell limit prices set equal to  $p$ ".

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<sup>15</sup>In this regard, the tremendous rise in the use of passive investment products, such as index funds, by retail and professional investors, supports this view.

Price taking behavior: requirement 2. Without loss of generality, fix a player  $f \in F$  and a risky asset  $j \in J_{-0}$ . Suppose that  $x_{fj}^{**} > \theta_{fj}$  for  $j \in J_{-0}$ . In this case, we can model requirement 2) by restricting the types of  $f$  restrict  $T_f$  to those certain of

$$Q_j^{s,f}(a_{-f}, p^{**}) - Q_j^{d,f}(a_{-f}, p^{**}) = -Q_j^{z,f}(a_{-f}, p^{**}) \geq x_{fj}^{**} - \theta_{fj}.$$

In other words, we require a buyer of  $j$  at  $x^{**}$  to be certain to buy up to  $q \leq x_{fj}^{**} - \theta_{fj}$  units of  $j$  with an order of equal size and bid price at least equal to  $p_j^{**}$ . Similarly, suppose that  $x_{fj}^{**} < \theta_{fj}$  for  $j \in J \setminus \{0\}$ . By the same argument, we can model requirement 2) by restricting the types of  $f$  restrict  $T_f$  to those certain of

$$Q_j^{d,f}(a_{-f}, p^{**}) - Q_j^{z,f}(a_{-f}, p^{**}) = Q_j^{s,f}(a_{-f}, p^{**}) \geq \theta_{fj} - x_{fj}^{**}.$$

Consequently, the event “ $f$  believes that the market is deep” is  $B_f(S \times D_{-f} \times T_{-f})$ , where  $D_{-f}$  is defined as,

$$(5.2) \quad D_{-f} = \left\{ a_{-f} \in A_{-f} : \forall j \in J \setminus \{0\}, \text{sign} \left( Q_j^{z,f}(a_{-f}, p_j^{**}) + x_{fj}^{**} - \theta_{fj} \right) = \text{sign}(\theta_{fj} - x_{fj}^{**}) \right\}.$$

In other words, at  $D_{-f}$ , the player  $f$  does not need to resort to penny jumps or over-subscriptions to implement her Walrasian asset allocation. Hence, the event  $B_f(S \times D_{-f} \times T_{-f})$  is sufficient to model requirement 2).

Mutual  $\pi$ -belief in Walrasian markets. A player  $f \in F$  is said to believe in Walrasian markets if she believes in events  $S \times L_{-f}^{p^{**}}$  and  $D_{-f}$ , that is, if she believes to be a price-taker, barring her intention to deliberately affect asset prices. By the conjunction property of the belief operator (see **Section 3.2 of Chapter 1**), for every  $f \in F$ , the event

$$W_f = B_f \left( S \times \left( L_{-f}^{(p^{**})} \cap D_{-f} \right) \times T_{-f} \right)$$

restricts the set of states of the world to those in which  $f$  believes in Walrasian markets. Accordingly, the event “players mutually  $\pi$ -believe in Walrasian markets” is

$$W^\pi = W \cap V^\pi = S \times \left( \times_{f \in F} (W_f \cap V_f^\pi) \right).$$

In the following section, we shall consider the implications of rationality and mutual  $\pi$ -belief in Walrasian markets for the possible outcome of  $\Gamma$ . That is, we impose mutual  $\pi$ -belief in Walrasian

market as an epistemic assumption for rational traders. Before proceeding, we summarize the relevant definitions below.

DEFINITION 3.10. Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . Suppose that, given a profile  $\pi \in \Delta(S)^F$ , there exists a unique and interior WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ .

- The event “prices are competitive” is

$$(5.3) \quad P = S \times \left\{ (a, t) \in A \times T : \forall j \in J \setminus \{0\}, \min \left\{ (a_{fj2})_{\{f \in F: \tilde{q}_{fj}^d(a_{fj}) > 0\}} \right\} = p_j^{**} \right\},$$

since, for every  $j \in J \setminus \{0\}$ ,  $p_j(a) = \min \left\{ (a_{fj2})_{\{f \in F: \tilde{q}_{fj}^d(a_{fj}) > 0\}} \right\}$ .

- The event “players have a limit price  $p \in \mathbb{R}^J$ ” is

$$L^{(p)} = \{a \in A : \forall f \in F, p_{fj}^d(a) = p_{fj}^s = p(a)\}.$$

- The event “player  $f \in F$  believes in Walrasian markets” is

$$W_f = B_f \left( S \times \left( L_{-f}^{(p^{**})} \cap D_{-f} \right) \times T_{-f} \right).$$

- The event “players believe in Walrasian markets and have priors  $\pi = (\pi_f)_{f \in F}$  on  $S$ ”, or, equivalently, “players mutually  $\pi$ -believe in Walrasian markets”, is

$$W^\pi = W \cap V^\pi = S \times \left( \times_{f \in F} (W_f \cap V_f^\pi) \right).$$

**5.2. Results.** As a first result, we show that, regardless of the specific payment rule of  $\Gamma$ , rationality and mutual  $\pi$ -belief in Walrasian markets imply actions intended to implement Walrasian allocations. That is, for every  $f \in F$ , a pair  $(a_f, t_f) \in R_f \cap W_f^\pi$  must be certain of implementing  $x_f^{**}$ .

PROPOSITION 3.11. Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . Suppose that, given a profile  $\pi \in \Delta(S)^F$ , there exists a unique and interior WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ .

- (1) Let  $f \in F$  be a generic player and consider a type  $t_f \in \text{proj}_{T_f} W_f^\pi$ ; for every  $x_f \in X_{a_f, t_f}$ , there exists  $x'_f \in BS_f(p^{**})$  such that

$$\mathbb{E}_{\pi_f}(U_{f, x'_f}) \geq \mathbb{E}_{\pi_f}(U_{f, x_f}).$$

Furthermore, there exists  $a_f^{**} \in A_f$  such that  $\mu_{a_f^{**}, t_f}(\{x_f^{**}\}) = 1$ .

- (2) For every player  $f \in F$  and for every pair  $(a_f, t_f) \in R_f \cap W_f^\pi$ ,

$$\mathbb{E}_{\eta_f(t_f)}(u_{f, a_f}) = \mathbb{E}_{\eta_f(t_f)}(u_{f, a_f^{**}}) = \mathbb{E}_{\pi_f}(U_{f, x_f^{**}}).$$

**PROOF. Point 1.** Type  $t_f$  assigns probability one to the occurrence of  $a_{-f} \in L_{-f}^{p^{**}} \cap D_{-f}$ . By definition, at  $L^{p^{**}}$ , a buyer of a risky asset  $j$  cannot pay less than  $p_j^{**}$ , while a seller cannot receive more than  $p_j^{**}$ . Hence, for every  $j \in J \setminus \{0\}$ , a pair  $(a_f, t_f) \in W_f^\pi$  attributes a positive probability to implement  $x_j > \theta_j$  only if  $a_{fj1} > 0$  and  $a_{fj2} \geq p_j^{**}$ . Likewise, for every  $j \in J \setminus \{0\}$ , a pair  $(a_f, t_f) \in W_f^\pi$  attributes a positive probability to implement  $x_j < \theta_j$  only if  $a_{fj3} > 0$  and  $a_{fj4} \leq p_j^{**}$ . In addition, if buyers pay the price they quote, a pair  $(a_f, t_f) \in W_f^\pi$  such that  $a_{fj2} > p_j^{**}$  is sure to pay a price for  $j$  higher than the corresponding market price. In this regard, a pair  $(a_f, t_f) \in W_f^\pi$  is also certain that  $p_j(a) = p_j^{**}$ . Thus, for every  $a \in A_f \times \text{proj}_A(L_{-f}^{p^{**}} \cap D_{-f})$  we have

$$\tilde{p}_{fj}^s(a) \tilde{q}_{fj}^s(a) - \tilde{p}_{fj}^d(a) \tilde{q}_{fj}^d(a) \leq p_j^{**} (\tilde{q}_{fj}^s(a) - \tilde{q}_{fj}^d(a)) = p_j^{**} (\theta_{fj} - x_{fj}),$$

where  $x_f = g_f^P(a_f, a_{-f})$ . In this regard, consider a hypothetical asset allocation  $x'_f$  such that, for every  $j \in J \setminus \{0\}$ ,  $x'_{fj} = x_{fj}$  and  $x'_{f0} = \theta_{f0} + \sum_{j \in J \setminus \{0\}} p_j^{**} (\theta_{fj} - x_{fj})$ , and observe that, by definition,  $x'_f \in BS_f(p^{**})$ . Since  $U_f$  is strictly increasing in  $x_{f0}$ , and, for every risky asset  $j \in J \setminus \{0\}$ ,

$$\tilde{p}_{fj}^s(a) \tilde{q}_{fj}^s(a) - \tilde{p}_{fj}^d(a) \tilde{q}_{fj}^d(a) \leq p_j^{**} (\theta_{fj} - x_{fj}),$$

we have

$$x_{f0}(a) = \theta_{f0} + \sum_{j \in J \setminus \{0\}} (\tilde{p}_{fj}^s(a) \tilde{q}_{fj}^s(a) - \tilde{p}_{fj}^d(a) \tilde{q}_{fj}^d(a)) \leq \theta_{f0} + \sum_{j \in J \setminus \{0\}} (p_j^{**} (\theta_{fj} - x_{fj})).$$

Hence, for every  $s \in S$ ,  $U_f(s, x'_f) \geq U_f(s, x_f)$ , and, we conclude that

$$\mathbb{E}_{\pi_f}(U_{f, x'_f}) \geq \mathbb{E}_{\pi_f}(U_{f, x_f}).$$

Consider the profile of actions  $a^{**} \in A$  defined in **Section 3.3**. By the definition of  $L_{-f}^{p^{**}} \cap D_{-f}$ , and condition 2) in the definition of a WE of  $\mathcal{E}$ , we have  $g_f^p(a_f^{**}, a_f) = x_f^{**}$  for every  $a_{-f} \in \text{proj}_{A_{-f}}(L_{-f}^{p^{**}} \cap D_{-f})$ . Therefore,  $\mu_{a_f^{**}, t_f}(\{x_f^{**}\}) = 1$ .

**Point 2.** Fix a generic player  $f \in F$ , and consider the function

$$\phi_f : \mathbb{R}^J \rightarrow \mathbb{R}^J$$

$$x_f \mapsto \left( \sum_{j \in J \setminus \{0\}} p_j^{**}(\theta_{fj} - x_{fj}), x_{f,-0} \right).$$

By point 1) above, for every  $t_f \in \text{proj}_{T_f} W_f^\pi$  and every  $x_f \in X_{a_f, t_f}$ ,

$$\mathbb{E}_{\pi_f}(U_{f, \phi_f(x_f)}) \geq \mathbb{E}_{\pi_f}(U_{f, x_f})$$

and  $\phi_f(x_f) \in BS_f(p^{**})$ . In addition, by condition 1) in the definition of a WE of  $\mathcal{E}$ , the assumptions on the players' preferences ensure that, for every  $x_f \in BS_f(p^{**}) \setminus \{x_f^{**}\}$ ,

$$\mathbb{E}_{\pi_f}(U_{f, x_f^{**}}) > \mathbb{E}_{\pi_f}(U_{f, x'_f}).$$

Hence, for every  $(a_f, t_f) \in W_f^\pi$  such that  $\mu_{a_f, t_f}(\{x_f \in \mathbb{R}^J : x_f \neq x_f^{**}\}) > 0$ ,

$$\mathbb{E}_{\eta_f(t_f)}(u_{f, a_f}) = \int_{X_{a_f, t_f}} \left[ \sum_{s \in S} \pi_f(s) U(s, x_f) \right] d\mu_{a_f, t_f}(x_f) \leq$$

$$\int_{X_{a_f, t_f}} \left[ \sum_{s \in S} \pi_f(s) U(s, \phi_f(x_f)) \right] d\mu_{a_f, t_f}(x_f) < \mathbb{E}_{\pi_f}(U_{f, x_f^{**}}),$$

that is, if  $t_f \in \text{proj}_{T_f} W_f^\pi$ ,  $\nexists a_f \in A_f$  such that  $\mathbb{E}_{\eta_f(t_f)}(u_{f, a_f}) > \mathbb{E}_{\pi_f}(U_{f, x_f^{**}})$ . By point 1) above, for every  $t_f \in \text{proj}_{T_f} W_f^\pi$ ,

$$\mathbb{E}_{\eta_f(t_f)}(u_{f, a_f^{**}}) = \mathbb{E}_{\pi_f}(U_{f, x_f^{**}}).$$

Therefore, for every  $(a_f, t_f) \in R_f \cap W_f^\pi$ , it must be the case that

$$\mathbb{E}_{\eta_f(t_f)}(u_{f, a_f}) = \mathbb{E}_{\eta_f(t_f)}(u_{f, a_f^{**}}) = \mathbb{E}_{\pi_f}(U_{f, x_f^{**}}).$$

□

We showed that rationality and mutual  $\pi$ -belief in Walrasian markets require players to submit orders that, given their beliefs, imply the certainty of implementing Walrasian asset allocations. In this regard, observe that, at  $R \cap W^\pi$ , players are certain that assets will be priced competitively. For

the sake of clarity, notice that every  $t_f \in W_f^\pi$  is certain that if  $f$  does not participate in the trading of risky assets, asset prices will be competitive. Instead, at  $R_f \cap W_f^\pi$ ,  $(a_f, t_f) \in W_f^\pi$  is certain that asset prices will be competitive. As we shall see, whether this belief is correct or not essentially depends on the payment rule of the game. Nevertheless, we can also conclude for the moment that, at  $R \cap W^\pi$ , the players are certain that it is not possible to manipulate prices to induce profitable arbitrage opportunities. This further motivates  $W^\pi$  as a correct representation of a state in which players mutually believe that markets are efficient. The next proposition shows that a rational player can choose different orders to implement her Walrasian allocation at  $W^\pi$ , depending on her type and the payment rule of the game. Anticipating one of the main results of this section, the possibility of incautious actions in the case of pay-as-clear auctions may prevent asset prices from being competitive at  $R \cap W^\pi$ .

PROPOSITION 3.12. *Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game*

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

*equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . Suppose that, given a profile  $\pi \in \Delta(S)^F$ , there exists a unique and interior WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ . For every  $f \in F$ , every  $(a_f, t_f) \in R_f \cap W_f^\pi$ , and every  $j \in J \setminus \{0\}$ ,*

- (1) *if  $x_{fj}^{**} > \theta_{fj}$ ,*
  - $\mathcal{P} = \text{Pay} - \text{as} - \text{bid} \implies a_{fj1} \geq x_{fj}^{**} - \theta_{fj}, a_{fj2} = p_{fj}^{**};$
  - $\mathcal{P} = \text{Pay} - \text{as} - \text{clear} \implies a_{fj1} \geq x_{fj}^{**} - \theta_{fj}, a_{fj2} \geq p_{fj}^{**};$
- (2) *regardless of the payment rule,*
  - $x_{fj}^{**} < \theta_{fj} \implies a_{fj3} \geq \theta_{fj} - x_{fj}^{**}, a_{fj2} \leq p_{fj}^{**}.$
- (3) *regardless of the payment rule,*
  - $a_{fj1} = a_{fj3} = 0 \implies x_{fj}^{**} = \theta_{fj};$
  - $a_{fj1} > 0, a_{fj2} < p_j^{**} \implies x_{fj}^{**} = \theta_{fj};$
  - $a_{fj3} > 0, a_{fj4} > p_j^{**} \implies x_{fj}^{**} = \theta_{fj}.$

PROOF. Fix a pair  $(a_f, t_f) \in R_f \cap W_f^\pi$  and a risky asset  $j \in J \setminus \{0\}$ . and recall that, by **Proposition 3.11**, every  $(a_f, t_f) \in R_f \cap W_f^\pi$  is such that  $g_f^P(a_f, a_{-f}) = x_f^{**}$  for every  $a_{-f} \in L_{-f}^{p^{**}} \cap D_{-f}$ .

**Point 1.** By the definition of  $W_f^\pi$ ,  $\text{supp} \left[ \text{marg}_{A_{-f}} \eta_f(t_f) \right] \subseteq L_{-f}^{p^{**}} \cap D_{-f}$ . Hence, if  $x_{fj}^{**} > \theta_{fj}$ , it must

be either the case that

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{s,f}(p_j^{**}) - Q_j^{d,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj3} - a_{fj1}) > x_{fj}^{**} - \theta_{fj}\} \right) > 0,$$

or

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{s,f}(p_j^{**}) - Q_j^{d,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj3} - a_{fj1}) = x_{fj}^{**} - \theta_{fj}\} \right) = 1.$$

Since  $a_f$  must be such that  $g_f^{\mathcal{P}}(a_f, a_{-f})_j = x_{fj}^{**}$  for every  $a_{-f} \in \text{supp}[\text{marg}_{A_{-f}} \eta_f(t_f)]$ . Hence,  $a_{fj1} > 0$  and  $a_{fj2} \geq p_{fj}^{**}$ , otherwise  $(a_f, t_f)$  would be certain to remain stuck with her endowment of  $j$ . Furthermore, if buyers pay the price they quote ( $\mathcal{P} = \text{pay} - \text{as} - \text{bid}$ ), or, in the case of a pay-as-clear auction,  $f$  is the only buyer of  $j$ , it must be necessary the case that  $a_{fj2} = p_{fj}^{**}$ . Otherwise,  $(a_f, t_f)$  would be sure to pay more than  $p_{fj}^{**}$  without affecting the allocation in risky assets, that is, at  $\left((a_{fj}^{**}, a_{f,-j}), t_f\right)$  the type  $t_f$  would be certain that  $g_{f,0}\left((a_{fj}^{**}, a_{f,-j}), a_{-f}\right) > g_{f,0}(a_f, a_{-f})$ , and, for every  $j \in J \setminus \{0\}$ ,  $g_{f,j}\left((a_{fj}^{**}, a_{f,-j}), a_{-f}\right) = g_{f,j}(a_f, a_{-f})$ , contradicting the rationality of  $(a_f, t_f)$  at  $W^\pi$ . Finally, we have to determine the size of  $a_{f1j}$ . If  $a_{fj1} < x_{fj}^{**} - \theta_{fj}$ , then  $(a_f, t_f)$  is certain to buy less than  $x_{fj}^{**} - \theta_{fj}$  and therefore not to implement  $x_{fj}^{**}$ . If  $a_{fj1} = x_{fj}^{**} - \theta_{fj}$ , then  $(a_f, t_f)$  is certain to buy  $x_{fj}^{**} - \theta_{fj}$  and therefore implement  $x_{fj}^{**}$ . Now, suppose  $a_{fj1} > x_{fj}^{**} - \theta_{fj}$ . If

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{s,f}(p_j^{**}) - Q_j^{d,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj3} - a_{fj1}) > x_{fj}^{**} - \theta_{fj}\} \right),$$

then  $(a_f, t_f)$  attributes a non-null probability to buy more than  $x_{fj}^{**} - \theta_{fj}$  and therefore to not implement  $x_{fj}^{**}$ . However, if  $t_f$  is such that

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{s,f}(p_j^{**}) - Q_j^{d,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj3} - a_{fj1}) = x_{fj}^{**} - \theta_{fj}\} \right),$$

and, in addition,  $\eta_f(t_f)(\{a_{-f} \in A_{-f} : \forall i \in F \setminus \{f\}, a_{fj1} = 0\}) = 1$ , that is,  $t_f$  is certain to be the only possible buyer of the asset  $j$ , a pair  $(a_f, t_f)$  that  $a_{fj1} > x_{fj}^{**} - \theta_{fj}$  would be certain to buy  $x_{fj}^{**} - \theta_{fj}$  of asset  $j$  and therefore implement  $x_{fj}^{**}$ . Hence, we cannot exclude over-subscription buy orders.

**Point 2.** By definition of  $W_f^\pi$ ,  $\text{supp}[\text{marg}_{A_{-f}} \eta_f(t_f)] \subseteq L_{-f}^{p^{**}} \cap D_{-f}$ . Hence, if  $x_{fj}^{**} < \theta_{fj}$ , it must be either the case that

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{d,f}(p_j^{**}) - Q_j^{s,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj1} - a_{fj3}) > \theta_{fj} - x_{fj}^{**}\} \right) > 0,$$

or,

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{d,f}(p_j^{**}) - Q_j^{s,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj1} - a_{fj3}) = \theta_{fj} - x_{fj}^{**}\} \right) = 1.$$

Since  $a_f$  must be such that  $g_f^P(a_f, a_{-f})_j = x_{fj}^{**}$  for every  $a_{-f} \in L_{-f}^{p_{**}} \cap D^{-f}$ , it follows that  $a_{fj3} > 0$  and  $a_{fj2} \leq p_{fj}^{**}$ , otherwise  $(a_f, t_f)$  would be certain to remain stuck with her endowment of  $j$ . Since the market price of  $j$  is determined by the least active bidder,  $(a_f, t_f)$  is certain to receive a price equal to  $p_{fj}^{**}$  for every unit of  $j$  sold regardless of the specific payment rule characterizing the game. Hence, for every  $a_{fj2} \leq p_{fj}^{**}$ ,  $(a_f, t_f)$  is certain to receive  $p_{fj}^{**}$  for every unit of  $j$  sold, and we are only left to evaluate the size of  $a_{fj3}$ . If  $0 < a_{fj3} < \theta_{fj} - x_{fj}^{**}$ , then  $(a_f, t_f)$  is certain to sell less than  $\theta_{fj} - x_{fj}^{**}$  and therefore not to implement  $x_{fj}^{**}$ . If  $a_{fj3} = \theta_{fj} - x_{fj}^{**}$ , then  $(a_f, t_f)$  is certain to sell  $\theta_{fj} - x_{fj}^{**}$  and therefore implement  $x_{fj}^{**}$ . Suppose, instead, that  $\theta_{fj} \geq a_{fj3} > \theta_{fj} - x_{fj}^{**}$ . If

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{d,f}(p_j^{**}) - Q_j^{s,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj1} - a_{fj3}) > \theta_{fj} - x_{fj}^{**}\} \right) > 0,$$

then  $(a_f, t_f)$  attributes a non-null probability to sell more than  $\theta_{fj} - x_{fj}^{**}$  and therefore not to implement  $x_{fj}^{**}$ . However, if  $t_f$  is such that

$$\eta_f(t_f) \left( \{a_{-f} : Q_j^{d,f}(p_j^{**}) - Q_j^{s,f}(p_j^{**}) = \sum_{i \in F \setminus \{f\}} (a_{fj1} - a_{fj3}) = \theta_{fj} - x_{fj}^{**}\} \right) = 1,$$

and  $\eta_f(t_f) (\{a_{-f} \in A_{-f} : \forall i \in F \setminus \{f\}, a_{fj3} = 0\}) = 1$ , that is,  $t_f$  is certain to be the only seller of the asset  $j$ , a pair  $(a_f, t_f)$  such that  $a_{fj3} > \theta_{fj} - x_{fj}^{**}$  is certain to sell only an amount  $\theta_{fj} - x_{fj}^{**}$  of the asset  $j$  and implement  $x_{fj}^{**}$ . Hence, we cannot exclude over-subscription sell orders.

**Point 3.** As said, every  $(a_f, t_f) \in R_f \cap W_f^\pi$  must be certain to implement the asset allocation  $x_f^{**}$ . If  $a_{fjj} = a_{fj3} = 0$ , the pair  $(a_f, t_f)$  is certain to implement an allocation equal to her initial endowment of the asset  $j$ ; hence, if  $a_{fjj} = a_{fj3} = 0$  and  $(a_f, t_f) \in R_f \cap W_f^\pi$ , it must be necessary the case that  $x_{fj}^{**} = \theta_{fj}$ . Likewise, every  $t_f \in W_f^\pi$  is certain that  $\nexists j \in J \setminus \{0\}$  can be bought (resp. sold) if to bid less (resp. ask more) than  $p_j^{**}$ . Hence, for every  $(a_f, t_f) \in \cap W_f^\pi$  such that, either  $a_{fj1} > 0$  and  $a_{fj2} < p_j^{**}$ , or  $a_{fj3} > 0$  and  $a_{fj4} > p_j^{**}$ , entails an out of market order for  $j$ . Therefore, since  $(a_f, t_f) \in R_f \cap W_f^\pi$  must be certain to implement a Walrasian asset allocation, it must be the case that  $x_{fj}^{**} = \theta_{fj}$ .  $\square$

We showed that rational players  $\pi$ -believing in Walrasian markets may submit over-subscription orders to implement their Walrasian allocations, and, in the case buyers pay the market price, also

over-bids; clearly, as the proof of the **Proposition 3.12** shows, over-bid orders are not possible at  $R \cap W^\pi$  for risky assets demanded by a single buyer, hence the possibility of over-bids must be always understood with this minor caveat. These are notable examples of incautious actions. As we shall see in the next proposition, the presence of over-subscriptions or over-bids orders, or both, may prevent one or more players from implementing their Walrasian allocations at  $R \cap W^\pi$ . Before proceeding, we make three important considerations. First, by **Propositions 3.11-12**, we conclude that  $a^{**} \in \text{proj}_A(R \cap W^\pi)$ , and, by **Proposition 3.7**,  $a_f^{**}$  supports competitive prices and asset allocation. Second, a pair  $(a_f, t_f) \in W_f^\pi$  is not necessarily sure of the quantities demanded and supplied by her opponents, that is, at  $W^\pi$  players do not necessarily hold point conjectures, and, even so, they would not necessarily be correct. As an immediate application of **Proposition 3.11** we have that for every  $f \in F$ , every  $a_f \in A_f$  is weakly dominated by  $a_f^{**}$  if  $a_{-f} \in L_{-f}^{p^{**}} \cap D_{-f}$ . In other words, over-subscriptions and over-bids are weakly dominated actions when the action space is restricted to  $\text{proj}_A(R \cap W^\pi)$ . Third, suppose that  $x_{fj}^{**} = \theta_{fj}$  for a given pair  $(f, j) \in F \times J \setminus \{0\}$ . In such a case,  $f$  does not need trade  $j$  to implement the corresponding Walrasian asset allocation; therefore, at  $W_f^\pi$ , an action-type pair of  $f$  would rationally choose an order with zero buy and sell quantities or an out-of-market order. For this reason, we shall first consider the case in which, for every pair  $(f, j) \in F \times J \setminus \{0\}$ ,  $x_{fj}^{**} \neq \theta_{fj}$ . The implications of dropping this assumption, which we denote as condition TR, will be clear from the corollary of the following proposition, which contains the most relevant result of this chapter.

**PROPOSITION 3.13.** *Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game*

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

*equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . Suppose that, given a profile  $\pi \in \Delta(S)^F$ , there exists a unique and interior WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ , and that condition TR holds.*

- (1) *If  $\mathcal{P} = \text{Pay} - \text{as} - \text{bid}$ ,  $a \in \text{proj}_A(R \cap W^\pi) \implies p_j(a) = p_j^{**} \forall j \in J \setminus \{0\}$ .*
- (2) *If  $\mathcal{P} = \text{Pay} - \text{as} - \text{clear}$ ,  $a \in \text{proj}_A(R \cap W^\pi) \implies p_j(a) \geq p_j^{**} \forall j \in J \setminus \{0\}$ .*

**PROOF.** Without loss of generality, we fix a generic risky asset  $j \in J \setminus \{0\}$ , as, once we fix  $\mathcal{P}$ , the resulting trading protocol is the same for every risky asset  $j' \in J \setminus \{0\}$ .

**Point 1.** Suppose that  $a \in \text{proj}_A(R \cap W^\pi)$ . By **Proposition 3.12**, if buyers pay the price they

quote for every  $f \in F$  such that  $x_{fj}^{**} > \theta_{fj}$ , it must be the case that  $a_{fj1} \geq x_{fj}^{**} - \theta_{fj}$ , and  $a_{fj2} = p_{fj}^{**}$ . Similarly, for every  $f \in F$  such that  $x_{fj}^{**} < \theta_{fj}$ , it must be the case that  $a_{fj3} \geq \theta_{fj} - x_{fj}^{**}$  and  $a_{fj4} \leq p_{fj}^{**}$ . Hence, we have

$$Q_j^d(a, p) = \begin{cases} 0 & p > p_j^{**} \\ Q_j^d \geq \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj}) & p \leq p_j^{**}, \end{cases}$$

and there exist  $p'_j \leq p_j^{**}$  such that  $Q_j^s(a, p'_j) = Q_j^s \geq \sum_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**})$ . In addition, and  $Q_j^s(a, p_j) < Q_j^s(a, p'_j)$  for every  $p_j < p'_j$  and  $Q_j^s(a, p'_j) = Q_j^s(a, p''_j)$  for every  $p''_j > p'_j$ . We still have to apply **Algorithm 3.1** to compute  $p_j(a)$ . In this regard, we have just shown that

$$\begin{aligned} \max(a_{fj2})_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} &= \min(a_{fj2})_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} = p_j^{**} \geq \\ \max(a_{fj3})_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}} &\geq \min(a_{fj3})_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}}, \end{aligned}$$

therefore we initialize the search for the market price of the asset  $j$  at  $p_j^{**}$ , as there are no buy orders with a limit price above  $p_j^{**}$ . Recall that the highest bidding buyers are served by the lowest asking sellers. At  $p_j^{**}$  we have,

$$Q_j^d(a, p_j^{**}) = Q_j^d \neq Q_j^s = Q_j^s(a, p_j^{**}) = Q_j^s(a, p_j^{**}),$$

or

$$Q_j^d(a, p_j^{**}) = Q_j^d = Q_j^s = Q_j^s(a, p_j^{**}) = Q_j^s(a, p_j^{**}).$$

Starting from the latter case, since  $Q_j^s(a, p_j) < Q_j^s(a, p'_j) = Q_j^s$  for every  $p_j < p'_j \leq p_j^{**}$ , no additional trades can be matched at a tentative market price lower than  $p_j^{**}$ ; consequently, **Algorithm 3.1** stops at  $p_j(a) = p_j^{**}$ . In the former case, we have  $Q_j^d > Q_j^s$  or  $Q_j^d < Q_j^s$ . Suppose  $Q_j^d > Q_j^s$ ; at  $p_j(a) = p_j^{**}$  buyers are rationed and since  $Q_j^s(a, p_j) < Q_j^s(a, p'_j) = Q_j^s$  for every  $p_j < p'_j \leq p_j^{**}$ , no additional trades can be matched at a tentative market price lower than  $p_j^{**}$ . Similarly, if  $Q_j^d < Q_j^s$ , at  $p_j(a) = p_j^{**}$  the sellers are rationed, and since, for every  $p_j \leq p_j^{**}$ ,  $Q_j^d(a, p) = Q_j^d(a, p_j^{**}) = Q_j^d$ , no additional trades can be matched at a tentative market price lower than  $p_j^{**}$ . Therefore, even in the case of  $Q_j^d \neq Q_j^s$ , **Algorithm 3.1** stops at  $p_j(a) = p_j^{**}$ .

**Point 2.** If buyers pay the market price, **Proposition 3.12** ensures that, for every  $a \in \text{proj}_A(R \cap W^\pi)$ ,

and every  $f \in F$ ,

$$x_{fj}^{**} > \theta_{fj} \implies a_{fj1} \geq x_{fj}^{**} - \theta_{fj}, a_{fj2} \geq p_{fj}^{**}$$

$$x_{fj}^{**} < \theta_{fj} \implies a_{fj3} \geq \theta_{fj} - x_{fj}^{**}, a_{fj4} \leq p_{fj}^{**}$$

If,  $\forall f \in F$ ,  $a_{fj2} = p_{fj}^{**}$ ,

$$Q_j^d(a, p) = \begin{cases} 0 & p > p_j^{**} \\ Q_j^d \geq \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj}) & p \leq p_{fj}^{**}, \end{cases}$$

and there exists  $p'_j \leq p_{fj}^{**}$  such that  $Q_j^s(a, p'_j) = Q_j^s \geq \sum_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**})$ . Hence, by the proof of **Point 1**, we conclude that there exists  $a \in \text{proj}_A(R \cap W^\pi)$  such that  $p_j(a) = p_j^{**}$ . Therefore, to complete the proof of this point, we must show that there exists  $a \in \text{proj}_A(R \cap W^\pi)$  such that  $p_j(a) > p_j^{**}$ . In this regard, consistent with **Proposition 3.12**, consider  $a \in \text{proj}_A(R \cap W^\pi)$  such that, for every  $f \in F$ ,

$$x_{fj}^{**} > \theta_{fj} \implies a_{fj1} = x_{fj}^{**} - \theta_{fj}, a_{fj2} = p_{fj}^{**} + \epsilon$$

$$x_{fj}^{**} = \theta_{fj} \implies a_{fj3} = \theta_{fj} - x_{fj}^{**}, a_{fj4} \leq p_{fj}^{**},$$

where  $\epsilon \in \{\epsilon \in \mathbb{R}_+ : p_j^{**} + \epsilon \leq 1\}$ . The maximum bid price of buyers is clearly  $p_{fj} = p_j^{**} + \epsilon$ , and, at the same price level, the demand for the asset  $j$  is  $Q_j^d(a, p^{**}) = \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj})$ . The minimum ask price is clearly  $p_{fj}^{**}$ , and the corresponding supply of the asset is  $Q_j^s(a, p^{**}) = -\sum_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**})$ . Consequently, we initiate **Algorithm 3.1** at  $p_j(a) = p_j^{**} + \epsilon$ . By condition ii) in the definition of WE,

$$\sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj}) + \sum_{\{f \in F: x_{fj}^{**} < \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**}) = 0,$$

and, consequently,  $Q_j^d(a, p_j^{**}) = Q_j^s(a, p_j^{**})$ . Since, for every  $p_j \in [p_j^{**}, p_j^{**} + \epsilon]$  we have  $Q_j^d(a, p_j^{**}) = Q_j^d(a, p_j)$ , and, for every  $p_j < p_j^{**}$ ,  $Q_j^s(a, p_j) = 0$ , the execution of **Algorithm 3.1** stops at  $p_j(a) = p_j^{**} + \epsilon$ , and there are no rationed buyers or sellers of  $j$ .  $\square$

Unlike the case in which buyers pay the price they quote, rationality and mutual  $\pi$ -belief in Walrasian markets are not sufficient to ensure competitive prices when buyers pay the market price. As a consequence of over-bid orders, asset prices may be arbitrarily higher than their corresponding Walrasian levels, even if players are rational and  $\pi$ -believe in Walrasian markets. This is not the only unpleasant consequence of weakly dominated actions. In fact, as we show in the next proposition,

at  $R \cap W^\pi$  asset prices may be competitive but asset allocation may not be Walrasian, and if we drop condition TR, asset prices may also fall below their corresponding Walrasian equilibrium levels (cf. point 2 of **Proposition 3.14**).

**PROPOSITION 3.14.** *Consider an economy  $\mathcal{E} = \langle S, F, (\theta_f, U_f)_{f \in F} \rangle$  and a trading game*

$$\Gamma = \langle F, S, (A_f, U_f, g_f^P)_{f \in F} \rangle,$$

*equipped with the corresponding universal epistemic structure  $\mathcal{T} = \langle F, (S, A_{-f}, T_f, \eta_f)_{f \in F} \rangle$ . Suppose that, given a profile  $\pi \in \Delta(S)^F$ , there exists a unique and interior WE  $(x^{**}, p^{**})$  of  $\mathcal{E}$ .*

- (1) *Regardless of the payment rule of the game,  $\exists a \in \text{proj}_A(R \cap W^\pi)$  such that  $j \in J \setminus \{0\}$ ,  $p_j(a) = p_j^{**}$  and  $g_{f,j}(a_f, a_{-f}) \neq x_{f,j}^{**}$  for at least a pair  $(f, j) \in F \times J \setminus \{0\}$ .*
- (2) *Regardless of the payment rule of the game, if  $\exists i \in F$  is such that  $\theta_i = x_i^{**}$ , for every  $p \in \times_{j \in J \setminus \{0\}} (0, p_j^{**})$ , there exists  $a \in \text{proj}_A(R \cap W^\pi)$  such that for every  $j \in J \setminus \{0\}$ ,  $p_j(a) = p_j < p_j^{**}$ .*

**PROOF.** Deferred to the **Appendix** of this chapter. □

In pay-as-clear auctions, there is no natural way to prevent the possibility that, for a risky asset  $j$ , all buyers decide to submit a limit price above  $p_j^{**}$ . In particular, imposing the additional requirement that every  $f$  believes in the event  $R_{-f} \cap W_{-f}^\pi$  would be useless. Although this result may look a defeat at first, we believe that it provides additional insights on real-world financial markets, reconciling long-standing opposite views on their state of "equilibrium" or "efficiency". Our results show that in pay-as-clear auctions with limit orders and price priority, even if investors believe in Walrasian markets, asset prices may be arbitrary high as a consequence of "incautious" buy orders, such as market orders. From a normative perspective, this result suggests that pay-as-bid auctions are more likely to support competitive prices and that market orders may exacerbate the risk of rational bubbles in pay-as-clear auctions.

## 6. Conclusion

In this chapter, we extended Dubey's (1982) model of an auction with limit orders and price priority to the case of trading in financial assets with digital payments contingent on the realization of an exogenous state of nature. To ensure consistency between pay-as-bid (i.e. buyers pay the price they quote) and pay-as-clear auctions (i.e. buyers pay the market price), we considered a

trading protocol in which the market price of every asset is determined as the bid price of the lowest filled buyer. For simplicity, we also assumed common knowledge of the game being played as well as symmetric information on the state of nature. Without the need to impose ad-hoc penalty functions in the players' utilities, we showed that, given a profile of priors  $\pi$  on the state of nature, an interior Walrasian equilibrium (WE) of the corresponding market economy is always supported by a strategic equilibrium (SE) of the game, regardless of its specific payment rule. However, every profile of actions is rationalizable and  $\Delta^\pi$ -rationalizable, where by  $\Delta^\pi$ -rationalizability we mean rationalizability directed by a profile of priors  $\pi \in \Delta(S)^F$  on the state of nature (cf. Chapter 8 in Battigalli et al., 2024). While striking at first, this result follows from the trivial observation that, if a player is sure that the others have zero demand and zero supply (i.e., they do not trade in risky assets), then any action has the same expected payoff. For this reason, we fixed an arbitrary profile  $\pi \in \Delta(S)^F$ , and, considering the case of an economy with a unique WE associated to  $\pi$ , we restricted our attention to an epistemic state consistent with the idea of price-taking behavior in GET, although not necessarily implying knowledge or belief in a SE supporting the unique WE associated to  $\pi$ . We show that this state occurs if every player: i) forms a conjecture independent from the state of nature, ii) believes that in her absence from the market, assets prices would be competitive, and iii) believes that there are sufficient orders, posted by her opponents, to implement her Walrasian asset allocation at competitive limit prices. Together, these conditions provide the definition of mutual  $\pi$ -beliefs in Walrasian markets, and we denote with  $W^\pi$  the action-type pairs consistent with it. Intuitively, one may expect that, if players are rational, imposing mutual  $\pi$ -beliefs in Walrasian markets leads to an SE supporting the unique WE of the economy. Instead, we show that this is not generally the case, and the possibility of competitive asset prices at  $R \cap W^\pi$  depends on the payment rule of the game and the initial distribution of assets. In this regard, we show that for every player  $f \in F$ , every action-type pair in  $R_f \cap W_f^\pi$  induces a push-forward probability measure attributing probability one to implementing her Walrasian allocation and to observe competitive asset prices. In other words, a rational player  $\pi$ -believing in Walrasian markets takes an action that, given her beliefs, implies the certainty of implementing a Walrasian asset allocation. However, for every player, we observe that there are more types (i.e. hierarchies of beliefs) consistent with  $\pi$ -belief in Walrasian markets, and some of these types may support incautious rational behavior, that is, weakly dominated actions. Assuming that no player is endowed with the Walrasian allocation of any risky asset (condition TR), we show that, in pay-as-bid auctions (i.e., when buyers pay the price they quote), assets are

priced competitively at  $R \cap W^\pi$ . However, players do not necessarily implement their Walrasian asset allocations, as a result of over-subscription orders, which is an example of weakly dominated actions. Under the same conditions, in pay-as-clear auctions (i.e. when buyers pay the market price), assets are priced at or above their corresponding Walrasian equilibrium prices. In particular, the presence of over-bid orders, which is another example of weakly dominated actions, may support arbitrary high asset prices. Several considerations can be made in light of these theoretical results, which we discuss below.

*Mutual  $\pi$ -belief in Walrasian markets: price conjectures and strategic equilibrium.* A price conjecture is defined as the push-forward measure of (the type of) a player over the possible risky asset prices obtained from her first-order beliefs. By submitting a penny-jump order, at  $W^\pi$ , the buyer of a risky asset is certain that the market price of the same asset will be equal to its bid, which, by definition of  $W^\pi$  and of a penny-jump, must be higher than the WE price. In other words, at  $W^\pi$ , players do not necessarily believe that assets will be priced competitively. As we showed in **Section 5**, this is instead a necessary consequence of imposing the additional assumption of rational behavior on top of that of mutual  $\pi$ -belief in Walrasian markets. Nevertheless, these beliefs may be wrong. In fact, if condition TR does not hold, or buyers pay the market price, there is no guarantee that assets will be priced competitively at  $R \cap W^\pi$ . It follows that, at  $W^\pi$ , players do not necessarily hold correct beliefs about the market price of any risky asset. In addition, it is important to observe that, at  $W^\pi$ , (the type of) a player does not necessarily believe that her opponents are behaving as in a SE consistent with the unique WE of the economy. In fact, the notion of mutual  $\pi$ -belief in Walrasian markets does not restrict (the type) of a player to necessarily believe that her opponents are submitting SE actions.

*Absence of arbitrage opportunities, existence of a stochastic discount factor and relation to investment games.* By standard results in asset pricing theory, given our assumptions on preferences, the price vector associated with an interior WE is strictly positive and does not allow for arbitrage opportunities. Suppose condition TR holds, that is, no player is endowed with the Walrasian allocation of any risky asset. In a pay-as-bid auction, although asset allocations may not be Walrasian, assets are always priced competitively at  $R \cap W^\pi$ . Hence, a stochastic discount factor (SDF) exists due to the lack of arbitrage opportunities and we can still use WE allocations to obtain the marginal utility of a representative investor and the SDF. In contrast, in a pay-as-clear auction, asset prices may be not Walrasian and allow for arbitrage opportunities at  $R \cap W^\pi$ . However, it is important

to observe that arbitrage opportunities exist only *ex-post*, that is, after players have submitted their orders, and, since the action space of a trading game is bounded, they cannot generate unbounded profits. In addition, since at  $R \cap W^\pi$  every player chooses an action that, according to her own beliefs, ensures that assets will be competitively priced, we can also conclude that at  $R \cap W^\pi$  every player is certain that she cannot manipulate prices to induce a profitable arbitrage opportunity. As an application of the results obtained in this chapter, under the assumptions of a common prior, complete markets, suitable quadratic utility functions, and  $\mathcal{P} = \text{pay} - \text{as} - \text{bid}$ , we can provide a epistemic game theoretic motivation to the assumption of a positive linear SDF in investment games (cf. **Section 2.1.3** of **Chapter 2**) with a finite set of states of nature (i.e. a finite set of values for the future profitability of each firm).

*Rationality and the states of the market.* In the class of auctions with limit orders that we considered, every profile of orders is rationalizable. This result carries the unfortunate conclusion that rationality is not a testable hypothesis, as every market outcome is the result of justifiable actions. However, in a trading game, rationality simply means self-interested behavior, which is a quite agreeable hypothesis on financial markets. Since rationality allows for any market outcome, a repeated observation of assets prices consistent with their fundamentals may be interpreted as a natural tendency of traders to form beliefs supporting equilibrium outcomes. In this regard, we observe contrasting empirical evidence in the asset pricing literature for stock markets, of which a notable example is the *Samuelson dictum* examined in Jung and Shiller (2007).<sup>16</sup> Since most of these studies consider closing prices to compute returns, which are usually determined in pay-as-clear auctions with limit orders and price priority, our results provide a possible explanation under the additional assumptions of a common prior on the state of nature and complete markets, which allow to identify competitive equilibrium prices as "commonly agreed" fundamental values. Firstly, we note the coexistence of two alternative epistemic states, one in which traders try to take speculative positions on the book and one in which, instead, they decide to buy or sell an asset depending on its fundamental value. The latter is characterized by the notion of mutual  $\pi$ -belief in Walrasian markets. Secondly, in the case of rational traders that mutually  $\pi$ -believe in Walrasian markets, we can observe either fundamental (i.e., Walrasian equilibrium) prices or asset pricing bubbles, that is, situations in which one or more assets are priced above its fundamental price level. In other

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<sup>16</sup>In short, Jung and Shiller (2007) observe that the cross-section of returns tend to be consistent with equilibrium asset pricing (micro-efficiency), while there is a substantial inconsistency at macro-level, that is, in the time series of aggregate returns (i.e. index returns)

words, in pay-as-clear auctions, at  $R \cap W^\pi$  we may observe the coexistence of two market states, one in which assets are priced competitively and one in which one or more assets experience rational bubbles. In light of these considerations, we believe that our theoretical framework may provide guidance to refine empirical methods in asset pricing, particularly models of stock market multiples such as Tobin Q. This may be part of our future research agenda.

*No-trade and heterogeneous priors.* A general version of the no-trade theorem of Milgrom and Stockey (1982) states that, in a Bayesian game, if the players' endowments are Pareto efficient and there is a common prior on the state of nature, every Bayesian Nash Equilibrium in which trading occurs is payoff equivalent to one in which players do no trade with each other. In this regard, condition TR, that is, no player is endowed with the Walrasian allocation of any risky asset, serves to avoid outcomes consistent with  $R \cap W^\pi$  and either no-trading or the presence of out-of-market orders. In other words, a no-trade result is obtained when the players' endowments constitute a WE allocation.

We also note that, contrary to Aumann (1976), we allow players to have different priors on the state of nature. In such a case, the prior of each player could be interpreted as a mixture of possible data generating processes, consistent with the representation theorem discussed in **Section 1.1.6**. To clarify this point, we can imagine a room full of portfolio managers, each with different opinions on the consequence of some exogenous event (e.g; a new trade policy, the possibility of a war, etc.). Although they might agree that such an event can be attributed to a hidden mechanism, they might not agree on its specific functional form or parameterization. To the best of our knowledge, this state of disagreement is common in the investment industry. We also stress that we do not need to require that the players priors on the state of nature be transparent at  $W^\pi$ . However, this is instead necessary to establish, from an epistemic perspective, that a SE is a special case of Bayesian equilibrium. By the assumption of a universal type structure, for every  $\pi \in \Delta(S)^F$ , common belief in  $D^\pi$  is always possible at  $W^\pi$ , and, therefore, at  $R \cap W^\pi$ .

*Penalty functions, cash-in-advance constraints and wash sales.* In Dubey (1982), a player may borrow any amount of money, just as in our model. Since money bears no utility as in Dubey (1982), a reduced-form penalty function is imposed. Otherwise, the setting would be equivalent to one in which a player can borrow as much as she wants from a central clearing authority, without ever repaying such debt. In the trading game considered, money is equivalent to the allocation of the risk-free asset, which positively affects the utility of every player. Hence, we can directly assume

the same preferences for the player as in the GE model, since there is no need to impose a penalty function to avoid excessive borrowing. Our modeling strategy is close to that of Shubik (1977), except that we do not need to impose a cash-in-advance constraint. The role of a cash-in-advance constraint in Shubik (1977) is that of ruling out buy orders of arbitrary large size. We achieve the same objective by limiting the size of a buy order for a given asset to the corresponding amount in circulation, and requiring bid and ask prices to lie within the interval  $[0, 1]$ , which is without particular loss of generality given that risky assets are digital contingent claims. We believe that our approach is more realistic since traders do not necessarily face cash-in-advance constraints, whereas trading venues and market authorities generally impose limits on maximum trade sizes and limit prices. Going a bit more into the details of why a penalty function is not required in our setting, observe that, in the case a player spends more than her original amount of money plus the proceeds from sales (if any), the overdraft in which she incurs corresponds to a negative allocation of the risk-free asset. Since unbounded allocations are ruled out by construction and there are no redundant assets, the strict concavity of players' utility function is sufficient to rule out over-borrowing schemes. Another notable difference from Dubey (1982) is the absence of wash sales, that is, we do not allow a player to be simultaneously a buyer and a seller of the same asset. We avoid wash sales to simplify the epistemic analysis of the game, but we believe that this is not a limitation of our model. In fact, the simultaneous placement of buy and sell orders by non-dealers is often considered as a market manipulation practice, and, as such, forbidden by the majority of market authorities.

*On the realism of the trading protocol.* Most, if not all, stock exchanges adopt a refined version of our trading protocol to determine the official opening and closing price of listed securities in pay-as-clear auctions. However, there are some aspects that are worth mentioning. First, the majority of stock exchanges have introduced rules to prevent excessive price swings. Our model takes into account these provisions implicitly, by limiting the maximum bid price of every traded asset. However, although our trading protocol is designed to maximize the traded volume and ensure strict price priority in order execution, the determination of the market price is not fully consistent with the practice of most stock exchanges which, instead of considering the bid price of lowest active buyer, typically consists in the application of different rules depending on the direction of the demand-supply imbalance. In this regard, it is interesting to analyze the alternative case in which the market price is determined as the arithmetic average of the bid price of the lowest active buyer and the ask price of the highest active seller, the so-called *mid-price*. Suppose that rationality and

mutual  $\pi$ -belief in Walrasian markets holds. It is immediate to verify that **Propositions 3.11-12** remain valid under the mid-price rule. Having said that, consider the case of a single risky asset and Bob, Serena and Simon being the players of the game. Suppose Bob's endowment of the risky asset is less than the corresponding Walrasian allocation and that the opposite is true for Serena and Simon, and that all the three players are rational and  $\pi$ -believe in Walrasian markets. Let  $p^{**}$  be the Walrasian price of the risky asset; by definition on mutual  $\pi$ -belief in Walrasian markets, Bob believes Serena and Simon to be sellers with ask price equal to  $p^{**}$  and supply in excess of what he needs to implement his Walrasian asset allocation. Likewise, Serena and Simon believe Bob to be a buyer with bid price equal to  $p^{**}$  and demand in excess of what they need to implement their respective Walrasian asset allocations. Since Bob is the only buyer of the risky asset, he will bid  $p^{**}$ , to avoid increasing the market price and lower his allocation of risk-free asset (i.e. "cash"). However, both Serena and Simon may submit orders with ask price strictly lower than  $p^{**}$ , since, by definition of mutual  $\pi$ -belief in Walrasian markets, Serena believes Simon to ask  $p^{**}$  and vice-versa. As a result, under the *mid-price* rule, the market price of the risky asset could be lower than the corresponding Walrasian level.<sup>17</sup> This example shows that, in pay-as-clear auctions, a difference convention to determine market prices may support a broader range of asset prices consistent with rationality and mutual  $\pi$ -belief in Walrasian markets.

*Payment rules and pricing efficiency.* Recently, Woodward and Pycia (2025) have made a case in favor of pay-as-bid auctions. Considering a Bayesian trading game with asymmetric information, they prove the existence of a unique Bayesian Nash equilibrium that ensures greater price transparency and more efficient allocations. While we assume symmetric information, we show that a similar result is obtained when we allow players to be uncertain on their respective opponents' strategies, differently than in an "equilibrium" context. In particular, in pay-as-clear auctions, rationality and mutual  $\pi$ -belief in Walrasian markets is not sufficient to ensure competitive prices. Besides, as we argued in the previous paragraph, if the market price is no longer determined as the limit price of the lowest active buyer, rationality and mutual  $\pi$ -belief in Walrasian markets may result also in risky assets being priced above and below their respective WE levels. From a normative perspective, our results provide additional support to the view that pay-as-bid auctions are more likely to result in Walrasian market prices.

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<sup>17</sup>It is also worth noting that if there had been instead a unique seller of the asset, under-ask orders would have been clearly ruled out.

*Extensions: asymmetric information and lexicographic beliefs.* Two evident limitations are the assumptions of common knowledge of the game and symmetric information on the state of nature. Extending the analysis to the inclusion of asymmetric information is challenging and might be an object of future research. Our results also call for investigating the role of lexicographic beliefs (Blume et al. 1991a, 1991b), as the possibility of non-Walrasian asset prices is inherently related to the presence of weakly dominated actions at  $W^\pi$ , which may be excluded restricting beliefs to *assumptions*; we refer to Brandenburger et al. (2008) and Dekel et al. (2016) for the relevant literature on *admissibility* and *assumptions*. Our preliminary investigation of the topic shows that, if we equip a trading game with a universal lexicographic type structure as in Cantonini and De Vito (2014, 2017), and restrict players' types to those *assuming*, at least topologically (cf. TWD-assumption in Dekel et al., 2016), that markets are deep, rationality and mutual  $\pi$ -assumption in Walrasian markets supports competitive asset prices regardless of the payment rule of the game.

### Appendix to Chapter 3

#### Proof of Proposition 3.14.

**PROOF. Point 1.** Let  $k \in J \setminus \{0\}$  be a risky asset, and  $i \in F$  be a player such that  $x_{ik}^{**} > \theta_{ik}$ . Consider the profile of actions  $a$  such that, for every  $f \in F \setminus \{i\}$ ,  $a_f = a_f^{**}$ , and

$$a_i : J \setminus \{0\} \mapsto \mathbb{R}^{[4]}$$

$$j \mapsto \begin{cases} a_j^{**} & j \in J_{-\{0,k\}} \\ (x_{ij}^{**} - \theta_{ij} + \epsilon, 0, 0) & j = k, \end{cases}$$

where  $\epsilon$  is a sufficiently small positive number, in the sense that  $x_{ij}^{**} - \theta_{ij} + \epsilon \in A_{ik1}$ . By **Proposition 3.12**, we have  $a \in \text{proj}_A(R \cap W^\pi)$ , regardless of the specific payment rule of the game considered, and by the proof of **Proposition 3.13**, we also have for every  $j \in J \setminus \{0\}$ ,  $p_j(a) = p_j^{**}$ . For every risky asset  $j \in J_{-\{0,k\}}$ , we have, in addition,  $Q_k^d(a, p_j^{**}) = Q_j^s(a, p_j^{**})$ , and consequently,  $g_{J_{-\{0,k\}}}(a) = \left(x_{fj}^{**}\right)_{F \times J_{-\{0,k\}}}$ . In contrast, for asset  $k$  we have  $Q_k^d(a, p_j^{**}) - Q_j^s(a, p_j^{**}) = \epsilon$ , and, consequently, the buyers of  $k$  are rationed. As a result,  $g_k(a) \neq \left(x_{fk}^{**}\right)_{f \in F}$ .

**Point 2.** Let  $a \in A$  be a profile of actions such that, for every  $f \in F \setminus \{i\}$ , and every  $j \in J \setminus \{0\}$ ,

- $a_{fj1} = (x_j^{**} - \theta_{fj}) \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$ ,
- $a_{fj3} > (\theta_{fj} - x_j^{**}) \mathbb{I}(x_{fj}^{**} < \theta_{fj})$ ,

- $a_{fj2} = p_j^{**} \mathbb{I}(x_{fj}^{**} \geq \theta_{fj})$ ,
- $a_{fj4} = p'_j \mathbb{I}(x_{fj}^{**} < \theta_{fj})$ ,

where  $0 < p'_j < p_j^{**}$ , while for the player  $i \in F$ , and for every  $j \in J \setminus \{0\}$ ,

- $a_{ij1} \in A_{ij1} \setminus \{0\}$ ;
- $a_{ij2} = p'_j$ ;
- $a_{ij3} = a_{ij4} = 0$ .

By **Proposition 3.12**, it is immediate to verify that  $a_{-i} \in \text{proj}_{A_{-i}}(R_{-i} \cap W_{-i}^\pi)$ . Regarding the player  $i$ , it is sufficient to consider a type  $t_i$  that is certain that, for every  $j \in J \setminus \{0\}$ ,  $\sum_{\{f \in F\}} (x_{fj}^{**} - \theta_{fj}) = 0$ , which clearly exists by the definition of  $L_{-i}^{p_{-i}^{**}} \cap D_{-i}$  and condition 2) in the definition of a WE of  $\mathcal{E}$ . In addition, for every  $j \in J \setminus \{0\}$ , we have

$$Q_j^d(a, p) = \begin{cases} 0 & p > p_j^{**} \\ \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj}) & p = p_j^{**} \\ Q_j^d > \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj}) & p = a_{ij2} = p'_j < p_j^{**} \end{cases}$$

and,

$$Q_k^s(a, p) = \begin{cases} Q_j^s > \sum_{\{f \in F: x_{fj}^{**} > \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**}) & p \geq a_{ij2} \\ 0 & p < a_{ij2}. \end{cases}$$

Hence, for every risky asset  $j \in J \setminus \{0\}$ , the highest buyers of  $j$  are condensed to  $p = p_j^{**}$ , demanding a quantity equal to

$$\sum_{\{f \in F \setminus \{i\}: x_{fj}^{**} > \theta_{fj}\}} (x_{fj}^{**} - \theta_{fj})$$

of  $j$ , and the only buyer with a bid price below  $p_j^{**}$  is the player  $i$ , who demands a quantity  $a_{ij1} > 0$ .

At the same time, for every risky asset  $j \in J \setminus \{0\}$ , all sellers of  $j$  are asking a price equal to  $a_{ij2} < p_j^{**}$ , for a total of

$$Q_j^s > \sum_{\{f \in F \setminus \{i\}: x_{fj}^{**} > \theta_{fj}\}} (\theta_{fj} - x_{fj}^{**})$$

units of  $j$ . Therefore, consistent with **Algorithm 3.1**, for every  $j \in J \setminus \{0\}$ , we initialize the search for  $p_j(a)$  at  $p_j^{**}$ . From condition 2.) in the definition of a WE, we know that

$$\sum_{\{f \in F: x_{fk}^{**} > \theta_{fk}\}} (x_{fk}^{**} - \theta_{fk}) = \sum_{\{f \in F: x_{fk}^{**} < \theta_{fk}\}} (\theta_{fk} - x_{fk}^{**}),$$

and consequently  $Q_j^s > Q_j^d(a, p_j^{**})$ . Hence, the highest bidding buyers can serve only a fraction of the quantity supplied by the sellers of the asset  $j$ . Thus, consistent with **Algorithm 3.1**, we proceed with the search for additional buyers to match the seller's residual orders. As noted above, the only additional buyer is the player  $i$ , which has a bid price equal to the ask price of all sellers. Hence, the player  $i$  will act as the marginal buyer of the asset  $j$ , and, consequently,  $p_j(a) = a_{ij2} < p_j^{**}$ .  $\square$

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